Covariance Control of Discrete-Time Gaussian Linear Systems Using Affine Disturbance Feedback Control Policies

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Abstract— **In this paper, we present a new control policy parametrization for the finite-horizon covariance steering problem for discrete-time Gaussian linear systems (DTGLS) via which we can reduce the latter stochastic optimal control problem to a tractable optimization problem. We consider two different formulations of the covariance steering problem, one with hard terminal LMI constraints and another one with soft terminal constraints in the form of a terminal cost which corresponds to the squared Wasserstein distance between the actual terminal state (Gaussian) distribution and the desired one. We propose a solution approach that relies on the affine disturbance feedback parametrization for both problem formulations. We show that this particular parametrization allows us to reduce the hard-constrained covariance steering problem into a semi-definite program (SDP) and the soft-constrained covariance steering problem into a difference of convex functions program (DCP). Finally, we show the advantages of our approach over other covariance steering algorithms in terms of computational complexity and computation time by means of theoretical analysis and numerical simulations.**

I. Introduction

In this work, we consider the problem of characterizing computationally tractable control policies that will steer the mean and covariance of the terminal state of a discrete-time linear stochastic system "close" to respective goal quantities. This type of problems are referred to as *covariance steering* (or *covariance control*) in the literature of stochastic control. We will consider two variations of the covariance steering problem. The goal in the first problem formulation is to steer the mean of the terminal state to a prescribed vector and have the terminal state covariance satisfy a certain LMI-type constraint; we refer to this problem as the hard constrained covariance steering (HCCS) problem. In the second formulation of the covariance steering problem, we seek for a control policy that will minimize the distance between the terminal (Gaussian) distribution of the state and a desired goal (Gaussian) distribution measured in terms of the (squared) Wasserstein distance between the two distributions while satisfying the probabilistic input and state constraints; we refer to this problem as the soft-constrained covariance steering (SCCS) problem.

Literature Review: Infinite-horizon covariance control problems for both continuous-time and discrete-time stochastic linear systems have been addressed in [1]–[3]. Finite-horizon covariance steering problems have recently received significant attention for both the continuous-time case [4], [5] and the discrete-time case [6]–[8]. Covariance control problems for the partial information case have been studied in [9]–[12]. Approaches that consider soft-constrained formulations of the covariance steering problem (based on appropriate terminal costs) can be found in [13], [14] and [15], in which the terminal cost is defined in terms of, respectively, the squared Wasserstein distance and the squared $\mathcal{L}_2$ spatial norm between the goal distribution and the distribution attained by the terminal state.

In our previous work, we have addressed covariance steering problems for discrete-time stochastic (Gaussian) linear systems under both full state and partial state information based on techniques from convex optimization [6], [8]–[10] and difference of convex functions programming [14]. In these references, the reduction of the stochastic optimal control problems to tractable optimization problems relied on the use of the state history feedback control parametrization [16]. By using the state feedback control policy parametrization, one can reduce the covariance steering problem into a convex program via a bilinear transformation. [17]. Because, the whole history of states is used in this method, the dimension of the resulting optimization problem can be prohibitive for problems with long time horizons.

Main Contribution: In this paper, we present a new solution approach to the covariance steering problem (for both the hard-constrained and the soft-constrained problem formulations) in the case of full state information. Our approach is based on a control policy parameterization which can be interpreted as a stochastic version of the affine disturbance feedback control parametrization [18] tailored to the covariance steering problem. We show that by using this particular control policy parametrization, one can directly reduce the HCCS problem into a convex optimization problem and the SCCS problem into a difference of convex functions program, whose decision variables are essentially the controller parameters. This is in sharp contrast with the state history feedback control parametrization which requires significant pre-processing in order to associate the controller parameters (decision variables of the stochastic optimal control problem) with the decisions variables of the corresponding (finite-dimensional) optimization program by means of bilinear transformations. The fact that in our proposed approach the decision variables of the control and optimization problems are in direct correspondence also allows us to consider modified control policies which are based on truncated histories of the disturbances which have acted upon the system. Using these modified policies lead to more computationally tractable optimization problems

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(problems with fewer decision variables).

Structure of the paper: The rest of the paper is organized as follows. In Section II, we formulate the two variations of the covariance steering problem and introduce our proposed policy parametrization. The reduction of the stochastic optimal control problem into a convex semidefinite program (for the HCCS problem) and a difference of convex functions program (for the SCCS problem) are described in Sections III and IV, respectively. In Section V, we present numerical simulations and finally, we conclude the paper in Section VI.

