

A Steiner formula in the L_p Brunn Minkowski theory *

Kateryna Tatarko and Elisabeth M. Werner †

Abstract

We prove an analogue of the classical Steiner formula for the L_p affine surface area of a Minkowski outer parallel body for any real parameter p . We show that the classical Steiner formula and the Steiner formula of Lutwak's dual Brunn Minkowski theory are special cases of this new Steiner formula. This new Steiner formula and its localized versions lead to new curvature measures that have not appeared before in the literature. They have the intrinsic volumes of the classical Brunn Minkowski theory as well as the dual quermassintegrals of the dual Brunn Minkowski theory as special cases.

Properties of these new quantities are investigated, a connection to information theory among them. A Steiner formula for the s -th mixed L_p affine surface area of a Minkowski outer parallel body for any real parameters p and s is also given.

1 Introduction and Results

1.1 Introduction

The Brunn Minkowski theory whose classical building block is also called the theory of *mixed volumes* is the very core of convex geometric analysis. It centers around the study of geometric invariants and geometric measures associated with convex bodies. A central part of the theory is the classical Steiner formula. It is via this formula that the elementary mixed volumes, or intrinsic volumes, are defined as the coefficients in the polynomial expansion in t of the volume of the outer parallel body $K + tB_2^n$ of the convex body K with the Euclidean unit ball B_2^n ,

$$\text{vol}_n(K + tB_2^n) = \sum_{i=0}^n \binom{n}{i} W_i(K) t^i = \sum_{i=0}^n \text{vol}_{n-i}(B_2^{n-i}) V_i(K) t^{n-i}. \quad (1)$$

The coefficients $W_i(K)$ and $V_i(K)$ are called quermassintegrals and intrinsic volumes, respectively. Note that the intrinsic volumes $V_i(K) = \binom{n}{i} \frac{W_{n-i}(K)}{\text{vol}_{n-i}(B_2^{n-i})}$ are just differently normalized quermassintegrals. Intrinsic volumes, respectively quermassintegrals, are the principal geometric invariants in the Brunn Minkowski theory and include volume, surface area and mean width. We refer readers with further interest in this topic to the books [13, 20]. Various generalizations and applications of Steiner's formula can be found in [15, 42, 44]. Results from the classical Brunn Minkowski theory have been applied to and have connections with numerous mathematical disciplines [16, 47].

A theory analogous and dual to the Brunn Minkowski theory, called the theory of *dual mixed volumes* or *dual Brunn Minkowski theory*, was introduced by Lutwak [30]. The main geometric invariants in the dual Brunn Minkowski theory are the dual quermassintegrals $\widetilde{W}_i(K)$. As proved in [30], they appear as the coefficients in the Steiner formula of the dual Brunn Minkowski theory

*Keywords: Steiner formula, curvature measures, (Dual) Brunn Minkowski theory, L_p Brunn Minkowski theory, 2010 Mathematics Subject Classification: 52A39 (Primary), 28A75, 52A20, 53A07 (Secondary)

†Partially supported by NSF grant DMS-1811146

where Minkowski addition “+” of convex bodies is replaced by the radial addition “ $\tilde{+}$ ” of star bodies. Indeed, for a star body K and $t > 0$, we have that

$$\text{vol}_n(K \tilde{+} tB_2^n) = \sum_{i=0}^n \binom{n}{i} \widetilde{W}_i(K) t^i = \sum_{i=0}^n \binom{n}{i} \widetilde{V}_i(K) t^{n-i}.$$

The details are given in Section 2.1. Investigations in the dual Brunn Minkowski theory led to isoperimetric inequalities and kinematic formulas involving dual mixed volumes, as summarized in [16, 47].

A localization of the quermassintegrals gives rise to curvature measures for Borel sets in $\mathbb{R}^n$, respectively area measures for Borel sets on the Euclidean unit sphere. The *classical Minkowski problem* asks to characterize those measures. Similarly, a localization of the dual quermassintegrals leads to the dual curvature measures [25, 30], and analogously to the *dual Minkowski problem*. Much work has been devoted to these problems. We refer to, e.g., [4, 5, 6, 11, 25, 43, 51, 62] for background and progress.

An extension of the classical Brunn Minkowski theory, the L_p Brunn Minkowski theory, was initiated by Lutwak in the groundbreaking paper [33]. This theory evolved rapidly over the last years and due to a number of highly influential works, see, e.g., [18, 21, 22, 23, 28, 29], [39] - [41], [48] - [57], it is now a central part of modern convex geometry. The L_p Brunn Minkowski theory centers around the study of affine invariants associated with convex bodies. In fact, this theory redirected much of the research about convex bodies from the Euclidean aspects to the study of the affine geometry of these bodies, and some questions that had been considered Euclidean in nature turned out to be affine problems. For example, the famous Busemann-Petty Problem (finally laid to rest in [17, 19, 59, 60]), was shown to be an affine problem with the introduction of intersection bodies by Lutwak in [32]. Central objects in the L_p Brunn Minkowski theory are the L_p affine surface areas,

$$as_p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, \nu(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(x),$$

where $\nu(x)$ denotes the outer unit normal at $x \in \partial K$, the boundary of K , $H_{n-1}(x)$ is the Gauss curvature at x and $\mathcal{H}^{n-1}$ is the usual surface area measure on ∂K .

In this paper we provide an analogue of the Steiner formula for the L_p affine surface area and s -th mixed p -affine surface area (see Section 5) of a Minkowski outer parallel body for any real parameters $p \neq -n$ and s , i.e. we investigate

$$as_p(K + tB_2^n) \quad \text{and} \quad as_{p,s}(K + tB_2^n).$$

A different Steiner formula for the L_p Brunn Minkowski theory, involving Blaschke sum instead of Minkowski sum, was put forward by Lutwak in [31].

Our new Steiner formula, presented in Theorem A and 3.1, covers a whole range of Steiner formulas for all $-\infty \leq p \leq \infty$. It includes the classical Steiner formula and the Steiner formula from the dual L_p Brunn Minkowski theory as special cases.

We call the coefficients in our Steiner formula L_p Steiner coefficients. In their general form they do not seem to have appeared before in the literature. Special cases of those include not only the classical quermassintegrals and the dual quermassintegrals, but also variants of the Willmore energy. This is explained in Section 4. We also observe a connection of the L_p Steiner coefficients to information theory. Analogously to the curvature measures of the classical theory and the dual curvature measures of the dual Brunn Minkowski theory, the L_p Steiner coefficients lead to new curvature measures and area measures, respectively. They are discussed in more detail in Section 3. The Steiner formula for the s -th mixed L_p affine surface area is treated in Section 5.

1.2 Results

For a convex body K in $\mathbb{R}^n$ (that is, a compact, convex set with non-empty interior), we define

$$\beta(K) = \min_{u \in S^{n-1}} h_K(u). \quad (2)$$

If K is such that its centroid is at the origin, which we will assume throughout, there is $\lambda(K) > 0$ such that $B^n(0, \lambda(K)) \subset K$. As K is bounded, there is $\lambda(K) \leq \Lambda(K) < \infty$ such that $K \subset B^n(0, \Lambda(K))$. Thus,

$$\lambda(K) \leq \beta(K) \leq \Lambda(K).$$

Theorem A *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(K)$. For all $p \in \mathbb{R}$, $p \neq -n$,*

$$as_p(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} t^k \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m d\sigma(u) \right]. \quad (3)$$

When integrating over the sphere, we write $\sigma(u) = \mathcal{H}^{n-1}(u)$. The coefficients A_p^m represent a sum of mixed products of the elementary symmetric functions of the principal curvatures $H_i = H_i(\xi_K(u))$ (see Section 2.1 below), with corresponding multinomial coefficients. The detailed statement and local versions of it will be given in Section 3.

We call the coefficients in the formula (3) the L_p Steiner coefficients

$$\mathcal{W}_{m,k}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m d\sigma(u). \quad (4)$$

The expression $f_K(u)h_K(u)^{1-p}$ is called the L_p curvature function $f_p(K, \cdot) : S^{n-1} \rightarrow [0, \infty)$. With this notation we can write

$$\mathcal{W}_{m,k}(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}} h_K(u)^{m-k} A_p^m d\sigma(u).$$

We want to point out that the first coefficient in the expansion (3) represents the L_p affine surface area of the body K ,

$$\mathcal{W}_{0,0}(K) = as_p(K).$$

As other special cases, we get mixed affine surface areas, defined in Section 5. More properties of the coefficients are discussed in Section 4, a connection to information theory among them.

