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Finite-time Controllers for a Class of Planar Nonlinear Systems with Mismatched Disturbances*

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Abstract— **This paper considers the problem of globally finite-time stabilizing a planar nonlinear system with a mismatched unknown disturbance. First, a nonlinear integral dynamic is constructed to compensate the disturbance. For finite-time convergence, we use the system states and integral state to construct a new controller which not only is homogeneous with a negative degree but also contains a linear combination of states to handle the unmatched disturbance. The finite-time integral controller originally obtained for a linear planar system is appropriately scaled to regulate a nonlinear planar system within the framework of homogeneous domination. In addition, for a faster convergence speed for any initial condition, a dual-mode integral controller is introduced with better performances demonstrated by simulation studies.**

I. INTRODUCTION

Since uncertainties and/or disturbances exist in almost all control systems, countering their effects has become one of the main problems in the field of control theory. In this paper, we consider a class of planar nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 + \theta + g_1(t, x_1), \\ \dot{x}_2 = u + g_2(t, x_1) \end{cases} \tag{1}$$

where $x = [x_1, x_2]^T \in R^2$, $u \in R$, and $y = x_1$ are the system state, control input and output, respectively. In addition, $g_1(t, x_1)$ and $g_2(t, x_1)$ are unknown and vanishing functions at the origin and θ is an unknown, mismatched, and non-vanishing constant disturbance. The main aim of this paper is to solve the problem of Global Finite-Time Stabilization by constructing a controller under which the states of nonlinear planar system (1) are globally bounded and converge to the equilibrium in a finite time. More specifically, for the closed-loop system there is a finite time t^* such that $x_1(t) = 0$ and $x_2(t) = -\theta$ when $t \geq t^*$.

Based on the controllability of nominal linear system, global stabilization of nonlinear system under low-triangular uncertainties that satisfy a linear growth condition has been realized in [1] by a linear feedback controller. Under the same condition, an output feedback controller has been developed in [2]. However, those results are only applicable to the case when the disturbances vanish at the origin. Estimating non-vanishing disturbances using observers and then canceling their effects using appropriate controllers has become the main feature of many methods [3] such as active disturbance rejection control (ADRC), disturbance-observer based control (DOBC) and generalized extended state observer based control (GESOC). More recently, the unobservable and mismatched disturbances have been transformed into total disturbances which are observable and matched in [4] and then the ability of ADRC is achieved using extended state observer based control (ESOC). With appropriate compensation of the disturbance, the standard ESOC for SISO systems with matched disturbances has been further generalized to MIMO systems with mismatched disturbances in [5]. With the help of nonlinear disturbance observers, nonlinear DOBC has also been proposed in [6] for dealing with mismatched disturbances. However, only asymptotic results can be obtained using those methods.

Due to the faster convergence speed as well as better disturbance rejection performance, finite-time control of systems with mismatched disturbances has also received a lot of attention. With the help of finite-time disturbance observer (FTDOB), a continuous non-singular terminal sliding mode control approach has been proposed in [7] for mismatched disturbance attenuation. Still with the help of FTDOB, finite-time stabilization of systems with low-triangular disturbances is solved in [8] through compositing a finite time controller and FTDOB. To relieve the computation burden in calculating derivative when using method of backstepping, adaptive finite-time controller and adaptive sliding mode observer have been utilized [9] in finite-time stabilization of systems with mismatched disturbances. In the results mentioned above, an additional dynamic observer should be firstly constructed and then composited with some appropriate controllers. Thus the controller design procedure is not very intuitive especially when the separation principle in terms of controller design and observer design is not satisfied.

The main aim of this paper is constructing finite-time controllers for a class of nonlinear planar system under mismatched and non-vanishing disturbances. Based on the inspiration of integral action in PID controllers, a nonlinear integral mode is firstly employed to compensate the effects of mismatched disturbances. Then, we use the system states and integral state to construct a new controller which is homogeneous with a negative degree to guarantee finite-time convergence. This nonlinear controller also has a specific structure containing a linear combination of two states to

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handle mismatched disturbances. The finite-time integral controller is first developed for a linear planar system. Then, by adding an appropriate scaling gain, the controller can be used to handle a nonlinear planar system within the framework of homogeneous domination. Finally, for a faster convergence speed throughout the whole state-space, we introduce a dual-mode integral controller with a lower-order mode to handle the region close to the equilibrium and a higher-order mode to handle the region far away form the equilibrium.