II. Problem Formulation

A. Notation

We denote by $\mathbb{R}^n$ the set of n-dimensional real vectors and by $\mathbb{Z}$ the set of integers. We write $\mathbb{E}[\cdot]$ to denote the expectation functional. $[\tau_1, \tau_2]$ denotes the set $\{\tau_1, \tau_1 + 1, \ldots, \tau_2\}$. Given a finite sequence of vectors $\mathscr{X}$, we denote by $\mathrm{vertcat}(\mathscr{X})$ the concatenation of its vectors, that is, $\mathrm{vertcat}(\mathscr{X}) := [x_1^{\mathrm{T}}, \ldots, x_m^{\mathrm{T}}]^{\mathrm{T}}$. Given a matrix $\mathbf{A}$, we denote its Frobenius norm by $\|\mathbf{A}\|_F$ and its nuclear norm by $\|\mathbf{A}\|_*$square matrix $\mathbf{A}$; if in addition $\mathbf{A}$ is a square matrix, we denote its trace by $\mathrm{tr}(\mathbf{A})$. We write $\mathbf{0}$ and I_n to denote the zero matrix (of suitable dimensions) and the $n\times n$ identity matrix, respectively. We will denote the convex cone of $n\times n$ symmetric positive semi-definite (symmetric positive definite) matrices by $\mathbb{S}_n^+$ ($\mathbb{S}_n^{++}$). Given $A \in \mathbb{S}_n^+$, we denote by $A^{1/2} \in \mathbb{S}_n^+$ its (unique) square root, that is, $A^{1/2}A^{1/2} = A$. We write $\mathrm{bdiag}(A_1, \ldots, A_\ell)$ to denote the block diagonal matrix formed by the matrices A_i, $i \in \{1, \ldots, \ell\}$. Finally, we denote by μ_z and var_z the mean and the variance of a random vector z, respectively, that is, $\mu_z := \mathbb{E}[z]$ and $\mathrm{var}_z := \mathbb{E}[(z-\mu_z)(z-\mu_z)^{\mathrm{T}}] = \mathbb{E}[zz^{\mathrm{T}}] - \mu_z\mu_z^{\mathrm{T}}$.

B. Squared Wasserstein Distance

The Wasserstein distance defines a valid metric (i.e., satisfies all relevant axioms of a metric) in the space of probability distributions. Although, in general, it is not possible to find a closed-form expression for the Wasserstein distance between two arbitrary probability distributions, the Wasserstein distance between two Gaussian distributions admits a closed form expression [19]. In particular, given two multivariate Gaussian distributions $\mathcal{N}_1(\mu_1, \Sigma_1)$ and $\mathcal{N}_2(\mu_2, \Sigma_2)$, with $\mu_1, \mu_2 \in \mathbb{R}^n$ and $\Sigma_1, \Sigma_2 \in \mathbb{S}_n^{++}$, the squared Wasserstein distance between them is given as follows:

$$W^2(\mathcal{N}_1, \mathcal{N}_2) = \|\mu_1 - \mu_2\|_2^2 + \mathrm{tr}\left(\Sigma_1 + \Sigma_2 - 2\left(\Sigma_2^{1/2}\Sigma_1\Sigma_2^{1/2}\right)^{1/2}\right). \quad (1)$$

For more details, the reader may refer to [13], [14].

C. Problem Setup

We consider the following discrete-time stochastic linear system

$$x(t+1) = A(t)x(t) + B(t)u(t) + w(t), \quad (2a)$$
$$x(0) = x_0, \qquad x_0 \sim \mathcal{N}(\mu_0, \Sigma_0), \quad (2b)$$

for $t \in [0, T-1]_d$, where $\mu_0 \in \mathbb{R}^n$ and $\Sigma_0 \in \mathbb{S}_n^{++}$ are given. Let $\mathscr{X}_{0:t} := \{x(\tau) \in \mathbb{R}^n : \tau \in [0,t]_d\}$, for $t \in [0,T]_d$, $\mathscr{U}_{0:t} := \{u(\tau) \in \mathbb{R}^m : \tau \in [0,t]_d\}$, for $t \in [0, T-1]_d$, and $\mathscr{W}_{0:t} := \{w(\tau) \in \mathbb{R}^n : \tau \in [0,t]_d\}$, for $t \in [0, T-1]_d$. We assume that the noise process $\mathscr{W}_{0:t}$ corresponds to a sequence of independent and identically distributed normal random variables with

$$\mathbb{E}[w(t)] = \mathbf{0}, \qquad \mathbb{E}\left[w(t)w(t)^{\mathrm{T}}\right] = \delta(t,\tau)W, \quad (3)$$

for all $t, \tau \in [0, T-1]_d$, where $W \in \mathbb{S}_n^+$ and $\delta(t,\tau) := 1$, when $t = \tau$, and $\delta(t,\tau) := 0$, otherwise. Furthermore, x_0 is independent of $W_{0:T-1}$, that is,

$$\mathbb{E}\left[x_0 w(t)^{\mathrm{T}}\right] = \mathbf{0}, \qquad \mathbb{E}\left[w(t)x_0^{\mathrm{T}}\right] = \mathbf{0}, \quad (4)$$

for all $t \in [0, T-1]_d$.

