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Graphs with prescribed radius, diameter, and center

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Abstract

Among other things, it is shown that for every pair of positive integers r, d, satisfying $1 < r < d \leq 2r$, and every finite simple graph H, there is a connected graph G with diameter d, radius r, and center H.

1 Introduction

All graphs referred to will be finite and simple. The vertex and edge sets of a graph G will be denoted $V(G)$ and $E(G)$, respectively. If G is connected and $u, v \in V(G)$, $dist_G(u, v)$ is the length of a shortest walk in G from one of u, v to the other; a geodesic under the shortest-walk metric. As every shortest walk is a path, $dist_G(u, v)$ may also be formulated as the length of a shortest path in G with end-vertices u and v.

If G is connected and $v \in V(G)$, the *eccentricity* of v in G, denoted $\varepsilon_G(v)$, is:

$$\varepsilon_G(v) = \max_{u \in V(G)} \{dist_G(u, v)\}.$$

The *radius* of a connected graph G is:

$$rad(G) = \min_{u \in V(G)} \{\varepsilon_G(u)\},$$

and its *diameter* is:

$$diam(G) = \max_{u \in V(G)} \{\varepsilon_G(u)\}.$$

Equivalently,

$$diam(G) = \max_{u,v \in V(G)} \{dist_G(u, v)\}.$$

It is easy to see that $rad(G) \leq diam(G) \leq 2rad(G)$. It is a standard exercise in a first course in graph theory to show that for any positive integers satisfying $r \leq d \leq 2r$, there is a connected graph G such that $rad(G) = r$ and $diam(G) = d$. (A more challenging, but still elementary, exercise would be to determine, for pairs r, d constrained as above, the values of n such that there exists a connected graph G with $rad(G) = r$, $diam(G) = d$, and $|V(G)| = n$.)

A vertex $v \in V(G)$ is a *central vertex* in G if and only if $\varepsilon_G(v) = rad(G)$. The *center* of G, denoted $C(G)$, is the subgraph of G induced by the set of centers of G. (Therefore, that set is $V(C(G))$.)

The question broached in [1] is: which graphs can be installed as the center of another graph? That is, given a graph H, can you find a connected graph G such that $C(G) \cong H$?

As reported in [1], this question in full generality was killed at its birth as a question meriting research by a brilliant observation of Stephen T. Hedetniemi, encapsulated in Figure 1.

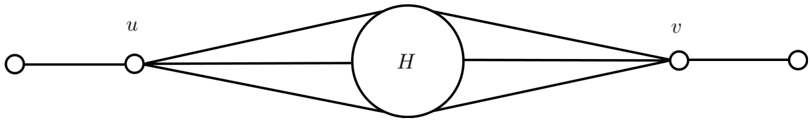

Figure 1: A connected graph G with an arbitrary graph H as its center. Each vertex of H is adjacent to both u and v, in G.

The authors of [1] resurrect the problem by asking: for a distinguished family $\mathcal{F}$ of connected graphs, which graphs H can be the center of a graph $G \in \mathcal{F}$? And, for such H and $\mathcal{F}$, how small can $|V(G)| - |V(H)|$ be, if $G \in \mathcal{F}$? These questions have borne fruit, but we are going in a different direction.

The graph G in Figure 1 has diameter 4 and radius 2. The set of central vertices of G is precisely $V(H)$, regardless of what H is. If the paths leading away from H from u and v are each lengthened to have length $t > 1$, the result is a graph with center H, radius $t+1$, and diameter $2t+2$.

Our aim here is to answer the question: for which positive integers d, r, satisfying $r \leq d \leq 2r$, and graphs H, does there exist a connected graph G such that $rad(G) = r$, $diam(G) = d$, and $C(G) \simeq H$? The extension of the observation of Hedetniemi just above shows that there is such a G for every H, $r > 1$, and $d = 2r$. Our main result, in 3, is that there is such a G for every H, $r > 1$, and $r < d \leq 2r$. In the next section we deal with extremes, and alternative solutions to that in Section 3, in some cases.

