PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 149, Number 8, August 2021, Pages 3569–3574
https://doi.org/10.1090/proc/15473
Article electronically published on May 18, 2021

METRIC RIGIDITY OF KÄHLER MANIFOLDS WITH LOWER RICCI BOUNDS AND ALMOST MAXIMAL VOLUME

VED DATAR, HARISH SESHADRI, AND JIAN SONG

(Communicated by Jia-Ping Wang)

ABSTRACT. In this short note we prove that a Kähler manifold with lower Ricci curvature bound and almost maximal volume is Gromov-Hausdorff close to the projective space with the Fubini-Study metric. This is done by combining the recent results on holomorphic rigidity of such Kähler manifolds (see Gang Liu [Asian J. Math. 18 (2014), 69–99]) with the structure theorem of Tian-Wang (see Gang Tian and Bing Wang [J. Amer. Math. Soc 28 (2015), 1169–1209]) for almost Einstein manifolds. This can be regarded as the complex analog of the result on Colding on the shape of Riemannian manifolds with almost maximal volume.

1. INTRODUCTION

In this note we wish to study metric rigidity of Kähler manifolds (M^n, ω) satisfying

$$\mathrm{Ric}(\omega) \geq \omega, \tag{1.1}$$

and with almost maximal volume. Recently Zhang [14] proved that any Kähler manifold satisfying (1.1) must have

$$\mathrm{Vol}(M, \omega) := \int_M \omega^n \leq \mathrm{Vol}(\mathbb{CP}^n, \omega_{\mathbb{CP}^n}).$$

Here $\omega_{\mathbb{CP}^n}$ is the Fubini-Study metric on $\mathbb{CP}^n$ with $\mathrm{Ric}(\omega_{\mathbb{CP}^n}) = \omega_{\mathbb{CP}^n}$. Moreover, Zhang proved that the maximal volume is attained if and only if (M, ω) is isometric to $(\mathbb{CP}^n, \omega_{\mathbb{CP}^n})$. For Kähler-Einstein Fano manifolds such optimal bounds were proved earlier by Berman-Berndtsson [1] in presence of a $\mathbb{C}^*$ action with finitely many fixed points, and unconditionally by Fujita [5]. On the other hand Colding in [3] proved that an n-dimensional Riemannian manifold with Ricci curvature greater or equal to $(n-1)$ and almost maximal volume is close to the round sphere in Gromov-Hausdorff distance. The main purpose of this note is to establish the following metric rigidity result as the complex analogue of Colding's theorem.

Theorem 1.1. *For all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, n) > 0$ such that if (M^n, ω) is a Kähler manifold satisfying (1.1) and*

$$\mathrm{Vol}(M, \omega) > (1-\delta)\mathrm{Vol}(\mathbb{CP}^n, \omega_{\mathbb{CP}^n}),$$

Received by the editors October 30, 2020.
2020 *Mathematics Subject Classification.* Primary 53C24, 53C55.
Research supported in part by National Science Foundation grant DMS-1711439, UGC (Govt. of India) grant no. F.510/25/CAS- II/2018(SAP-I), and the Infosys Young investigator award.

©2021 American Mathematical Society

Licensed to Rutgers Univ-New Brunswick. Prepared on Thu Jan 27 11:28:34 EST 2022 for download from IP 128.6.45.205.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

then

$$d_{GH}\Big((M,\omega),(\mathbb{CP}^n,\omega_{\mathbb{CP}^n})\Big) < \varepsilon,$$

where d_{GH} is the Gromov-Hausdorff distance.

The starting point for this paper is the almost holomorphic rigidity proved by Liu in the appendix of [14]. Liu proved that if (M^n,ω) is a Kähler manifold satisfying (1.1), M must be biholomorphic to $\mathbb{CP}^n$ if the volume of (M^n,ω) is sufficiently close to that of $(\mathbb{CP}^n,\omega_{\mathbb{CP}^n})$. This can be regarded as a complex version of Perelman's result in [8]. In particular, the Kähler manifold M in Theorem 1.1 must be $\mathbb{CP}^n$ and indeed Theorem 1.1 is a metric extension of Liu and Zhang's theorem. The proof of Theorem 1.1 relies on the structure theorem of Tian and Wang [12] on Gromov-Hausdorff limits of almost Einstein manifolds. We also offer an alternate proof relying on the recent results of Liu and Szekelyhidi on structure of non-collapsed Gromov-Hausdorff limits of Kähler manifolds with a Ricci curvature lower bound.