Equation (2) can be written more compactly as follows:

$$\boldsymbol{x} = \mathbf{G}_{\boldsymbol{u}}\boldsymbol{u} + \mathbf{G}_{\boldsymbol{w}}\boldsymbol{w} + \mathbf{G}_0 x_0, \quad (5)$$

where $\boldsymbol{x} := \mathrm{vertcat}(\mathscr{X}_{0:T}) \in \mathbb{R}^{(T+1)n}$, $\boldsymbol{u} := \mathrm{vertcat}(\mathscr{U}_{0:T-1}) \in \mathbb{R}^{Tm}$ and $\boldsymbol{w} := \mathrm{vertcat}(\mathscr{W}_{0:T-1}) \in \mathbb{R}^{Tn}$. The exact expressions for $\mathbf{G_u}, \mathbf{G_w}$ and $\mathbf{G_0}$ are omitted due to space constraints. The reader can refer to [14] for further details. Additionally, we have

$$\mathbb{E}[\boldsymbol{w}] = \mathbf{0}, \quad \mathbb{E}\big[\boldsymbol{w}\boldsymbol{w}^{\mathrm{T}}\big] = \mathbf{W}, \quad (6)$$

where in light of (3)

$$\begin{aligned}\mathbf{W} &:= \mathbb{E}\big[\mathrm{bdiag}(w(0)w(0)^{\mathrm{T}}, \ldots, w(T-1)w(T-1)^{\mathrm{T}}\big] \\ &= \mathrm{bdiag}(W, \ldots, W).\end{aligned}$$

D. Affine Disturbance Feedback Controller Parametrization

Under the assumption of perfect state information, one can recover at each stage the disturbance terms that have acted upon the system at all previous stages. Thus, one can use all these past distrurbances to compute the control input that will be applied to the system at each stage. Next, we propose a modified version of the so-called affine disturbance feedback control policy parametrization, which we denote by $\kappa(t, W_{0:t-1}, x(0))$ which is defined as follows:

$$\kappa(t) = \begin{cases} \bar{u}(t) + L_t\left(x(0) - \mu_0\right) & \\ \quad + \sum_{\tau=0}^{t-1} K_{(t-1,\tau)} w(\tau) & \text{if } t \in [1, T-1]_d, \\ \bar{u}(0) + L_t\left(x(0) - \mu_0\right) & \text{if } t = 0, \end{cases} \quad (7)$$

where $L_t, K_{(t,\tau)} \in \mathbb{R}^{m\times n}$, $\forall t, \tau \in [0, T-1]_{\mathrm{d}}$.

It is worth noting that the parametrization we propose is different from the standard affine disturbance feedback parametrization due to the presence of the extra term $L_t(x(0) - \mu_0)$ for all $t \in [0, T-1]_d$. To understand the necessity of this extra term, one can think of a case in which the system is not acted upon any disturbances, that is, $w(t) = 0$ for all $t \in [0,T]_d$ (although there are no disturbances, the initial state is still uncertain), and also $L_t \equiv 0$. In this case, $\kappa(t) = baru(t)$, which means that there is not feedback terms to control the evolution of the state covariance.

Remark 1 To reduce the computational burden, one can truncate the disturbance history feedback policy (7) up to a desired number, that is, to use only a portion of the past disturbances that have acted upon the system to compute the control input. We denote by $\gamma \in [0, T]_{\mathrm{d}}$ a parameter that determines the length of the truncated history of disturbances such that the term $\sum_{\tau=0}^{t-1} K_{(t-1,\tau)} w(\tau)$ that appears in (7) is replaced by the term $\sum_{\tau=t-(1+\gamma)}^{t-1} K_{(t-1,\tau)} w(\tau)$. To solve the SCCS and HCCS problems based on the truncation of the disturbance history, the optimization (matrix) variable $\boldsymbol{\mathcal{K}}$ which is defined in (15) will have to be revised by setting the blocks $K_{(t-1,\tau)}$ equal to $\mathbf{0}$ for all $\tau < t-(1+\gamma)$.

E. Problem Formulation

Next, we provide the precise formulations for the two variants of the covariance steering problems based on the control policy parametrization given in (7).

Problem 1 (Hard Constrained Covariance Steering)**.** *Let $\mu_{\mathrm{d}}, \mu_0 \in \mathbb{R}^n$, $\Sigma_{\mathrm{d}} \in \mathbb{S}_n^{++}$, $\rho \in \mathbb{R}^+$ be given. Consider the system described by* (2)*. Then, find the collection of matrix gains $\mathscr{K} := \{K_{(t,\tau)},\ L_t \in \mathbb{R}^{m \times n} : (t,\tau) \in [0, T-2] \times [0, T-2],\ t \geq \tau\}$ and the sequence of vectors $\overline{\mathscr{U}} := \{\bar{u}(0), \ldots, \bar{u}(T-1)\}$ that minimize the following performance index:*

$$J_1(\mathscr{U}, \mathscr{K}) := \mathbb{E}\Bigg[\sum_{t=0}^{T-1} u(t)^{\mathrm{T}} u(t)\Bigg] \tag{8}$$

subject to the boundary condition on the terminal mean and covariance:

$$\mu_{x(T)} = \mu_{\mathrm{d}}, \tag{9}$$

$$\mathrm{var}_{x(T)} \preceq \Sigma_{\mathrm{d}}. \tag{10}$$

Remark 2 The objective function defined in (8) represents the expected value of the total control effort. The constraint in (9) dictates that the terminal state mean be equal to the desired mean and the constraint in (10) dictates that terminal state covariance be upper bounded, with respect to the Lowner partial order, by the desired covariance matrix.