The case $p = 0$ of Theorem A reduces to the classical Steiner formula which involves the classical quermassintegrals $W_i(K)$ (see (7) below).

Corollary 1.1 *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 . Then*

$$as_0(K + tB_2^n) = n \sum_{i=0}^n \binom{n}{i} W_i(K) t^i.$$

In the case $p = \pm\infty$, Theorem A reduces to the Steiner formula from the dual Brunn Minkowski theory involving Lutwak's dual quermassintegrals $\widetilde{W}_i(K)$ (see (9) below).

Corollary 1.2 *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(K)$. Then*

$$\begin{aligned} as_{\pm\infty}(K + tB_2^n) &= n \operatorname{vol}((K + tB_2^n)^\circ) = n \widetilde{W}_0((K + tB_2^n)^\circ) \\ &= n \sum_{i=0}^{\infty} \binom{-n}{i} \widetilde{W}_{-i}(K^\circ) t^i. \end{aligned}$$

Here, $K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}$ is the polar body of the convex body K .

Thus Theorem A covers a whole range of Steiner formulas in the L_p Brunn Minkowski theory for all $-\infty \leq p \leq \infty$, including the classical case and a case related to the dual Brunn Minkowski theory.

2 Background

2.1 Background from differential geometry

For more information and the details in this section we refer to e.g., [16, 47].

Let K be a convex body of class C^2 . For a point x on the boundary ∂K of K we denote by $\nu(x)$ the unique outward unit normal vector of K at x . The map $\nu : \partial K \rightarrow S^{n-1}$ is called the spherical image map or Gauss map of K and is of class C^1 . Its differential is called the Weingarten map. The eigenvalues of the Weingarten map are the principal curvatures $k_i(x)$ of K at x .

The j -th normalized elementary symmetric functions of the principal curvatures are denoted by H_j . They are defined as follows

$$H_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \dots < i_j \leq n-1} k_{i_1} \cdots k_{i_j} \quad (5)$$

for $j = 1, \dots, n-1$ and $H_0 = 1$. Note that

$$H_1 = \frac{1}{n-1} \sum_{1 \leq i \leq n-1} k_i$$

is the mean curvature, that is the average of principal curvatures, and

$$H_{n-1} = \prod_{i=1}^{n-1} k_i$$

is the Gauss curvature.

We say that K is of class C_+^2 if K is of class C^2 and the Gauss map ν is a diffeomorphism. This means in particular that ν has a smooth inverse. This assumption is stronger than just C^2 , and is equivalent to the assumption that all principal curvatures are strictly positive, or that the Gauss curvature $H_{n-1} \neq 0$. It also means that the differential of ν , i.e., the Weingarten map, is of maximal rank everywhere.

Let K be of class C_+^2 . For $u \in \mathbb{R}^n \setminus \{0\}$, let $\xi_K(u)$ be the unique point on the boundary of K at which u is an outward normal vector. The map ξ_K is defined on $\mathbb{R}^n \setminus \{0\}$. Its restriction to the sphere S^{n-1} , the map $\bar{\xi}_K : S^{n-1} \rightarrow \partial K$, is called the reverse spherical image map, or reverse Gauss map. The differential of $\bar{\xi}_K$ is called the reverse Weingarten map. The eigenvalues of the reverse Weingarten map are called the principal radii of curvature $r_1, \dots, r_{n-1}$ of K at $u \in S^{n-1}$.

The j -th normalized elementary symmetric functions of the principal radii of curvature are denoted by s_j . In particular, $s_0 = 1$, and for $1 \leq j \leq n-1$ they are defined by

$$s_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \dots < i_j \leq n-1} r_{i_1} \cdots r_{i_j}. \quad (6)$$

Note that the principal curvatures are functions on the boundary of K and the principal radii of curvature are functions on the sphere.

Now we describe the connection between H_j and s_j . For a body K of class C_+^2 , we have for $u \in S^{n-1}$ that $\bar{\xi}_K(u) = \nu^{-1}(u)$. In particular, the principal radii of curvature are reciprocals of the principal curvatures, that is

$$r_i(u) = \frac{1}{k_i(\bar{\xi}_K(u))}.$$

This implies that for $x \in \partial K$ with $\nu(x) = u$,

$$s_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \dots < i_j \leq n-1} \frac{1}{k_{i_1}(\bar{\xi}_K(u)) \cdots k_{i_j}(\bar{\xi}_K(u))} = \frac{H_{n-1-j}(\bar{\xi}_K(u))}{H_{n-1}}$$

and

$$H_j = \frac{s_{n-1-j}}{s_{n-1}}(\nu(x)),$$

for $j = 1, \dots, n-1$.

The mixed volumes $W_i(K)$ of the classical Steiner formula (1) can be expressed with the help of the elementary symmetric functions of the principal curvatures. By definition, $W_0(K) = \text{vol}_n(K)$ and

$$W_i(K) = \frac{1}{n} \int_{\partial K} H_{i-1} d\mathcal{H}^{n-1}, \quad (7)$$

for $i = 1, \dots, n$. Moreover, we have the following formulae, which in integral geometry and differential geometry are known as *Minkowskian integral formulae*,

$$\int_{S^{n-1}} s_j d\mathcal{H}^{n-1} = \int_{S^{n-1}} h_K s_{j-1} d\mathcal{H}^{n-1},$$

and

$$\int_{\partial K} H_{j-1} d\mathcal{H}^{n-1} = \int_{\partial K} \langle x, \nu(x) \rangle H_j d\mathcal{H}^{n-1} \quad (8)$$

for $j = 1, \dots, n-1$.

The dual mixed volume of the convex bodies K and L that contain 0 in its interior is defined for all real i by

$$\tilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} \rho_L(u)^i d\sigma(u),$$

where $\rho_K(u) = \max\{\lambda \geq 0 \mid \lambda u \in K\}$ is the radial function of K . In particular, if $L = B_2^n$, then

$$\widetilde{W}_i(K) = \tilde{V}_i(K, B_2^n) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} d\sigma(u) \quad (9)$$

are called *dual quermassintegrals* of order i . The corresponding Steiner formula in the dual Brunn Minkowski theory is

$$\text{vol}_n(K \widetilde{+} t B_2^n) = \sum_{i=0}^n \binom{n}{i} \widetilde{W}_i(K) t^i.$$

A well known change of integral formula, e.g., [47], which we use frequently, says that for a C_+^2 convex body K and a continuous function $g : \partial K \rightarrow \mathbb{R}$

$$\int_{S^{n-1}} g(u) f_K(u) d\sigma(u) = \int_{\partial K} g(x) d\mathcal{H}^{n-1}(x), \quad (10)$$

where $u \in S^{n-1}$ and $x \in \partial K$ are related via the Gauss map, i.e., $\nu(x) = u$.

2.2 Background from affine geometry

From now on we will always assume that the centroid of a convex body K in $\mathbb{R}^n$ is at the origin. For real $p \neq -n$, we define the L_p affine surface area $as_p(K)$ of K as in [30] ($p > 1$) and [49] ($p < 1$, $p \neq -n$) by

$$as_p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, \nu(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(x) \quad (11)$$

and

$$as_{\pm\infty}(K) = \int_{\partial K} \frac{H_{n-1}(x)}{\langle x, \nu(x) \rangle^n} d\mathcal{H}^{n-1}(x). \quad (12)$$

In particular, for $p = 0$,

$$as_0(K) = \int_{\partial K} \langle x, \nu(x) \rangle d\mathcal{H}^{n-1}(x) = n \operatorname{vol}_n(K). \quad (13)$$

The case $p = 1$,

$$as_1(K) = \int_{\partial K} H_{n-1}(x)^{\frac{1}{n+1}} d\mathcal{H}^{n-1}(x)$$

is the classical affine surface area which is independent of the position of K in space. For dimensions 2 and 3 and sufficiently smooth convex bodies, its definition goes back to Blaschke [3].

If the boundary of K is sufficiently smooth, then (11) and (12) can be written as integrals over the boundary $\partial B_2^n = S^{n-1}$ of the Euclidean unit ball B_2^n in $\mathbb{R}^n$,

$$as_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u), \quad (14)$$

where $f_K(u)$ is the curvature function, i.e. the reciprocal of the Gaussian curvature $H_{n-1}(x)$ at this point $x \in \partial K$ that has u as outer normal. In particular, for $p = \pm\infty$,

$$as_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n \operatorname{vol}_n(K^\circ), \quad (15)$$

where $K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}$ is the polar body of K .