II. PRELIMINARIES

This section includes the definition of homogeneity and some useful properties of homogeneous systems and functions.

Definition 1 (Weighted Homogeneity): [10] A vector field $f(x) = [f_1(x),\ f_2(x),\ \cdots,\ f_n(x)]^T$ is said to be δ^r-homogeneous of degree k if the component f_i is δ^r-homogeneous of degree $k + r_i$ for each i, i.e.,

$$f_i(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \cdots, \varepsilon^{r_n}x_n) = \varepsilon^{k+r_i} f_i(x)\ \forall x \in R^n, \forall \varepsilon > 0$$

where the one-parameter family of dilation $\delta^r_\varepsilon(x) := (\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \cdots, \varepsilon^{r_n}x_n)$ and $r = (r_1, r_2, \cdots, r_n)$ be a n-uplet of positive real numbers. The homogeneous norm is defined as $\|x\|_\delta = \sqrt[2]{\sum_{i=1}^n |x_i|^{2/r_i}}$.

Lemma 1: [10] If $V(x)$ is a homogeneous function of degree l with respect to the weight $r = (r_1, r_2, \cdots, r_n)$, then $\frac{\partial V(x)}{\partial x_i}$ is homogeneous of degree $l - r_i$.

Lemma 2: [10] If $V_1(x)$ is homogeneous function of degree k_1 and $V_2(x)$ is homogeneous function of degree k_2 with respect to the same weight $r = (r_1, r_2, \cdots, r_n)$, then $V_1(x) \cdot V_2(x)$ is homogeneous of degree $k_1 + k_2$.

Lemma 3: [11] Suppose that V is continuous real-valued functions on R^n, homogeneous of degrees $l > 0$ with respect to δ^r. Then there is a positive constant $\bar{c}$ such that

$$V(x) \le \bar{c}\, \|x\|^l_\delta\,, \quad \forall x \in R^n.$$

In addition, if $V(x)$ is positive definite, there is a positive constant $\underline{c}$ such that

$$\underline{c}\, \|x\|^l_\delta \le V(x), \quad \forall x \in R^n.$$

Lemma 4: [12] For the following continuous system

$$\dot{x}(t) = f(x(t)), \tag{2}$$

suppose $f(x)$ is homogeneous of degree τ with respect to $\delta^r_\varepsilon(x)$. Then the origin is a finite-time-stable equilibrium if the origin is an asymptotically stable equilibrium of system (2) and $\tau < 0$.

Lemma 5: [10] Suppose f is homogeneous of degree $\tau < 0$ with respect to δ. If the origin is a finite-time stable equilibrium of (2), then there exist a real number $c > 0$ and a C^1 positive-definite function V that is homogeneous of degree $l > |\tau|$ such that

$$\dot{V}(x) = \frac{\partial V}{\partial x} f \le -cV(x)^\alpha, \quad \forall\, x \in R^n$$

where $\alpha := \frac{l+\tau}{l} \in (0, 1)$.

Definition 2: For any positive constant n, define

$$\lfloor x \rceil^n := \text{sign}(x)|x|^n, \quad \forall x \in R.$$

III. MAIN RESULTS

To solve the finite-time stabilization problem of system (1), we impose the following condition on the nonlinearities $g_1(t, x_1)$ and $g_2(t, x_1)$.

Assumption 1: There exist positive constants c_1, c_2 and $q \in (0, 1)$ such that $|g_1(t, x_1)| \le c_1 |x_1|^{\frac{q+1}{2}}$, $|g_2(t, x_1)| \le c_2 |x_1|^q$.

Under Assumption 1, a scaled controller can be constructed to regulate the nonlinear system in a finite time.

Theorem 1: Under Assumption 1, there is a large enough $L \ge 1$ such that the following homogeneous integral controller

$$\begin{cases} \dot{x}_0 = L\lfloor x_1 \rceil^q, \\ u = -L^2 \left(\lfloor a x_0 + c\dfrac{x_2}{L} \rceil^{\frac{2q}{q+1}} + b\lfloor x_1 \rceil^q \right), \end{cases} \tag{3}$$

with positive constants q satisfying $0 < q < 1$ and a, b, c satisfying $bc > a$, globally stabilizes the nonlinear system (1) in a finite time.

