On the Convexity of Discrete Time Covariance Steering in Stochastic Linear Systems with Wasserstein Terminal Cost

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Abstract— **We revisit the covariance steering problem for discrete-time Gaussian linear systems with a squared Wasserstein distance terminal cost and analyze the properties of its solution in terms of existence and uniqueness. Specifically, we derive the first and second order conditions for optimality and provide analytic expressions for the gradient and the Hessian of the performance index by utilizing specialized tools from matrix calculus. Subsequently, we prove that the optimization problem always admits a global minimizer, and finally, we provide a sufficient condition for the performance index to be a strictly convex function. In particular, we show that when the terminal state covariance is lower bounded, with respect to the Löwner partial order, by the covariance matrix of the desired terminal normal distribution, then the objective function is strictly convex.**

I. Introduction

We study the existence and uniqueness of solutions to the covariance steering problem for discrete-time Gaussian linear systems with a squared Wasserstein distance terminal cost. This instance of stochastic optimal control problem seeks for a feedback control policy that will steer the probability distribution of the state of the uncertain system, close to a goal multivariate normal distribution over a finite time horizon, where the closeness of the two distributions is measured in terms of the squared Wasserstein distance between them. In our previous work [1], we have shown that the latter problem can be reduced into a difference of convex functions program (DCP) provided that the control policy conforms to the so-called state feedback control parametrization according to which the control input can be expressed as an affine function of the current state and all past states visited by the system. Whereas the focus in [1] was on the control design problem, in this work we focus on the analysis of the problem and in particular, addressing questions about the existence and uniqueness of solutions and the convexity (or lack thereof) of the performance index.

Literature review: Early works on covariance control problems can be attributed to Skelton and his co-authors who mainly examined infinite-horizon problems in a series of papers (refer to, for instance, [2]–[4]). Recently, finite-horizon covariance control problems for Gaussian linear systems have received significant attention; the reader may refer to [5]–[7] for the continuous-time case and [8]–[13] for the discrete-time case. The covariance steering problem for continuous-time Gaussian linear systems with a Wasserstein distance terminal cost was first studied in [14] whereas the same problem but for the discrete-time case was studied in [1]. Both of these references present numerical algorithms (shooting method in [14] and convex-concave procedure in [1]) for control design but do not address theoretical questions regarding the existence and uniqueness of solutions, or investigate convexity properties of the performance index.

Main contributions: First, we establish the existence of at least one global minimizer to the optimization problem. Subsequently, we derive first and second order conditions of optimality, and provide analytic expressions for the gradient and the Hessian of the performance index by utilizing specialized tools from matrix calculus (these analytic expressions may also facilitate the implementation of numerical optimization algorithms, and thus improve in practice the speed of convergence). Finally, we present a sufficient condition for the performance index to be a strictly convex function under which the optimization problem admits a unique solution. In particular, we show that when the terminal state covariance is bounded from above, with respect to the Löwner partial order over the cone of positive semidefinite matrices, by the covariance matrix of the goal normal distribution, then the Hessian of the performance index becomes a strictly positive definite matrix, which in turn implies that the performance index is a strictly convex function.

II. Preliminaries

Set and inequality notations: We denote the set of nonnegative integers as $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and for any positive integer ν, let $\mathbb{N}_0[\nu] := \{0, 1, \ldots, \nu\}$. We use the inequalities $\succeq$ and $\succ$ in the sense of Löwner partial order. Given a square matrix M, we define $\mathrm{sym}(M) := (M + M^{\mathrm{T}})/2$.

Kronecker product, Kronecker sum, and the vec *operator:* The basic properties of Kronecker product will be useful in the sequel, including

$$(M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1 M_3 \otimes M_2 M_4), \quad (1)$$

and that matrix transpose and inverse are distributive w.r.t. the Kronecker product. The vectorization operator $\mathrm{vec}(\cdot)$ and the Kronecker product are related through

$$\mathrm{vec}(M_1 M_2 M_3) = \left(M_3^{\mathrm{T}} \otimes M_1\right) \mathrm{vec}(M_2). \quad (2)$$

Furthermore,

$$\mathrm{trace}\left(M_1^{\top} M_2\right) = \mathrm{vec}(M_1)^{\mathrm{T}} \mathrm{vec}(M_2). \quad (3)$$

We need the Kronecker sum $M_1 \oplus M_2 := M_1 \otimes I + I \otimes M_2$, where I is an identity matrix of commensurate dimension.

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For matrices M, L of appropriate size and L non-singular, we have

$$(L \otimes L)(M \oplus M)(L^{-1} \otimes L^{-1}) = LML^{-1} \oplus LML^{-1} \quad (4)$$

which is easy to verify using the definition of Kronecker sum and (1), and will be useful later.

Commutation matrix: The commutation matrix K_0 is the unique symmetric permutation matrix such that $\mathrm{vec}\,(M) = K_0 \,\mathrm{vec}\left(M^{\mathrm{T}}\right)$, see e.g., [15]. Being orthogonal, K_0 satisfies $K_0^{-1} = K_0^{\mathrm{T}} = K_0$. Therefore, K_0 is idempotent of order two. Two useful properties of K_0 are

$$K_0 \,\mathrm{vec}\,(I) = \mathrm{vec}\,(I)\,, \; K_0\,(M_1 \otimes M_2) = (M_2 \otimes M_1)\,K_0.$$

Notice that K_0 being symmetric orthogonal, its eigenvalues are ± 1. Consequently, the matrix $I + K_0$, which is also symmetric idempotent, has eigenvalues 0 and 2.

Another observation that will be useful is that $I + K_0$ commutes with "self Kronecker product or sum", i.e., for any square matrix M, we have

$$(I + K_0)\,(M \otimes M) = (M \otimes M)\,(I + K_0)\,, \quad (5a)$$
$$(I + K_0)\,(M \oplus M) = (M \oplus M)\,(I + K_0)\,, \quad (5b)$$

which follows from the property of K_0 mentioned before. We also have

$$(I + K_0)\,(M \oplus M)^{-1} = (M \oplus M)^{-1}\,(I + K_0)\,. \quad (6)$$

To see (6), notice that $K_0\,(M \oplus M)^{-1}$ equals

$$\left((M \oplus M)\,K_0^{-1}\right)^{-1} = ((M \oplus M)\,K_0)^{-1} = (K_0\,(M \oplus M))^{-1}$$
$$= (M \oplus M)^{-1}\,K_0^{-1} = (M \oplus M)^{-1}\,K_0.$$

Matrix differential and Jacobian: The matrix differential $\mathrm{d}(\cdot)$ and the vectorization $\mathrm{vec}(\cdot)$ are linear operators that commute with each other. We will frequently use the Jacobian identification rule [16, Ch. 9, Sec. 5], which for a given matrix function $F(X)$, is

$$\mathrm{d}\,\mathrm{vec}\,(F(X)) = \mathrm{D}F(X)\,\mathrm{d}\,\mathrm{vec}\,X, \quad (7)$$

where $\mathrm{D}F(X)$ is the Jacobian of F evaluated at X. In case F is independent of X, the Jacobain $\mathrm{D}F$ is a zero matrix. Some Jacobians of our interest are collected in the Appendix.