Problem 2 (Soft Constrained Covariance Steering)**.** *Let $\mu_{\mathrm{f}}, \mu_0 \in \mathbb{R}^n$, $\Sigma_{\mathrm{f}}, \Sigma_0 \in \mathbb{S}_n^{++}$ and $\rho \in \mathbb{R}^+$ be given. Consider the system described by* (2)*. Then, find the collection of matrix gains $\mathscr{K} := \{K_{(t,\tau)},\ L_t \in \mathbb{R}^{m \times n} : (t,\tau) \in [0, T-2] \times [0, T-2],\ t \geq \tau\}$ and the sequence of vectors $\overline{\mathscr{U}} := \{\bar{u}(0), \ldots, \bar{u}(T-1)\}$ that minimize the following performance index:*

$$J_2(\overline{\mathscr{U}}, \mathscr{K}) := W_2^2(\mathcal{N}_{\mathrm{f}}, \mathcal{N}_{\mathrm{d}}), \tag{11}$$

subject to input constraint $C_{total}(\overline{\mathscr{U}}, \mathscr{K}) \leq 0$, where

$$C(\overline{\mathscr{U}}, \mathscr{K}) := \mathbb{E}\Big[\sum_{t=0}^{T-1} u(t)^{\mathrm{T}} u(t)\Big] - \rho^2, \tag{12}$$

$\mathcal{N}_{\mathrm{f}} = \mathcal{N}(\mu_{x(T)}, \mathrm{var}_{x(T)})$ and $\mathcal{N}_{\mathrm{d}} = \mathcal{N}(\mu_d, \Sigma_d)$ represent the Gaussian probability distribution of the terminal state at $t = T$ and the desired (goal) Gaussian probability distribution, respectively.

III. Reduction of the HCCS Problem into a Semidefinite Program

To reduce Problems 1 and 2 into tractable optimization problems, first we need to express the control input vector $\boldsymbol{u}$ in terms of the decision variables in (7). In particular, the concatenated control input vector can be expressed as follows:

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{\mathcal{L}}(x_0 - \mu_0) + \boldsymbol{\mathcal{K}} \boldsymbol{w}, \tag{13}$$

where $\boldsymbol{\mathcal{K}} = \left[\begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{K} & \mathbf{0} \end{smallmatrix}\right]$ and

$$\bar{\boldsymbol{u}} := [\bar{u}(0)^{\mathrm{T}}, \bar{u}(1)^{\mathrm{T}} \ldots, \bar{u}(T-1)^{\mathrm{T}}]^{\mathrm{T}}, \tag{14}$$

$$\boldsymbol{K} := \begin{bmatrix} K_{(0,0)} & \mathbf{0} & \cdots & \mathbf{0} \\ K_{(1,0)} & K_{(1,1)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ K_{(T-2,0)} & K_{(T-2,1)} & \cdots & K_{(T-2,T-2)} \end{bmatrix}, \tag{15}$$

$$\boldsymbol{\mathcal{L}} := [L(0)^{\mathrm{T}}, L(1)^{\mathrm{T}} \ldots, L(T-1)^{\mathrm{T}}]^{\mathrm{T}}. \tag{16}$$

By plugging equation (13) into equation (5), we get

$$\boldsymbol{x} = \mathbf{G_u} \bar{\boldsymbol{u}} + (\mathbf{G_w} + \mathbf{G_u} \boldsymbol{\mathcal{K}}) \boldsymbol{w} + \mathbf{G_0} x_0 + \mathbf{G_u} \boldsymbol{\mathcal{L}} \tilde{x}(0), \tag{17}$$

where $\tilde{x}(0) = x(0) - \mu_0$. Now, $x(t)$ can be expressed as $x(t) = \mathbf{P}_{t+1} \boldsymbol{x}$, where $\mathbf{P}_{t+1} := [\mathbf{0}, \ldots, I_n, \ldots, \mathbf{0}]$ is a block row vector whose t^{th} block is equal to I_n whereas all other blocks are equal to the zero matrix.

Next we provide analytical expressions for the mean and the variance of $\boldsymbol{x}$ and the state $x(t)$ for all $t \in [0, T]_d$.

Proposition 1. *The mean and the variance of the random vector $\boldsymbol{x}$ which satisfies equation* (17) *is given by:*

$$\mu_{\boldsymbol{x}} = \mathfrak{f}(\bar{\boldsymbol{u}}), \qquad \mathrm{var}_{\boldsymbol{x}} = \mathfrak{h}(\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}}), \tag{18}$$

where

$$\mathfrak{f}(\bar{\boldsymbol{u}}) := \mathbf{G_u} \bar{\boldsymbol{u}} + \mathbf{G_0} \mu_0, \tag{19a}$$

$$\begin{aligned} \mathfrak{h}(\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}}) := & (\mathbf{G_0} + \mathbf{G_u} \boldsymbol{\mathcal{L}}) \Sigma_0 (\mathbf{G_0} + \mathbf{G_u} \boldsymbol{\mathcal{L}})^{\mathrm{T}} \\ & + (\mathbf{G_w} + \mathbf{G_u} \boldsymbol{\mathcal{K}}) \mathbf{W} (\mathbf{G_w} + \mathbf{G_u} \boldsymbol{\mathcal{K}})^{\mathrm{T}}. \end{aligned} \tag{19b}$$

