Journal of Combinatorics
Volume 12, Number 4, 563–569, 2021

Cliques with many colors in triple systems

Dhruv Mubayi* and Andrew Suk†

Erdős and Hajnal constructed a 4-coloring of the triples of an N-element set such that every n-element subset contains 2 triples with distinct colors, and N is double exponential in n. Conlon, Fox and Rödl asked whether there is some integer $q \geq 3$ and a q-coloring of the triples of an N-element set such that every n-element subset has 3 triples with distinct colors, and N is double exponential in n. We make the first nontrivial progress on this problem by providing a q-coloring with this property for all $q \geq 9$, where N is exponential in n^{2+cq} and $c > 0$ is an absolute constant.

AMS 2000 subject classifications: Primary 05D10, 05C55.
Keywords and phrases: Ramsey theory, hypergraphs.

1. Introduction

The Ramsey number $r_k(n;q)$ is the minimum integer N such that for any q-coloring of the k-tuples of an N-element set V, there is a subset $A \subset V$ such that all of the k-tuples of A have the same color. Estimating $r_3(n;2)$ is one of the most central problems in combinatorics. The best known bounds, due to Erdős, Hajnal and Rado [5, 4], state that there are positive constants c and c' such that

$$2^{cn^2} < r_3(n;2) < 2^{2^{c'n}}. \tag{1}$$

Erdős conjectured that the upper bound is closer to the truth, namely, $r_3(n;2)$ grows double exponentially in $\Theta(n)$, and he even offered a \$500 reward for a proof. His conjecture is supported by the fact that a double exponential growth rate is known when we have 4 colors [3, 4], that is, for fixed $q \geq 4$

$$r_3(n;q) = 2^{2^{\Theta(n)}}. \tag{2}$$

arXiv: 2005.03078
*Research partially supported by NSF grant DMS-1763317.
†Research partially supported by an NSF CAREER award, NSF award DMS-1952786, and an Alfred Sloan Fellowship.

In this paper, we study the following generalization of $r_3(n;q)$. For integers $n > q \geq t \geq 2$, let $f(n;q,t)$ denote the maximum integer N such that there is a q-coloring of the triples of an N-element set V with the property that every subset of V of size n induces at least t distinct colors. Thus when $t = 2$, we have

$$f(n;q,2) = r_3(n;q) - 1,$$

and for $q \geq t \geq 3$, we have $f(n;q,t) < r_3(n;q)$. When $t = 3$, Conlon, Fox, and Rödl raised the following problem [2].

Problem 1.1 (Conlon-Fox-Rödl)**.** *Is there an integer $q \geq 3$ and a positive constant c such that $f(n;q,3) > 2^{2^{cn}}$ holds for all $n > 2$?*

A simple application of the Probabilistic Method (see [1]) shows that $f(n;q,3) > 2^{cn^2}$, where $c = c(q)$. Our main result is the following.

Theorem 1.2. *There is an absolute constant $c > 0$ such that for all integers $n > q \geq 9$,*

$$f(n;q,3) \geq 2^{n^{2+c\cdot q}}.$$

For larger values of t, we show the following.

Theorem 1.3. *Given integers $q \geq t \geq 2$, there is an $n_0 = n_0(q,t)$ such that for all integers $n > n_0$,*

$$f(n;q,t) \geq 2^{n^{\log(q/(t-1))}/4}.$$

Both proofs are based on a stepping-up argument introduced by Erdős and Hajnal [3]. We start with the proof of Theorem 1.3 in the next section, as it is a direct application of the stepping-up method. The proof of Theorem 1.2 combines a more general stepping-up argument with induction, and is given in Section 3. Throughout this paper, all logarithms are in base 2.

2. Forcing many colors

In this section, we prove Theorem 1.3. We will need the following lemma.