After this note appeared online, we were informed by Bing Wang that he has also independently proved our main Theorem [13, Corollary 7.10], once again by using the results in [14]. The results of [14] rely on recent works on stability thresholds and K-stability from algebraic geometry. It will be interesting to obtain an independent differential geometric proof for the holomorphic rigidity of $\mathbb{CP}^n$.

2. Proof of the main theorem

We will first prove the following general result.

Theorem 2.1. *Let (M^n,ω_{KE}) be a Kähler-Einstein manifold. Let $\delta_i \to 0$ and $\omega_i \in c_1(M)$ such that*

$$\mathrm{Ric}(\omega_i) \geq (1-\delta_i)\omega_i.$$

Then

$$(M,\omega_i) \xrightarrow{d_{GH}} (M,\omega_{KE}).$$

Before we begin the proof, let us recall the definition of almost Kähler-Einstein manifolds from [12]. A sequence of closed Kähler manifolds (M_i^n,ω_i,p_i) is said to be *almost Kähler-Einstein* if the following conditions are satisfied.

- $\mathrm{Ric}(\omega_i) \geq -\omega_i$
- $p_i \in M_i$ and
$$|B_{\omega_i}(p_i,1)| \geq \kappa > 0.$$
-
$$F_i := \int_{M_i} |\mathrm{Ric}(\omega_i) - \lambda_i\omega_i|\omega_i^n \xrightarrow{i\to\infty} 0.$$
- For some $\lambda_i \in [-1,1]$, the flow
$$\frac{\partial\omega_i}{\partial t} = -\mathrm{Ric}(\omega_i) + \lambda_i\omega_i$$
has a solution on $M_i \times [0,1]$. Moreover,
$$E_i := \int_0^1 \int_{M_i} |S_{g_i} - n\lambda_i|\, \omega_i(t)^n\, dt \xrightarrow{i\to\infty} 0.$$

Theorem 2.2 (Theorem 2 in [12]). *Let (M_i^n,ω_i,p_i) be a sequence of pointed almost Káhler-Einstein manifolds of (complex) dimension n with $\lambda_i = 1$. Let (Z,d) be a subsequential Gromov-Hausdorff limit. Then there exist a regular-singular decomposition $Z = \mathcal{R}\cup\mathcal{S}$ such that*

Licensed to Rutgers Univ-New Brunswick. Prepared on Thu Jan 27 11:28:34 EST 2022 for download from IP 128.6.45.205.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

- *$\mathcal{R}$ is a smooth convex, open Kähler manifold with complex structure J_∞ and Kähler form ω_∞ satisfying*
$$\mathrm{Ric}(\omega_\infty) = \omega_\infty.$$
- $\dim \mathcal{S} \leq 2n-4$.

Proof of Theorem 2.1. First we need the following observation from [12], whose proof we reproduce for the convenience of the reader.

Lemma 2.3 (Theorem 6.2 in [12])**.** *The sequence (M, ω_i, p) from the statement of Theorem 2.1 forms a sequence of almost-Einstein manifolds with $\lambda_i = 1$.*

Proof. The Ricci lower bound is from the hypothesis, and the volume lower bound follows from Bishop-Gromov inequality, Myers theorem and the fact that the volume of ω_i is constant. Moreover, it is well known through the work of Perelman that the Ricci flow
$$\begin{cases} \frac{\partial \omega_i(t)}{\partial t} = -\mathrm{Ric}(\omega_i(t)) + \omega_i(t) \\ \omega_i(0) = \omega_i \end{cases}$$
exists for all time. All we need to prove is that F_i and E_i converge to zero.

