STEM UNDERGRADUATES' STRUCTURAL CONCEPTION OF RATE OF CHANGE

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Using data from teaching experiments and theories from quantitative reasoning, we built second-order accounts of students' mathematics with regards to how they conceived rate of change through operating on existing quantities. In this report, we explain three different ways STEM undergraduates structurally conceive rate of change as they constructed mathematical models for real-world scenarios.

Keywords: Quantitative Reasoning, Mathematical Modeling, Rate of Change

Understanding *rate of change* (RoC) is central to the understanding of mathematical concepts such as the fundamental theorem of calculus (Thompson, 1994b; Thompson, 2008), derivative (Zandieh, 2000), limits and Function (Tall, 1986), and differential equations. Beyond mathematical concepts, an understanding of RoC is necessary to model dynamic situations (e.g., Arleback et al, 2013; Ellis, 2007). Generally, the main goal of studies surrounding RoC is to understand students' mathematical conceptions of RoC. That is, the field is reporting accounts of students' understanding of RoC as they are working on tasks that help them to understand RoC as a mathematical concept. We still need an account of how students are conceiving RoC as they are developing mathematical models for real world scenarios that aren't necessarily set out teach the concept of RoC. This is important because educators of the field will get an idea of how students apply their understanding of RoC into practice. Accordingly, we ask: how do STEM undergraduates conceive RoC while constructing mathematical models for real-world scenarios?

Theoretical Framing and Background

Our research lies within the cognitive perspective of mathematical modeling (Kaiser, 2017). In this perspective, mathematical modeling is the cognitive processes involved in constructing a mathematical model for real-world scenarios. We define a mathematical model to be a representation of the relations among conceived quantities.

A *quantity* is a mental construct of a measurable attribute of an object. It consists of three inter-dependent entities: an object, a measurable attribute, and a *quantification*. Quantification involves conceiving a measurable attribute of an object and a unit of measure and forming a proportional relationship between an attribute's measure and the unit of measure (Thompson, 2011). Quantification takes place in the mind of an individual and we can only use an individual's externalized actions to infer whether she has quantified an attribute. Czocher & Hardison (2021) presented eight quantification criteria that can be taken as indication that a student has conceived measurement process for a measurable attribute of an object.

Quantitative reasoning is the mental operations involved in conceiving a situation through quantities and relation among those conceived quantities (Thompson, 2011). A relation among conceived quantities is established through operation on quantities. Thompson (1994) defines *quantitative operation* as the "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (p.10). As a result of a quantitative operation a *relationship* is created: the quantities operated upon along with the quantitative operation are in relation to the result of operating (Thompson, 1994). Examples of quantitative operations include combining two quantities additively or multiplicatively and comparing two quantities additively or multiplicatively. A *quantitative structure* is a network of quantitative relationships (Thompson, 1990).

Reasoning about quantities may also entail reasoning about how the quantities can vary. While *variational reasoning* involves reasoning about varying quantities independently (Thompson & Carlson, 2017), *co-variational reasoning* involves "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al, 2002, p.354). Scholars have extended the work of covariational reasoning to *multivariational reasoning*, which is reasoning about more than two quantities changing in conjunction with each other (Jones 2018; Jones & Jeppson,2020). Recently, Ellis et al (2020) defined *scaling continuous covariation* as "imagining the continuum as infinitely zoomable, couple with the understanding that one can re-scale to any arbitrarily small increment for x and coordinate that scaling with associated values for y" (p. 88). When an individual engages in covariational reasoning she constructs a *multiplicative object* in her mind. Constructing a multiplicative object entails coupling of two quantities (Saldanha & Thompson, 1998) that yields a new conceptual object "formed by mentally uniting the attributes of two quantities, so that the new conceptual object is simultaneously one and the other" (Thompson & Carlson, 2017, p. 433). For example, the rate at which the size of a bank account changes with respect to time is a multiplicative object that may be formed by coupling *change in size of the account* and the *length of time elapsed*.

