QUNATITATIVE REASONING AND COVARIATIONAL REASONING AS THE BASIS FOR MATHEMATICAL STRUCTURE FOR REAL-WORLD SITUATIONS

Sindura Subanemy Kularajan
Texas State University
S_k252@txstate.edu

Jennifer A. Czocher
Texas State University
Czocher.1@txstate.edu

In this paper we address the question, how do quantitative reasoning and covariational reasoning present as students build structural conceptions of real-wrold situations. We use data from an exploratory teaching experiment with an undergraduate STEM major to illustrate the explanatory roles quantitative reasoning and covariational reasoning play in, (a) coordinating more than two interdependent quantities, (b) conceiving of real-world situations in more than one way, (c) constructing networks of quantitative relationships, and (d) creating a mathematical expression. We conclude that quantitative and covariational reasoning influences a modeler's structuring for a real-world situation as he conceives quantities, operates on them, reasons about how they vary with one another, and construct a mathematical representation.

Keywords: Mathematical Modeling, Quantitative Reasoning, Covariational Reasoning

Mathematical Modeling plays a central role in supporting students' views of relevance of mathematics to the real world. Many scholars as well as curricular materials have advocated for the importance of including mathematical modeling into the mathematics curriculum because it would motivate the use of mathematics in the world outside of a classroom, for students (Blum & Niss, 1991, Zibek & Connor,2006; CCSSI,2010). In addition, empirical studies have also shown that incorporating mathematical modeling in the teaching of mathematics can positively impact both students' learning of mathematics and affective factors which influence students' learning such as interest, motivation, and self-efficacy (e.g Czocher, 2017; Rasmussen & Blumenfeld, 2007; Schukajlow et al., 2012; Zbiek & Conner, 2006). Despite the field's desire to motivate the learning of mathematical modeling and its inclusion to curricula, mathematical modeling remains highly challenging for students (Stillman, Brown, & Galbraith, 2010; Jankvist & Niss, 2017; Blum, 2011; Blum & Leiss, 2007).

The challenges students experience while engaging in mathematical modeling are multifaceted. Mathematical modeling involves translating between real-world and mathematics in both directions simultaneously. This translation requires the appropriate mathematical and real-world knowledge (Blum, 2011), so that the modeler can associate the appropriate mathematics with the real-world situation. Existing research on mathematical modeling investigates students' mathematical modeling activities, informing the field about challenges students face in given content areas or real-world scenarios. These studies focus on how students simplify the real-world situation, identify important parameters and variables from the simplified situation, transform these identifications into a mathematical representation, and check the validity of the mathematical representation created against real world constraints. However, studies focusing on such modeling processes are of limited utility for describing how students may conceive of a real-world situation in ways supportive of choosing the appropriate mathematics to represent the real-world situation.

The literature on mathematical modeling is clear that mathematical knowledge alone is insufficient for choosing viable mathematics to represent a real-world situation. The modeler, would also need an understanding the entities present in the real-world situation, how these entities contribute to the aspects that needs to be modeled, and relations among these entities. That is, the modeler may have to conceive the real-world situation through quantities and relations among those conceived quantities. Before the field of mathematical modeling can promote this kind of understanding among modelers, the field would first need an idea of how quantitative relations are established during model construction. Conceiving of a real-world situation through quantities and establishing relations among those quantities involves quantitative reasoning and covariational reasoning. Therefore, the purpose of this paper is to discuss how covariational reasoning and quantitative reasoning is present in the construction of mathematical models of real-world situations. We first present research surrounding mathematical modeling and quantititative reasoning, Next, we provide our theoretical orientation that was used towards analysis of data. Next we present four examples of how quantitative reasoning covariational reasoning is present in model construction. Finally, we discuss implications.

Mathematical Modeling and Quantitative Reasoning

Thompson (2011) claimed that “mathematical modeling is simply mathematics in the context of quantitative reasoning” (p. 52). By this Thompson means, in mathematical modeling a modeler uses mathematical notation and methods to express a relationship among quantities that were constructed by the modeler. Larson (2013) in her study with linear algebra college students explored the role of quantities and quantitative reasoning in mathematical model construction. Larsen operationalized a mathematical model as a system that consists of elements, relationship among elements, and operations that describe how these elements interact. Larsen made the case that quantities act as “elements” in students’ mathematical model. Further, she stated that quantitative reasoning provides a language to describe (i) how students consider quantities that are relevant, (ii) how students’ express the relationship among these quantities, (iii) use these relationships to operate on these quantities, and finally (iv) how these operations would give rise to new quantities that are also “elements” of the students’ model. Larsen claims that quantitative reasoning is a central mechanism in model development because *products* (derivation of new quantities by operating on identified quantities) of one stage at model development become the *objects* at the next stage.