Furthermore, the mean and the variance of the state $x(t)$ are given by

$$\mu_{x(t)} = \mathbf{P}_{t+1} \mathfrak{f}(\bar{\boldsymbol{u}}) \tag{20a}$$

$$\mathrm{var}_{x(t)} = \mathbf{P}_{t+1} \mathfrak{h}(\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}}) \mathbf{P}_{t+1}^{\mathrm{T}}. \tag{20b}$$

The proofs of the main theoretical results of this paper are omitted due to space constraints and can be found in [20].

Next, we obtain an expression for the performance index of the HCCS (Problem 1) in terms of the decision variables $(\bar{\boldsymbol{u}}, \boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}})$.

Proposition 2. *The performance index $J_1(\overline{\mathscr{U}}, \mathscr{K})$ which is defined in* (8) *is equal to $\mathcal{J}_1(\bar{\boldsymbol{u}}, \boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}})$, where*

$$\mathcal{J}_1(\bar{\boldsymbol{u}}, \boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}}) := \bar{\boldsymbol{u}}^{\mathrm{T}} \bar{\boldsymbol{u}} + \mathrm{tr}(\boldsymbol{\mathcal{K}} \mathbf{W} \boldsymbol{\mathcal{K}}^{\mathrm{T}}) + \mathrm{tr}(\boldsymbol{\mathcal{L}} \Sigma_0 \boldsymbol{\mathcal{L}}^{\mathrm{T}}) \tag{21}$$

provided that the pairs of decision variables $(\overline{\mathscr{U}}, \mathscr{K})$ and $(\bar{\boldsymbol{u}}, \boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}})$ are related by (15)*. Furthermore, $\mathcal{J}_1(\bar{\boldsymbol{u}}, \boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{K}})$ is a convex function.*

The next proposition shows that terminal covariance constraint (10) can be written as a positive semidefinite constraint.

Proposition 3. *The positive semi-definite constraint* $\Sigma_{\mathrm{d}} \succeq \mathrm{var}_{x(T)}$ *is satisfied iff* $\boldsymbol{\mathcal{V}}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) \in \mathbb{S}_n^+$ *where*

$$\boldsymbol{\mathcal{V}}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) := \begin{bmatrix} \Sigma_{\mathrm{d}} & \boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) \\ \boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})^{\mathrm{T}} & I_n \end{bmatrix} \tag{22}$$

and $\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})$ *is defined as in* (23).

To show that the terminal covariance constraint in (10) can be written as the positive semidefinite constraint (22), let

$$\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) := \mathbf{P}_{T+1}\left[(\mathbf{G}_0 + \mathbf{G}_{\boldsymbol{u}}\boldsymbol{\mathcal{L}}) \quad (\mathbf{G}_{\boldsymbol{w}} + \mathbf{G}_{\boldsymbol{u}}\boldsymbol{\mathcal{K}})\right]\mathbf{R} \tag{23}$$

where $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \left[\begin{smallmatrix}\Sigma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}\end{smallmatrix}\right]$ and $\mathbf{P}_{T+1}\mathfrak{h}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\mathbf{P}_{T+1}^{\mathrm{T}} = \boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})^{\mathrm{T}}$.

Theorem 1. *Problem 1 is equivalent to the following semi-definite program:*

$$\min_{\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}} \quad \mathcal{J}_1(\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) \tag{24a}$$

$$\text{subject to} \quad \mathbf{P}_{T+1}\mathfrak{f}(\bar{\boldsymbol{u}}) = \mu_d \tag{24b}$$

$$\boldsymbol{\mathcal{V}}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) \in \mathbb{S}_n^+ \tag{24c}$$

Remark 3 Theorem 2 is a direct consequence of Propositions 1, 2 and 3. In view of this theorem, Problem 1 reduces into a semi-definite program.

IV. Reduction of the SCCS Problem into a Difference of Convex Functions Program

In this section, we associate the SCCS problem with a difference of convex functions program (DCP). In order to do that, we utilize the control policy parametrization in (7) and use the results from Section III.