Theorem 1 is proved in three steps. First, we consider the finite-time stabilization problem of a three-dimensional nonlinear system. Then we construct a finite-time stabilizer for the nominal part of system (1) without considering the nonlinearities. Finally, we introduce a scaling gain into the stabilizer constructed for the nominal system and show that by adjusting the gain the proposed controller (3) will globally stabilize the nonlinear system (1) in a finite time.

A. Global finite-time stabilization via homogeneous controllers

We first consider the following nonlinear system

$$\begin{cases} \dot{z}_1 = \lfloor z_2 \rceil^q \\ \dot{z}_2 = \theta z_3 \\ \dot{z}_3 = u \end{cases} \tag{4}$$

where the constant $q \in (0, 1)$ and θ is an unknown positive constant. With the help of homogeneous system theory, a finite-time stabilizer can be explicitly constructed as shown in the following theorem.

Theorem 2: The nonlinear system (4) is globally finite-time stabilized by the following controller

$$u = -\lfloor a z_1 + c z_3 \rceil^{\frac{2q}{q+1}} - b\lfloor z_2 \rceil^q \tag{5}$$

where a, b, c are positive constants satisfying $bc > a$, $q \in (0, 1)$.

Proof: The closed-loop system of (4) under the controller (5) can be described as

$$\dot{z} = F(z) = \begin{pmatrix} \lfloor z_2 \rceil^q \\ \theta z_3 \\ -\lfloor a z_1 + c z_3 \rceil^{\frac{2q}{q+1}} - b\lfloor z_2 \rceil^q \end{pmatrix}. \tag{6}$$

Construct the following Lyapunov function

$$V(z_1, z_2, z_3) = \frac{a}{2}(b z_1 + z_3)^2 + \frac{b(bc-a)}{(q+1)\,\theta} |z_2|^{q+1} + \frac{(bc-a)}{2} z_3^2, \tag{7}$$

which is positive definite and radically unbounded since $bc > a$. The derivative of $V(z_1, z_2, z_3)$ along system (6) can be computed as follows

$$\begin{aligned}\dot{V}(z_1,z_2,z_3) =& a(bz_1+z_3)\Big(b\lfloor z_2\rceil^q - \lfloor az_1+cz_3\rceil^{\frac{2q}{q+1}} - b\lfloor z_2\rceil^q\Big)\\ &+ b(bc-a)\lfloor z_2\rceil^q z_3\\ &+ (bc-a)z_3\left(-\lfloor az_1+cz_3\rceil^{\frac{2q}{q+1}} - b\lfloor z_2\rceil^q\right)\\ =& -(abz_1+az_3)\lfloor az_1+cz_3\rceil^{\frac{2q}{q+1}}\\ &-(bc-a)z_3\lfloor az_1+cz_3\rceil^{\frac{2q}{q+1}}\\ =& -b(az_1+cz_3)\lfloor az_1+cz_3\rceil^{\frac{2q}{q+1}} \le 0.\end{aligned}$$

Define $S = \Big\{z \in R^n |\ \dot{V}(z_1, z_2, z_3) = 0\Big\}$ as the LaSalle's invariant set. Notice that

$$\dot{V}(z_1, z_2, z_3) = 0 \Longrightarrow az_1 + cz_3 = 0,$$

which implies $S = \{z \in R^n |\ az_1 = -cz_3\}$. For the solution z of (6) in S, we have the following

$$\begin{aligned}0 =& a\dot{z}_1 + c\dot{z}_3\\ =& -c\lfloor az_1+cz_3\rceil^{\frac{2q}{q+1}} - (bc-a)\lfloor z_2\rceil^q\\ =& -(bc-a)\lfloor z_2\rceil^q.\end{aligned} \tag{8}$$

From (8), we can conclude that $z_2 \equiv 0$ in S. In addition, we can obtain $z_3 \equiv 0$ by $0 = \dot{z}_2 = \theta z_3$ and consequently $z_1 \equiv 0$ (recall $az_1 = -cz_3$) in S.

Therefore, the only solution that stays identically in S is the trivial solution $z(t) \equiv 0$. Thus the origin of system (6) is globally asymptotically stable.