Matrix geometric mean: Given two symmetric positive definite matrices A and B, their geometric mean (see, for example, [17]) is the symmetric positive definite matrix

$$A\#B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}. \quad (8)$$

It satisfies intuitive properties such as $A\#A = A$, $A\#B = B\#A$, $(A\#B)^{-1} = A^{-1}\#B^{-1}$.

Function composition and normal distribution: We use the symbol $\circ$ to denote function composition. We write $z \sim \mathcal{N}(\mu, \Sigma)$ to denote that the random vector z has normal distribution with mean vector μ and covariance matrix Σ.

Preparatory lemmas (proofs omitted):

Lemma 1. *Let $F(X) := AXB$. Then $\mathrm{D}F(X) = B^{\mathrm{T}} \otimes A$.*

Lemma 2. *Let $F(X) := XX^{\mathrm{T}}$. Then $\mathrm{D}F(X) = (I + K_0)\,(X \otimes I)$.*

Lemma 3. *Let $F(X) := X\tilde{S}X^{\mathrm{T}}$ where $\tilde{S}$ is a given symmetric positive definite matrix. Then $\mathrm{D}F(X) = (I + K_0)\left(X\tilde{S} \otimes I\right)$.*

Lemma 4. *For X nonsingular, let $F(X) := X^{-1}$. Then $\mathrm{D}F(X) = -\left(X^{-\mathrm{T}} \otimes X^{-1}\right)$.*

Lemma 5. *Let $A \in \mathbb{R}^{n\times n}$ and $B = B^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. If $\mathrm{sym}(A) \succeq \mathbf{0}$ and $B \succ \mathbf{0}$ then both $\mathrm{sym}(AB) \succeq 0$, $\mathrm{sym}(BA) \succeq 0$ hold.*

III. Problem Set up

We consider a discrete-time stochastic linear system

$$x_{k+1} = A_k x_k + B_k u_k + G_k w_k, \quad k \in \mathbb{N}_0, \quad (9)$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, and $w_k \in \mathbb{R}^{n_w}$ denote the state, control input, and disturbance vectors at time $t = k$, respectively. It is assumed that the initial state is a normal vector and in particular, $x_0 \sim \mathcal{N}(\mu_0, S_0)$, where $\mu_0 \in \mathbb{R}^n$ and $S_0 \succ \mathbf{0}$, and in addition, the disturbance process is a sequence of independent and identically distributed random vectors $w_k \sim \mathcal{N}(0, S_w)$ for all $k \in \mathbb{N}_0$ and $S_w \succ \mathbf{0}$. We suppose that x_0 and w_k are mutually independent for all $k \in \mathbb{N}_0$, from which it follows that $\mathbb{E}[x_0 w_k^{\mathrm{T}}] = 0$ for all $k \in \mathbb{N}_0$, where $\mathbb{E}\left[\cdot\right]$ denotes the expectation functional. We assume that the matrices G_k are full rank for all $k \in \mathbb{N}_0[N-1]$.

For $N \in \mathbb{N}_0$, let $\boldsymbol{x} := [x_0^{\mathrm{T}}, x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n_x}$, $\boldsymbol{u} := [u_0^{\mathrm{T}}, u_1^{\mathrm{T}}, \ldots, u_{N-1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn_u}$ and $\boldsymbol{w} := [w_0^{\mathrm{T}}, w_1^{\mathrm{T}}, \ldots, w_{N-1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn_w}$. Then, we can write

$$\boldsymbol{x} = \boldsymbol{\Gamma} x_0 + \boldsymbol{H_u u} + \boldsymbol{H_w w}, \quad (10)$$

where the block (column) vector

$$\boldsymbol{\Gamma} := [I_{n_x}^{\mathrm{T}} \;\; \Phi^{\mathrm{T}}(1,0) \;\; \Phi^{\mathrm{T}}(2,0) \;\; \ldots \;\; \Phi^{\mathrm{T}}(N,0)]^{\mathrm{T}}, \quad (11)$$

and for all $k, n \in \mathbb{N}_0$ with $k \geq n$, the matrices $\Phi(k,n) := A_{k-1} \ldots A_n$, and $\Phi(n,n) := I$ (note that $\Phi(n{+}1,n) = A_n$). Furthermore,

$$\boldsymbol{H_u} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ B_0 & \mathbf{0} & \ldots & \mathbf{0} \\ \Phi(2,1)B_0 & B_1 & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \Phi(N,1)B_0 & \Phi(N,2)B_1 & \ldots & B_{N-1} \end{bmatrix}, \quad (12)$$

and $\boldsymbol{H_w}$ is defined likewise by replacing the matrices $\{B_k\}_{k=0}^{N-1}$ in (12) with the matrices $\{G_k\}_{k=0}^{N-1}$.

The problem of interest is to perform minimum energy feedback control synthesis for (9) over a time horizon of length N, such that the distribution of the terminal state x_N goes close to desired distribution $\mathcal{N}\left(\mu_d, S_d\right)$ where $\mu_d \in \mathbb{R}^n$, $S_d \succ \mathbf{0}$ are given. The mismatch between the desired distribution and the distribution of the actual terminal state x_N is penalized as a terminal cost quantified using the squared 2-Wasserstein distance $W_2^2\left(\cdot,\cdot\right)$ between those two distributions. We refer the readers to [1, Sec. II] for the details on problem formulation.

To recover the statistics of the terminal state x_N from the concatenated state $\boldsymbol{x}$, the following relation will be useful: $x_N = \boldsymbol{F}\boldsymbol{x}$, where $\boldsymbol{F} := [\mathbf{0}, \ldots, \mathbf{0}, I_{n_x}]$.

It was shown in [1] that the problem of discrete time covariance steering with Wasserstein terminal cost subject to (9) (or equivalently (10)), can be reduced to a difference of convex functions program, provided the control policy is parameterized as

$$u_k = u_{\text{ff},k} + \sum_{t=0}^{k} K_{(k,t)}(x_t - \bar{x}_t) \tag{13}$$

where $\bar{x}_t := \mathbb{E}[x_t]$, and the parameters of the control policy are $u_{\text{ff},k} \in \mathbb{R}^{n_x}$, $K_{(k,t)} \in \mathbb{R}^{n_u \times n_x}$ for all $\{(k,t) \in \mathbb{N}_0 \mid k \geq t\}$. The concatenated control input $\boldsymbol{u}$ can be written as $\boldsymbol{u} := \boldsymbol{u}_{\text{ff}} + \boldsymbol{K}(\boldsymbol{x} - \bar{\boldsymbol{x}})$ where $\bar{\boldsymbol{x}} := \mathbb{E}[\boldsymbol{x}]$, $\boldsymbol{K} := \begin{bmatrix}\tilde{\boldsymbol{K}} & \mathbf{0}\end{bmatrix}$, and

$$\tilde{\boldsymbol{K}} := \begin{bmatrix} K_{(0,0)} & \mathbf{0} & \cdots & \mathbf{0} \\ K_{(1,0)} & K_{(1,1)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ K_{(N-1,0)} & K_{(N-1,1)} & \cdots & K_{(N-1,N-1)} \end{bmatrix}. \tag{14}$$

The controller synthesis thus amounts to computing the optimal feedforward control and feedback gain pair $(\boldsymbol{u}_{\text{ff}}, \boldsymbol{K})$.