By setting $\mu_1 = \mu_{x(T)}$ and $\Sigma_1 = \mathrm{var}_{x(T)}$, where $\mu_{x(T)}$ and $\Sigma_1 = \mathrm{var}_{x(T)}$ are defined in (20a) and (20b) for $t = T$, respectively, and also $\mu_2 = \mu_{\mathrm{f}}$ and $\Sigma_2 = \Sigma_{\mathrm{d}}$ into the expression of the squared Wasserstein distance given in (1), we obtain the following expression of the objective function (11) in terms of the new decision variables:

$$\begin{aligned}\mathcal{J}_3(\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) &:= \|\mathbf{P}_{T+1}\mathfrak{f}(\bar{\boldsymbol{u}}) - \mu_{\mathrm{d}}\|_2^2 \\ &+ \mathrm{tr}\left(\mathbf{P}_{T+1}\mathfrak{h}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\mathbf{P}_{T+1}^{\mathrm{T}} + \Sigma_{\mathrm{d}}\right) \\ &- 2\,\mathrm{tr}\left((\sqrt{\Sigma_{\mathrm{d}}}\mathbf{P}_{T+1}\mathfrak{h}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\mathbf{P}_{T+1}^{\mathrm{T}}\sqrt{\Sigma_{\mathrm{d}}})^{1/2}\right).\end{aligned} \tag{25}$$

To show that the function defined in (25) is a difference of two convex functions we can define the objective function as a function of $\bar{\boldsymbol{u}}$ and $\boldsymbol{\zeta}$, where $\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})$ is an affine function which is defined in (23). In view of (25) and (23), we define the new objective function as follows:

$$\begin{aligned}\tilde{\mathcal{J}}_3(\bar{\boldsymbol{u}},\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})) &:= \|\mathbf{P}_{T+1}\mathfrak{f}(\bar{\boldsymbol{u}}) - \mu_{\mathrm{d}}\|_2^2 \\ &+ \|\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\|_F^2 + \mathrm{tr}(\Sigma_{\mathrm{d}}) - 2\|\sqrt{\Sigma_{\mathrm{d}}}\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\|_*.\end{aligned} \tag{26}$$

Proposition 4. *The performance index* $J_2(\overline{\mathscr{U}},\mathscr{K})$ *that is defined in equation* (11) *is equivalent to* $\tilde{\mathcal{J}}_3(\bar{\boldsymbol{u}},\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}))$ *which is defined in* (26). *Also, the function defined in the equation* (26) *is the difference of two convex functions in variables* $\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}$.

Recall that, Problem 2 has additional constraints compared with Problem 1. In the next proposition, we show that these constraints correspond to convex constraints in terms of the decision variables $(\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})$.

Proposition 5. *The constraint function* $C_{total}(\bar{\mathscr{U}},\mathscr{K})$ *which is defined in equation* (12) *can be expressed in terms of the decision variables* $(\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})$ *as*

$$\begin{aligned}\mathcal{C}(\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) := \bar{\boldsymbol{u}}^{\mathrm{T}}\bar{\boldsymbol{u}} &+ \mathrm{tr}(\boldsymbol{\mathcal{K}}\boldsymbol{W}\boldsymbol{\mathcal{K}}^{\mathrm{T}}) \\ &+ \mathrm{tr}(\boldsymbol{\mathcal{L}}\Sigma_0\boldsymbol{\mathcal{L}}^{\mathrm{T}}) - \rho^2,\end{aligned} \tag{27}$$

and the set of all $(\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})$ *that satisfy the constraint* $\mathcal{C}(\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) \leq 0$ *defines a convex set.*

The next theorem which is a direct consequence of Propositions 4 and 5 will allow us to reduce Problem 2 into a difference of convex functions program.

Theorem 2. *Problem 2 is equivalent to the following optimization problem:*

$$\min_{\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}},\boldsymbol{\xi}} \quad \tilde{\mathcal{J}}(\bar{\boldsymbol{u}},\zeta) \tag{28a}$$

$$\text{subject to} \quad \boldsymbol{\xi} = \boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) \tag{28b}$$

$$\boldsymbol{\mathcal{C}}(\bar{\boldsymbol{u}},\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}) \leq 0 \tag{28c}$$

where $\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})$ *is defined in Equation* (23).

One can exploit the structure of the performance index $\mathcal{J}_3$ in (26) to improve computational tractability, by using the convex-concave procedure (CCP) [21]. In the CCP, the objective and constraint functions that can be expressed as a difference of two convex functions are convexified by linearizing the difference function around the solution of the previous iteration. Then, the convexified problem is solved using convex optimization techniques. The procedure is terminated after the difference in the optimal values of convex sub-problems between iterations are sufficiently close. To use the CCP as computational scheme for the DCP defined in Theorem 2, the derivative of the term $-2\|\sqrt{\Sigma_{\mathrm{d}}}\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\|_*$ in equation (26) is required. Since the nuclear norm is a non-smooth function, this derivative may not exist in general but in this particular case, the derivative has a closed form expression which is given by the following proposition.

Proposition 6. *If* $\boldsymbol{\zeta}\boldsymbol{\zeta}^{\mathrm{T}} \in \mathbb{S}_n^{++}$, *holds* $\forall \boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}}$ *then the gradient of* $\|\sqrt{\Sigma_{\mathrm{d}}}\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\|_*$ *is well-defined and given by:*

$$\begin{aligned}\nabla_{\boldsymbol{\zeta}}\|\sqrt{\Sigma_{\mathrm{d}}}\boldsymbol{\zeta}(\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{K}})\|_* &:= \\ &\sqrt{\Sigma_{\mathrm{d}}}(\sqrt{\Sigma_{\mathrm{d}}}\boldsymbol{\zeta}\boldsymbol{\zeta}^{\mathrm{T}}\sqrt{\Sigma_{\mathrm{d}}})^{-1/2}\sqrt{\Sigma_{\mathrm{d}}}\boldsymbol{\zeta}.\end{aligned} \tag{29}$$