Czocher & Hardison (2020) presented methodological approaches for understanding the quantities that modelers identify as situationally relevant in a given modeling tasks and how the conception of these quantities are manifested as observations through external inscriptions and utterances the students’ make. They formulated eight observable criteria that can be used as indications that the modeler engaged in the process of quantification. Further, they defined the construct *modeling space* (Czocher & Hardison, 2019) as the set of mathematical models the modeler constructs within a given modeling task to conceptualize students’ mathematical model. Collectively, contributions lead the mathematical modeling field in a new path to trace the genesis of students’ quantities and to understand how the meanings students attribute to these quantities may change over time in the context of mathematical modeling. In this paper we contribute to the existing conversation around mathematical modeling and quantitative reasoning. In particular, we address the following question: Given that quantitative and covariational reasoning are foundational to mathematical modeling, how do they present in students’ conceiving of mathematical structure within a real-world situation?

Theoretical Orientation

We take on the cognitive perspective on mathematical modeling (Kaiser, 2017). In this perspective, mathematical modeling is considered to be the cognitive processes involved in constructing a mathematical model of real-world scenarios. We view *mathematical model* an individual constructs as an external representation of the relationship between the quantities the individual conceived as relevant to the real-world situation she is given to model. We take on Thompson's (2011) view on quantity, where quantity is a mental construction of a measurable attribute of an object. Thompson (2011) defines *quantification*, the mental construction of quantities, as "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship with its unit" (p.37). *Quantitative reasoning* (QR) refers to conceiving and reasoning about quantities and the relations among the conceived quantities. A *quantitative operation* is a "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (Thompson, 1994, p.10). Examples of quantitative operations include combining two quantities additively or multiplicatively and comparing two quantities additively or multiplicatively.
As a result of a quantitative operation, a relationship is created: the quantities that were operated on along with the quantitative operation in relation to the result of operating.
Each quantitative operation creates a relationship: The quantities operated upon with the quantitative operation in relation to the result of operating. As Thompson (1994) noted, Conception of complex situations are built by constructing networks of quantitative relationships" (p.11).

Carlson and colleagues (2002) define *covariational reasoning* (CR), a form of quantitative reasoning, to be "the cognitive activities involved in coordinating varying quantities while attending to the ways in which they change in relation each other" (p. 354). Carlson et al (2002) identified five mental actions students exhibited when engaging in covariational reasoning. These five mental actions include, *coordination* of quantities, coordinating the *direction of change* of quantities, coordinating the *amounts of change* of quantities, coordinating *average rate of change* of one quantity with respect to the other quantity, and coordinating the *instantaneous rate of change* of one quantity over the interval of the domain. Similarly, Thompson and Carlson (2017) proposed six major levels of covariational reasoning. These levels are smooth continuous variation, chunky continuous variation, coordination of values, gross coordination of values, pre coordination of values, and no coordination of values. Moore et al (2020) extend the work of QR and CR by introducing the construct *abstracted quantitative structure*: "a system of quantitative relationships a person has interiorized to the extent they can operate as if it is independent of specific figurative material" (p.752, Moore et al, 2020) as means to explain the construction of a concept. Borrowing ideas from these constructs, we define a *structure for real-world situation* to be the network of quantitative relations among the quantities one constructed as relevant to model a real-world situation. By network of quantitative relations, we mean the set of quantitative relations that was created as a result of operating on conceived quantities. We call this way of understanding about the real-world situation as *a structural conception for the real-world situation.* Students' structural conception of a real- world situation will be analyzed by seeking instances of quantitative and covariational reasoning while they engage in mathematical modeling activities.

Methods

We present data from an exploratory teaching experiment (Steffe & Thompson, 2000) conducted with an undergraduate STEM major at a large university in the United States of America. We used a teaching experiment methodology because it afforded us means to build explanatory accounts of students' mathematical reasoning while they conceive of real-world situations through quantities and relations among those quantities. The overall goal of the exploratory teaching experiment was to test an initial research hypothesis, tasks and task sequence that will be used to investigate how students use quantitative reasoning and covariational reasoning to construct mathematical models of real-world scenarios.