For $p = -n$, the L_{-n} affine surface area was introduced in [39] as

$$as_{-n}(K) = \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}}. \quad (16)$$

3 The Steiner formula of the L_p Brunn Minkowski theory

In this section, we state our main theorems and discuss some of their consequences. The proofs are given in Section 6.

3.1 The general case

We will need the generalized binomial coefficients. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, they are defined as

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0, \\ \frac{\alpha(\alpha-1) \cdots (\alpha-k+1)}{k!} & \text{if } k > 0. \end{cases} \quad (17)$$

Also, we will need the multinomial coefficients from the multinomial formula

$$(a_1 + \dots + a_r)^q = \sum_{\substack{i_1, \dots, i_r \geq 0 \\ i_1 + \dots + i_r = q}} \binom{q}{i_1, \dots, i_r} a_1^{i_1} \cdots a_r^{i_r}, \quad (18)$$

where

$$\binom{q}{i_1, i_2, \dots, i_r} = \frac{q!}{i_1! i_2! \cdots i_r!}$$

is the multinomial coefficient. Note that

$$\binom{q}{i_1, i_2, \dots, i_r} = 0 \quad \text{if } i_j < 0 \text{ or } i_j > q. \quad (19)$$

The sum in the multinomial formula is taken over all nonnegative integer indices $i_1, \dots, i_r$ such that the sum of all i_j is q . In the case $r = 2$, we get the binomial theorem.

To give the precise statement of Theorem A, we introduce the following coefficients, which are sums of mixed products of the elementary symmetric functions of the principal curvatures, up to some multinomial coefficients. For any real $p \neq -n$,

$$A_p^m = A_p^m(\bar{\xi}_K(u)) = \sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{\frac{n}{n+p}}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j}(\bar{\xi}_K(u)) \right\}. \quad (20)$$

For $p = -n$, we have

$$A_{-n}^m = A_{-n}^m(u) = \sum_{\substack{i_1, \dots, i_{2n} \geq 0 \\ i_1 + 2i_2 + \dots + 2ni_{2n} = m}} \binom{\frac{1}{2}}{i_1 + \dots + i_{2n}} \binom{i_1 + \dots + i_{2n}}{i_1, \dots, i_{2n}} \prod_{q=1}^{2n} B_q^{i_q} \quad (21)$$

and

$$B_q = B_q(u) = \sum_{k+i=q} \left[\binom{n-1}{k} \binom{n+1}{i} \frac{H_k(\bar{\xi}_K(u))}{h_K(u)^i} \right].$$

Recall that the L_p Steiner coefficients are defined by

$$\mathcal{W}_{m,k}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p} - k + m} A_p^m d\sigma(u).$$

Theorem 3.1 *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(K)$. For all $p \in \mathbb{R}$, $p \neq -n$,*

$$as_p(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}_{m,k} t^k \right]. \quad (22)$$

In particular,

$$as_1(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} A_1^m d\sigma(u) \right] t^m = \sum_{m=0}^{\infty} \mathcal{W}_{m,m} t^m. \quad (23)$$

The cases $p = 0$ and $p = \pm\infty$ are Corollary 1.1 and Corollary 1.2, respectively.

The next Theorem 3.2 describes the case $p = -n$.

Theorem 3.2 *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(K)$. Then*

$$as_{-n}(K + tB_2^n) = \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \sum_{m=0}^{\infty} A_{-n}^m t^m.$$

Observe that the first coefficient in the expansion is $as_{-n}(K)$.

Remark on the polytopal case

When $K = P$ is a polytope, we denote by $\text{vert } P$ the set of its vertices and for $v \in \text{vert } P$,

$$\text{Norm}(v) = \{u \in \mathbb{R}^n : \langle u, z - v \rangle \leq 0 \text{ for all } z \in P\}$$

is the normal cone to P at v , see [47]. Then the following Steiner formula for polytopes holds.

Theorem 3.3 *Let P be a convex polytope in $\mathbb{R}^n$ and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(P)$. For all $p \in \mathbb{R}$ such that*

$$\begin{aligned} (i) \quad & p \notin [-n, 0], \\ & as_p(P + tB_2^n) = \sum_{m=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{m} \sum_{v \in \text{vert } P} \int_{u \in \text{Norm}(v)} h_P(u)^{\frac{n(1-p)}{n+p}-m} d\sigma(u) t^{m+\frac{n(n-1)}{n+p}}; \\ (ii) \quad & p \in [-n, 0), \quad as_p(P + tB_2^n) = \infty; \\ (iii) \quad & p = 0, \quad as_0(P + tB_2^n) = n \text{vol}_n(P + tB_2^n). \end{aligned}$$

Note that by definition (16), $as_{-n}(P)$ is trivially infinite, since the curvature function of the flat part is infinite. Of course, $as_{-n}(P + tB_2^n)$ is also infinite by the same reasoning.

3.2 Local version

In this section, we introduce new curvature and area measures for Borel sets on the boundary of a convex body K and for Borel sets on the Euclidean sphere S^{n-1} , and state local Steiner formula of the L_p Brunn Minkowski theory.

When K is a C_+^2 convex body, the curvature, respectively area, measures for Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$ and $\omega \in \mathcal{B}(S^{n-1})$ are defined by

$$\begin{aligned} C_i(K, B) &= \int_{\partial K \cap B} H_{n-1-i}(x) d\mathcal{H}^{n-1}(x); \\ S_i(K, \omega) &= \int_{\omega} s_i d\sigma(u) \end{aligned}$$

for $i = 0, \dots, n-1$, e.g., [47]. Note that for general convex bodies these measures replace the elementary symmetric functions of principal curvatures and the elementary symmetric functions of the principal radii of curvature. In the extreme cases $i = 0$ and $i = n-1$, we obtain $C_{n-1}(K, B) = \mathcal{H}(\partial K \cap B)$ and $S_0(K, \omega) = \sigma(\omega)$. If $B = \mathbb{R}^n$ and $\omega = S^{n-1}$, we get the classical quermassintegrals

$$W_i(K) = \frac{1}{n} C_{n-i}(K, \mathbb{R}^n) = \frac{1}{n} S_{n-i}(K, S^{n-1})$$

which were introduced in (7).

Our approach in Theorem 3.1 leads to new curvature and area measures. Namely, for a Borel set $B \in \mathcal{B}(\mathbb{R}^n)$ and for a Borel set $\omega \in \mathcal{B}(S^{n-1})$,

$$\mathcal{C}_{m,k}(K, B) = \int_{\partial K \cap B} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, \nu(x) \rangle^{\frac{n(p-1)}{n+p}+k-m}} A_p^m(x) d\mathcal{H}^{n-1}$$

and

$$\mathcal{S}_{m,k}(K, \omega) = \int_{\omega} s_{n-1}(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m(\bar{\xi}(u)) d\sigma(u)$$

for $k \geq m$ and $m \geq 0$. Taking $B = \mathbb{R}^n$ and $\omega = S^{n-1}$, we recover the L_p Steiner coefficients, i.e.

$$\mathcal{W}_{m,k}(K) = \mathcal{C}_{m,k}(K, \mathbb{R}^n) \quad \text{and} \quad \mathcal{W}_{m,k}(K) = \mathcal{S}_{m,k}(K, S^{n-1}).$$

When $m = k = 0$, $B = \mathbb{R}^n$ and $\omega = S^{n-1}$, we obtain the L_p affine surface area, i.e.

$$\mathcal{C}_{0,0}(K, \mathbb{R}^n) = \mathcal{S}_{0,0}(K, S^{n-1}) = as_p(K).$$

Now we can state the local Steiner formula for these newly introduced measures.

Theorem 3.4 (Local Steiner formula) *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 , $B \in \mathcal{B}(\mathbb{R}^n)$ and $\omega \in \mathcal{B}(S^{n-1})$. Let t be such that $0 \leq t < \beta(K)$. For all $p \in \mathbb{R}^n$, $p \neq -n$, we have*

$$\mathcal{C}_{0,0}(K + tB_2^n, B) = \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{C}_{m,k}(K, B) t^k$$

and

$$\mathcal{S}_{0,0}(K + tB_2^n, \omega) = \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{S}_{m,k}(K, \omega) t^k.$$

We recover the Steiner formula (22) of the L_p Brunn Minkowski theory when $B = \mathbb{R}^n$ or $\omega = S^{n-1}$.