Under the selection of $r_1 = 1, r_2 = \frac{2}{q+1}, r_3 = 1$ and $\tau = \frac{q-1}{q+1}$, we can verify

$$\begin{aligned}&F(\varepsilon^{r_1}z_1, \varepsilon^{r_2}z_2, \varepsilon^{r_3}z_3)\\ =&\begin{pmatrix}\varepsilon^{\tau+r_1}\lfloor z_2\rceil^q\\ \varepsilon^{\tau+r_2}\theta z_3\\ \varepsilon^{\tau+r_3}\left(-\lfloor az_1+cz_3\rceil^{\frac{2p}{p+1}} - b\lfloor z_2\rceil^q\right)\end{pmatrix}.\end{aligned}$$

Therefore, by the weighted homogeneity in Definition 1, it can be concluded that $F(z)$ is homogeneous of degree $\frac{q-1}{q+1} < 0$. Then based on Lemma 4, the origin of system (4) under controller (5) is globally finite-time stable. ■

Remark 1: It is worth pointing out that if we set $q = 1$, the closed-loop system (6) becomes a linear system whose characteristic equation is

$$s^3 + cs^2 + \theta bs + \theta a = 0$$

for a positive θ. By Routh-Hurwitz stability criterion, the sufficient and necessary condition for asymptotic stability of the linear case is $bc > a$ which is the exactly same as the sufficient condition introduced in Theorem 2 for the nonlinear system (6).

B. Finite-time control of linear planar systems with mismatched disturbances

In this subsection, we consider the finite-time stabilization problem of the following linear system

$$\begin{cases}\dot{x}_1 = \theta_0 x_2 + \theta_1,\\ \dot{x}_2 = u,\end{cases} \tag{9}$$

where $\theta_0 > 0$ and θ_1 are unknown constants. The unknown constant θ_1 represents a mismatched disturbance. Not only we want to stabilize the output, i.e., to drive the output to zero, but also we will try to make it converge to zero in a finite time.

Theorem 3: The equilibrium of system (9) can be globally stabilized in a finite time by the following integral controller

$$\begin{cases}\dot{x}_0 = \lfloor x_1\rceil^q,\\ u = -\lfloor ax_0 + cx_2\rceil^{\frac{2q}{q+1}} - b\lfloor x_1\rceil^q,\end{cases} \tag{10}$$

for positive constants a, b, c, and q, satisfying $bc > a$ and $q < 1$.

Proof: The closed-loop system of (9)-(10) can be described as

$$\begin{cases}\dot{x}_0 = \lfloor x_1\rceil^q,\\ \dot{x}_1 = \theta_0 x_2 + \theta_1,\\ \dot{x}_2 = -\lfloor ax_0 + cx_2\rceil^{\frac{2q}{q+1}} - b\lfloor x_1\rceil^q.\end{cases} \tag{11}$$

Under the following coordinate transformation

$$z_1 = x_0 - \frac{c\theta_1}{a\theta_0}, \quad z_2 = x_1, \quad z_3 = x_2 + \frac{\theta_1}{\theta_0},$$

system (11) can be rewritten as

$$\begin{cases}\dot{z}_1 = \lfloor z_2\rceil^q,\\ \dot{z}_2 = \theta_0 z_3,\\ \dot{z}_3 = -\lfloor az_1 + cz_3\rceil^{\frac{2q}{q+1}} - b\lfloor z_2\rceil^q.\end{cases} \tag{12}$$

Under the conditions $bc > a$ and $q < 1$, by Theorem 2 the origin of system (12) is globally finite-time stable and the equilibrium $(0, -\theta_1/\theta_0)$ of system (9) is also globally finite-time stable under the integral controller (10). ■

In what follows, we use an example to show how a finite-time integral controller can be designed.

Example 1: Consider

$$\begin{cases}\dot{x}_1 = x_2 + \theta,\\ \dot{x}_2 = u,\end{cases} \tag{13}$$

with an unknown constant θ. Based on Theorem 3, the following integral controller can be constructed

$$\begin{cases}\dot{x}_0 = \lfloor x_1\rceil^{\frac{3}{7}},\\ u = -\lfloor ax_0 + cx_2\rceil^{\frac{3}{5}} - b\lfloor x_1\rceil^{\frac{3}{7}}.\end{cases} \tag{14}$$

For simulation study, we set the initial condition as $[x_0(0), x_1(0), x_2(0)] = [1, 2, 3]$, and $\theta = 1$. First, we choose $a = 1, b = 2$ and $c = 1$ to obtain Figure 1, from which we can see the convergence of $x_1(t)$ and $x_2(t)$ has been realized in a finite-time by the integral controller (14).