2 Extremes and alternative solutions

2.1 $r = d$

If $rad(G) = diam(G)$, then G is its own center. Therefore, $H = C(G)$ and $rad(G) = diam(G)$ if and only $H \simeq G$ and $rad(H) = diam(H)$.

2.2 $r = 1, d = 2$

If $rad(G) = 1$, then each central vertex of G is adjacent to every other vertex of G. Therefore, if $H \cong C(G)$ then H must be a complete graph, and each vertex of H must be adjacent to each vertex of $V(G) \setminus V(H)$. Furthermore, since all central vertices of G are in $V(H)$, it must be that every $v \in V(G) \setminus V(H)$ has a non-neighbor in G in $V(G) \setminus V(H)$.

Let "$\vee$" stand for the *join* of two graphs: $X \vee Y$ is formed by taking disjoint copies of X and Y and then adding in every edge xy, $x \in V(X), y \in V(Y)$. By the paragraph above, when $r = 1$, $d = 2$, the only H for which a solution G can exist are $H = K_t, t > 0$, and the only possible solutions are $K_t \vee Y$ in which Y is a graph with $|V(Y)| > 1$ and for each $y \in V(Y)$, the degree $\deg(y)$ of $y \in V(Y)$ satisfies $\deg_Y(y) < |V(Y)| - 1$.

Every such $G = K_t \vee Y$ satisfies $rad(G) = 1$, $diam(G) = 2$, and $C(G) = K_t$, so we have completely characterized the values of $H(H = K_t)$ for which our problem with $r = 1, d = 2$ has a solution, and all possible solutions ($G = K_t \vee Y$, as above).

2.3 A standard method

Proposition 2.1. *Suppose that X is a connected graph with $|V(X)| > 1$, $rad(X) > 1$, and $V(C(X)) = \{h\}$; i.e., there is a single central vertex in X. For an arbitrary graph H, if G is formed by replacing h by H, with every vertex of H adjacent in G to every vertex in X to which h is adjacent, then $rad(G) = rad(X)$, $diam(G) = diam(X)$, and $C(G) \cong H$.*

The proof is straightforward. Note that the assumption that $rad(X) = \varepsilon_X(h) \geq 2$ plays a role in the proof that $H \cong C(G)$.

For instance, the graph in Figure 1 is obtained from $X = P_5$, the path on 5 vertices, by the device of Proposition 2.1. The generalization to the solution of our problem for all H when $d = 2r \geq 4$ uses the device of Prop. 2.1 with $X = P_{2r+1}$.

In Figure 2 we have a graph X with a single central vertex h such that $rad(X) = r$, $diam(G) = 2r - 1$, for arbitrary $r \geq 2$. By Proposition 2.1, this shows that every graph H can be the center of a graph G of radius r and diameter $2r - 1$, for every $r \geq 2$.

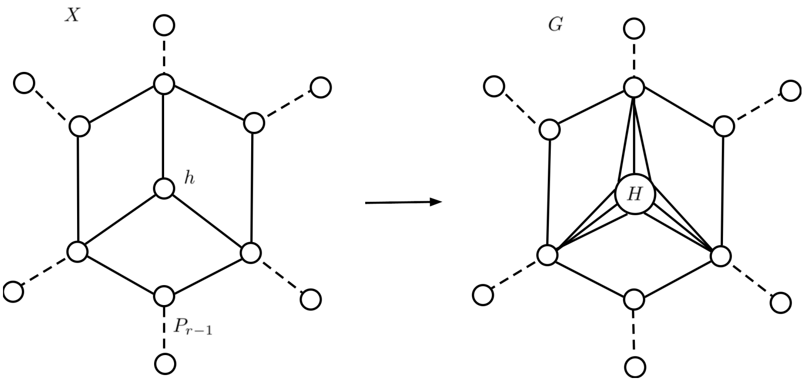

Figure 2: A graph X with radius $r \geq 2$, diameter $2r-1$, and a single central vertex h; and a graph G with $rad(G) = r$, $diam(G) = 2r-1$, and $C(G) \simeq H$. The paths hanging off the vertices of C_6 are all P_{r-1}, paths of length $r-2$. In the case $r = 2$, they are not there, and $|V(X)| = 7$.