In [1], the authors proposed a bijective mapping $\boldsymbol{K} \mapsto \boldsymbol{\Theta}$ and back, given by

$$\boldsymbol{\Theta} := \boldsymbol{K}(I - \boldsymbol{H_u}\boldsymbol{K})^{-1}, \quad \boldsymbol{K} := \boldsymbol{\Theta}(I + \boldsymbol{H_u}\boldsymbol{\Theta})^{-1}.$$

With the new feedback gain parameterization $\boldsymbol{\Theta}$, it was deduced in [1] that the optimal pair $(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta})$ minimizes the objective $J : \mathbb{R}^{Nn_u} \times \mathbb{R}^{Nn_u \times (N+1)n_x} \mapsto \mathbb{R}_{\geq 0}$, given by

$$J(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta}) = J^{\text{cost-to-go}}(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta}) + \lambda W_2^2(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta}), \tag{16}$$

where $\lambda > 0$ is given, and

$$J^{\text{cost-to-go}}(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta}) = \text{trace}\left(\boldsymbol{\Theta}\tilde{S}\boldsymbol{\Theta}^{\mathrm{T}}\right) + \|\boldsymbol{u}_{\text{ff}}\|_2^2, \tag{17}$$

$$\begin{aligned} W_2^2(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta}) &= \|\mu_d - (\boldsymbol{\Gamma}\mu_0 + \boldsymbol{H_u}\boldsymbol{u}_{\text{ff}})\|_2^2 \\ &+ \text{tr}\Big(\boldsymbol{F}(I + \boldsymbol{H_u}\boldsymbol{\Theta})\tilde{S}(I + \boldsymbol{H_u}\boldsymbol{\Theta})^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}} + S_d\Big) \\ &- 2\text{tr}\Big((\sqrt{S_d}\boldsymbol{F}(I + \boldsymbol{H_u}\boldsymbol{\Theta})\tilde{S}(I + \boldsymbol{H_u}\boldsymbol{\Theta})^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\sqrt{S_d})^{1/2}\Big), \end{aligned}$$

where $\tilde{S} := \boldsymbol{\Gamma}S_0\boldsymbol{\Gamma}^{\mathrm{T}} + \boldsymbol{H_w}\mathbf{W}\boldsymbol{H_w}^{\mathrm{T}}$ and the block diagonal matrix $\mathbf{W} := \text{blkdiag}(S_w, \ldots, S_w)$.

Proposition 1. $\tilde{S} := \boldsymbol{\Gamma}S_0\boldsymbol{\Gamma}^{\mathrm{T}} + \boldsymbol{H_w}\mathbf{W}\boldsymbol{H_w} \succ \mathbf{0}$.

Proof. It is clear that $\tilde{S} \succeq \mathbf{0}$. Suppose for the sake of contradiction, that $\tilde{S}$ is singular. Then there exists $(N+1)n_x \times 1$ vector $v \neq 0$ such that $v^{\mathrm{T}}\tilde{S}v = v^{\mathrm{T}}(\boldsymbol{\Gamma}S_0\boldsymbol{\Gamma}^{\mathrm{T}} + \boldsymbol{H_w}\mathbf{W}\boldsymbol{H_w}^{\mathrm{T}})v = 0$, which in turn, is possible iff $\Gamma^{\mathrm{T}}v = 0$ and $\boldsymbol{H_w}^{\mathrm{T}}v = 0$, since $S_0 \succ \mathbf{0}$, $S_w \succ \mathbf{0}$.

Now let $v := \left[v_0^{\mathrm{T}}, v_1^{\mathrm{T}}, \ldots, v_N^{\mathrm{T}}\right]^{\mathrm{T}}$ where the sub-vector $v_i \in \mathbb{R}^{n_x}$ for all $i \in \mathbb{N}_0[N]$. From $\boldsymbol{H_w}^{\mathrm{T}}v = 0$, we get $v_1 = v_2 = \ldots = v_N = 0$ since the matrices $\{G_k\}_{k=0}^{N-1}$ are full rank per our assumption. In $\Gamma^{\mathrm{T}}v = 0$, substituting $v_1 = v_2 = \ldots = v_N = 0$, yields $v_0 = 0$. Thus, $v = 0$ which contradicts our hypothesis. Therefore, the positive semidefinite matrix $\tilde{S}$ is nonsingular, i.e., $\tilde{S} \succ \mathbf{0}$. ∎

Remark 1. *An important consideration is that in order to ensure causality of the control policy, the matrix* $\boldsymbol{\Theta} \in \mathbb{R}^{Nn_u \times (N+1)n_x}$ *should be constrained to be block lower triangular of the form*

$$\boldsymbol{\Theta} := \begin{bmatrix} \theta_{0,0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \theta_{1,0} & \theta_{1,1} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{N-1,0} & \theta_{N-1,1} & \cdots & \theta_{N-1,N-1} & \mathbf{0} \end{bmatrix} \tag{18}$$

where $\theta_{i,j} \in \mathbb{R}^{n_u \times n_x}$ *for all index pairs* (i,j).

The block lower triangular condition on $\boldsymbol{\Theta}$ in Remark 1 can be equivalently expressed as $\theta_{i,j} = \mathbf{0}\; \forall\, (i,j) \in \mathbb{N}_0[N-1] \times \mathbb{N}_0[N]$ s.t. $j > i$. We transcribe this constraint in terms of the decision variable $\boldsymbol{\Theta}$ as

$$\mathcal{E}_{u,i}\boldsymbol{\Theta}\mathcal{E}_{x,j}^{\mathrm{T}} = \mathbf{0} \quad \forall (i,j) \in \mathcal{H} \tag{19}$$

where $\mathcal{H} = \{(i,j) | (i,j) \in \mathbb{N}_0[N-1] \times \mathbb{N}_0[N]$ and $j > i\}$ $\mathcal{E}_{u,i} \in \mathbb{R}^{n_u \times n_u N}$ and $\mathcal{E}_{x,i} \in \mathbb{R}^{n_x \times n_x(N+1)}$ are defined as block vectors whose $(i+1)^{\text{th}}$ and $(j+1)^{\text{th}}$ blocks are equal to the identity matrices of suitable dimensions; all the other blocks are equal to the zero matrix.