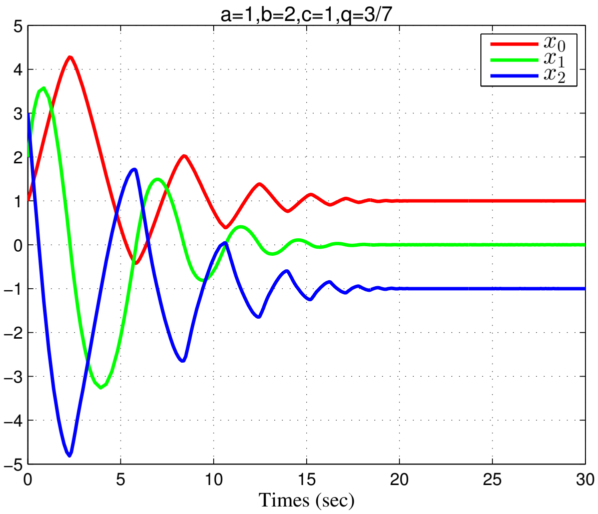

Fig. 1. States of system (13) under the controller (14)

In order to improve the transition response of the closed-loop system, a different choice of control gains $a = b = c = 5$ in controller (14) have been used in the simulation of Figure 2. Compared with the simulation obtained in Figure 1, the settling time has been reduced with larger values of control coefficients a, b and c.

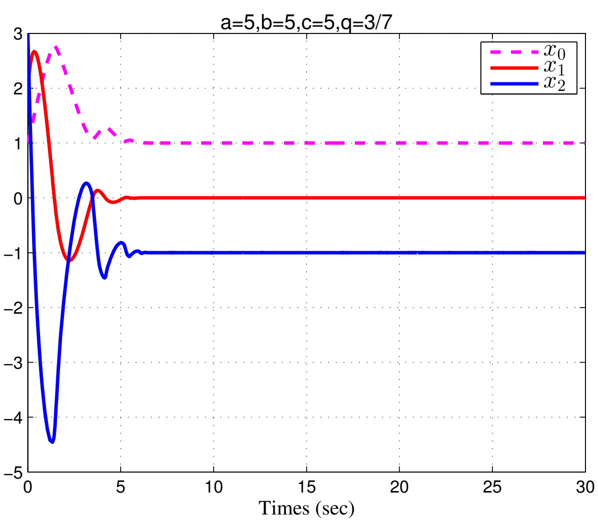

Fig. 2. States of system (13) under the controller (14)

Remark 2: To handle system (9), different from the complex techniques such as DOB (disturbance observer)[13], ESO (extended state observer)[13] or GPIO (generalized proportional-integral observer)[14] in the previous studies, we have used a nonlinear integral action and constructed a homogeneous controller with a linear combination, i.e., $ax_0 + cx_2$, to counter the effects of mismatched constant disturbances. With the controllers proposed in Theorem 3, globally finite-time stabilization of system (9) has been successfully realized.

C. Finite-time control of nonlinear planar systems with mismatched disturbances

Based on the results obtained for the linear system (9), now are can readily prove Theorem 1.

Proof of Theorem 1: The closed-loop consisting of system (1) and the homogeneous integral controller (3) is

$$\begin{cases} \dot{x}_0 = L\lfloor x_1\rceil^q, \\ \dot{x}_1 = x_2 + \theta + g_1(t,x_1), \\ \dot{x}_2 = -L^2\Big(\lfloor ax_0 + c\dfrac{x_2}{L}\rceil^{\frac{2q}{q+1}} + b\lfloor x_1\rceil^q\Big) + g_2(t,x_1). \end{cases} \quad (15)$$

Under the following change of coordinates

$$z_1 = x_0 - \frac{c\theta}{aL}, \quad z_2 = x_1, \quad z_3 = \frac{x_2+\theta}{L}, \quad (16)$$

system (15) can be rewritten as

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{pmatrix} = L\underbrace{\begin{pmatrix}\lfloor z_2\rceil^q\\ z_3\\ -\lfloor az_1+cz_3\rceil^{\frac{2q}{q+1}} - b\lfloor z_2\rceil^q\end{pmatrix}}_{\Xi(Z)} + \underbrace{\begin{pmatrix}0\\ g_1(t,z_2)\\ \frac{g_2(t,z_2)}{L}\end{pmatrix}}_{\Psi(Z)} \quad (17)$$