Many scholars have posited ways in which students may develop a conceptual understanding of RoC based on *rates*. For example, using a quantitative reasoning orientation, Thompson (1994a) defined *rate* as a reflectively abstracted constant ratio, where a ratio is a result of comparing two quantities multiplicatively. Thompson (1994b) argued that developing images of rate starts with student imagining a change in some quantity, next coordinating the changes two quantities, and finally developing images of the two quantities covarying so that they remain in constant ratio. Confrey & Smith (1994) used a covariational reasoning approach as a way to learn the concept of function through exploring the concept of RoC. They discovered that students employed three different ways to describe RoC in the context of cell splitting: an additive RoC, a multiplicative RoC, and a 'proportional new to old' RoC. Through these investigations, they proposed an analytic approach to RoC as a "unit per unit comparison" (p. 37) where a unit is a "invariant relationship between a successor and its predecessor" (p. 142), as means to instill a multiplicative unit approach viewing RoC for exponential functions. Johnson (2015) investigated how students quantify *rate* through "forming and interpreting relationship between varying quantities" (p. 66). Johnson discovered that students quantified rate by associating amounts of change in extensive quantities, using a single extensive quantity that represents an association of two quantities, and coordination of intensive and extensive quantity through the quantitative operations *coordination* and *comparison*. Through *coordination,* rate is quantified as an intensive quantity and through *comparison,* rate is quantified as an extensive quantity. While Johnson reported distinct ways students quantify RoC, it excludes ways in which students may conceive RoC that do not necessarily involve quantitative operations as how Thompson defined. We postulate one such operation in this report.

Borrowing ideas from the aforementioned constructs we define *structural conception* of *RoC* as conceiving RoC as a measurable attribute of an object through forming a relation among constituent quantities (i.e., operating on quantities). We use the term *operating on quantities* to include Thompson's quantitative operations, mental operations involved in reasoning about varying quantities, and other operations on quantities that could yield a new quantity. The research question addressed in this report is: In what ways do STEM undergraduates structurally conceive RoC while constructing mathematical models for real-world scenarios?

Methods

We present data from a pair of 10hr individual teaching experiments (Steffe & Thompson, 2000) conducted with undergraduate STEM majors at a large university. The overall goal of the teaching experiment was to investigate how students conceive real-world situations through quantities and relations among quantities. Our students, Szeth and Pai were both enrolled in differential equations at the time of the interviews. Throughout the teaching experiment, Pai and Szeth worked on 9 and 10 tasks, respectively, that were based on real-world scenarios. In this report we present data from the *CI8 Account Task* and *The Cats & Birds Task*. We focus on these tasks because they exemplify how Pai and Szeth operated differently on similar quantities, that they had previously constructed, to conceive RoC.

CI8 Account: The competing Amtrak Trust has introduced a modification to City bank's SI8, which they call the CI8 account. Like the SI8 account, the CI8 earns 8% of the "initial investment". However, at the end of each year Amtrak Trust recalculates the "initial investment" of the CI8 account to include all the interest that the customer has earned up to that point. Create an expression that gives the value of the CI8 account at any time t (Castillo-Garsow, 2010).

Cats & Birds: Cats, our most popular pet, are becoming our most embattled. A national debate has simmered since a 2013 study by the Smithsonian's Migratory Bird Center and the U.S. Fish and Wildlife Service concluded that cats kill up to 3.7 billion birds and 20.7 billion small mammals annually in the United States. The study blamed feral "unowned" cats but noted that their domestic peers "still cause substantial wildlife mortality" (Raasch,2013). Consider a backyard habitat, where cats are the natural predators of birds. Model the rate of decrease of the population of birds due to predation by cats.

The interviews were retrospectively analyzed to construct second-order accounts (Steffe & Thompson, 2000) of students' reasonings via inferences made from students' observable activities such as verbal descriptions, language, written work, discourse, and gestures. The retrospective analysis consisted of multiple passes of through the data to arrive at examples that illustrate the different ways Szeth and Pai structurally conceived RoC. First, we watched the videos in MAXQDA in chronological order and paraphrased each interview by chunks. Next, we created accounts of students' mathematics and the reasons they attributed to their mathematics. Next, we went over the accounts we created and the videos at the same time and refined our accounts by adding details using quantitative and covariational reasoning lenses. We credited a student to have instantiated a quantity if we were able to infer from his reasonings that he had conceived an object, attribute, and a measurement process for the attribute. As evidence of student to have conceived a measurement process, we checked if at least one of the quantifications criteria (see Czocher&Hardison, 2021) was met. We used segments of transcripts, where the students engaged in quantitative reasoning along with inscriptions and gestures as evidence for our claims. Next, from these accounts, we selected instances where the students engaged in conceiving RoC. Next, while watching the videos and going over our accounts of students' RoC, we further refined our accounts by adding details of how Pai and Szeth operated on constituent quantities to form a relation in order to conceive RoC. Next, we went through our accounts of Pai's and Szeth's conception of RoC and observed any patterns, in terms of operation on constituent quantities, that were consistent throughout the sessions. We made note of them by summarizing the pattern. We watched the videos in their entirety again and made sure all of the different ways of conceiving RoC were recorded and refined our second-order accounts by seeking clarification on utterance and gestures to support our claims on Pai's and Szeth's structural conception of RoC. We present some of these second-order accounts below.