It is clear that (17) is a convex quadratic function in its arguments. The squared Wasserstein distance is a difference of convex functions in $(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta})$. Thus, the objective J in (16) is a difference of convex functions in the decision variables, and as such, it is unclear when it might in fact be convex. In [1], we used the convex-concave procedure [18] to numerically compute the optimal solution. In our numerical experiments, we observed multiple local minima which motivates investigating the conditions of optimality for (16). This is what we pursue in Sections IV and V. Before doing so, we show that the objective J in (16) is not convex in general but there exists a global minimizer.

Proposition 2. *The problem of minimizing the objective* J *in* (16) *subject to the constraints* (19)*, admits a global minimizing pair* $(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta})$.

Proof. The objective J in (16) is continuous and coercive (i.e., $\lim_{\|\boldsymbol{u}_{\text{ff}}\|_2 \to \infty, \|\boldsymbol{\Theta}\|_2 \to \infty} J = \infty$) in its arguments.

That J is continuous in $(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta})$ is immediate. To establish coercivity, following [1, see equation (26)], we write

$$J(\boldsymbol{u}_{\text{ff}}, \boldsymbol{\Theta}) = J_1(\boldsymbol{u}_{\text{ff}}) + J_2(\boldsymbol{\Theta}) + J_3(\boldsymbol{\Theta}) - J_4(\boldsymbol{\Theta}), \tag{20}$$

$$J_1(\boldsymbol{u}_{\text{ff}}) := \|\boldsymbol{u}_{\text{ff}}\|_2^2 + \lambda\|\boldsymbol{F}(\boldsymbol{\Gamma}\mu_0 + \boldsymbol{H_u}\boldsymbol{u}_{\text{ff}}) - \mu_d\|_2^2, \tag{21a}$$

$$J_2(\boldsymbol{\Theta}) := \text{trace}\left(\boldsymbol{\Theta}\tilde{S}\boldsymbol{\Theta}^{\mathrm{T}}\right), \tag{21b}$$

$$J_3(\boldsymbol{\Theta}) := \lambda\,\text{tr}\Big(\boldsymbol{F}(I + \boldsymbol{H_u}\boldsymbol{\Theta})\tilde{S}(I + \boldsymbol{H_u}\boldsymbol{\Theta})^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}} + S_d\Big), \tag{21c}$$

$$J_4(\boldsymbol{\Theta}) := 2\lambda\,\text{tr}\left(\left(S_d^{1/2}\boldsymbol{F}(I + \boldsymbol{H_u}\boldsymbol{\Theta})\tilde{S}(I + \boldsymbol{H_u}\boldsymbol{\Theta})^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}S_d^{1/2}\right)^{1/2}\right). \tag{21d}$$

Since $J_1(\boldsymbol{u}_{\text{ff}})$ in (21a) is strictly convex quadratic in $\boldsymbol{u}_{\text{ff}}$, it is clear that $J_1(\boldsymbol{u}_{\text{ff}}) \to \infty$ as $\|\boldsymbol{u}_{\text{ff}}\|_2 \to \infty$.

We note that $J_2(\boldsymbol{\Theta})$ equals $\text{trace}\big(\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\Theta}\tilde{S}\big)$ due to invariance of the trace operator under cyclic permutation. Using (2) and (3), we then write

$$J_2(\boldsymbol{\Theta}) = \text{vec}(\boldsymbol{\Theta})^{\mathrm{T}}(\tilde{S} \otimes I)\text{vec}(\boldsymbol{\Theta}). \tag{22}$$

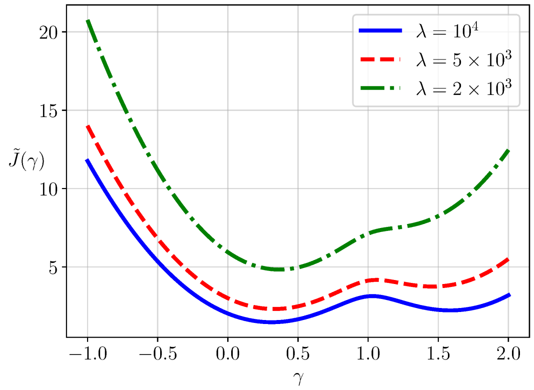

Fig. 1: $\tilde{J}(\gamma)$ versus γ for Example 1.

Since $I \succ \mathbf{0}, \tilde{S} \succ \mathbf{0}$ (by Proposition 1), we have $\tilde{S} \otimes I \succ \mathbf{0}$. Thus, $J_2(\mathbf{\Theta})$ is a strictly convex quadratic function and $J_2(\mathbf{\Theta}) \to \infty$ as $\|\mathbf{\Theta}\|_2 \to \infty$.

Finally, since $J_3(\mathbf{\Theta}) - J_4(\mathbf{\Theta})$ equals to a squared Wasserstein distance between two zero mean Gaussians, it is lower bounded by zero. Thus, $\lim_{\|\boldsymbol{u}_{\rm ff}\|_2 \to \infty, \|\mathbf{\Theta}\|_2 \to \infty} J = \infty$, i.e., the function $J(\boldsymbol{u}_{\rm ff}, \mathbf{\Theta})$ in (20) is coercive.

Moreover, the constraint set

$$\{(\boldsymbol{u}_{\rm ff}, \mathbf{\Theta}) \in \mathbb{R}^{Nn_u} \times \mathbb{R}^{Nn_u \times (N+1)n_x} \mid (19) \text{ holds}\}$$

is closed. Thus, minimizing the objective J in (16) subject to the constraints (19), amounts to minimizing a continuous coercive function over a closed set. Hence, there exists a global minimizing pair $(\boldsymbol{u}_{\rm ff}, \mathbf{\Theta})$ for this problem. ■

Notice that Proposition 2 only guarantees the existence of a global minimizer; it does not guarantee uniqueness. The following example shows that in general, J is nonconvex, and there might be multiple local minima which makes it challenging to find the global minimizer.

Example 1. *(Nonconvexity of J) Consider system matrices*

$$A_k = \begin{bmatrix} 1.0 & 0.1 \\ 0.0 & 1.0 \end{bmatrix}, B_k = \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix}, G_k = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \forall k \in \mathbb{N}_0,$$

with time horizon $N = 10$. The initial and desired mean vectors are $\mu_0 = [0.0, 0.0]^{\rm T}$, $\mu_d = [10.0, 5.0]^{\rm T}$, respectively. The initial covariance is $S_0 = I_2$. With this data, for two different desired distributions $\mathcal{N}(\mu_d, S_{d1}), \mathcal{N}(\mu_d, S_{d2})$ with

$$S_{d1} = \begin{bmatrix} 4.0 & -2.0 \\ -2.0 & 2.0 \end{bmatrix}, \quad S_{d2} = \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.1 \end{bmatrix},$$

we numerically computed (using the convex-concave procedure, see [1, Sec. IV]) the minimizers $(\boldsymbol{u}_{\rm ff}^1, \mathbf{\Theta}^1)$ and $(\boldsymbol{u}_{\rm ff}^1, \mathbf{\Theta}^2)$. Since the desired mean vector μ_d is the same in both cases, $\boldsymbol{u}_{\rm ff}^1 = \boldsymbol{u}_{\rm ff}^2$.