It is clear that

$$\dot{Z} = \Xi(Z) \quad (18)$$

is the exactly same as (12) with $\theta = 1$. Based on Theorem 2, for $bc > a$ system (18) is globally finite-time stable and is homogeneous of negative degree $\tau = \frac{q-1}{q+1}$ with respect to δ^r, where $r_1 = 1, r_2 = \frac{2}{q+1}, r_3 = 1$. By Lemma 5, there exist a C^1 positive-definite function V that is homogeneous of degree $l > |\tau|$, and constants $\alpha = \frac{l+\tau}{l} \in (0,1)$ and $k > 0$ such that

$$\dot{V}(Z) = \frac{\partial V}{\partial Z}\Xi(Z) \le -kV(Z)^\alpha, \forall Z \in R^n. \quad (19)$$

The derivative of $V(Z)$ along system (17) can be computed as

$$\dot{V}(Z) \le L\frac{\partial V}{\partial Z}\Xi(Z) + \frac{\partial V}{\partial z_2}g_1(t,z_2) + \frac{\partial V}{\partial z_3}\frac{g_2(t,z_2)}{L}. \quad (20)$$

Next we can use the nice properties of homogeneous functions to estimate the terms on the right side of (20). First, under Assumption 1, the following estimations can be obtained

$$\left|\frac{\partial V}{\partial z_2}g_1(\cdot)\right| \le c_1\left|\frac{\partial V}{\partial z_2}\right||z_2|^{\frac{q+1}{2}}, \left|\frac{\partial V}{\partial z_3}\frac{g_2(\cdot)}{L}\right| \le \frac{c_2}{L}\left|\frac{\partial V}{\partial z_3}\right||z_2|^q.$$

By Lemma 1, $\frac{\partial V}{\partial z_2}$ and $\frac{\partial V}{\partial z_3}$ are homogeneous functions with homogeneous degree $l-r_2$ and $l-r_3$, respectively. Due to the fact that both $|z_2|^{\frac{q+1}{2}}$ and $|z_2|^q$ are homogeneous functions with homogeneous degree 1 and $\frac{2q}{q+1}$, by Lemma 2 we have that $\left|\frac{\partial V}{\partial z_2}\right||z_2|^{\frac{q+1}{2}}$ and $\left|\frac{\partial V}{\partial z_3}\right||z_2|^q$ are also homogeneous functions of a homogeneous degree $l+\tau$ with respect to the dilation $\delta_\varepsilon^r(x)$, where $r_1 = 1, r_2 = \frac{2}{q+1}, r_3 = 1$. This, together with the fact that the positive definite function

$V(x)^\alpha$ is homogeneous of degree $l \cdot \alpha = l + \tau$, implies that the following inequalities hold by Lemma 3

$$\left|\frac{\partial V(Z)}{\partial z_2}\right| |z_2|^{\frac{q+1}{2}} \le \bar{c}_1 V(Z)^\alpha, \left|\frac{\partial V(Z)}{\partial z_3}\right| |z_2|^q \le \bar{c}_2 V(Z)^\alpha,$$

for positive constants $\bar{c}_1$ and $\bar{c}_2$.

Substituting the above estimations into the right side of (20), we can have the following

$$\begin{aligned}\dot{V}(Z) &\le -LkV(Z)^\alpha + (c_1\bar{c}_1 + c_2\bar{c}_2)V(Z)^\alpha \\ &= -(kL - c_1\bar{c}_1 - c_2\bar{c}_2)V(Z)^\alpha. \end{aligned} \quad (21)$$

Thus, with a sufficiently large L, there exists a positive constant c_3 such that the derivative of $V(Z)$ along system (17) becomes

$$\dot{V}(Z) \le -c_3 V(Z)^\alpha. \quad (22)$$

Therefore, the origin of closed-loop system (17) is globally finite-time stable. According to the coordinate transformation (16), the equilibrium of system (1) under the homogeneous integral controller (3) is also globally finite-time stable. ■

Remark 3: Different from the previous results on stabilization in finite-time [15], [16], the nonlinear integral action $\dot{z}_0 = \lfloor z_1 \rceil^q$ with $q < 1$ has been employed to handle the mismatched and non-vanishing disturbances. Under the framework of homogeneous domination [17], a scaled integral controller has been constructed to solve the finite-time stabilization of nonlinear planar systems with mismatched disturbances.

Remark 4: As shown in [10], based on the inequality (22) the upper bound of settling time can be estimated as $\frac{1}{c_3(1-\alpha)} V(Z(0))^{1-\alpha}$, provided that we have the explicit construction of $V(Z)$.