Lemma 2.1. *Given integers $q \geq t \geq 2$, there is an integer m_0 such that the following holds. For every $m \geq m_0$, there is a q-coloring ϕ of the pairs of $U = \{0, 1, \ldots, \lfloor (q/(t-1))^{m/4} \rfloor - 1\}$ such that every subset of size m induces at least t distinct colors.*

Proof. Given $q \geq t \geq 2$, let $m_0 = m_0(q,t)$ be a sufficiently large integer that will be determined later. Color the pairs of $U = \{0, 1, \ldots, \lfloor (q/(t-1))^{m/4} \rfloor$ uniformly independently at random with colors $\{\alpha_1, \ldots, \alpha_q\}$. Let X denote the number of subsets $A \subset U$ of size m that have less than t distinct colors among their pairs. Then we have

$$\begin{aligned}\mathbb{E}[X] &\le \binom{|U|}{m}\binom{q}{t-1}\left(\frac{t-1}{q}\right)^{\binom{m}{2}}\\ &\le \left(\frac{q}{t-1}\right)^{m^2/4} q^{t-1}\left(\frac{t-1}{q}\right)^{m^2/2}\\ &= q^{t-1}\left(\frac{q}{t-1}\right)^{-m^2/4}.\end{aligned}$$

By setting $m_0 = m_0(q,t)$ sufficiently large, we have for all $m \ge m_0$, $\mathbb{E}[X] < 1$. Hence, there is a q-coloring $\phi : \binom{U}{2} \to \{\alpha_1,\dots,\alpha_q\}$ such that every subset $A \subset U$ of size m has at least t distinct colors among its pairs. □

Proof of Theorem 1.3. Given $q \ge t \ge 2$, let $n_0 = n_0(q,t)$ be a sufficiently large integer that will be determined later. Set $M = \lfloor (q/(t-1))^{m/4} \rfloor$, $U = \{0,1,\dots,M-1\}$, and let $\phi : \binom{U}{2} \to \{\alpha_1,\dots,\alpha_q\}$ be a q-coloring of the pairs of U with the properties described in Lemma 2.1. Set $V = \{0,1,\dots,2^M-1\}$. In what follows, we will use ϕ to define a q-coloring $\chi : \binom{V}{3} \to \{\alpha_1,\dots,\alpha_q\}$ of the triples of V with the desired properties.

For each $v \in V$, write $v = \sum_{i=0}^{M-1} v(i)2^i$ with $v(i) \in \{0,1\}$ for each i. For $u \ne v$, let $\delta(u,v) \in U$ denote the largest i for which $u(i) \ne v(i)$. Notice that we have the following stepping-up properties (see [6]).

Property I: For every triple $u < v < w$, $\delta(u,v) \ne \delta(v,w)$.
Property II: For $v_1 < \dots < v_r$, $\delta(v_1,v_r) = \max_{1\le j\le r-1} \delta(v_j,v_{j+1})$.

Using $\phi : \binom{U}{2} \to \{\alpha_1,\dots,\alpha_q\}$, we define $\chi : \binom{V}{3} \to \{\alpha_1,\dots,\alpha_q\}$ as follows. For vertices $v_1 < v_2 < v_3$ in V and $\delta_i = \delta(v_i,v_{i+1})$, we define $\chi(v_1,v_2,v_3) = \alpha_j$ if and only if $\phi(\delta_1,\delta_2) = \alpha_j$. We now need the following lemma.

Lemma 2.2. *For $m \ge 2$ set $n = 2^m$. Then for any set of n vertices $v_1,\dots,v_n \in V$, where $v_1 < \dots < v_n$, there is a subset $B \subset \{\delta(v_i,v_{i+1}) : 1 \le i \le n-1\}$ with at least m distinct elements such that for each pair $(\delta_r,\delta_s) \in \binom{B}{2}$, there is a triple $v_i < v_j < v_k$ in $\{v_1,\dots,v_n\}$ such that $\chi(v_i,v_j,v_k) = \phi(\delta_r,\delta_s)$.*