For $\gamma \in \mathbb{R}$, define an affine function $g(\gamma) := (\boldsymbol{u}_{\rm ff}^1 + \gamma(\boldsymbol{u}_{\rm ff}^2 - \boldsymbol{u}_{\rm ff}^1),\ \mathbf{\Theta}^1 + \gamma(\mathbf{\Theta}^2 - \mathbf{\Theta}^1))$ and let $\tilde{J}(\gamma) := J \circ g(\gamma)$. The function J is convex iff its restriction to a line, $\tilde{J}$, is convex. Fig. 1 shows that the function $\tilde{J}(\gamma)$ has multiple local minima, thus the function $J(\boldsymbol{u}_{\rm ff}, \mathbf{\Theta})$ is nonconvex.

IV. First Order Conditions for Optimality

Recall the function $J(\boldsymbol{u}_{\rm ff}, \mathbf{\Theta})$ in (20) and (21). We define the index set $\mathcal{I} := \{(i,j) \in \mathbb{N}_0[N-1] \times \mathbb{N}_0[N] \mid j > i\}$. Now consider the Lagrangian

$$\mathcal{L}(\boldsymbol{u}_{\rm ff}, \mathbf{\Theta}, \boldsymbol{N}) = J(\boldsymbol{u}_{\rm ff}, \mathbf{\Theta}) + \sum_{(i,j)\in\mathcal{I}} \langle \mathbf{\Psi}_{i,j}, \mathcal{E}_{u,i} \mathbf{\Theta} \mathcal{E}_{x,j}^{\rm T} \rangle \quad (23)$$

where $\mathbf{\Psi}_{i,j}$ is the Lagrange multiplier matrix associated with the $(i,j)^{\rm th}$ linear equality constraint (19) for all $(i,j) \in \mathcal{I}$, and $\langle \cdot, \cdot \rangle$ denotes the Frobenius inner product. Let us denote the optimal pair as $(\boldsymbol{u}_{\rm ff}^\star, \mathbf{\Theta}^\star)$. The first order necessary conditions for optimality are

$$\left.\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}_{\rm ff}}\right|_{(\boldsymbol{u}_{\rm ff}^\star, \mathbf{\Theta}^\star)} = \mathbf{0}, \quad \left.\frac{\partial \mathcal{L}}{\partial \mathbf{\Theta}}\right|_{(\boldsymbol{u}_{\rm ff}^\star, \mathbf{\Theta}^\star)} = \mathbf{0}, \quad \mathcal{E}_{u,i} \mathbf{\Theta}^\star \mathcal{E}_{x,j}^{\rm T} = \mathbf{0}.$$

We next compute the gradients of $\mathcal{L}$ w.r.t. the vector variable $\boldsymbol{u}_{\rm ff}$ and the matrix variable $\mathbf{\Theta}$, respectively, and use them to determine the pair $(\boldsymbol{u}_{\rm ff}^\star, \mathbf{\Theta}^\star)$.

A. The optimal feedforward control

The optimal feedforward control input $\boldsymbol{u}_{\rm ff}^\star$ solves $\partial \mathcal{L} / \partial \boldsymbol{u}_{\rm ff} = \mathbf{0}$, which results in a linear matrix-vector equation with unique solution

$$\boldsymbol{u}_{\rm ff}^\star = \left(I + \lambda \boldsymbol{H}_{\boldsymbol{u}}^{\rm T} \boldsymbol{F}^{\rm T} \boldsymbol{F} \boldsymbol{H}_{\boldsymbol{u}}\right)^{-1} \lambda \boldsymbol{H}_{\boldsymbol{u}}^{\rm T} \boldsymbol{F}^{\rm T} (\mu_d - \boldsymbol{F} \mathbf{\Gamma} \mu_0).$$

The uniqueness is a consequence of the term $\left(I + \lambda \boldsymbol{H}_{\boldsymbol{u}}^{\rm T} \boldsymbol{F}^{\rm T} \boldsymbol{F} \boldsymbol{H}_{\boldsymbol{u}}\right)$ being non-singular.

B. The optimal feedback gain

From (20), (21) and (23), we have

$$\frac{\partial \mathcal{L}}{\partial \mathbf{\Theta}} = \frac{\partial J_2}{\partial \mathbf{\Theta}} + \frac{\partial J_3}{\partial \mathbf{\Theta}} - \frac{\partial J_4}{\partial \mathbf{\Theta}} + \sum_{(i,j)\in\mathcal{I}} \frac{\partial}{\partial \mathbf{\Theta}} \text{trace}\left(\mathcal{E}_{x,j}^{\rm T} \mathbf{\Psi}_{i,j}^{\rm T} \mathcal{E}_{u,i} \mathbf{\Theta}\right)$$
$$= \frac{\partial J_2}{\partial \mathbf{\Theta}} + \frac{\partial J_3}{\partial \mathbf{\Theta}} - \frac{\partial J_4}{\partial \mathbf{\Theta}} + \sum_{(i,j)\in\mathcal{I}} \mathcal{E}_{u,i}^{\rm T} \mathbf{\Psi}_{i,j} \mathcal{E}_{x,j}. \quad (24)$$

Notice that

$$\frac{\partial J_2}{\partial \mathbf{\Theta}} = \frac{\partial}{\partial \mathbf{\Theta}} \text{trace}\left(\mathbf{\Theta} \tilde{S} \mathbf{\Theta}^{\rm T}\right) = 2 \mathbf{\Theta} \tilde{S}, \quad (25)$$

which follows from the invariance of trace under cyclic permutation, and from the use of Jacobian identification rule (7). Furthermore, let $J_{31}(\mathbf{\Theta}) := \mathbf{\Theta}\mathbf{\Theta}^{\rm T}$, $J_{32}(\mathbf{\Theta}) := \boldsymbol{F} \boldsymbol{H}_{\boldsymbol{u}} \mathbf{\Theta} \tilde{S}^{1/2}$, $\boldsymbol{Y}_3 := \dfrac{\partial}{\partial \mathbf{\Theta}} \text{trace}\left(J_{31} \circ J_{32}(\mathbf{\Theta})\right)$, and notice that

$$\frac{\partial J_3}{\partial \mathbf{\Theta}} = \lambda \left(2 \boldsymbol{H}_{\boldsymbol{u}}^{\rm T} \boldsymbol{F}^{\rm T} \boldsymbol{F} \tilde{S} + \boldsymbol{Y}_3\right). \quad (26)$$