Findings

Pai and Szeth structurally conceived RoC in the following ways: (1) as how quickly change is happening, (2) as a multiplicative object, (3) as the derivative, (4) as the net change during a time interval, and (5) as the additive comparison of rate of increase and rate of decrease. Due to space constraints, we share examples only for (1)-(3).

RoC as How Quickly Change is Happening

In the CI8 Account task**,** Szeth constructed the following models for the value of the account at the end of 1st, 2nd, and nth years: $S_1 = \frac{8}{100} \cdot S_0 \cdot t + S_0$; $S_2 = \frac{8}{100} \cdot S_1 \cdot t + S_1$;$S_n = \frac{8}{100} \cdot S_{n-1} \cdot t + S_{n-1}$, where t is the length of the compounding period. We then started to discuss how the account would grow when the compounding periods gets smaller. Szeth reasoned:

> Szeth: gut reaction is it [rate] should increase faster because the compounding is more frequent. Then giving it some thought, I'd agree. If we did one month, make this calculation [Figure 1], it's slightly less than like a year, but you're doing that calculation 12 times over the year. Yeah, yeah, if you decrease the time interval, your rate should increase quicker … I mean, the final amount of money in the account should increase quicker, which I guess means that our rate would be of a greater value over a shorter interval. Yeah, that makes sense.

In the excerpt above Szeth was coordinating three inter-dependent quantities: the length of the time interval, the amount of money in the account, and the rate at which the account grows. Through coordinating the length of the time interval with the rate at which the account grows, Szeth was engaging in *scaling continuous co-variation*. When asked to elaborate further, Szeth coordinated the value of the account for compounding periods 1 year and 1 month (Figure 1). He started with $500 and showed how the value of the account would increase each compounding period when the length of the compounding period is 1 year. He indicated that the increase in the account's value would be the *difference* between the sizes of the account for consecutive years. Next, he considered the increase in the size of the account when the compounding periods are of length 1 month. He reasoned: "Then actually they should all be the same numbers, but the point being that's one year… But that's one month" (Figure 1). Here Szeth is comparing the values of the account for compounding periods of different lengths to discuss how quickly the value of the account changes. We infer that, for Szeth, a bank account in which the size of the account changes over a small time period grows at a greater rate over time than one for which the same change in the size of the account happens over a larger time period. In this scenario, Szeth constructed a mental image of how quickly the size of the account changes in order to reason about the rate at which the account would grow.

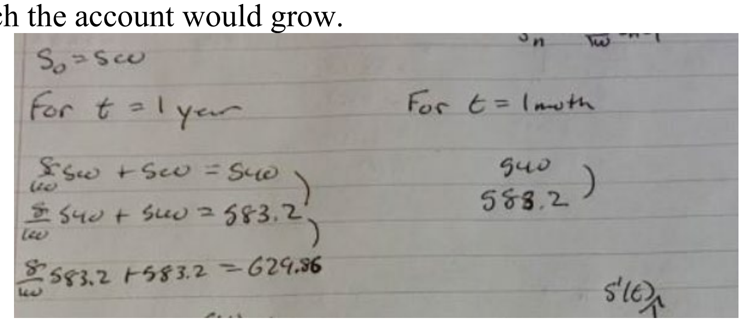


Figure 1: Szeth evaluates the size of the account for different compounding periods

Rate of Change as a Multiplicative Object

In the CI8 Account Task, Pai constructed the following models to indicate the size of the account at the end of 1, 2, and 3 years: $S_1 = S_0 + 0.08(S_0)(t)$; $S_2 = S_1 + 0.08(S_1)(t)$; $S_3 = S_2 + 0.08(S_2)(t)$, were $t = 1$ year. He drew the graph in Figure 2, and we proceeded have a discussion on how during each year the account grows linearly at a rate proportional to the size of the account at the beginning of that year.