4 Properties of the coefficients

In this section, we discuss some properties of the new coefficients which appeared in Theorem 3.1, i.e. the L_p Steiner coefficients,

$$\mathcal{W}_{m,k}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p} - k + m} A_p^m d\sigma(u)$$

for $k \geq m$ and $m \in \mathbb{N} \cup \{0\}$, where A_p^m is given by (20). We will see that the s -mixed p -affine surface areas, defined in Section 5, appear as special cases of L_p Steiner coefficients. As it is known that mixed affine surface areas in general are not affine invariant quantities, we cannot expect affine invariance for L_p Steiner coefficients either.

4.1 Willmore energy

Firstly, we restrict ourselves to three-dimensional space. Recall that the Willmore energy of a compact surface Σ in $\mathbb{R}^3$ is given by

$$W_E(\Sigma) = \int_{\Sigma} H_1^2 d\mathcal{H}^2,$$

where $H_1 = (k_1 + k_2)/2$ is the mean curvature. The Willmore energy naturally appears in mathematical biology and physics, and has been widely studied. In the 1960s Willmore [58] conjectured the lower bound on the Willmore energy of a torus immersed in $\mathbb{R}^3$. This conjecture was proved only recently in [38]. The choice of the exponent 2 of the mean curvature and dimension $n = 3$ in the definition of the Willmore energy is the proper fit in the context of differential geometry as W_E is invariant under conformal maps. The natural generalization of the type $\int_{\Sigma} H_1^n d\mathcal{H}^{n-1}$ of the Willmore energy to higher dimensional hypersurfaces is called Willmore-Chen functional. When $n = 2$, it coincides with the Willmore energy. The Willmore energy and Willmore-Chen functionals have been studied in [1, 10, 26] and references therein. If one considers integrals of the type $\int_{\Sigma} H_1^\alpha d\mathcal{H}^{n-1}$, then they lose their conformal invariance and become much more difficult to study. For any dimension n , we recover such integrals as the L_p Steiner coefficients in the Steiner formula (22).

4.2 The case $p = 1$

We start analyzing the coefficients from expansion (23). We observe that among them there are mixed affine surface areas which will be introduced in Section 5.

If $m = 0$, then $A_1^0 = 1$ and we have

$$\mathcal{W}_{0,0}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} A_1^0 d\sigma(u) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} d\sigma(u) = as_1(K).$$

Hence, the first term in (23) is the classical affine surface area of a body K .

If $m = l(n-1)$, $l \in \mathbb{N}$, then $A_1^{l(n-1)} = \left(\frac{n}{n+1}\right) H_{n-1}^l$. Thus,

$$\begin{aligned} \mathcal{W}_{l(n-1), l(n-1)}(K) &= \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} A_1^{l(n-1)} d\sigma(u) = \left(\frac{n}{n+1}\right) \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}-l} d\sigma(u) \\ &= \left(\frac{n}{n+1}\right) \int_{S^{n-1}} f_K(u)^{\frac{n-l(n+1)}{n+1}} d\sigma(u) = \left(\frac{n}{n+1}\right) as_{1, l(n+1)}(K), \end{aligned}$$

where $as_{1, l(n+1)}(K)$ is the $l(n+1)$ -mixed 1-affine surface area of K . Therefore, in (23) we have mixed affine surface areas as coefficients in front of powers of t , which are multiples of $n-1$.

4.3 The general case

Now we move to analyzing the coefficients appearing in (22). We note that again mixed affine surface areas appear.

If $m = 0$, then $A_p^0 = 1$ and we have

$$\begin{aligned} \mathcal{W}_{0,k}(K) &= \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k} A_p^0 d\sigma(u) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k} d\sigma(u) \\ &= as_{p+\frac{k}{n}(n+p), -k}(K), \end{aligned}$$

for $k \in \mathbb{N} \cup \{0\}$. When $k = 0$, we get the L_p affine surface area $as_p(K)$ of a body K .

If $m = l(n-1)$, $l \in \mathbb{N}$, then $A_p^{l(n-1)} = \left(\frac{n}{n+p}\right) H_{n-1}^l$. Thus,

$$\begin{aligned} \mathcal{W}_{l(n-1), k}(K) &= \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+l(n-1)} A_p^{l(n-1)} d\sigma(u) = \\ &= \left(\frac{n}{n+p}\right) \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}-l} h_K(u)^{\frac{n(1-p)}{n+p}-k+l(n-1)} d\sigma(u) \\ &= as_{\frac{np+(n+p)(k-ln)}{n-l(n+p)}, 2nl-k}(K). \end{aligned}$$

The L_p Steiner coefficients consist of combinations of mixed products of the elementary symmetric functions of the principal curvatures H_i and the support function. Applying Hölder's inequality, we can bound those integrals from above. We present one typical example and the other cases can be dealt accordingly.

For instance, consider the case when only one symmetric function of the principal curvatures H_j appears in A_p^m , that is, if $m = lj$, $l \in \mathbb{N}$, i.e., all $i_s = 0$, $s \neq j$ and $i_j = l$ for $1 \leq j \leq n-2$, then $A_p^{lj} = \left(\frac{n}{n+p}\right) \binom{n-1}{j}^l H_j^l$. Using Hölder's inequality, we get an upper bound

$$\begin{aligned} \mathcal{W}_{lj, k}(K) &= \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k} A_p^{lj} d\sigma(u) \\ &= \left(\frac{n}{n+p}\right) \binom{n-1}{j}^l \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+lj} H_j^l d\sigma(u) \\ &\leq \left(\frac{n}{n+p}\right) \binom{n-1}{j}^l \left(\int_{S^{n-1}} H_j^{2l} d\sigma(u) \right)^{\frac{1}{2}} as_{p+\frac{(k-lj)(n+p)}{n}, 2lj-2k-n}(K)^{\frac{1}{2}}. \end{aligned}$$

Applying Hölder's inequality a necessary number of times, we can obtain similar bounds for any term in the expansion (22).

The following theorem gives the inequality for the L_p Steiner coefficients which is similar to the inequality for the mixed affine surface areas given in [31, Theorem 3] when the second body is taken to be a ball.

Theorem 4.1 *Let K be a convex body and $i, j, k \in \mathbb{R}$ such that $m \leq i < j < k$. Then for a fixed $m \in \mathbb{N} \cup \{0\}$,*

$$\mathcal{W}_{m,k}(K)^{j-i} \mathcal{W}_{m,i}(K)^{k-j} \geq \mathcal{W}_{m,j}(K)^{k-i}$$

with equality if and only if K is a ball.

Proof We use Hölder's inequality

$$\int_{S^{n-1}} g_1(u) \cdots g_q(u) d\sigma(u) \leq \prod_{i=1}^q \left(\int_{S^{n-1}} g_i(u)^{a_i} d\sigma(u) \right)^{\frac{1}{a_i}}$$

with $q = 2$ and $a_1 = \frac{k-i}{j-i}$, $a_2 = \frac{k-i}{k-j}$. We take

$$\begin{aligned} g_1(u) &= \left(f_p(K, u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}} A_p^m \right)^{\frac{1}{a_1}} h_K(u)^{\frac{1}{a_1}(m-k)}, \\ g_2(u) &= \left(f_p(K, u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}} A_p^m \right)^{\frac{1}{a_2}} h_K(u)^{\frac{1}{a_2}(m-i)}. \end{aligned}$$

Then using the definition of the L_p Steiner coefficients (4), we get the desired result.

Equality holds in Hölder inequality if and only if $g_1^{a_1}$ is proportional to $g_2^{a_2}$. This leads to the condition that the support function of K must be a constant, i.e. $h_K(u) = \text{const}$. Thus, K must be a ball, which follows from the fact that the support function uniquely determines a convex body.

As we already shown above that some $\mathcal{W}_{m,k}$ are mixed affine surface areas, they satisfy the inequality of Theorem 4.1.

4.4 Connections to information theory

We would like to point out a connection between the L_p Steiner coefficients and information theory. To do so we need some background.

Let (X, μ) be a measure space and let $dP = p d\mu$ and $dQ = q d\mu$ be measures on X that are absolutely continuous with respect to the measure μ . Then the Rényi divergence of order α , introduced by Rényi [46] for $\alpha > 0$ and $\alpha \neq 1$, is defined by

$$D_\alpha(P\|Q) = \frac{1}{\alpha - 1} \log \int_X p^\alpha q^{1-\alpha} d\mu. \quad (24)$$

It is the convention to put $p^\alpha q^{1-\alpha} = 0$, if $p = 0$ or $q = 0$, even if $\alpha < 0$ and $\alpha > 1$. The integrals

$$\int_X p^\alpha q^{1-\alpha} d\mu \quad (25)$$

are also called *Hellinger integrals*. See e.g. [27] for those integrals and additional information. Rényi divergences and Hellinger integrals and their related inequalities are important tools in information theory, statistics, probability theory, machine learning and convex geometry, see e.g., [2, 8, 9, 12, 24, 27, 45].