From the chain rule of Jacobians, we have

$$\text{d trace}\left(J_{31} \circ J_{32}(\mathbf{\Theta})\right) = (\text{vec}(\boldsymbol{Y}_3))^{\rm T} \, \text{d vec}(\mathbf{\Theta}), \quad (27)$$

wherein using Lemma 1 and 2, we get

$$\text{D}J_{32}(\mathbf{\Theta}) = \tilde{S}^{1/2} \otimes \boldsymbol{F} \boldsymbol{H}_{\boldsymbol{u}}, \quad (28a)$$
$$\text{D}J_{31}(\mathbf{\Theta}) = (I + K_0)(\mathbf{\Theta} \otimes I). \quad (28b)$$

Combining (27) and (28), we obtain

$$\text{vec}(\boldsymbol{Y}_3) = (\text{D}J_{32}(\mathbf{\Theta}))^{\rm T} (\text{D}J_{31}(J_{32}(\mathbf{\Theta})))^{\rm T} \text{vec}(I)$$
$$= \left(\tilde{S}^{1/2} \otimes \boldsymbol{H}_{\boldsymbol{u}}^{\rm T} \boldsymbol{F}^{\rm T}\right) \left(\tilde{S}^{1/2} \mathbf{\Theta}^{\rm T} \boldsymbol{H}_{\boldsymbol{u}}^{\rm T} \boldsymbol{F}^{\rm T} \otimes I\right) (I + K_0) \, \text{vec}(I)$$
$$= 2 \, \text{vec}\left(\boldsymbol{H}_{\boldsymbol{u}}^{\rm T} \boldsymbol{F}^{\rm T} \boldsymbol{F} \boldsymbol{H}_{\boldsymbol{u}} \mathbf{\Theta} \tilde{S}\right), \quad (29)$$

where we used the identity: $(I+K_0)\,\mathrm{vec}\,(I) = 2\,\mathrm{vec}\,(I)$, and (29) follows from (2).

From (29), we identify $\boldsymbol{Y}_3 = 2\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta}\tilde{S}$, which together with (26), yields

$$\frac{\partial J_3}{\partial \boldsymbol{\Theta}} = 2\lambda \boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\left(I+\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta}\right)\tilde{S}. \tag{30}$$

Next, let

$$J_{41}(\boldsymbol{\Theta}) := \left(\sqrt{S}_d\boldsymbol{\Theta}\sqrt{S}_d\right)^{1/2},$$
$$J_{42}(\boldsymbol{\Theta}) := \boldsymbol{\Theta}\boldsymbol{\Theta}^{\mathrm{T}}, \quad J_{43}(\boldsymbol{\Theta}) := \boldsymbol{F}\left(I+\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta}\right)\tilde{S}^{1/2},$$
$$\boldsymbol{Y}_4 := \frac{\partial}{\partial \boldsymbol{\Theta}}\mathrm{trace}\left(J_{41}\circ J_{42}\circ J_{43}(\boldsymbol{\Theta})\right),$$

and notice that $\dfrac{\partial J_4}{\partial \boldsymbol{\Theta}} = 2\lambda \boldsymbol{Y}_4$. Therefore, writing

$$\begin{aligned}&\mathrm{d}\,\mathrm{trace}\left(J_{41}\circ J_{42}\circ J_{43}(\boldsymbol{\Theta})\right)\\ &= \left(\mathrm{vec}\,(I)\right)^{\mathrm{T}}\mathrm{D}J_{41}\left(J_{42}\left(J_{43}(\boldsymbol{\Theta})\right)\right)\mathrm{D}J_{42}(J_{43}\left(\boldsymbol{\Theta}\right))\\ &\qquad\qquad\qquad\qquad \mathrm{D}J_{43}\left(\boldsymbol{\Theta}\right)\,\mathrm{d}\,\mathrm{vec}(\boldsymbol{\Theta})\\ &= \left(\mathrm{vec}(\boldsymbol{Y}_4)\right)^{\mathrm{T}}\mathrm{d}\,\mathrm{vec}(\boldsymbol{\Theta}),\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{vec}\left(\boldsymbol{Y}_4\right) = &\left(\mathrm{D}J_{43}(\boldsymbol{\Theta})\right)^{\mathrm{T}}\left(\mathrm{D}J_{42}\left(J_{43}(\boldsymbol{\Theta})\right)\right)^{\mathrm{T}}\\ &\left(\mathrm{D}J_{41}\left(J_{42}\left(J_{43}(\boldsymbol{\Theta})\right)\right)\right)^{\mathrm{T}}\mathrm{vec}\left(I\right).\end{aligned} \tag{31}$$

To proceed further, we utilize the following results:

$$\mathrm{D}J_{43}(\boldsymbol{\Theta}) = \mathrm{D}\left(\boldsymbol{F}\tilde{S}^{1/2} + \boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta}\tilde{S}^{1/2}\right) = \tilde{S}^{1/2}\otimes \boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}, \tag{32a}$$
$$\mathrm{D}J_{42}(\boldsymbol{\Theta}) = (I+K_0)\left(\boldsymbol{\Theta}\otimes I\right), \tag{32b}$$
$$\begin{aligned}\mathrm{D}J_{41}(\boldsymbol{\Theta}) = &\left(\left(S_d^{1/2}\boldsymbol{\Theta}S_d^{1/2}\right)^{1/2}\oplus\left(S_d^{1/2}\boldsymbol{\Theta}S_d^{1/2}\right)^{1/2}\right)^{-1}\\ &\times\left(S_d^{1/2}\otimes S_d^{1/2}\right).\end{aligned} \tag{32c}$$

The result (32a) follows from Lemma 1 while (32b) follows from Lemma 2, and (32c) follows from [14, equation (30)].

Let $\Omega(\boldsymbol{\Theta}) := \boldsymbol{F}\left(I+\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta}\right)$, which is a linear function of $\boldsymbol{\Theta}$. Substituting (32) in (31), and then using (1), (5a), (6), and recalling $(I+K_0)\,\mathrm{vec}(I) = 2\,\mathrm{vec}(I)$, we obtain

$$\begin{aligned}\mathrm{vec}\left(\boldsymbol{Y}_4\right) = 2&\left(\tilde{S}\Omega^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}S_d^{1/2}\otimes \boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}S_d^{1/2}\right)\\ &\left(\mathbf{M}^{1/2}\oplus\mathbf{M}^{1/2}\right)^{-1}\mathrm{vec}(I).\end{aligned} \tag{33}$$

where $\mathbf{M} := S_d^{1/2}\Omega\tilde{S}\Omega^{\mathrm{T}}S_d^{1/2}$. Therefore, following similar steps as in [14, Appendix B, equations (32)-(35)], we arrive at[1] $\boldsymbol{Y}_4 = \boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}(S_d\#(\Omega\tilde{S}\Omega^{\mathrm{T}})^{-1})\Omega\tilde{S}$, where $\#$ denotes the matrix geometric mean as in (8). Hence

$$\frac{\partial J_4}{\partial \boldsymbol{\Theta}} = 2\lambda\boldsymbol{Y}_4 = 2\lambda\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\left(S_d\,\#\left(\Omega\tilde{S}\Omega^{\mathrm{T}}\right)^{-1}\right)\Omega\tilde{S}. \tag{34}$$

Combining (24), (25), (30) and (34), with $\left.\dfrac{\partial\mathcal{L}}{\partial\boldsymbol{\Theta}}\right|_{(\boldsymbol{u}_{\mathrm{ff}}^{\star},\boldsymbol{\Theta}^{\star})} = \mathbf{0}$, we arrive at a nonlinear matrix equation in $\boldsymbol{\Theta}^{\star}$ given by (35). Thus, the primal feasibility (19) and the Lagrangian gradient (35) together give the first order optimality conditions for $\boldsymbol{\Theta}^{\star}$.