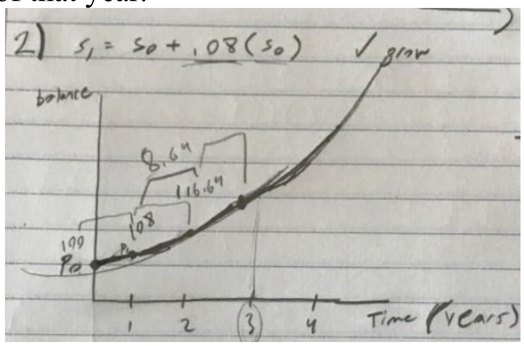


Figure 2. Pai's graph of account value vs time in years

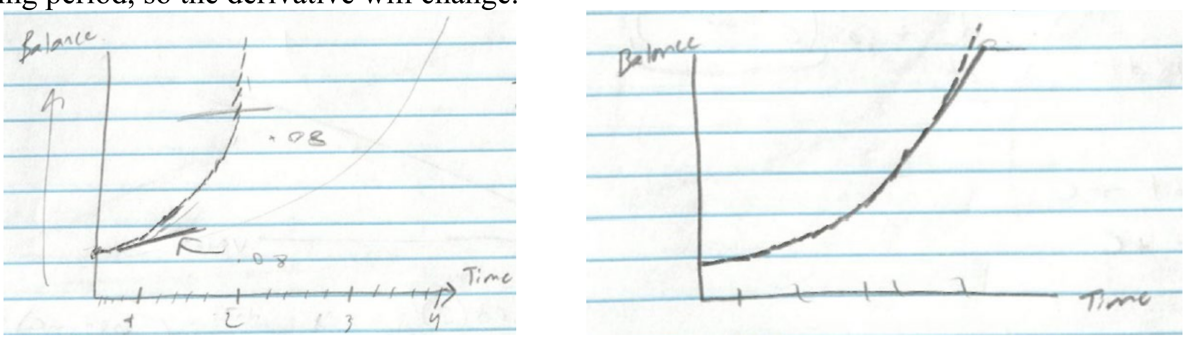
We asked Pai if his expression for S_i would work for any time period of length Δt and not just $t = 1$ year. Pai indicated that it would work as long as all of the S_i's are the account's value at the beginning of each of the compounding period. He rewrote the expression as $S = S_0 + (0.08)(S_0)(\Delta t)$, where Δt was the length of the compounding period and S_0 was the size of the account at the beginning of the compounding period. We asked him to interpret his graph as Δt gets smaller. For example, when we discussed the possibility of having four compounding periods instead of one in a year, Pai gave the following reasoning:

> Pai: "It would change, not the overall direction, but it would change the number of straight pegs, I guess, straight increases, because there's now 16 compounded periods in our time period [4 years]. So it'd be more round, but still not perfectly round, but it'd be my thinking of a circle, circle has infinite edges. It would be like that [tracing a curve along his graph (Figure 3b)] …that's what it would be. It'd be more like a curve"

In the above excerpt, Pai was thinking of the graph taking a curve like shape as Δt becomes smaller and smaller. We next asked Pai the effect smaller Δts would have on the RoC of the account's value, specifically its proportional relationship with the money at the beginning of the compounding period. During this discussion Pai reasoned:

> Pai: That should be proportional still. It's still proportional, but time is increasing so much that, or so fast, or the compounded periods are so small that it's going to change instantly…Like S is going to change. Like S_0 is going to be extremely different than the first... S_0 without… a very shortchange… like a very rapid change. It's still proportional to what was initially there, which just happening much, much faster because the t is much smaller than when we started"

Pai thought the rate at which the account's value changes would still be proportional to the account's value at the beginning of the compounding period. However, this time, he argued that the account's value at the beginning of the compounding period would change rapidly because Δt was much smaller. Pai was *coordinating* the quantities Δt, RoC of the account's value, and account's value at the beginning of the compounding period. We interpret Pai's reasoning as indicative of Pai imagining the quantities change in the size of the account and Δt to be happening simultaneously. We infer that Pai formed a multiplicative object of (change in size of the account, length of time elapsed). We also infer that Pai's formation of a multiplicative object had consequences beyond enabling him to coordinate the quantities change in size of the account and time taken for the change. He also used this imagery to help him in reasoning about the rate at which the size of the account changes. This was further evidenced when we asked him to write an expression for the RoC of the size of the account. Pai wrote $\frac{dS}{dt} = 0.08(S_0)$, where S_0 is the starting value at any compounding period. He reasoned as "[S_0] is just any starting period. Not just the literal start, but any starting period of these pegs, as *t* [Δt] gets smaller, it's 0.08 of that starting period, so the derivative will change."