Usually, in the literature, $\alpha \geq 0$. However, we will also consider $\alpha < 0$, provided the expressions exist. Following the ideas of [55], where Rényi divergences for convex bodies K were introduced, we consider the measure space $(\partial K, \mathcal{H}^{n-1})$ and densities p_K, q_K on ∂K ,

$$p_K(x) = \frac{H_{n-1}(x)}{\langle x, \nu(x) \rangle^n}, \quad q_K(x) = \langle x, \nu(x) \rangle. \quad (26)$$

Then

$$P_K = p_K \mu_K \quad \text{and} \quad Q_K = q_K \mu_K \quad (27)$$

are measures on ∂K that are absolutely continuous with respect to μ_K . We remark that those measures are the *cone measures* of K and K° , respectively, see e.g. [41].

By the change of integration formula (10), the L_p Steiner coefficients can be written as

$$\begin{aligned} \mathcal{W}_{m,k}(K) &= \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m d\sigma(u) \\ &= \int_{\partial K} H_{n-1}(x)^{\frac{p}{n+p}} \langle x, \nu(x) \rangle^{\frac{n(1-p)}{n+p}-k+m} A_p^m d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial K} \left(\frac{H_{n-1}(x)}{\langle x, \nu(x) \rangle^n} \right)^{\frac{p}{n+p}} \langle x, \nu(x) \rangle^{1-\frac{p}{n+p}} \langle x, \nu(x) \rangle^{m-k} A_p^m d\mathcal{H}^{n-1}(x). \end{aligned} \quad (28)$$

Thus the L_p Steiner coefficients are weighted (by the weight $\langle x, \nu(x) \rangle^{m-k} A_p^m$) Hellinger integrals of the measures P_K and Q_K , respectively,

$$\frac{1}{\alpha - 1} \log \mathcal{W}_{m,k}(K)$$

are weighted α -Rényi divergences with weight $\langle x, \nu(x) \rangle^{m-k} A_p^m$ and $\alpha = \frac{p}{n+p}$.

5 Mixed affine surface areas

For all $p \geq 1$ and all real s , the s -th mixed L_p affine surface area of K is defined in [31]. We use generalization of this definition to all $p \neq -n$ and all real s , which is given in [53] by

$$as_{p,s}(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n-s}{n+p}} d\sigma(u),$$

where $f_p(K, u) = f_K(u) h_K(u)^{1-p}$.

Theorem 5.1 *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(K)$. For all $p \in \mathbb{R}$, $p \neq -n$, for all $s \in \mathbb{R}$,*

$$as_{p,s}(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n-s}{n+p}(1-p)}{k-m} \int_{S^{n-1}} f_K(u)^{\frac{n-s}{n+p}} h_K(u)^{\frac{n-s}{n+p}(1-p)-k+m} A_{p,s}^m d\sigma(u) \right] t^k,$$

where

$$A_{p,s}^m = A_{p,s}^m(\bar{\xi}_K(u)) = \sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{\frac{n-s}{n+p}}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\}.$$

When $s = 0$, we recover Theorem 3.1 and when $p = 1$, we get the expansion for the s -mixed affine surface areas.

Remark on the polytopal case

The next proposition gives the corresponding Steiner formula for the s -mixed L_p affine surface area of the polytope P .

Theorem 5.2 *Let P be a convex polytope in $\mathbb{R}^n$. Let t be such that $0 \leq t < \beta(P)$. For all $s \in \mathbb{R}$, for all $p \in \mathbb{R}$ such that if $n < s$, then $p \notin (-s, -n]$ and if $n > s$, then $p \notin [-n, -s)$,*

$$as_{p,s}(P + tB_2^n) = \sum_{m=0}^{\infty} \binom{\frac{n-s}{n+p}(1-p)}{m} \sum_{v \in \text{vert } P} \int_{u \in \text{Norm}(v)} h_P(u)^{\frac{(n-s)(1-p)}{n+p} - m} d\sigma(u) t^{m + \frac{(n-1)(n-s)}{n+p}}.$$

6 Proofs

We start by showing that Corollary 1.1 and 1.2 are consequences of Theorem 3.1.

6.1 Proof of Corollary 1.1 and 1.2

Proof of Corollary 1.1

By (13), we get for the left hand side of (22),

$$as_0(K + tB_2^n) = n \text{vol}_n(K + tB_2^n).$$

The right hand side of (22) becomes

$$\begin{aligned} & \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{1}{k-m} t^k \int_{S^{n-1}} f_K(u) h_K(u)^{1-k+m} A_0^m d\sigma(u) \right] = \\ & \sum_{m=0}^{\infty} \left[t^m \int_{S^{n-1}} f_K(u) h_K(u) A_0^m d\sigma(u) + t^{m+1} \int_{S^{n-1}} f_K(u) A_0^m d\sigma(u) \right], \end{aligned}$$

where

$$A_0^m = \sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{1}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\}.$$

By (17), we only get a contribution for A_0^m if either $i_1 + i_2 + \dots + i_{n-1} = 0$, i.e., all $i_j = 0$, or if $i_1 + i_2 + \dots + i_{n-1} = 1$, which means that $i_j = 1$ and $i_k = 0$ for all $k \neq j$. Then $m = j$ as the summation is over all $i_1, \dots, i_{n-1} \geq 0$ such that $i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m$. We also use (10) and then we get for the right hand side of (22),

$$\begin{aligned} & \sum_{m=0}^{n-1} \binom{n-1}{m} \left[t^m \int_{\partial K} \langle \nu_K(x), x \rangle H_m d\mathcal{H}^{n-1} + t^{m+1} \int_{\partial K} H_m d\mathcal{H}^{n-1} \right] \\ &= \int_{\partial K} \langle \nu_K(x), x \rangle d\mathcal{H}^{n-1}(x) \\ &+ \sum_{m=1}^{n-1} \binom{n-1}{m} \left[t^m \int_{\partial K} H_{m-1}(x) d\mathcal{H}^{n-1}(x) + t^{m+1} \int_{\partial K} H_m(x) d\mathcal{H}^{n-1}(x) \right]. \end{aligned}$$

In the last equality we have used (8). Collecting terms and using the recursive identity for binomial coefficients

$$\binom{n}{k} = \binom{n-1}{m} + \binom{n-1}{m-1}$$

for all integers n, m such that $1 \leq m \leq n-1$, and the identity (7), we get for the right hand side of (22)

$$n \operatorname{vol}_n(K) + \sum_{m=1}^n \binom{n}{m} \int_{\partial K} H_{m-1}(x) d\mathcal{H}^{n-1}(x) t^m = n \sum_{m=0}^n \binom{n}{m} W_m t^m.$$

This shows that the classical Steiner formula is a corollary of Theorem 3.1.

Proof of Corollary 1.2

By (12) and (9), we get for the left hand side of (22),

$$as_{\pm\infty}(K + tB_2^n) = n \operatorname{vol}_n((K + tB_2^n)^\circ) = n \widetilde{W}_0((K + tB_2^n)^\circ).$$

The right hand side of (22) becomes

$$\sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{-n}{k-m} t^k \int_{S^{n-1}} h_K(u)^{-n-k+m} A_{\pm\infty}^m d\sigma(u) \right],$$

where

$$A_{\pm\infty}^m = \sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{0}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\}.$$

We only get a contribution for $A_{\pm\infty}^m$ if $i_1 + i_2 + \dots + i_{n-1} = 0$, i.e., if $i_j = 0$ for all j . This means that we only get a contribution for $m = 0$. As $A_{\pm\infty}^0 = 1$, the right hand side of (22) becomes

$$\sum_{k=0}^{\infty} \binom{-n}{k} t^k \int_{S^{n-1}} h_K(u)^{-n-k} d\sigma(u).$$

If K is a convex body such that $0 \in \operatorname{int}(K)$, then

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)} \quad \text{for all } u \in S^{n-1}.$$

Hence, with (9),

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{-n}{k} t^k \int_{S^{n-1}} h_K(u)^{-n-k} d\sigma(u) &= \\ \sum_{k=0}^{\infty} \binom{-n}{k} t^k \int_{S^{n-1}} \rho_{K^\circ}(u)^{n+k} d\sigma(u) &= n \sum_{k=0}^{\infty} \binom{-n}{k} \widetilde{W}_{-k}(K^\circ) t^k. \end{aligned}$$

Before we prove the general case of Theorem 3.1, it will be helpful to treat a special case.