Note that since $S_{\omega_i} - n \geq -n\delta_i$, by the maximum principle for the scalar curvature under Ricci flow, we have the bound
$$S_{\omega_i(t)} - n \geq -n\delta_i e^t$$
for all i and for all t. On the other hand, since the Kähler class remains fixed,
$$\int_M (S_{\omega_i(t)} - n)\frac{\omega_i(t)^n}{n!} = 0,$$
and hence
$$\int_M |S_{\omega_i(t)} - n|\frac{\omega_i(t)^n}{n!} \leq n\delta_i e^t (c_1(M))^n. \tag{2.1}$$
Integrating in t, we obtain the required decay on E_i. Next, since $\mathrm{Ric}(\omega_i) - (1 - \delta_i)\omega_i \geq 0$, we have
$$\begin{aligned} F_i &:= \int_M |\mathrm{Ric}(\omega_i) - \omega_i|\omega_i^n \\ &\leq \int_M |\mathrm{Ric}(\omega_i) - (1-\delta_i)\omega_i|\omega_i^n + n\delta_i c_1(M)^n \\ &\leq 10n^{3/2}\delta_i c_1(M)^n \xrightarrow{i\to\infty} 0, \end{aligned}$$
where we used (2.1) at $t = 0$ to estimate the first integral. □

After passing to a subsequence,
$$(M, \omega_i, p) \xrightarrow{d_{GH}} (Z, d, p_\infty)$$
where Z has the regular-singular decomposition as in Theorem 2.2. From the proof of Theorem 2.2 in [12] it follows that any tangent cone is a metric cone $C(Y)$ over a link Y with singular set $\mathcal{S}_Y$ of real co-dimension at least four. Moreover, on the regular part of $C(Y)$, the cone metric $g_{C(Y)} = dr^2 + r^2 g_Y$ is Ricci flat and its Kähler form is given by
$$\omega_{C(Y)} = \frac{\sqrt{-1}}{2}\partial\bar{\partial} r^2,$$

Licensed to Rutgers Univ-New Brunswick. Prepared on Thu Jan 27 11:28:34 EST 2022 for download from IP 128.6.45.205.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

where r is the distance from the vertex (cf. Proposition 5.2 and Lemma 5.2 in [12]). The arguments in [4] now apply and we have the following.

Lemma 2.4.

(1) *For sufficiently large $k \in \mathbb{N}$, there is a sequence of embeddings $T_i : M \to \mathbb{CP}^N$ by sections of $H^0(M, -K_M^k)$ which are orthonormal with respect to the metric induced by ω_i such that the flat limit $W = \lim_{i\to\infty} T_i(M)$ is a normal $\mathbb{Q}$-Fano variety.*

(2) *The limiting Káhler metric ω_∞ extends globally to a weak Kähler-Einstein metric on W.*

By a weak Kähler-Einstein metric we mean that $\omega_\infty = \sqrt{-1}\partial\overline{\partial}\varphi_\infty$ where $e^{-r\varphi_\infty}$ is a continuous hermitian metric on K_W^{-r}, and φ_∞ satisfies the following Monge-Ampere equation

$$(\sqrt{-1}\partial\overline{\partial}\varphi_\infty)^n = e^{-\varphi_\infty}.$$

Continuing with our proof of the Theorem 2.1, since W admits a weak Kähler-Einstein metric, the Futaki invariant vanishes identically and $\mathrm{Aut}(W)$ is reductive. Then, by the Luna slice theorem, there is a test configuration of $(M, -K_M^r)$ with $(W, \mathcal{O}_{\mathbb{CP}^N}(1))$ as the central fiber. Since M is K-stable (by virtue of admitting a Kähler-Einstein metric), this forces W to be biholomorphic to M and ω_∞ to be a smooth Kähler-Einstein metric. But then by the uniqueness of Kähler-Einstein metrics, ω_∞ is isometric to ω_{KE}, and hence $(M, \omega_i) \xrightarrow{d_{GH}} (M, \omega_{KE})$. □