[1]The matrix $\Omega\tilde{S}\Omega^{\mathrm{T}}$ is the right bottom corner block of size $n_x\times n_x$ from the $(N+1)n_x\times(N+1)n_x$ symmetric positive definite matrix $(I+\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta})\tilde{S}(I+\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta})^{\mathrm{T}}$, and is thus symmetric positive definite.

V. Second Order Conditions

We start by noting that:

$$\mathrm{Hess}\,(J) = \mathrm{Hess}\,(J_2) + \mathrm{Hess}\,(J_3) - \mathrm{Hess}\,(J_4)\,. \tag{36}$$

The following derives closed form expression of the Hessian.

Theorem 1. ***(Hessian of*** J***)*** *Consider* $\Omega = \Omega(\boldsymbol{\Theta})$. *Let* $M := \left(S_d^{-1/2}(\Omega\tilde{S}\Omega^{\mathrm{T}})^{-1}S_d^{-1/2}\right)^{1/2}$, *and* $\tilde{M} := S_d^{1/2}MS_d^{1/2} = S_d\#(\Omega\tilde{S}\Omega^{\mathrm{T}})^{-1}$. *Then, the Hessian of* J *in* (16) *is given by*

$$\begin{aligned}\mathrm{Hess}(J) = &\,2\left(\tilde{S}\otimes I\right) + 2\lambda\left(\tilde{S}\otimes\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\right)\\ &+2\lambda\left(\Omega\tilde{S}\otimes\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\right)^{\mathrm{T}}\left(S_d^{1/2}MS_d^{-1/2}\oplus S_d^{1/2}MS_d^{-1/2}\right)\\ &\left(\left(\Omega\tilde{S}\Omega^{\mathrm{T}}\right)^{-1}\otimes\left(\Omega\tilde{S}\Omega^{\mathrm{T}}\right)^{-1}\right)(I+K_0)\left(\Omega\tilde{S}\otimes\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\right)\\ &-2\lambda\left(\tilde{S}\otimes\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\tilde{M}\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\right).\end{aligned} \tag{37}$$

Proof. Combining (36), (7) with Lemma 1, we get

$$\begin{aligned}\mathrm{Hess}(J) = &\,2\left(\tilde{S}\otimes I\right) + 2\lambda\left(\tilde{S}\otimes\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\right)\\ &-2\lambda\left(\tilde{S}\otimes\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\right)\mathrm{D}P\left(\Omega(\boldsymbol{\Theta})\right)\mathrm{D}\Omega(\boldsymbol{\Theta}).\end{aligned} \tag{38}$$

From the definition of $\Omega(\boldsymbol{\Theta})$ and Lemma 1, we have $\mathrm{D}\Omega(\boldsymbol{\Theta}) = I\otimes\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}$. Using (7) along with Lemma 3 and Lemma 4, we also have:

$$\begin{aligned}\mathrm{D}P(\Omega) \overset{(4)}{=} &-\left(\Omega^{\mathrm{T}}\otimes I\right)\left(S_d^{1/2}MS_d^{-1/2}\oplus S_d^{1/2}MS_d^{-1/2}\right)\\ &\left(\left(\Omega\tilde{S}\Omega^{\mathrm{T}}\right)^{-1}\otimes\left(\Omega\tilde{S}\Omega^{\mathrm{T}}\right)^{-1}\right)(I+K_0)(\Omega\tilde{S}\otimes I) + (I\otimes\tilde{M}).\end{aligned}$$

In (38), substituting the above for $\mathrm{D}P(\Omega)$ and $\mathrm{D}\Omega(\boldsymbol{\Theta})$, and then substituting Ω, yields (37) as claimed. ■

With the formula (37) in hand, $\mathrm{Hess}(J)\succ\mathbf{0}$ (strictly positive definite) is a sufficient condition for the unique solution for $\boldsymbol{\Theta}^{\star}$ (since constraints (19) are linear). In Theorem 2 below, we deduce a simpler sufficient condition involving the terminal covariance (i.e., covariance of the random vector $x(N)$) that guarantees $\mathrm{Hess}(J)\succ\mathbf{0}$. It was shown in [1, Sec. III] that the covariance of $x(N)$ equals $\Omega\tilde{S}\Omega^{\mathrm{T}}$.

Lemma 6. *Let* $\mathbf{M} = (\mathbf{S}^{-1/2}\mathbf{A}\mathbf{S}^{-1/2})^{1/2}$, $\mathbf{S}\succ\mathbf{0}$, $\mathbf{A}\succ\mathbf{0}$. *Then,* $\mathrm{sym}\left(\left(\mathbf{S}^{1/2}\mathbf{M}\mathbf{S}^{-1/2}\oplus\mathbf{S}^{1/2}\mathbf{M}\mathbf{S}^{-1/2}\right)(I+K_0)\right)\succeq\mathbf{0}$ *where* K_0 *is the commutation matrix with suitable size.*

Proof. $\mathbf{A}\succ\mathbf{0}$ implies that $\mathbf{S}^{-1/2}\mathbf{A}\mathbf{S}^{-1/2}\succ\mathbf{0}$ (congruence transformation by $\mathbf{S}^{-1/2}$ which is non-singular). Therefore, $\mathbf{M}\succ\mathbf{0}$ as the square root of a (symmetric) positive definite matrix. Due to similarity transform $\mathbf{S}^{-1/2}\mathbf{M}\mathbf{S}^{1/2} = \bar{\mathbf{M}}$, $\mathrm{sym}(\bar{\mathbf{M}})\succ\mathbf{0}$ and therefore $\mathrm{sym}(\bar{\mathbf{M}}\oplus\bar{\mathbf{M}})\succ\mathbf{0}$ by the properties of the Kronecker sum.