Our participant Baxil, a non-native English speaker, participated in a total of ten clinical interview sessions comprising the teaching experiment. Baxil was an undergraduate mathematics major who at the time of the experiment was enrolled in a differential equations course. Each session was approximately an hour long. In this report, we draw data from three teaching sessions where Baxil engaged in the *Baker's Yeast Task*, *The population Dynamics Task*, and *The Fruit Ripening Task*. We focus on these tasks because they provide illustrations of how conceiving of quantities, operating on conceived quantities, reasoning about conceived quantities that change in tandem are present in the structuring for real-world scenarios and because they offer insights into networks of quantitative relations that can be built up into models.

> *The Baker's Yeast Task:* Baker's yeast is a type of fungus that reproduces through budding. Each cell reproduces once every 30 minutes. To grow yeast for baking bread, you have to proof it first (allow it to form a colony) in a bowl of warm water. Suppose that in a particular bowl, after six hours, the surface of the water is covered in yeast cells. Can you come up with an expression that gives the number of cells present after 6 hours if we start with n cells initially?
>
> *The Population Dynamics Task:* Suppose in a laboratory setting, we are looking at large populations of breeding stock in which species give birth to new offspring but also die after some time. Suppose that the given population has a birth rate of α% and, the death rate of the population due to natural causes is β%. If P is the population of species at any given time, write a mathematical expression for the rate at which the population changes with time.
>
> *The Fruit Ripening Task:* There is a surprising effect in nature where a tree or bush will suddenly ripen all of its fruit or vegetables, without any visible signal. If we look at an apple tree, with many apples, seemingly overnight they all go from unripe to ripe to overripe. This will begin with the first apple to ripen. Once ripe, it gives off a gas known as ethylene (C_2H_4) through its skin. When exposed to this gas, the apples near to it also ripen. Once ripe, they too produce ethylene, which continues to ripen the rest of the tree in an effect much like a wave. This feedback loop is often used in fruit production, with apples being exposed to manufactured ethylene gas to make them ripen faster. Develop a mathematical model that captures the dynamics of the ethylene gas produced.

The primary goal of the exploratory teaching experiment was to build accounts of Baxil's mental activities as he reasoned quantitatively and covariationally to construct mathematical models of the task scenarios. Since we did not have direct access to Baxil's mental activities, we created second-order accounts (Steffe & Thompson, 2000) of inferences we made from Baxil's observable activities including his language, verbal descriptions and discourse, written work, and his mathematically salient gestures. Each episode was video recorded, and his written work was digitized.

We conducted both ongoing analysis and retrospective analysis (Steffe & Thompson, 2000). The ongoing analysis involved testing and formulating hypothesis during the teaching experiment based on ways Baxil was reasoning with the quantities he conceived as relevant to model the situations. After the completion of the teaching experiment, we revisited the data to perform an in-depth retrospective analysis. Our retrospective analysis consisted of two phases: observing and describing Baxil's mathematical modeling activities and constructing and refining accounts of Baxil's use of quantitative reasoning and covariational reasoning to mathematically structure the task scenarios. The two phases of the retrospective analysis of Baxil's engagement in the three tasks, comprised of five rounds of data analysis to arrive at examples that could serve for theory-building. First, we watched the videos or subsets of videos without interruption to observe patterns in Baxil's activities. Second, we paid closed attention to Baxil's utterances, gestures, and written work and described his mathematical modeling activities for the three tasks. Third, we identified instances where Baxil was reasoning with conceived quantities (Thompson, 2011; Author 2, Year 3), operating on the conceived quantities (Thompson, 1994), and engaging in covariational reasoning (Carlson et al, 2002; Thompson and Carlson, 2016). Fourth, we constructed annotated transcripts of such instances that provided rich descriptions of Baxil's mathematical modeling activities. Finally, we constructed and refined explanatory models of Baxil's structuring of the three task scenarios.