(a) **(b)**

Figure 3: Pai's graph for as Balance vs time as Δt gets smaller

Rate of Change as the Derivative

In the Cats & Birds task, Pai and Szeth both conceived RoC as a measurable attribute that can be measured by taking the derivative of the function. For example, both Pai and Szeth conceived RoC of the bird population due to cats as a measurable attribute that could be measured by taking the derivative of the function bird population at time t.

Taking the derivative. Szeth arrived at the expression the $I(t) = \alpha \cdot \beta \cdot C(t) \cdot B(t)$ as the number of birds killed due to cat-bird interactions. We asked he could use the expression to model the rate of decrease of bird population due to predation by cats. Szeth Solved the above expression for $B(t)$ as $B(t) = \frac{I(t)}{\alpha \cdot \beta \cdot C(t)}$. Since he intended model the rate of decrease of the bird population, he mentioned that he first wanted to isolate $B(t)$ from the expression and thus solved for it. Szeth said, to find the rate he "would take the derivative [of $B(t)$]". Even though we had asked him to construct an expression for the rate of decrease of bird population due to just cats, Szeth tried to evaluate $B'(t)$, which the RoC of bird population due to all causes, and not just cats - a different attribute of the object bird population. Szeth proceeded with taking the derivative of $B(t)$ using the quotient rule.

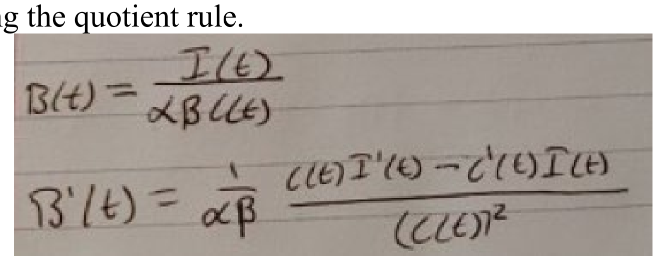
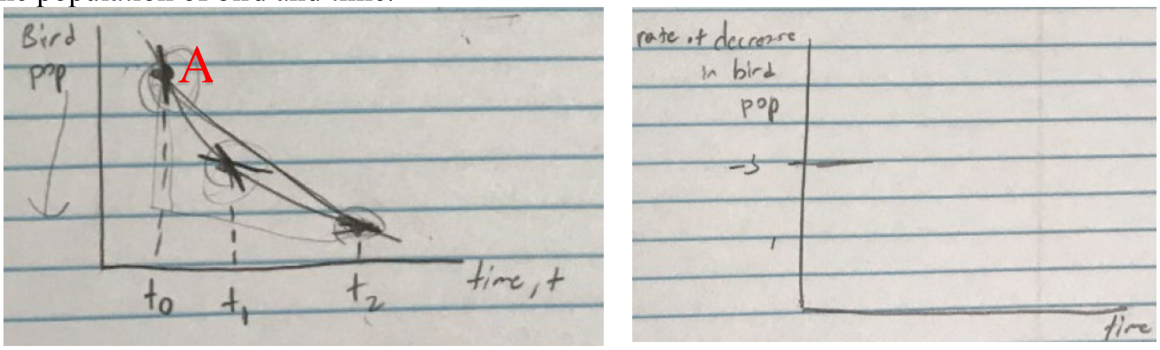


Figure 4: Szeth takes the derivative of B(t)

Imagining the slope of the tangent. In the Cats & Birds Task, After Pai constructed an expression for "the number of bird-cat interaction that resulted in a kill" (which for Pai, is also the "number of birds dead at time t") as $f_k(t) = [\alpha B(t) C(t)]\beta$, we asked what the rate of decrease of the bird population due to cat predation be. Pai's first reaction to this prompt was to draw a graph of bird population vs time [Figure 5]., where he *grossly coordinated* the quantities the population of bird and time.



(a) **(b)**

Figure 5: (a) bird population vs time and (b) rate of decrease of bird population vs time

According to the graph above, for Pai, as time goes on, the bird population decreases due to cat predation. After drawing the graph Pai reasoned:

> Pai: I think this question is trying to plot the function for the derivative of my model, because it wants the rate of decrease. I think we're asking for decrease in bird pop would be the y axis…over time t. The rate would be whatever this slope is [pointing at A], called, I don't know. Negative three. It would be like that [Figure 5b].