We would like to remark that Theorem 2.1 can also be proved by using the result in [6] and we sketch the proof below. By the assumption of Theorem 2.1, (M, ω_i) converges to a metric space (Z, d) after passing to a subsequence since the diameter is uniformly bounded above by volume comparison. The main result of [6] states that Z is an n-dimensional normal projective variety. For sufficiently large $k > 0$, the L^2-orthonormal basis $\{\sigma_0^{(i)}, ..., \sigma_{N_k}^{(i)}\}$ of $H^0(M, (-K_M)^k)$ with respect to ω_i and its induced hermitian metric on $-K_X$ converge to an orthonormal basis of $H^0(Z, (-K_Z)^k)$ thanks to the partial C^0-estimate from [6]. The basis $\{\sigma_0^{(i)}, ..., \sigma_{N_k}^{(i)}\}$ induces a sequence of Fubini-Study metrics

$$\theta_i = k^{-1}\sqrt{-1}\partial\overline{\partial}\log\left(|\sigma_0^{(i)}|^2 + ... + |\sigma_{N_k}^{(i)}|^2\right)$$

and $\omega_i = \theta_i + \sqrt{-1}\partial\overline{\partial}\varphi_i$ with φ_i uniformly bounded in $L^\infty(M)$. Furthermore, if we let $\Omega_i = \left(|\sigma_0^{(i)}|^2 + ... + |\sigma_{N_k}^{(i)}|^2\right)^{-1/k}$ be the induced volume form on M, then φ_i satisfies the following complex Monge-Ampere equation

$$(\theta_i + \sqrt{-1}\partial\overline{\partial}\varphi_i)^n = e^{-(1-\delta_i)\varphi_i - \delta_i f_i}\Omega_i,$$

where $\theta_i + \sqrt{-1}\partial\overline{\partial}f_i \geq 0$ and $\int_M e^{-\delta_i f_i}\Omega_i$ is uniformly bounded for all i. After letting $i \to \infty$, the limiting equation is given by

$$(\theta_\infty + \sqrt{-1}\partial\overline{\partial}\varphi_\infty)^n = e^{-\varphi_\infty + F_\infty}\Omega_\infty,$$

for some global plurisubharmonic function F_∞ on Z. The reader can refer to [9] for more details (cf. Section 3). This immediately implies that F_∞ is a constant and $\omega_\infty = \theta_\infty + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$ is a Kähler-Einstein metric with bounded local potentials. This replaces the proof of Lemma 2.4 and Theorem 1.1 is proved by the same argument as above using K-stability to show that Z must be biholomorphic to M and ω_∞ is the unique Kähler-Einstein metric on Z up to an automorphism of Z.

Licensed to Rutgers Univ-New Brunswick. Prepared on Thu Jan 27 11:28:34 EST 2022 for download from IP 128.6.45.205.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We argue by contradiction. By choosing δ small enough, by the appendix of [14], we may assume that M is biholomorphic to $\mathbb{CP}^n$. In particular, $[\omega]$ is a multiple of $c_1(M)$. Suppose there exists an $\varepsilon > 0$, a sequence $\delta_i \to 0$ and a sequence of metrics ω_i on $\mathbb{CP}^n$ such that

$$\mathrm{Ric}(\omega_i) \geq \omega_i,$$
$$\mathrm{Vol}(\mathbb{CP}^n, \omega_i) \geq (1-\delta_i)\mathrm{Vol}(\mathbb{CP}^n, \omega_{\mathbb{CP}^n}),$$

but

$$d_{GH}\Big((\mathbb{CP}^n, \omega_i), (\mathbb{CP}^n, \omega_{\mathbb{CP}^n})\Big) \geq \varepsilon. \tag{2.2}$$

Consider the rescaled metrics $\tilde{\omega}_i = \frac{\mathrm{Vol}(\mathbb{CP}^n, \omega_{\mathbb{CP}^n})^{1/n}}{\mathrm{Vol}(\mathbb{CP}^n, \omega_i)^{1/n}}\omega_i$. Then, $\mathrm{Vol}(\mathbb{CP}^n, \tilde{\omega}_i) = \mathrm{Vol}(\mathbb{CP}^n, \omega_{\mathbb{CP}^n})$, and so $\tilde{\omega}_i \in c_1(\mathbb{CP}^n)$. Moreover, we also have

$$\omega_i \leq \tilde{\omega}_i \leq \frac{1}{1-\delta_i}\omega_i$$
$$\mathrm{Ric}(\tilde{\omega}_i) \geq (1-\delta_i)\tilde{\omega}_i.$$