Proof. We proceed by induction on m. The base case $m = 2$ follows from Property I. For the inductive step, assume that the statement holds for all $m' < m$. Let $v_1,\dots,v_n \in V$ such that $v_1 < \dots < v_n$ and $n = 2^m$. Let $\delta_i = \delta(v_i,v_{i+1})$, for $i = 1,\dots,n-1$. Set $\delta_w = \max\{\delta_i : 1 \le i \le n-1\}$

and notice that, by Properties I and II above, $\delta_w > \delta_i$ for all $i \neq w$. Set $S = \{v_1, \ldots, v_w\}$ and $T = \{v_{w+1}, \ldots, v_n\}$. Then either $|S|$ or $|T|$ has size at least 2^{m-1}. Without loss of generality, we can assume that $|S| \geq 2^{m-1}$ since a symmetric argument would follow otherwise. By the induction hypothesis, there is a subset $B_0 \subset \{\delta_1, \ldots, \delta_{w-1}\} \subset U$ with at least $m-1$ distinct elements and for each pair $(\delta_r, \delta_s) \in \binom{B_0}{2}$, there is a triple $v_i < v_j < v_k$ in S such that

$$\chi(v_i, v_j, v_k) = \phi(\delta_r, \delta_s).$$

Set $B = \{\delta_w\} \cup B_0$, which implies $|B| \geq m$. Then notice that for each pair (δ_w, δ_r), where $\delta_r \in B_0$, by Property I above, we have

$$\chi(v_r, v_{r+1}, v_{w+1}) = \phi(\delta_w, \delta_r).$$

Hence $B \subset U$ has the desired properties, and this completes the proof of the claim. □

Set $n_0 = \lceil 2^{m_0} \rceil$ where m_0 is defined in Lemma 2.1. Then for all $n > n_0$ we have $m > m_0$. Thus, by Lemma 2.1 and Lemma 2.2, any set of n vertices in V induces at least t distinct colors with respect to χ. Since $|V| = 2^{(q/(t-1))^{m/4}}$ and $n = 2^m$, we have $|V| = 2^{n^{\log(q/(t-1))}/4}$. □

3. Forcing three colors

In this section, we prove Theorem 1.2. We will need the following lemma.

Lemma 3.1. *Let $r > 3$ and set $V_3 = \{0, 1, \ldots, \lfloor 2^{r^2/24} \rfloor - 1\}$. Then there is a 3-coloring $\phi_3 : \binom{V_3}{3} \to \{\beta_1, \beta_2, \beta_3\}$ of the triples of V_3 such that every subset of size r induces at least three distinct colors.*

We omit the proof of Lemma 3.1 as it follows by the same probabilistic argument used for Lemma 2.1. Hence, Lemma 3.1 implies that $f(n; 3, 3) \geq 2^{n^2/24}$. Together with the following recursive formula, Theorem 1.2 quickly follows.

Theorem 3.2. *For integers $n > q \geq 9$, we have*

$$f(n; q, 3) \geq (f(\lfloor n/\log n \rfloor, q-6, 3))^{n^{1/4}/2}.$$

We will also need the following lemma, whose proof is also omitted since it follows from the same probabilistic argument as in Lemma 2.1.

Lemma 3.3. *Let $s > 3$ and set $V_2 = \{0, 1, \ldots, \lfloor 2^{s/4} \rfloor\}$. Then there is a 3-coloring $\phi_2 : \binom{V_2}{2} \to \{\alpha_1, \alpha_2, \alpha_3\}$ of the pairs of V_2 such that every subset of size s induces at least three distinct colors.*

Proof of Theorem 3.2. Given $n > q \geq 9$, let $r = \lfloor n/\log n \rfloor$ and $s = \lfloor \log n \rfloor$. Set $N_2 = \lfloor 2^{s/4} \rfloor$, $N_3 = f(r; q-6, 3)$, and

$$V_2 = \{0, 1, \ldots, N_2 - 1\} \quad \text{and} \quad V_3 = \{0, 1, \ldots, N_3 - 1\}.$$

Using Lemma 3.3, we obtain $\phi_2 : \binom{V_2}{2} \to \{\alpha_1, \alpha_2, \alpha_3\}$ such that every subset of V_2 of size s induces at least three colors. Likewise, by definition of $f(r, q-6, 3)$, we obtain $\phi_3 : \binom{V_3}{3} \to \{\beta_1, \ldots, \beta_{q-6}\}$ such that every subset of V_3 of size r induces at least three distinct colors. We now apply the following more general stepping-up procedure.