Remark 4 Since the objective function of the problem given in (28) corresponds to a difference of convex function and the constraints determine a convex set, the CCP heuristic is guaranteed to converge to a stationary point which satisfies the first order necessary conditions for optimality. [21]

V. Numerical Expeeriments

In this section, we present results obtained by numerical experiments in which we compare the proposed policy parametrization with the parametrization utilized in [8] in terms of performance and computational efficiency. All computations were performed on a laptop with 2.8 GHz Intel Core i7-7700HQ CPU and 16 GB RAM. We used CVXPY [22] for modelling with MOSEK [23] as the solver. To solve the SCCS problem, we used the convex-concave procedure by utilizing Proposition 6 to convexify the DCP objective function (11). The termination criteria was chosen as $|f_k - f_{k-1}| \leq \epsilon$ where f_k denotes the result of the optimization problem at k^{th} iteration.

In our numerical simulations, we consider two examples. One is based on a randomly generated linear dynamical system and the other corresponds to linearized model of the longitudinal dynamics of an aircraft.

A. Random Linear System

The parameters of the random linear system are taken as follows:

$$A(t) = \begin{bmatrix} 1.1 & -0.07 \\ 0.23 & -0.87 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad W = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

In addition, $\mu_0 = [1.0, 0.0]^{\mathrm{T}}$, $\Sigma_0 = I_2$, $\mu_{\mathrm{d}} = [10.0, 0.0]^{\mathrm{T}}$, $\Sigma_{\mathrm{d}} = \begin{bmatrix} 4.0 & -1.5 \\ -1.5 & 4.0 \end{bmatrix}$ and $T = 50$. In our simulations, we have truncated the disturbance history in order to decrease the number of decision variables of the optimization problem as explained in Remark 1.

Table I presents comparison results between the truncated affine disturbance feedback control policy and the state history feedback control policy [8]. The first column shows the truncation parameter γ used in the controller parametrization. The second and third columns show the objective value at the computed minimizer and the computation time of the HCCS problem whereas the last two columns show the corresponding results obtained for the SCCS problem. The last row of the table shows the results obtained by using the state history feedback control policy.

Based on these results, we can claim that the full disturbance history feedback policy achieves the same value for both the HCCS and the SCCS problems while reducing the computational cost. Furthermore, as we increase the truncation parameter γ, the optimal value does not decrease below a certain value whereas the computation time increases.

The results based on the experiments with the random linear system suggests that our policy parametrization may be equivalent to the one in [8] given that the two policies achieve the same objective value. However, more research is needed to establish rigorously the validity of the previous claim.

B. Linearized Longitudinal Aircraft Dynamics

The discrete-time model of the linearized longitudinal dynamics of an aircraft is taken from [12] and is obtained after the discretization of the continuous-time dynamics with sampling period ($\Delta T = 10$s). For our simulations, we used $\mu_0 = [0.0, 0.0, 0.0, 0.0, 0.0]^{\mathrm{T}}$ and $\mu_{\mathrm{d}} = [400.0, 0.0, 0.0, 0.0, 0.0]^{\mathrm{T}}$ whereas $\Sigma_0 = \mathrm{bdiag}(100.0, 25.0, 25.0, 1.0, 1.0)$ and $\Sigma_{\mathrm{d}} = \mathrm{bdiag}(10^4, 100.0, 4.0, 1.0, 1.0)$. For the SCCS simulations, we used $\rho = 4$ and $T = 40$.

In Figures 2 and 3, the statistics of the first component of the state $x(t)$, which is the deviation from steady flight altitude denoted as Δh, is shown along with sample trajectories and the desired mean and covariance. Figure 2 corresponds to

TABLE I: Comparison between affine disturbance feedback policy with different truncation lengths and state history feedback policy parametrization in terms of performance and computation time

	HCCS		SCCS	
γ	Value	Time (s)	Value	Time (s)
0	2839.93	1.32	2.00	2.19
1	2535.38	0.89	1.19	3.63
3	2383.88	1.34	0.74	6.17
5	2333.49	1.69	0.59	9.78
10	2288.79	3.29	0.46	15.80
20	2271.51	4.84	0.40	29.77
30	2269.66	6.32	0.39	36.94
40	2269.46	9.78	0.38	38.87
50	2269.44	9.61	0.38	37.38
[6]	2269.44	16.32	0.38	112.39

the HCCS problem whereas Figure 3 to the SCCP problem. The black curve corresponds to the trajectory of the mean and the blue shaded area illustrates the 2-σ confidence region, whereas the green line and green shaded region shows the desired mean and the 2-σ confidence region of the desired covariance respectively. The optimal policy is computed in 6.9s and 57.3s for the HCCS and the SCCS, respectively.