Let $\mathbf{T} = (\bar{\mathbf{M}}\oplus\bar{\mathbf{M}})(I+K_0)$, so $\mathbf{T}^{\mathrm{T}} = (I+K_0)^{\mathrm{T}}(\bar{\mathbf{M}}\oplus\bar{\mathbf{M}})^{\mathrm{T}} = (\bar{\mathbf{M}}^{\mathrm{T}}\oplus\bar{\mathbf{M}}^{\mathrm{T}})(I+K_0)$ from (5b) and the identity $K_0 = K_0^{\mathrm{T}}$. Thus, $\mathbf{T}+\mathbf{T}^{\mathrm{T}} = 2\,\mathrm{sym}(\mathbf{T}) = ((\bar{\mathbf{M}}^{\mathrm{T}}+\bar{\mathbf{M}})\oplus(\bar{\mathbf{M}}^{\mathrm{T}}+\bar{\mathbf{M}}))(I+K_0)$. Note that $I+K_0\succeq\mathbf{0}$ and $\bar{\mathbf{M}}^{\mathrm{T}}+\bar{\mathbf{M}}\succ\mathbf{0}$. Using Lemma 5, $2\,\mathrm{sym}(\mathrm{sym}(\mathbf{T})) = 2\,\mathrm{sym}(\mathbf{T})\succeq\mathbf{0}$. ■

Theorem 2. *If the terminal covariance* $\Omega\tilde{S}\Omega^{\mathrm{T}}\succeq S_d$, *then* $\mathrm{Hess}(J)\succ\mathbf{0}$.

Proof. Let us view the right-hand-side of (37) as linear combination of four terms. We refer to $2\big(\tilde{S}\otimes I\big)$ as term 1,

$$2\boldsymbol{\Theta}^\star\tilde{S} + 2\lambda\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\left(I+\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta}^\star\right)\tilde{S} - 2\lambda\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}S_d^{1/2}\left(S_d^{-1/2}\left(\boldsymbol{F}\left(I+\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta}^\star\right)\tilde{S}\left(I+\boldsymbol{\Theta}^{\star\mathrm{T}}\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\right)\boldsymbol{F}^{\mathrm{T}}\right)^{-1}S_d^{-1/2}\right)^{1/2}$$
$$\times S_d^{1/2}\boldsymbol{F}\left(I+\boldsymbol{H}_{\boldsymbol{u}}\boldsymbol{\Theta}^\star\right)\tilde{S} + \sum_{(i,j)\in\mathcal{I}}\mathcal{E}_{u,i}^{\mathrm{T}}\boldsymbol{\Psi}_{i,j}\mathcal{E}_{x,j} = \mathbf{0}. \quad (35)$$

the quantity $2\lambda\big(\tilde{S}\otimes\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\big)$ as term 2, and $2\lambda\big(\tilde{S}\otimes\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\tilde{M}\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\big)$ as term 4. The remaining term in (37) is referred to as term 3.

We now make use of two instances of the Löwner–Heinz theorem [19]–[21]; Specifically, since $X\mapsto X^{-1}$ is operator decreasing, we have

$$\Omega\tilde{S}\Omega^{\mathrm{T}}\succeq S_d \Rightarrow \left(\Omega\tilde{S}\Omega^{\mathrm{T}}\right)^{-1}\preceq S_d^{-1}$$
$$\Rightarrow S_d^{-1/2}\left(\Omega\tilde{S}\Omega^{\mathrm{T}}\right)^{-1}S_d^{-1/2}\preceq S_d^{-2}, \quad (39)$$

where the last line follows from the congruence transform by $S_d^{-1/2}$. Since $X\mapsto X^{1/2}$ is operator increasing, (39) gives

$$\underbrace{\left(S_d^{-1/2}\left(\Omega\tilde{S}\Omega^{\mathrm{T}}\right)^{-1}S_d^{-1/2}\right)^{1/2}}_{=M\text{ (defined in Thm. 1)}}\preceq S_d^{-1}\Rightarrow \underbrace{S_d^{1/2}MS_d^{1/2}}_{=\tilde{M}\text{ (defined in Thm. 1)}}\preceq I, \quad (40)$$

where the last inequality is due to congruence transform by $S_d^{1/2}$. Since $\tilde{S}\succ\mathbf{0}$, it follows from (40) that

$$\tilde{S}\otimes\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\tilde{M}\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}\preceq\tilde{S}\otimes\boldsymbol{H}_{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}},$$

and multiplying both sides of the above by $-2\lambda<0$, we get $-$ term 4 $\succeq$ $-$term 2. Since $\tilde{S}\succ\mathbf{0}$ (see Proposition 1), we have term 1 $\succ\mathbf{0}$.

On the other hand, since $M\succ\mathbf{0}$, the similarity transform with $S_d^{1/2}$ implies that $\mathrm{sym}(S_d^{1/2}MS_d^{-1/2})\succ\mathbf{0}$, and therefore, $\mathrm{sym}(S_d^{1/2}MS_d^{-1/2}\oplus S_d^{1/2}MS_d^{-1/2})\succ\mathbf{0}$. Furthermore, since $(\Omega\tilde{S}\Omega^{\mathrm{T}})^{-1}\succ\mathbf{0}$, we have $\mathbf{N}:=(\Omega\tilde{S}\Omega^{\mathrm{T}})^{-1}\otimes(\Omega\tilde{S}\Omega^{\mathrm{T}})^{-1}\succ\mathbf{0}$. Also, we have $\mathbf{N}(I+K_0)=(I+K_0)\mathbf{N}$, from (5a). Using lemma 6 and the fact that $I+K_0\succeq\mathbf{0}$, we show that $\mathrm{sym}\left((S_d^{1/2}MS_d^{-1/2}\oplus S_d^{1/2}MS_d^{-1/2})(I+K_0)\right)=\mathrm{sym}(T)\succeq\mathbf{0}$. Since $\mathbf{N}\succ\mathbf{0}$, we show that $\mathrm{sym}(T\mathbf{N})\succeq\mathbf{0}$ by Lemma 5. Also, it is clear that term 1, term 2 and term 4 are symmetric which implies that term 3 is also symmetric since $\mathrm{Hess}(J)$ is symmetric. This shows that $T\mathbf{N}=(T\mathbf{N})^{\mathrm{T}}$ because term 3 is the congruence transform of $T\mathbf{N}$ with $\Omega S\otimes\boldsymbol{F}\boldsymbol{H}_{\boldsymbol{u}}$. Thus, $T\mathbf{N}\succeq\mathbf{0}\Rightarrow$ term3 $\succeq\mathbf{0}$.

From (37), $\mathrm{Hess}(J)=$ term 1 + term 2 + term 3 − term 4 $\succeq$ term 1 + term 3 $\succ\mathbf{0}$, completing the proof. ∎

VI. CONCLUSIONS

In this paper, we showed that the covariance steering problem for discrete-time Gaussian linear systems with a squared Wasserstein distance terminal is in general non-convex, and may admit more than one local or global minimizers. We also derived the analytical expressions of the Jacobian and the Hessian of the objective function based on specialized tools from matrix calculus, and obtained the first-order and second-order conditions for optimality. Finally, we presented a sufficient condition for the strict convexity of the performance index, thereby guaranteeing the uniqueness of the solution under the same condition.

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