Throughout the paper, we will also need the facts that are listed next. First note that it is not difficult to see that the support function of $K + tB_2^n$ can be expressed in terms of the support function of K by

$$h_{K+tB_2^n}(u) = h_K(u) + t. \tag{29}$$

Now we write the expression for the curvature function $f_{K+tB_2^n}(u)$. Recall that the curvature function is reciprocal of the Gauss curvature, that is,

$$f_{K+tB_2^n}(u) = \frac{1}{H_{n-1}(\widetilde{\xi}_{K+tB_2^n}(u))}.$$

Since $\bar{\xi}_{K+tB_2^n}(u)$ is the point on $\partial(K+tB_2^n)$ that has u as unique outer unit normal, $\bar{\xi}_{K+tB_2^n}(u) = x + tu$, where x is this point on ∂K that has u as unique outer normal, i.e., $x = \bar{\xi}_K(u)$. We will also use the fact that the Gauss curvature $H_{n-1}(x+tu)$ is the product of the principal curvatures $k_1^t(x+tu), \dots, k_{n-1}^t(x+tu)$. A well-known fact from differential geometry provides the connection between the principal curvatures k_i^t of the outer parallel body $K+tB_2^n$ and principal curvatures k_i of the body K , namely

$$k_i^t(x+tu) = \frac{k_i(x)}{1 + tk_i(x)},$$

for $x \in \partial K$ and u the outer unit normal vector to K at the point x . Therefore, the Gauss curvature of the parallel body is

$$H_{n-1}(x+tu) = \prod_{i=1}^{n-1} k_i^t(x+tu) = \prod_{i=1}^{n-1} \frac{k_i(x)}{1 + tk_i(x)} = \frac{H_{n-1}(x)}{\prod_{i=1}^{n-1} (1 + tk_i(x))}.$$

Since u is a unit outward normal vector at x to K (and also a unit outer normal vector at $x+tu$ to $K+tB_2^n$), we derive an expression for the curvature function $f_{K+tB_2^n}(u)$,

$$\begin{aligned} f_{K+tB_2^n}(u) &= f_K(u) \prod_{i=1}^{n-1} (1 + tk_i(\bar{\xi}_K(u))) = f_K(u) \sum_{k=0}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \\ &= f_K(u) \left(1 + \sum_{k=1}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \right). \end{aligned} \quad (30)$$

6.2 The case when $\frac{n}{n+p}$ is a natural number

We consider case when $\frac{n}{n+p}$ is a natural number, that is

$$\frac{n}{n+p} = l \quad \text{where } l \in \mathbb{N},$$

or equivalently,

$$p = -\frac{n(l-1)}{l}, \quad \text{for } l \in \mathbb{N}.$$

Then $(1-p)\frac{n}{n+p} = l + n(l-1) \in \mathbb{N}$, since $l \in \mathbb{N}$.

Then, by (14), (29) and (30),

$$\begin{aligned} as_p(K+tB_2^n) &= \int_{S^{n-1}} f_K^l(u) \left(1 + \sum_{k=1}^{n-1} \binom{n-1}{k} H_k t^k \right)^l (h_K + t)^{l+n(n-1)} d\sigma(u) \\ &= \int_{S^{n-1}} f_K^l(u) \left[\binom{l}{0} + \binom{l}{1} \sum_{k=1}^{n-1} \binom{n-1}{k} H_k t^k + \binom{l}{2} \left(\sum_{k=1}^{n-1} \binom{n-1}{k} H_k t^k \right)^2 + \dots \right. \\ &\quad \left. + \binom{l}{l} \left(\sum_{k=1}^{n-1} \binom{n-1}{k} H_k t^k \right)^l \right] \\ &\quad \left[\binom{l+n(l-1)}{0} h_K^{l+n(l-1)} + \binom{l+n(l-1)}{1} h_K^{l+n(l-1)-1} t + \dots + \binom{l+n(l-1)}{l+n(l-1)} t^{l+n(l-1)} \right] d\sigma(u), \end{aligned}$$

where we used Taylor series expansion of the curvature term and support function term. Note that $\frac{t}{h_K(u)} \leq \frac{t}{\beta(K)} < 1$. Hence, the binomial series for the support function uniformly converges on S^{n-1} . Since K is C_+^2 , all curvature expressions are bounded from above and strictly positive, independently of x .

Expanding the summations, we can write the general term corresponding to the power t^m as

$$\begin{aligned} & \sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{l}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \binom{n-1}{1}^{i_1} \dots \binom{n-1}{n-1}^{i_{n-1}} H_1^{i_1} \dots H_{n-1}^{i_{n-1}} \\ &= \sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{l}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}. \end{aligned}$$

The multinomial coefficients here are coming from the multinomial formula (18).

Therefore, the following gives the general formula for the L_p affine surface area,

$$\begin{aligned} as_p(K + tB_2^n) &= \sum_{m=0}^{l(n-1)} \left[\sum_{k=m}^{l+n(l-1)+m} \binom{l+n(l-1)}{k-m} t^k \int_{S^{n-1}} f_K^l(u) h_K^{l+n(l-1)-k+m} \right. \\ & \quad \left. \left(\sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{l}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\} \right) d\sigma(u) \right]. \end{aligned}$$

The parameter m determines the number of sums inside and varies between 0 and $l(n-1)$, since the highest power of t is $l + n(l-1) + l(n-1)$.

For convenience, we denote the part of the expression under the integral by $A_{p,s}^m = A_{p,s}^m(\bar{\xi}_K(u))$:

$$A_{p,s}^m = \sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{\frac{n-s}{n+p}}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\} \quad (31)$$

for any real $p \neq -n$ and s . Note that in the current case $p = \frac{n(1-l)}{l}$, $s = 0$ and the expression $\frac{n}{n+p} = l$. Coefficients $A_p^m = A_{p,0}^m$ represent a sum of mixed products of symmetric functions of the principal curvatures H_i (5) with corresponding multinomial coefficients.

Thus,

$$as_{\frac{n(1-l)}{l}}(K + tB_2^n) = \sum_{m=0}^{l(n-1)} \left[\sum_{k=m}^{l+n(l-1)+m} \binom{l+n(l-1)}{k-m} t^k \int_{S^{n-1}} f_K^l(u) h_K^{l+n(l-1)-k+m} A_{\frac{n(1-l)}{l}}^m d\sigma(u) \right] \quad (32)$$

for $l \in \mathbb{N}$.

6.3 The case of real $p \neq -n$

The L_p affine surface area of $K + tB_2^n$ is given by (14). Using relation (30) for the curvature function of the parallel body, we can rewrite it in the following form,

$$as_p(K + tB_2^n) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} \left(1 + \sum_{k=1}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \right)^{\frac{n}{n+p}} (h_K(u) + t)^{\frac{n(1-p)}{n+p}} d\sigma(u).$$

Using Taylor series expansions of the curvature and support functions terms, we have

$$\begin{aligned} as_p(K + tB_2^n) &= \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} \sum_{i=0}^{\infty} \left\{ \binom{\frac{n}{n+p}}{i} \left[\sum_{k=1}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \right]^i \right\} \\ & \quad \cdot \sum_{j=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{j} h_K(u)^{\frac{n(1-p)}{n+p} - j} t^j d\sigma(u). \end{aligned}$$

Now, analogous to the procedure used in Section 6.2, we have the general formula

$$as_p(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} t^k \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} \left(\sum_{\substack{i_1, \dots, i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = m}} \binom{\frac{n}{n+p}}{i_1 + \dots + i_{n-1}} \binom{i_1 + \dots + i_{n-1}}{i_1, i_2, \dots, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\} \right) d\sigma(u) \right].$$

Using the notation A_p^m defined in (31), we can write the expression above in a more compact way:

$$as_p(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} t^k \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m d\sigma(u) \right].$$

We note that the first coefficient in this expansion represents the L_p affine surface area $as_p(K)$ of a body K .