Set $N = N_3^{N_2}$ and $V = \{0, 1, \ldots, N-1\}$. For each $v \in V$, write $v = \sum_{i=0}^{N_2-1} v(i)(N_3)^i$ with $v(i) \in V_3$ for each i. For $u, v \in V$ with $u < v$, let $\delta(u, v) \in V_2$ denote the largest i for which $u(i) \neq v(i)$. Notice that we no longer have Property I from the previous stepping-up procedure, but we do have the following properties.

Property II: For $v_1 < \cdots < v_r$, $\delta(v_1, v_r) = \max_{1 \leq j \leq r-1} \delta(v_j, v_{j+1})$.
Property III: For $v_1 < v_2 < v_3$ such that $\delta(v_1, v_2) = \delta(v_2, v_3) = i$, $v_1(i) < v_2(i) < v_3(i)$.

Using ϕ_2 and ϕ_3, we define $\chi : \binom{V}{3} \to \{\gamma_1, \ldots, \gamma_q\}$ as follows. For vertices $v_1 < v_2 < v_3$ in V, let $\delta_1 = \delta(v_1, v_2)$ and $\delta_2 = \delta(v_2, v_3)$. Then for $i \in \{1, 2, 3\}$,

- set $\chi(v_1, v_2, v_3) = \gamma_i$ if and only if $\delta_1 > \delta_2$ and $\phi_2(\delta_1, \delta_2) = \alpha_i$,
- set $\chi(v_1, v_2, v_3) = \gamma_{3+i}$ if and only if $\delta_1 < \delta_2$ and $\phi_2(\delta_1, \delta_2) = \alpha_i$,

and for $i \in \{1, \ldots, q-6\}$,

- set $\chi(v_1, v_2, v_3) = \gamma_{6+i}$ if and only if $\delta_1 = \delta_2 = j$ and we also have $\phi_3(v_1(j), v_2(j), v_3(j)) = \beta_i$,

Notice that $n \geq \max\{s \cdot r, 2^s\}$. We claim that any set of n vertices $v_1, \ldots, v_n \in V$ induces at least 3 distinct colors with respect to χ. For sake of contradiction, let $A = \{v_1, \ldots, v_n\} \subset V$ such that $v_1 < \cdots < v_n$ and $\chi(v_i, v_j, v_k) \in \{\gamma_x, \gamma_y\}$ for all triples $(v_i, v_j, v_k) \in \binom{A}{3}$. Set $\delta_i = \delta(v_i, v_{i+1})$ for $i = 1, \ldots, n-1$. The proof now falls into the following cases.

Case 1. Suppose $\gamma_x, \gamma_y \in \{\gamma_1, \gamma_2, \gamma_3\}$. Then we have $\delta_1 > \delta_2 > \cdots > \delta_{n-1}$. However, $\delta_i \in U = \{0, 1, \ldots, \lfloor 2^{s/4} \rfloor - 1\}$ and $n = 2^s$ which is a contradiction. A similar argument follows if $\gamma_x, \gamma_y \in \{\gamma_4, \gamma_5, \gamma_6\}$.

Case 2. Suppose $\gamma_x, \gamma_y \in \{\gamma_7, \ldots, \gamma_{q-6}\}$. Then we must have $\delta_1 = \cdots = \delta_{n-1} = i$ and $v_1(i) < \cdots < v_{n-1}(i)$. Since $n \geq r$, by definition of χ and ϕ_3, the set $\{v_1, \ldots, v_n\}$ induces at least three distinct colors, contradiction.

Case 3. Suppose $\gamma_x \in \{\gamma_1, \gamma_2, \gamma_3\}$ and $\gamma_y \in \{\gamma_4, \gamma_5, \gamma_6\}$. Then in this case, for any triple $v_i < v_j < v_k$, we have $\delta(v_i, v_j) \neq \delta(v_j, v_k)$ and

$$\phi_2(\delta(v_i, v_j), \delta(v_j, v_k)) = \alpha_z$$

for some fixed z. Set $\delta_w = \max\{\delta_i : 1 \leq i \leq n-1\}$ and notice that, by Property II above, $\delta_w > \delta_i$ for all $i \neq w$. Therefore, a straight-forward adaptation of Lemma 2.2 gives us the following claim.