Next we apply Theorem 1 to an example with uncertain nonlinearities.

Example 2: Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 + \sin(x_1)d(t), \\ \dot{x}_2 = u + \ln(x_1^2 + 1)d(t), \end{cases} \quad (23)$$

where $|d(t)| \le 1$. Instead of a direct mismatched disturbance, we assume there is an arbitrary constant drift in measurement of x_2, which is common in estimating attitude of UAV from its angular velocity that measured by gyroscope. Thus in this example, we assume that only the information of $\overline{x}_2 = x_2 + \vartheta$ not that of x_2 is available for controller design, where ϑ is an arbitrary constant drift. Under the measurable states x_1 and $\overline{x}_2$, system (23) can be rewritten as

$$\begin{cases} \dot{x}_1 = \bar{x}_2 - \vartheta + g_1(t, x_1), \\ \dot{\bar{x}}_2 = u + g_2(t, x_1) \end{cases}$$

where $g_1(t, x_1) = \sin(x_1)d(t)$, and $g_2(t, x_1) = \ln(x_1^2 + 1)d(t)$. It can be verified that $g_1(t, x_1)$ and $g_2(t, x_1)$ satisfy Assumption 1 for $q = \frac{3}{7}$. As a matter of fact, since $|\sin x| \le \max\{|x|, 1\}$, we have $|\sin(x_1)d(t)| \le |x_1|^{\frac{5}{7}}$. In addition, by the mean-vale theorem, we have $|g_2(t, x_1)| \le \left|\ln\left((|x_1|^{\frac{3}{7}})^{\frac{14}{3}} + 1\right)\right| \le \frac{14}{3}|x_1|^{\frac{3}{7}}$. Therefore, by Theorem 1 the following controller can be constructed

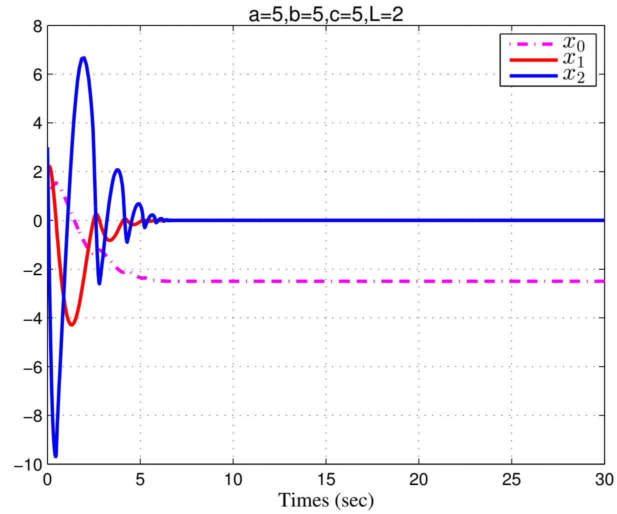

Fig. 3. States of system (23) under controller (24)

$$\begin{cases} \dot{x}_0 = L \lfloor x_1 \rceil^{\frac{3}{7}}, \\ u = -L^2 \left(\left\lfloor a x_0 + c\frac{\overline{x}_2}{L} \right\rceil^{\frac{3}{5}} + b \lfloor x_1 \rceil^{\frac{3}{7}} \right). \end{cases} \quad (24)$$

For simulation study, we set the initial condition as $[x_0(0), x_1(0), x_2(0)] = [1, 2, 3]$, $\vartheta = 5$ and $a = b = c = 5$. From the simulation results obtained in Figure 3, it can be seen that the states $x_1(t)$ and $x_2(t)$ converge to zero in a finite time.

D. Integral control with dual modes

In the previous subsections, to globally regulate the uncertain systems in a finite time, we have designed homogeneous controllers with a negative homogeneous degree. However, in the areas far away from the equilibrium, the homogeneous controllers with a negative homogeneous degree will not be as powerful as the linear controllers.

To address this issue, in this subsection we introduce a new integral controller with dual models. For two constants $q \in (0, 1)$ and $p \in [1, +\infty)$, we construct the following integral controller

$$\begin{cases} \dot{x}_0 = \lfloor x_1 \rceil^q + \lfloor x_1 \rceil^p, \\ u = -\lfloor ax_0 + cx_2 \rceil^{\frac{2q}{q+1}} - \lfloor ax_0 + cx_2 \rceil^{\frac{2p}{p+1}} - b\lfloor x_1 \rceil^q - b\lfloor x_1 \rceil^p, \end{cases} \quad (25)$$

with control coefficients a, b and c.