By Theorem 2.1, $(\mathbb{CP}^n, \tilde{\omega}_i) \xrightarrow{d_{GH}} (\mathbb{CP}^n, \omega_{\mathbb{CP}^n})$. Since $\frac{\mathrm{Vol}(\mathbb{CP}^n, \omega_{\mathbb{CP}^n})}{\mathrm{Vol}(\mathbb{CP}^n, \omega_i)}$ is almost one, we can make sure that for $i >> 1$,

$$d_{GH}\Big((\mathbb{CP}^n, \tilde{\omega}_i), (\mathbb{CP}^n, \frac{\mathrm{Vol}(\omega_{\mathbb{CP}^n})^{1/n}}{\mathrm{Vol}(\omega_i)^{1/n}}\omega_{\mathbb{CP}^n})\Big) \leq \frac{\varepsilon \mathrm{Vol}(\omega_{\mathbb{CP}^n})^{1/2n}}{2\mathrm{Vol}(\omega_i)^{1/2n}}.$$

This contradicts the inequality in (2.2). ☐

References

[1] Robert J. Berman and Bo Berndtsson, *The volume of Kähler-Einstein Fano varieties and convex bodies*, J. Reine Angew. Math. **723** (2017), 127–152, DOI 10.1515/crelle-2014-0069. MR3604979

[2] Xiuxiong Chen, Simon Donaldson, and Song Sun, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than* 2π, J. Amer. Math. Soc. **28** (2015), no. 1, 199–234, DOI 10.1090/S0894-0347-2014-00800-6. MR3264767

[3] Tobias H. Colding, *Shape of manifolds with positive Ricci curvature*, Invent. Math. **124** (1996), no. 1-3, 175–191, DOI 10.1007/s002220050049. MR1369414

[4] Simon Donaldson and Song Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math. **213** (2014), no. 1, 63–106, DOI 10.1007/s11511-014-0116-3. MR3261011

[5] Kento Fujita, *Optimal bounds for the volumes of Kähler-Einstein Fano manifolds*, Amer. J. Math. **140** (2018), no. 2, 391–414, DOI 10.1353/ajm.2018.0009. MR3783213

[6] Gang Liu, *Gromov-Hausdorff limits of Kähler manifolds with bisectional curvature lower bound*, Comm. Pure Appl. Math. **71** (2018), no. 2, 267–303, DOI 10.1002/cpa.21724. MR3745153

[7] Robert Molzon and Karen Pinney Mortensen, *A characterization of complex projective space up to biholomorphic isometry*, J. Geom. Anal. **7** (1997), no. 4, 611–621, DOI 10.1007/BF02921636. MR1669219

[8] G. Perelman, *Manifolds of positive Ricci curvature with almost maximal volume*, J. Amer. Math. Soc. **7** (1994), no. 2, 299–305, DOI 10.2307/2152760. MR1231690

[9] G. Szekelyhidi and J. Ross, *Twisted Kähler-Einstein metrics*, arXiv:1911.03442.

[10] Gábor Székelyhidi, *The partial* C^0*-estimate along the continuity method*, J. Amer. Math. Soc. **29** (2016), no. 2, 537–560, DOI 10.1090/jams/833. MR3454382

[11] Gang Tian, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), no. 7, 1085–1156, DOI 10.1002/cpa.21578. MR3352459

Licensed to Rutgers Univ-New Brunswick. Prepared on Thu Jan 27 11:28:34 EST 2022 for download from IP 128.6.45.205.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[12] Gang Tian and Bing Wang, *On the structure of almost Einstein manifolds*, J. Amer. Math. Soc. **28** (2015), no. 4, 1169–1209, DOI 10.1090/jams/834. MR3369910

[13] B. Wang, *The local entropy along Ricci flow—Part B: the pseudo-locality theorems*, arXiv:2010.09981.

[14] K. Zhang, *On the optimal volume upper bound for Kähler manifolds with positive Ricci curvature*, (with an appendix by Liu, Y.), arXiv:2001.04169v2

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
Email address: vvdatar@iisc.ac.in

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
Email address: harish@iisc.ac.in

Department of Mathematics, Rutgers University, Piscataway, New Jersey 08854
Email address: jiansong@math.rutgers.edu

Licensed to Rutgers Univ-New Brunswick. Prepared on Thu Jan 27 11:28:34 EST 2022 for download from IP 128.6.45.205.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use