Similarly, an expansion for the s -th mixed p -affine surface area $as_{p,s}(K + tB_2^n)$ is obtained for all real s . In fact, for a convex body K and $0 \leq t < \beta(K)$, one has

$$as_{p,s}(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n-s}{n+p}(1-p)}{k-m} t^k \int_{S^{n-1}} f_K(u)^{\frac{n-s}{n+p}} h_K(u)^{\frac{n-s}{n+p}(1-p)-k+m} A_{p,s}^m d\sigma(u) \right].$$

Note that the first coefficient in this expansion gives the s -mixed L_p affine surface area $as_{p,s}(K)$ of a body K . If $p = 1$, this gives the expansion for the s -th mixed affine surface area $as_{1,s}(K + tB_2^n)$ for any real s .

This finishes the proof of Theorem 3.1 and Theorem 5.1.

6.4 Proof of Theorem 3.2

The L_{-n} affine surface area of $K + tB_2^n$ is given by

$$as_{-n}(K + tB_2^n) = \max_{u \in S^{n-1}} f_{K+tB_2^n}(u)^{\frac{1}{2}} h_{K+tB_2^n}(u)^{\frac{n+1}{2}}.$$

Using relations (30) and (29), we can rewrite it as

$$\begin{aligned} as_{-n}(K + tB_2^n) &= \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \left(\left[1 + \sum_{k=1}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \right] (h_K(u) + t)^{n+1} \right)^{\frac{1}{2}} \\ &= \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \left(1 + \sum_{j=1}^{2n} \sum_{k+i=j} \left[\binom{n-1}{k} \binom{n+1}{i} \frac{H_k(\bar{\xi}_K(u))}{h_K(u)^i} \right] t^j \right)^{\frac{1}{2}}. \end{aligned}$$

For convenience, we denote the coefficients in front of powers of t as

$$B_j = B_j(u) = \sum_{k+i=j} \left[\binom{n-1}{k} \binom{n+1}{i} \frac{H_k(\bar{\xi}_K(u))}{h_K(u)^i} \right], \quad 0 \leq j \leq 2n.$$

Using Taylor series expansion, we obtain

$$\begin{aligned}
as_{-n}(K + tB_2^n) &= \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \left(1 + \sum_{j=1}^{2n} B_j t^j \right)^{\frac{1}{2}} \\
&= \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \sum_{m=0}^{\infty} \left[\sum_{\substack{i_1, \dots, i_{2n} \geq 0 \\ i_1 + 2i_2 + \dots + 2ni_{2n} = m}} \binom{\frac{1}{2}}{i_1 + \dots + i_{2n}} \binom{i_1 + \dots + i_{2n}}{i_1, \dots, i_{2n}} \prod_{q=1}^{2n} B_q^{i_q} \right] t^m.
\end{aligned}$$

Similarly to (31), we can introduce coefficients $A_{-n}^m = A_{-n}^m(u)$ by

$$A_{-n}^m = A_{-n}^m(u) = \sum_{\substack{i_1, \dots, i_{2n} \geq 0 \\ i_1 + 2i_2 + \dots + 2ni_{2n} = m}} \binom{\frac{1}{2}}{i_1 + \dots + i_{2n}} \binom{i_1 + \dots + i_{2n}}{i_1, \dots, i_{2n}} \prod_{q=1}^{2n} B_q^{i_q}.$$

Then

$$as_{-n}(K + tB_2^n) = \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \sum_{m=0}^{\infty} A_{-n}^m t^m.$$

6.5 Proof of Theorem 3.3

For $-n < p < 0$, the exponent $\frac{p}{n+p}$ of the Gauss curvature $H_{n-1}(x)$ in (11) is negative. Since for the set $\{y \in \partial(P + tB_2^n) : H_{n-1}(y) = 0\}$ Gauss curvature is equal to zero, we have that $as_p(P + tB_2^n) = \infty$.

Next, we consider the case when $p \notin [-n, 0]$. Note that in this case the exponent $\frac{p}{n+p}$ of $H_{n-1}(x)$ in (11) is positive. We have that $f_{P+tB_2^n}(u) \neq 0$ only for those $u \in S^{n-1}$ for which $f_{P+tB_2^n}(u) < \infty$ and then $f_{P+tB_2^n}(u)^{\frac{n}{n+p}} = t^{\frac{n(1-p)}{n+p}}$. By this, (14) and (30),

$$\begin{aligned}
as_p(P + tB_2^n) &= \int_{\{y \in \partial(P+tB_2^n) : H_{n-1}(y)=0\}} \frac{H_{n-1}(y)^{\frac{p}{n+p}}}{\langle y, \nu(y) \rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(y) \\
&+ \int_{\{y \in \partial(P+tB_2^n) : H_{n-1}(y) \neq 0\}} \frac{H_{n-1}(y)^{\frac{p}{n+p}}}{\langle y, \nu(y) \rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(y) \\
&= \int_{\{u : f_{P+tB_2^n}(u) < \infty\}} f_{P+tB_2^n}(u)^{\frac{n}{n+p}} h_{P+tB_2^n}(u)^{\frac{n(1-p)}{n+p}} d\sigma(u) \\
&= t^{\frac{n(1-p)}{n+p}} \int_{\{u : f_{P+tB_2^n}(u) < \infty\}} (h_P(u) + t)^{\frac{n(1-p)}{n+p}} d\sigma(u) \\
&= t^{\frac{n(1-p)}{n+p}} \int_{\{u : f_{P+tB_2^n}(u) < \infty\}} h_P(u)^{\frac{n(1-p)}{n+p}} \sum_{j=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{j} \frac{t^j}{h_P(u)^j} d\sigma(u) \\
&= \sum_{j=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{j} t^{j+\frac{n(1-p)}{n+p}} \int_{\{u : f_{P+tB_2^n}(u) < \infty\}} h_P(u)^{\frac{n(1-p)}{n+p}-j} d\sigma(u) \\
&= \sum_{j=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{j} \sum_{v \in \text{vert } P} \int_{u \in \text{Norm}(v)} h_P(u)^{\frac{n(1-p)}{n+p}-j} d\sigma(u) t^{j+\frac{n(1-p)}{n+p}}.
\end{aligned}$$

As by assumption $t < \beta(P)$, we have for all $u \in S^{n-1}$ that $\frac{t}{h_P(u)} \leq \frac{t}{\beta(P)} < 1$ and, therefore, the above infinite sum converges uniformly. Moreover, for all $u \in S^{n-1}$, $\lambda(P) \leq h_P(u) \leq \Lambda(P)$.

Thus, we can interchange integration and summation, which was done in the second last equality above.

For $p = 0$, we have a different situation, namely,

$$\begin{aligned}
as_0(P + tB_2^n) &= \int_{\{y \in \partial(P + tB_2^n) : H_{n-1}(y) = 0\}} \frac{1}{\langle y, \nu(y) \rangle^{-1}} d\mathcal{H}^{n-1}(y) \\
&+ \int_{\{y \in \partial(P + tB_2^n) : H_{n-1}(y) \neq 0\}} \frac{1}{\langle y, \nu(y) \rangle^{-1}} d\mathcal{H}^{n-1}(y) \\
&= \int_{\partial(P + tB_2^n)} \langle y, \nu(y) \rangle d\mathcal{H}^{n-1}(y) = n \operatorname{vol}_n(P + tB_2^n).
\end{aligned}$$

References

- [1] J. ARROYO, M. BARROS AND O.J. GARAY, *Willmore-Chen tubes on homogeneous spaces in warped product spaces*, Pacific J. Math. **188** (1999), 201–207.
- [2] A. R. BARRON, L. GYÖRFI AND E.C. VAN DER MEULEN, *Distribution estimation consistent in total variation and in two types of information divergence*, IEEE Trans. Inform. Theory **38** (1992), 1437–1454.
- [3] W. BLASCHKE, *Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie*, Springer, Berlin, 1923.
- [4] K. J. BÖRÖCZKY, M. HENK, AND H. POLLEHN, *Subspace concentration of dual curvature measures of symmetric convex bodies*, J. Differential Geom. **109** (2018), 411–429.
- [5] K. J. BÖRÖCZKY, E. LUTWAK, D. YANG, AND G. ZHANG, *The logarithmic Minkowski problem*, J. Amer. Math. Soc. (JAMS) **26** (2013), 831–852.
- [6] K. J. BÖRÖCZKY, E. LUTWAK, D. YANG, AND G. ZHANG, *Affine images of isotropic measures*, J. Differential Geom. **99** (2015), 407–442.
- [7] L. A. CAFFARELLI, *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampere equation*, Ann. of Math. **131** (1990), 135–150.
- [8] U. CAGLAR AND E.M. WERNER, *Divergence for s -concave and log concave functions*, Adv. Math. **257** (2014), 219–247.
- [9] U. CAGLAR AND E.M. WERNER, *Mixed f -divergence and inequalities for log-concave functions*, Proc. Lond. Math. Soc. (2) **110** (2015), 271–290.
- [10] B.Y. CHEN, *Some conformal invariants of submanifolds and their applications*, Boll. Un. Mat. Ital. **10** (1974), 380–385.
- [11] S.Y. CHENG AND S.T. YAU, *On the regularity of the solution of the n -dimensional Minkowski problem*, Commun. Pure Appl. Math. **29** (1976), 495–516.
- [12] T. COVER AND J. THOMAS, *Elements of information theory*, second ed., Wiley-Interscience, (John Wiley and Sons), Hoboken, NJ, 2006.
- [13] H. FEDERER, *Geometric Measure Theory*, Springer, Berlin, 1969.
- [14] W. J FIREY, *Blaschke sums of convex bodies and mixed bodies*, Proc. Coll. Convexity (Copenhagen, 1965), Københavns Univ. Mat. Inst (1967), pp. 94–101.
- [15] J.H.G. FU, *Curvature measures of subanalytic sets*, Amer. J. Math. **116** (1994), 819–880.