Theorem 4: The equilibrium of system (9) can be globally asymptotically stabilized by the integral controller (25), where a, b, c are positive constants satisfying $bc > a$.

Proof: Construct the following Lyapunov function

$$\begin{aligned} V(x_0, x_1, x_2) =& \frac{a}{2}(bx_0 + x_2)^2 + \frac{1}{2}(bc - a)x_2^2 \\ &+ \frac{b(bc - a)}{\theta_0} \left(\frac{|x_1|^{q+1}}{q+1} + \frac{|x_1|^{p+1}}{p+1} \right) \end{aligned} \quad (26)$$

which is positive definite and radically unbounded since $bc > a$. It can be verified that the derivative of the above Lyapunov function along the closed-loop system of (9) and (25) is

$$\dot{V}(x_0, x_1, x_2) = -b|ax_0 + cx_2|^{\frac{3q+1}{q+1}} - b|ax_0 + cx_2|^{\frac{3p+1}{p+1}}. \quad (27)$$

Clearly, the right hand side of (27) is semi-negative definite. Following the same line of the proof of Theorem 2, asymptotic stability of the closed-loop system of (9) and (25) can be proved using LaSalle's invariance principle. ■

Remark 5: It is clear that the closed-loop system (9) and (25) is not homogeneous if $p \neq q$. However, when the states are close to the equilibrium, the closed-loop system will be locally homogeneous of the negative degree $\frac{q-1}{q+1}$ which is smaller than $\frac{p-1}{p+1}$. Therefore, the closed-loop will be locally finite-time stable [15], [16]. In addition, when the states are far away from the equilibrium, the system is dominated by the homogeneous terms with the larger degree of $\frac{p-1}{p+1}$. Therefore, the closed-loop system will have faster convergence speeds guaranteed for states close to the equilibrium or far away form the equilibrium.

Next we apply the dual-mode controller (25) to stabilize (13).

Example 3: To conduct computer simulation of system (13) under the duel-mode controller (25), we use the same control coefficients a, b, c and initial conditions used for Figure 1, i.e., $a = 1$, $b = 2$, $c = 1$, $[x_0(0), x_1(0), x_2(0)] = [1, 2, 3]$, and $\theta = 1$. In addition to $q = 3/5$ used in Example 1, we use $p = 1$ for Figure 4.

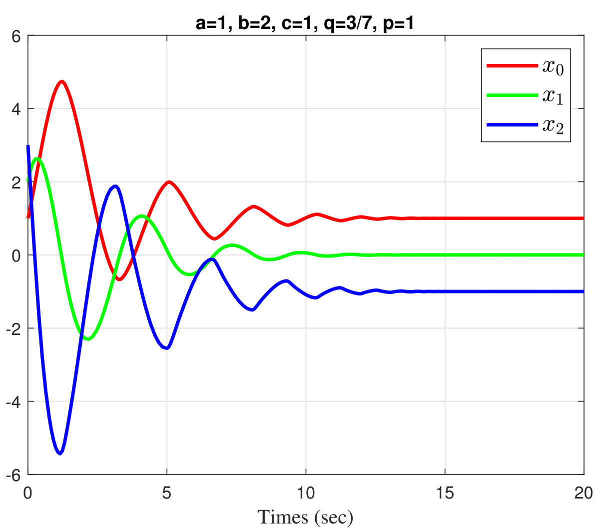

Fig. 4. States of system (13) under the controller (25)

Clearly, the states $x_1(t)$ and $x_2(t)$ in Figure 4 demonstrate faster convergence speeds compared to those in Figure 1. Therefore, in the practices, if faster convergence rates are desirable for both small and large initial conditions, the dual-mode controllers (25) will be more powerful than the lower-order controller (14) or linear controllers like PID controllers.

IV. Conclusions

Globally finite-time stabilization of a planar nonlinear system with a mismatched unknown disturbance has achieved in this paper. To compensate the disturbance, we have introduced a nonlinear integral dynamic and then constructed a finite-time integral controller using a linear combination of the system state and integral state. Under the framework of homogeneous domination, the obtained controllers for linear planar system is further scaled for global finite-time stabilization of a nonlinear planar system. For a faster convergence rate, we have also proposed a dual-model integral controller, under which the equilibrium is locally finite-time stable and the solutions are exponentially convergent in the large.

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