Findings

Coordination of three interdependent quantities

In the fruit ripening task, Baxil conceived ripeness as "readiness to eat" the fruit. When the TR asked to draw an ethylene gas production-time graph, Baxil sketched Figure 2. He reasoned "I would say increasing slowly at the beginning, then increasing faster as they are ready to eat because after you're ready to eat, it will produce more instead if it didn't ripe yet." Here, Baxil conceived of a relation among gas production and time where, as time goes on, the rate at which ethylene gas produced increases because as fruits are ripening, they produce more gas. The TR probed his rationale for why the ethelyne gas production would be faster as the fruit ripens. Baxil explained "When you're not ready to eat, it's just like a little bit amount of the gas, I would think, but after it's ready, it goes faster because everywhere have the gas". Baxil engaged in coordination of three interdependent quantities (amount of ethelyne gas produced, gas production, and time), while maintaining pairwise coordination between amount of gas vs. time and gas production vs. time, and production of gas vs. amount of gas.

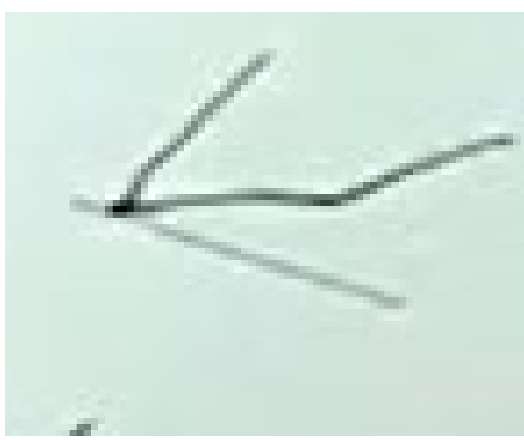


Figure 2: Baxil's graph for ethylene gas produced vs time.

There can be more than one way of conceiving a real-world scenario.
In the fruit ripening task, The TR asked Baxil to construct a mathematical expression for the amount of ethylene gas produced. He presented two expressions and discussed the merits of each:

i. Amount of gas produced by the apple which is ready to eat = $e^{rate\ of\ gas*\ time}$
ii. Amount of gas produced by the apple which is ready to eat = $rate^{\ time}$

In expression (i), Baxil conceived of rate of gas to be the "percentage of gas inside the apple". By that, Baxil meant the ripeness to ethylene conversion rate. Whereas in the second expression, he indicated that rate would be "the rate of gas that affect the (ripeness of the apple)". Baxil further indicated that the amount of gas, as represented in the first expression, would be increasing slowly. Whereas in the second expression, the amount of gas would increase quickly. This interpretation was evident in his following explanation:

> May I make an example like the raw apple there is a little bit of gas like I say 10% of them I guess, so it might be a 20% of them and the next there is something like that and there is a 40% then a 60% it doesn't add to 100% that's the second equation thinking and for the first equation I was thinking if it is 10% the rate won't be changing... I mean not the rate the like the amount then I say like its 10% it might be and depend on the tense it will be increasing by one-tenth, two-tenth, third-tenth, four-tenth... something like that.

Here Baxil conceived two distinct measurable attributes of the same object, apple. One was by how much the apple produces the gas and the other being by how much the gas affects the apple. As a result, he constructed two expressions that, despite being mathematically equivalent, behaved different to him in terms of quantities and quantitative operations.

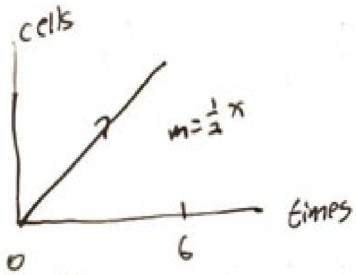
Constructing a network of quantitative operations
Baxil's initially conceived of the population dynamics scenario in terms of a birth rate and a death rate. Baxil was thinking about the population changing continuously along 1-second chunks of time, indicating chunky continuous variational reasoning. This was evident when he reasoned "every second have some people die and every second have people born". While he was reasoning about how the population changes, he was also coordinating the quantities population, people born, and people dead simultaneously. This was evident when he reasoned "Because if someone is born, so the population is growing as well. That means the principle is changing too. But when people die…the population, it's also going down." By principle, Baxil meant initial population during any time chunk. He also indicated that the quantities birth rate (α) and death rate (β) will be non-varying.