- [16] R. GARDNER, *Geometric Tomography*, second edition, Cambridge University Press, New York, 2006.
- [17] R. J. GARDNER, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. (2) **140** (1994), 435–447.
- [18] R. J. GARDNER, D. HUG, AND W. WEIL, *The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities*, J. Differential Geom. **97** (2014), 427–476.
- [19] R. J. GARDNER, A. KOLDOBSKY, AND T. SCHLUMPRECHT, *An analytical solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. (2) **149** (1999), 691–703.
- [20] A. GRAY, *Tubes. Second edition*, Progress in Mathematics **221**, Birkhäuser Verlag, Basel, 2004.
- [21] C. HABERL, *Blaschke valuations*, Amer. J. of Math. **133** (2011), 717–751.
- [22] C. HABERL AND F. SCHUSTER, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26.
- [23] C. HABERL, E. LUTWAK, D. YANG AND G. ZHANG, *The even Orlicz Minkowski problem*, Adv. Math. **224** (2010), 2485–2510.
- [24] P. HARREMOES AND F. TOPSØE, *Inequalities between entropy and the index of coincidence derived from information diagrams*, IEEE Trans. Inform. Theory **47** (2001), 2944–2960.
- [25] Y. HUANG, E. LUTWAK, D. YANG, AND G. ZHANG, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Math. **216** (2016), 325–388.
- [26] P. LI AND S.T. YAU, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. **69** (1982), 269–291.
- [27] F. LIESE AND I. VAJDA, *On Divergences and Information in Statistics and Information Theory*, IEEE Trans. Inform. Theory **52** (2006), 4394–4412.
- [28] M. LUDWIG, *Minkowski areas and valuations*, J. Differential Geom. **86** (2010), 133–162.
- [29] M. LUDWIG AND M. REITZNER, *A classification of $SL(n)$ invariant valuations*, Ann. of Math. **172** (2010), 1223–1271.
- [30] E. LUTWAK, *Dual mixed volume*, Pacific J. Math. **58**(2) (1975), 531–538.
- [31] E. LUTWAK, *Mixed affine surface area*, J. Math. Anal. Appl. **125** (1987), 351–360.
- [32] E. LUTWAK, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.
- [33] E. LUTWAK, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (2) (1996), 244–294.
- [34] E. LUTWAK, D. YANG AND G. ZHANG, *The Cramer–Rao inequality for star bodies*, Duke Math. J. **112** (2002), 59–81.
- [35] E. LUTWAK, D. YANG AND G. ZHANG, *Volume inequalities for subspaces of L_p* , J. Differential Geom. **68** (2004), 159–184.
- [36] E. LUTWAK, D. YANG AND G. ZHANG, *Moment-entropy inequalities*, Ann. Probab. **32** (2004), 757–774.

- [37] E. LUTWAK, D. YANG AND G. ZHANG, *Cramer-Rao and moment-entropy inequalities for Renyi entropy and generalized Fisher information*, IEEE Trans. Inform. Theory **51** (2005), 473–478.
- [38] FERNANDO C. MARQUES AND ANDRÉ NEVES, *Min-Max theory and the Willmore conjecture*, Ann. of Math. **179** (2) (2014), 683–782.
- [39] M. MEYER AND E. WERNER, *On the p -affine surface area*, Adv. Math. **152** (2000), 288–313.
- [40] F. NAZAROV, F. PETROV, D. RYABOGIN AND A. ZVAVITCH, *A remark on the Mahler conjecture: local minimality of the unit cube*, Duke Math. J. **154** (2010), 419–430.
- [41] G. PAOURIS AND E. WERNER, *Relative entropy of cone measures and L_p -centroid bodies*, Proc. Lond. Math. Soc. **104**(2) (2012), 253–286.
- [42] L. PARAPATITS AND F.E. SCHUSTER, *The Steiner formula for Minkowski valuations*, Adv. Math. **230** (2012), 978–994.
- [43] A.V. POGORELOV, *The Minkowski multidimensional problem*, V. H. Winston & Sons, Washington, D.C., 1978.
- [44] D. POKORNY AND J. RATAJ, *Normal cycles and curvature measures of sets with d.c. boundary*, Adv. Math. **248** (2013), 963–985.
- [45] M. D. REID AND R.C. WILLIAMSON, *Information, Divergence and Risk for Binary Experiments*, J. Mach. Learn. Res. **12** (2011), 731–817.
- [46] A. RÉNYI, *On measures of entropy and information*, Proceedings of the 4th Berkeley Symposium on Probability Theory and Mathematical Statistics, **1** (1961), 547–561.
- [47] R. SCHNEIDER, *Convex bodies: The Brunn-Minkowski theory*. 2nd expanded edition, Cambridge University Press, Cambridge, 2014.
- [48] C. SCHÜTT AND E. WERNER, *The convex floating body*, Math. Scand. **66** (1990), 275–290.
- [49] C. SCHÜTT AND E. WERNER, *Surface bodies and p -affine surface area*, Adv. Math. **187** (2004), 98–145.
- [50] F. SCHUSTER, *Crofton measures and Minkowski valuations*, Duke Math. J. **154** (2010), 1–30.
- [51] A. STANCU, *On the number of solutions to the discrete two-dimensional L_0 -Minkowski problem*, Adv. Math. **180** (2003), 290–323.
- [52] E. WERNER, *The p -affine surface area and geometric interpretations*, Rend. Circ. Math. Palermo Serie II **70** (2002), 367–382.
- [53] E. WERNER AND D. YE, *Inequalities for mixed p -affine surface area*, Math. Ann. **347** (2010), 703–737.
- [54] E. WERNER, *On L_p -affine surface areas*, Indiana Univ. Math. J. **56**(5) (2007), 2305–2324.
- [55] E. WERNER, *Rényi Divergence and L_p -affine surface area for convex bodies*, Adv. Math. **230** (2012), 1040–1059.
- [56] E. WERNER AND D. YE, *New L_p affine isoperimetric inequalities*, Adv. Math. **218**(3) (2008), 762–780.
- [57] E. WERNER AND D. YE, *Inequalities for mixed p -affine surface area*, Math. Ann. **347** (2010), 703–737.
- [58] T. WILLMORE, *Note on embedded surfaces*, An. Sti. Univ. “Al. I. Cuza” Iasi Sect. I a Mat. **11** (1965), 493–496.

- [59] G. ZHANG, *Intersection bodies and Busemann-Petty inequalities in $\mathbb{R}^4$* , Ann. of Math. **140** (1994), 331–346.
- [60] G. ZHANG, *A positive answer to the Busemann-Petty problem in four dimensions*, Ann. of Math. **149** (1999), 535–543.
- [61] G. ZHANG, *New Affine Isoperimetric Inequalities*, ICCM 2007, Vol. II, 239–267.
- [62] Y. ZHAO, *The dual Minkowski problem for negative indices*, Calc. Var. Partial Differential Equations, **56** (2):18 (2017).

Kateryna Tatarko,

Dept. of Math. and Stat. Sciences,
 University of Alberta,
 Edmonton, Alberta, Canada, T6G 2G1.
 e-mail: `tatarko@ualberta.ca`

Elisabeth Werner,

Department of Mathematics,
 Case Western Reserve University,
 Cleveland, Ohio, USA, 44106.
 e-mail: `elisabeth.werner@case.edu`