We infer that Pai is imagining the slope of the tangent line at different points in time ($t = t_0, t = t_1, t = t_2$), would give him the rate of decrease of bird due to cat predation. Here Pai, like Szeth, conceives the RoC of bird population due to cats by considering the derivative of the bird population. However, he wasn't very certain what the outcome of evaluating the slope of the tangent at different points in time would look like. Since Pai needed a "a model for the derivative" he thought $f_k'(t)$ would represent the tangent lines in figure 5a. He proceeded to take the derivative of $f_k(t)$ using the product rule and arrived at $f_k{}'(t) = \beta\alpha[B'(t)C(t) - B(t)C'(t)]$. However, he expressed that the result didn't make sense to him. When asked why it didn't make sense, Pai reasoned:

> Pai: $B(t)$ is birds at time t. I don't really know what $B'(t)$ would be. The change in birds at time t? The increase or decrease of their population? That's just a fact of, for bird, $B'(t)$, the birds, for that specific case, it would just be change in the natural bird population, not due to cats killing them? Because the bird population is changing, decreasing, because cats are constant killing them. I guess birds are also just die. They're birds... So, B' would be that change. Cats too, cats get run over and whatever, so it would be the change in their population [pointing at $C'(t)$].

We believe the biggest reason it wasn't making any sense to him was because he wasn't sure how to distinguish among the quantities $f_k'(t)$ and $B'(t)$.While reasoning out loud, he settled on $f_k'(t)$ as the rate of decrease of bird population due to predation by cats which he expressed in terms of $B'(t)$, where he attributed $B'(t)$ as change in bird population due to all causes. Even though f_k , for Pai, was "number of birds dead at time t", Pai's conception of RoC as a derivative was so strong, he overlooked the attribute $f_k{}'$ would be measuring.

Discussion

We presented three examples of differing ways that Pai and Szeth structurally conceived RoC in terms of operating on constituent quantities. Conceiving RoC as how quickly change is happening entails considering the differences in consecutive amount-values and coordinating it with the amount of time it took for the change to happen. Conceiving RoC as a multiplicative object entail uniting the attributes of two quantities so that their chnages both happen simultaneously. In both Pai's and Szeth's cases, conceiving RoC in these manners arose while they were both grappling with the thought: *How would the account grow as the compounding periods get smaller?* However, the ways in which they chose to operate on existing quantities differed. In Szeth's case, he first engaged in the *additive comparison* of the quantities amount of money at $t + 1$ and amount of money at t, and next considered how long it took for the change to happen. In Pai's case, he *coordinated* the change in the amount of money in the account and length of time duration for the change and envisioned those changes to be happening at the same time.

Conceiving RoC as a derivative entails considering the derivative of a function as a measurable attribute of an object. This way of conceiving the RoC is different from the first two examples presented here (and ways (4) and (5) not exemplified in this report). This is because envisioning the derivative or constructing an image of a derivative may or may not fall within the scope of quantitative operation. A derivative may be attributed as a quantitative operation if the meaning student has for derivative entailed the coordination of two quantities. For both students, there was no clear evidence that taking the derivative or envisioning the slope of the tangent carried situationally relevant quantitative meaning. Conceiving of RoC as the derivative might limit a modeler in terms of what he can model. For both students, the phrase RoC acted as a cue to take the derivative, dropping the situationally relevant meaning the derivative being taken on was measuring. Both Szeth and Pai, produced quantities that weren't measuring the attribute that was in question: *RoC of bird population due to cats*. However, for both students, considering derivative resulted in a quantified attribute, namely, the bird population. That is, Both Pai and Szeth conceived RoC as a measurable attribute that can be measured by considering the derivative with respect to time of the function bird population.

By adhering to the definition of quantitative operation and quantitative relation, instances like these, where a student may or may not have quantitative meanings for RoC as a derivative, would be missed. In our data, we found instances where the operation performed does not have clear evidence of a quantitative operation (as defined above), but the outcome of the operation is clearly a new quantity. We ask, *where ought these borderline instances be placed?* We do believe that even though taking the derivative in and of itself may not be credited as a quantitative operation, envisioning the derivative as yielding a new quantity is an image that is prevalent in undergraduate learners and the result of that action is often times useful for mathematical modeling. For example, in a simple interest context, Pai took the derivative of the size of the account at time t to describe the behavior of the growth of the account. Although the field has reported on students' conception of derivative through lenses of quantitative reasoning and covariational reasoning, we still need research on what it means when a student takes a derivative, as a procedure, through a quantity-oriented lens.

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