Baxil's final model for this real-world scenario was $P'(t) = P(t)\alpha - P(t)\beta,$ where $P(t)$ is the population over time, $P(t)\alpha$ is the "percentage of people to be born" and $P(t)\beta$ is the "percentage of people that will be dead." Here, he constructed the quantities "percentage of people be born" and "percentage of people be dead" via the multiplicative combination of the quantities (α, $P(t)$) and (β, $P(t)$) respectively. Baxil also gave evidence of thinking of rate of change of population as the net rate of change by meaningfully adding the percentage of people that can be born and subtracting the percentage of people that would die. By additive combination of two quantities, he created an expression for $P'(t)$, which itself is a multiplicative combination of population and time. Baxil was also aware that all of the quantities he constructed, were implicitly dependent on time itself. Therefore, Baxil expressed a network of quantitative operations as a network of arithmetic operations, by operating on the quantities $P(t), t, \alpha, \beta, \alpha P(t), \beta P(t)$, and $P'(t)$.

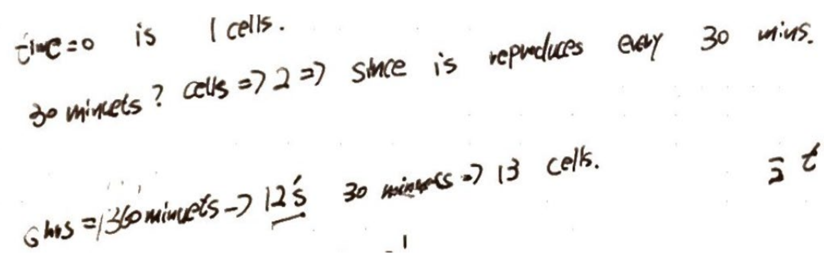
QR and CR supports in constructing a mathematical expression for a real-world situation.

Baxil's initial conception of the Baker's Yeast scenario was that the "number of cells" and "time" share a linear relationship. He drew the graph in Figure 1(a) reasoning as "because times is increasing…then the cells is also increasing". At this instance, there is evidence to claim that Baxil was reasoning covariationally, where not only did he coordinate the quantities "number of cells" and "time", but he also established a directional relationship between them.

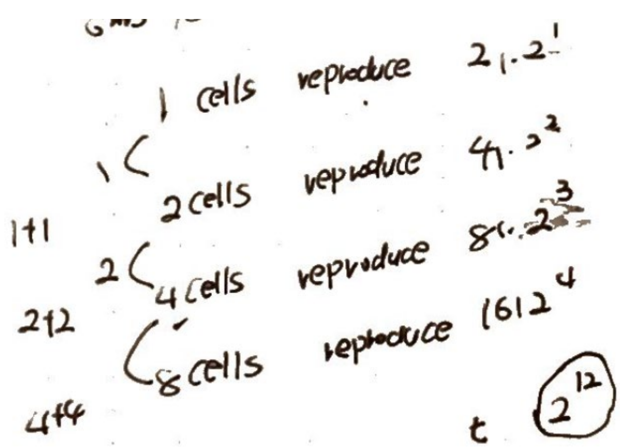
The teacher-researcher (TR) then asked him what happens to the number of cells at each 30-minute mark, to which Baxil constructed the representation in Figure 1(b). Although he reasoned that at the end of the first 30 minutes there would be 2 cells "because it reproduces every thirty minutes", at the end of the 6 hours, he reasoned as "six hours has 360 minutes, which is 12 of thirty minutes, there for 12+1=13 cells", attempting quantitatively coordinate how the number of cells would change at the end of every 30 minutes. The TR intervened, asking what happens to the 2 cells after at the end of an additional 30 minutes, Baxil answered that 2 cells become 4 cells "because both [cells] reproduce". As a result, Baxil produced the representation in Figure 1(c) and deduced that at the end of six hours there would be 2^{12} cells. He then wrote down 2^{2t} as the number of cells produced after t hours of time, given that he starts with 1 cell. He attained this structure through coordinating the amounts of change of time and number of cells. Baxil established a quantitative relationship between the number of cells and time through coordinating the direction of change and amounts of change of those quantities. Through his quantitative and covariational reasoning, Baxil was able to structure the Baker's yeast scenario.



(a)



(b)



(c)

Figure 1: (a) Baxil's graph for how number of cells varies with time from $t = 0$ to $t = 6$, (b) Baxil's explanation for the number of cells present at the end of 6 hours and (c) Baxil's final model when starting with 1 cell.