Claim 3.4. *For $s \geq 2$, any set of 2^s vertices $v_1, \ldots, v_{2^s} \in V$, with the properties described above, there is a subset $B \subset \{\delta(v_i, v_{i+1}) : 1 \leq i \leq 2^s - 1\}$ with at least s distinct elements such that $\phi_2(\delta_i, \delta_j) = \alpha_z$ for every pair $(\delta_i, \delta_j) \in \binom{B}{2}$.*

However, this contradicts Lemma 3.3.

Case 4. Suppose $\gamma_x \in \{\gamma_1, \ldots, \gamma_6\}$ and $\gamma_y \in \{\gamma_7, \ldots, \gamma_q\}$. Without loss of generality, we can assume that $\gamma_x = \gamma_1$ and $\gamma_y = \gamma_7$ since a symmetric argument would follow otherwise. Notice that there is an integer $w_1 \in \{1, \ldots, r\}$ such that $\delta(v_1, v_{w_1}) > \delta(v_{w_1}, v_{w_1+1})$. Indeed, otherwise if $\delta_1 = \cdots = \delta_r$, by the definition of χ and the properties of ϕ_3 described above, the set $\{v_1, \ldots, v_r\}$ induces at least three distinct colors with respect to χ, contradiction.

The same argument shows that there must be an integer $w_2 \in \{w_1 + 1 \ldots, w_1 + r\}$ such that $\delta(v_{w_1}, v_{w_2}) > \delta(v_{w_2}, v_{w_2+1})$. Since $n \geq s \cdot r$, a repeated application of the argument above shows that there are integers $w_1 < \cdots < w_{s-1}$, such that

$$\delta(v_1, v_{w_1}) > \delta(v_{w_1}, v_{w_2}) > \delta(v_{w_2}, v_{w_3}) > \cdots > \delta(v_{w_{s-1}}, v_{w_{s-1}+1}).$$

By Property II, χ colors every triple in $\{v_1, v_{w_1}, \ldots, v_{w_{s-1}}, v_{w_{s-1}+1}\}$ with color γ_1. However, this implies that the set

$$S = \{\delta(v_1, v_{w_1}), \delta(v_{w_1}, v_{w_2}), \ldots, \delta(v_{w_{s-2}}, v_{w_{s-1}}), \delta(v_{w_{s-1}}, v_{w_{s-1}+1})\} \subset U,$$

has the property that $|S| = s$ and $\phi_2 : \binom{S}{2} \to \alpha_1$, which is a contradiction. Since $|V| = N_3^{N_2}$,

$$f(n; q, 3) \geq |V| \geq \left(f(\lfloor n/\log n \rfloor; q-6, 3)\right)^{n^{1/4}/2}.$$

This completes the proof of Theorem 3.2. □

Combining Theorem 3.2 with the fact that $f(n;3,3) > 2^{n^2/24}$ gives the following.

Theorem 3.5. *For fixed $q \geq 3$ and for all $n > 3$ we have*

$$f(n;q,3) > 2^{n^{2+\frac{1}{4}\left\lfloor\frac{q-3}{6}\right\rfloor - o(1)}}.$$

References

[1] N. Alon, J. Spencer, The Probabilistic Method, John Wiley & Sons, New York, NY, 1992. MR1140703

[2] D. Conlon, J. Fox, V. Rödl, Hedgehogs are not colour blind, *J. Combin.* **8** (2017), 475–485. MR3668877

[3] P. Erdős and A. Hajnal, On Ramsey like theorems, problems and results, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), pp. 123–140, Inst. Math. Appl., Southhend-on-Sea, 1972. MR0337636

[4] P. Erdős, A. Hajnal, and R. Rado, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196. MR0202613

[5] P. Erdős and R. Rado, Combinatorial theorems on classifications of subsets of a given set, *Proc. Lond. Math. Soc.* **3** (1952), 417–439. MR0065615

[6] R. Graham, B. Rothschild, and J. Spencer, Ramsey Theory, 2nd ed., Wiley, New York, 1990. MR0591457

Dhruv Mubayi
Department of Mathematics, Statistics, and Computer Science
University of Illinois
Chicago
USA
E-mail address: mubayi@uic.edu

Andrew Suk
Department of Mathematics
University of California
San Diego
USA
E-mail address: asuk@ucsd.edu

Received May 23, 2020