For those who enjoy variety, we can vary X to the graph Y shown in Figure 3, which gives another solution to our problem when $d = 2r$ and H arbitrary.

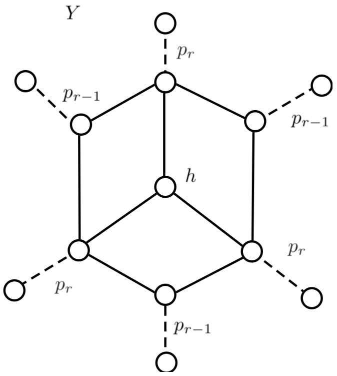

Figure 3: A graph with a single central vertex, radius $r \geq 2$, and diameter $2r$.

If you have been paying attention, you might exclaim: why do we need this? Hedetniemi's construction already gives us solutions of our problem in the case $d = 2r \geq 4$. Yes, but Figure 3 gives a *different* solution, and different solutions of our problem contribute to the solution of a problem that towers over ours: given positive integers r and d satisfying $1 < r < d \leq 2r$, and a graph H, find all possible graphs G satisfying $rad(G) = r$, $diam(G) = d$, and $C(G) \cong H$. In view of Proposition 2.1, in pursuit of this larger problem, it is appropriate to pose the following: given d and r as above, find all graphs X such that $rad(X) = r$, $diam(X) = d$, and $C(X) = K_1$.

Moreover, the alternative solutions to the $d = 2r$ case provide a related problem: what properties characterize those graphs with $d = 2r$ and center K_1? The majority of graphs constructed with center K_1 in fact had $d = 2r$, and the solution to this problem will considerably narrow down the larger problem.

In Figure 4, we have, for $r \geq 2$, a graph of radius r and diameter $r+1$, and a graph of radius r and diameter $r + \lceil \frac{r}{3} \rceil$, both with a single central vertex.

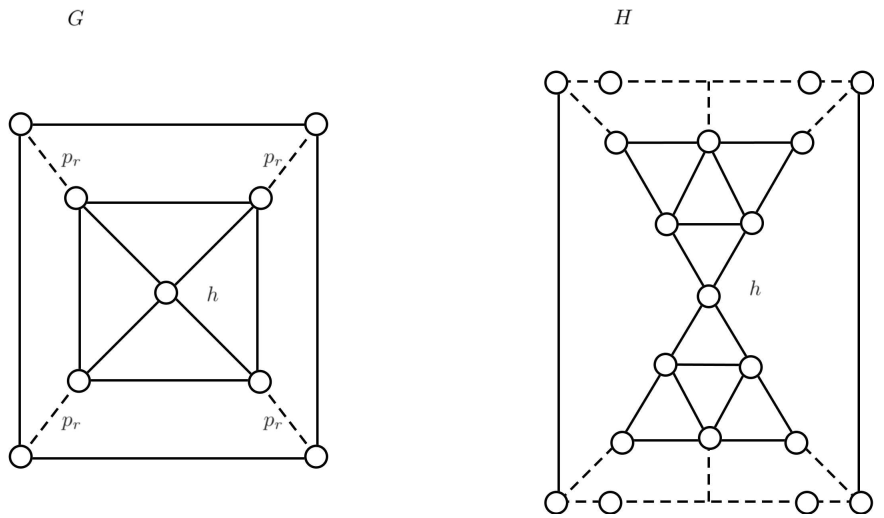

Figure 4: A graph G with radius r and diameter $r + 1$, and a graph H with radius r and diameter $r + \lceil \frac{r}{3} \rceil$. The "top" and the "bottom" of the drawing of H are P_{r+1}'s.

2.4 A non-standard strategy in special cases

The strategy referred to, applicable only when H is connected is: attach pairwise vertex-disjoint paths to the vertices of H. This trick appears to be of use only in a special class of cases.