Conclusion and Discussion

In the Baker's Yeast task, Baxil was able to establish a structure for the situation through coordinating the value and direction of change of quantities time and number of cells. Although initially his coordination of the amounts of change of these two quantities of these two wasn't representative of the situation (from the TR's perspective), through TR's intervention, he was able to conceive how the number of cells would increase every thirty minutes. Being able to internalize this quantitative coordination, helped him to create a mathematical expression that would predict the number of cells at time t. In the Population Dynamics task, Baxil was able to establish a structure for the situation through reasoning with the quantities: rate of change of prey population, the population of prey, the birth rate, and death rate of prey, through coordinating the direction of change of the quantities population of prey and time and engaging in quantitative operation to produce other quantities. And finally, In the fruit ripening task Baxil coordinated three interdependent quantities to reason how production of ethylene gas changes with time and brought in two distinct structures, in his perspective, to build a model for the scenario. In conclusion, Baxil structured these three scenarios through conceiving quantities, operating on those conceived quantities, and through engaging in covariational reasoning to discuss how those conceived quantities are related.

A primary goal of the work in QR and CR is describing students' learning of specific mathematical ideas and their attendant reasoning processes. We make the case that QR and CR influences a modelers' structuring for real-world situations. This is because, the manner in which one choses to operate on two or more already conceived quantities, establishes a relationship among the old quantities and the newly created quantity (that resulted by operating on the old ones). This newly constructed quantitative relation influences the construction of other new quantities and operations on those quantities. This way, multiple conceptions of real-world situations can originate through engaging in quantification and covariational reasoning. Therefore, having a structural conception of the real-world situation provides students the opportunity to realize how seemingly different real-wrold situations can be mathematically modeled using the same mathematics and how the same situation can be modeled using different mathematics, depending on the conceived quantities and the operations performed on them. Looking at QR and CR in model construction allows researchers to pay close attention to get an understanding of the quantities the modeler conceived and the modeler's reasoning about those quantities. This understanding will provide a better picture about the mathematical decisions the modeler makes to mathematize complex situations, particularly during the simplifying and specifying phases of modeling (Zbiek & Conner, 2006; Blum, 2011) that precede the formal mathematical expression of a model. We believe attending to the quantitative and covariational rationales for these decisions will open opportunities for appropriate intervention when necessary.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No.XXX.

References

Czocher, J. A. (2017). How can emphasizing mathematical modeling principles benefit students in a traditionally taught differential equations course? *Journal of Mathematical Behavior*, 45,78–94.

Czocher, J. A., & Hardison, H. (2019). *Characterizing Evolution of Mathematical Models* Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, St Louis, MO.

Czocher, J. A., & Hardison, H. L. (2020). Attending to Quantities through the Modelling Space. In F. Leung, G. A. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical Modelling Education in East and West*. Springer.

Blum, W. (2011). Can modelling be taught and learnt? Some answers from empirical research. In *Trends in teaching and learning of mathematical modelling* (pp. 15–30): Springer.

Blum, W., & Leiss, D. (2007). How do students and teachers deal with modelling problems? In *Mathematical modelling* (pp. 222–231): Elsevier.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to other subjects-State, trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22,* 37–68.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for research in mathematics education*, *33*(5), 352-378.

Common Core State Standards Initiative (2010), National Governors Association Center for Best Practices and Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Jankvist, U. T., & Niss, M. (2019). Upper secondary school students' difficulties with mathematical modelling. *International Journal of mathematical education in science and technology*, *51*(4), 467–496.

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 267–291). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Larson, C. (2013). Modeling and quantitative reasoning: The summer jobs problem. In *Modeling students' mathematical modeling competencies* (pp. 111–118). Springer, Dordrecht.

Moore, K., Liano, B., Stevens, I.E., Tasova, H.I., Paoletti, T., Yimg. Y. (2020) A quantitative reasoning framing of concept construction. In Karunakaran, S. S., Reed, Z., & Higgins, A. (Eds.). Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education. Boston, MA

Rasmussen, C., & Blumenfeld, H. (2007). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *The Journal of Mathematical Behavior*, 26(3), 195–210.

Schukajlow, S., Leiss, D., Pekrun, R., Blum,W., Müller, M., &Messner, R. (2012). Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. *Educational Studies in Mathematics*, 79, 215–237.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.

Stillman, G., Brown, J., & Galbraith, P. (2010). Identifying challenges within transition phases of mathematical modeling activities at year 9. In R. Lesh, P. Galbraith, C.R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies, ICTMA13* (pp. 385–398). New York, NY:Springer.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33–57). Laramie, WY: University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium for research in mathematics education*, 421-456.

Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics*, 63(1), 89–112.