VI. Conclusion

In this paper, we proposed new covariance steering algorithms for discrete-time Gaussian linear systems based on a new control policy parametrization. We have studied two variations of the covariance steering problem, one in which the constraints on the terminal state covariance are enforced as hard LMI constraints and another one in which they correspond to soft constraints encoded in an appropriately selected terminal cost. Our numerical experiments have demonstrated that the proposed algorithms perform better, in terms of computational efficiency, than algorithms which are based on the state (history) feedback policy parametrization, which have been employed in our previous work. In our future work, we plan to consider applications of the proposed covariance steering algorithms in trajectory optimization and distributed control problems.

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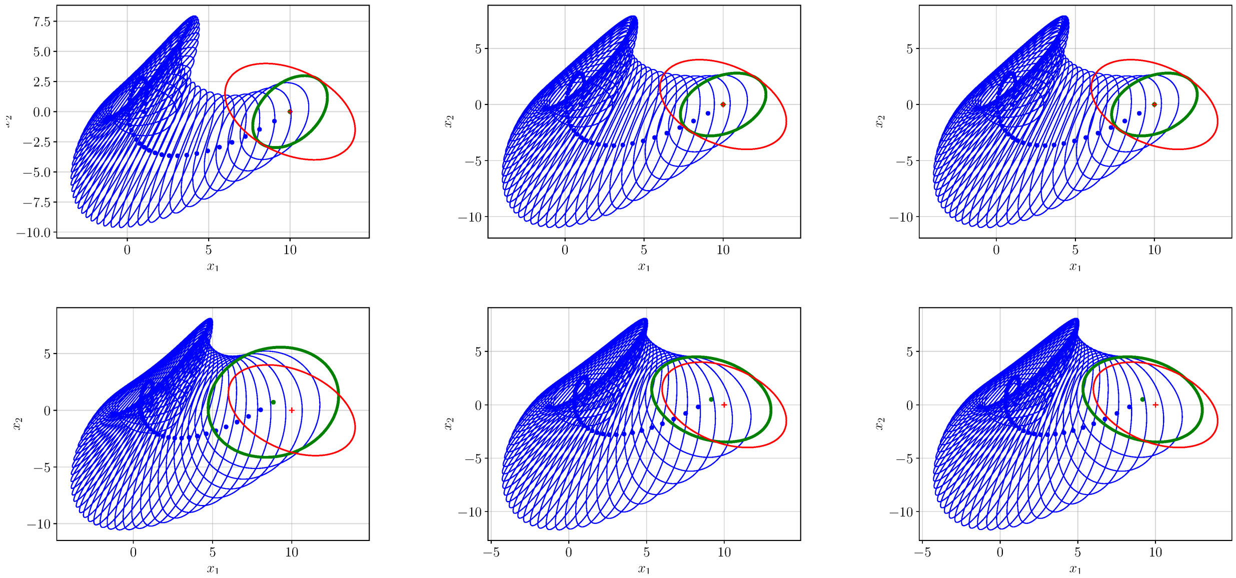

Fig. 1: Trajectories of the state mean and covariance of a random linear system: The top plots illustrate the results for the HCCS problem and the bottom plots those for the SCCS problem. Only the last disturbance term is used in the control policy for the left subfigures ($\gamma = 0$), the full disturbance history is used in the middle ones and the state history feedback policy is used on the right subfigures. Blue dots and ellipses show the evolution of the state mean and the 2-σ confidence ellipse over time. The green dot and ellipse show the final distribution and the red cross and ellipse show the desired mean and the 2-σ confidence ellipse of the desired terminal state distribution.

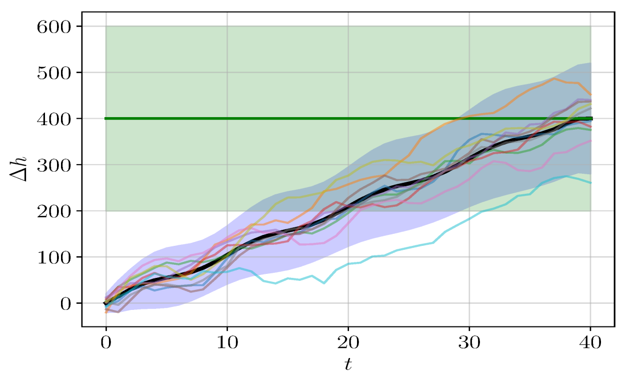

Fig. 2: Δh vs t, (HCCS problem).

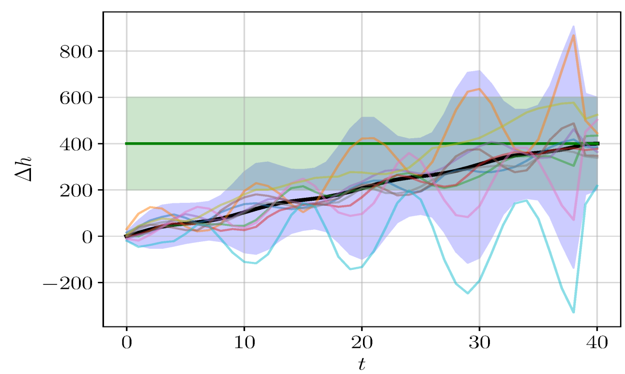

Fig. 3: Δh vs t, (SCCS problem).

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