Proposition 2.2. *Suppose that H is connected with $rad(H) = diam(H) =$*

z. Suppose that G is formed by attaching vertex-disjoint paths P_t to the vertices of H, with each vertex of H being an end of its attached path (when $t = 1$, nothing is attached, and $G = H$). Then $rad(G) = z + t - 1$, $diam(G) = 2(t-1) + z$, and $C(G) \cong H$.

The proof is straightforward.

Corollary 2.3. *If H is as in Proposition 2.2, then for all integers $r \geq z$ and $d = 2r - z$ there is a graph G, obtained as in Prop. 2.2 with $t = r - z + 1$ such that $rad(G) = r$, $diam(G) = d$, and $C(G) = H$.*

3 The main result

Lemma 3.1. *Let X be the graph depicted in Figure 5. Suppose that $n \geq 0$ and $r \geq \max\{2, n+1\}$. Then h is the unique central vertex of X, $rad(X) = r$, and $diam(X) = r + n + 1$.*

Proof. Clearly $\varepsilon_X(h) = \max\{r, n+1\} = r$. Checking shows that every other vertex of X has eccentricity $> r$ in X. For instance,

$$\varepsilon_X(v_{1,1}) = \max\{dist(v_{1,1}, w_n), dist(v_{1,1}, v_{r+1,2})\} = \max\{n+2, r+1\} = r+1.$$

Finally, it is easy to see that the vertices $v_{i,j}$, $i \in \{r, r+1\}$, $j \in \{1, 2\}$, have the greatest eccentricity; for instance, $\varepsilon_X(v_{r,1}) = dist(v_{r,1}, y_n) = r + n + 1 = diam(X)$. □

Theorem 3.2. *For all integers $r \geq 2$ and d satisfying $r < d \leq 2r$ and every graph H there is a graph G such that $rad(G) = r$, $diam(G) = d$, and $C(G) \cong H$. Furthermore, G is obtainable from some graph by the method of Proposition 2.1.*

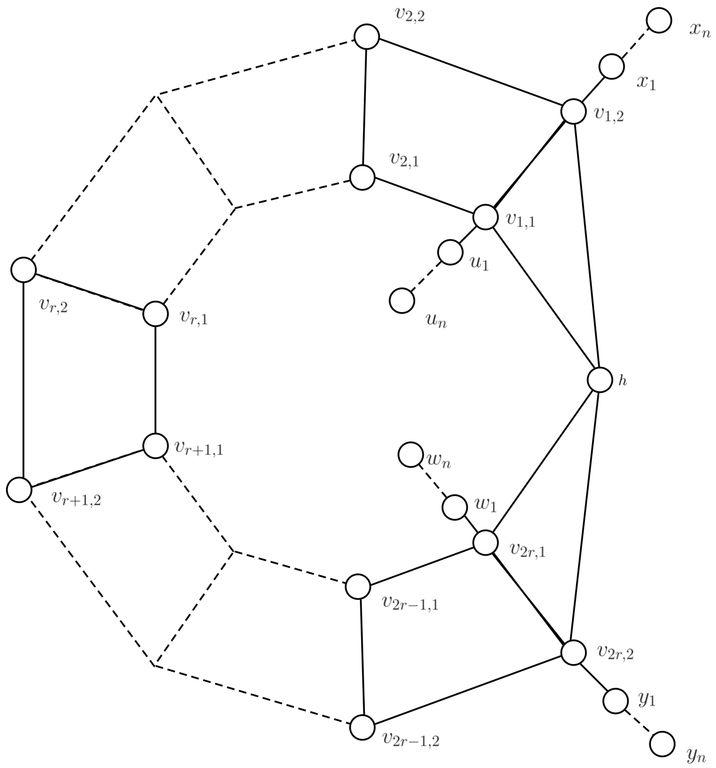

Figure 5: A graph X with a single central vertex h, radius r and diameter $r+n+1$, provided $r \geq n+1$.

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