Isometric flows of G_2-structures

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August 14, 2020

Abstract

We survey recent progress in the study of flows of isometric G_2-structures on 7-dimensional manifolds, that is, flows that preserve the metric, while modifying the G_2-structure. In particular, heat flows of isometric G_2-structures have been recently studied from several different perspectives, in particular in terms of 3-forms, octonions, vector fields, and geometric structures. We will give an overview of each approach, the results obtained, and compare the different perspectives.

1 Introduction

One of the most challenging problems in differential geometry is the question of existence conditions for torsion-free G_2-structures on smooth 7-dimensional manifolds. Such G_2-structures are precisely the ones that correspond to metrics with holonomy contained in G_2. One approach that has been pioneered by Robert Bryant [4] is to considered heat-like flows of G_2-structures with the hope that under certain conditions they may converge to a torsion-free G_2-structure. A difficulty that is encountered in such an approach is that in general, deformations of a G_2-structure also affect the corresponding metric, and so any heat equation for the G_2-structure becomes nonlinear. This is not unlike the situation for the Ricci flow, where the underlying geometry changes along the flow, however in the G_2 case, we have two separate but closely related objects, the G_2-structure and the metric, both of which vary along the flow. Given a Riemannian metric on a 7-manifold that admits G_2-structures, there is a family of G_2-structures that correspond to it, so a possible approach could be to separate as much as possible the deformations of the metric from the deformations of G_2-structures that preserve the metric. Indeed, as was shown by Karigiannis [13], given a decomposition of 3-forms according to representations of G_2, the deformations of the G_2-structure 3-form that preserve the metric are precisely the ones that lie in the 7-dimensional representation Λ_7^3. Bryant's

original Laplacian flow of closed G_2-structures has no component in Λ^3_7 [4], and as such is transverse to directions that preserve the metric. This allowed for more tractable analytic properties. In contrast, a similar flow for co-closed G_2-structures that was proposed in [15] does have a component in Λ^3_7, which, as shown in [9], causes non-parabolicity of the flow. This suggests that the freedom of G_2-structures to move in directions that preserve the metric is some kind of degeneracy and thus suitable gauge-fixing conditions within the metric class are needed to address it.

These considerations show that it is necessary to have a clearer picture of G_2-structures within a fixed metric class. In [4], Bryant observed that such G_2-structures are parametrized by sections of an $\mathbb{R}P^7$-bundle, or more concretely, by pairs (a, α) where a is a real-valued function and α is a vector field such that $a^2 + |\alpha|^2 = 1$, and $\pm (a, \alpha)$ define the same G_2-structure. If φ is a fixed G_2-structure, then any other G_2-structure $\sigma_{(a,\alpha)}(\varphi)$ within the same metric class is given by:

$$\sigma_{(a,\alpha)}(\varphi) = \left(a^2 - |\alpha|^2\right)\varphi - 2a\alpha\lrcorner\psi + 2\alpha \wedge (\alpha\lrcorner\varphi), \tag{1}$$

where $\psi = *\varphi$.

Given that the group G_2 may be defined as the automorphism group of the octonions, a G_2-structure defines an octonion structure on the manifold, and in [10], this observation was used to interpret the above pair (a, α) as a unit octonion V, and then (1) is just the 3-form that corresponds to a modified octonion product defined by V. Thus, a flow of isometric G_2-structures can be interpreted as a flow of the unit octonion section V. In particular, a natural heat flow of isometric G_2-structures was introduced in [10]. Given an octonionic covariant derivative D, constructed from the Levi-Civita connection and the torsion of the initial G_2-structure φ, the heat flow of isometric G_2-structures is then the semilinear, parabolic equation

$$\frac{\partial V}{\partial t} = \Delta_D V + |DV|^2 V \tag{2}$$

with some initial condition $V(0) = V_0$ and where $\Delta_D = -D^*D$ is the Laplacian operator corresponding to D. This was obtained as the negative gradient flow of an energy functional with respect to D. The critical points of the flow (2) correspond to G_2-structures for which the torsion tensor is divergence-free, i.e. satisfies $\operatorname{div} T = 0$, where divergence is taken with respect to the Levi-Civita connection. This is significant for several reasons. The divergence of torsion is precisely the term that causes the non-parabolicity of the Laplacian flow of co-closed G_2-structures from [15] as mentioned above, and $\operatorname{div} T = 0$ for closed G_2-structures. Thus, closed G_2-structures are automatically critical points of (2). Secondly, T has been interpreted in [10] as an imaginary octonion-valued 1-form, which is added to the Levi-Civita connection to obtain the octonionic covariant derivative D, hence the condition $\operatorname{div} T = 0$ is precisely analogous to the Coulomb gauge condition in gauge theory. This analogy makes this condition a reasonable candidate for a gauge-fixing condition within a fixed metric class.

Soon after the introduction of the flow (2) in [10], it was further studied from different perspectives by several authors: Bagaglini in [1]; Dwivedi, Gianniotis, and Karigiannis in [8]; the author in [11]; Loubeau and Sá Earp in [17].

Equivalently to the flow of octonions (2), one can consider directly the evolution of the 3-form φ via the equation

$$\frac{\partial \varphi}{\partial t} = 2 \left(\operatorname{div} T \right) \lrcorner \psi \tag{3}$$

where T is the torsion tensor that corresponds to the G_2-structure 3-form at time t. This is the way the flow was formulated in [1] and in [8] (although here we are following [10, 11] and added a factor of 2 in (3). In [17], a more general approach is taken and a harmonic heat flow of geometric structures is considered. In the case of G_2-structures, it is shown to reduce to (3). In this survey we will review the above approaches to the flow of isometric G_2-structures and outline the key analytic results.

Acknowledgements

2 Isometric G_2-structures

A G_2-structure on a 7-manifold is defined by a smooth *positive* 3-form φ [3, 12]. This is a nowhere-vanishing 3-form that defines a Riemannian metric g_φ, such that for any vectors u and v, the following holds

$$g_\varphi \left(u, v \right) \operatorname{vol}_\varphi = \frac{1}{6} \left(u \lrcorner \varphi \right) \wedge \left(v \lrcorner \varphi \right) \wedge \varphi. \tag{4}$$

At any point, the stabilizer of g_φ (along with orientation) is $SO\left(7\right)$, whereas the stabilizer of φ is $G_2 \subset SO\left(7\right)$. This shows that at a point, positive 3-forms forms that correspond to the same metric, i.e., are *isometric*, are parametrized by $SO\left(7\right)/G_2 \cong \mathbb{RP}^7 \cong S^7/\mathbb{Z}_2$. Therefore, on a Riemannian manifold, metric-compatible G_2-structures are parametrized by sections of an $\mathbb{RP}^7$-bundle, or alternatively, by sections of an S^7-bundle, with antipodal points identified. This is precisely the parametrization given by (1).

Alternatively, a G_2-structure in a fixed metric class can be interpreted as a reduction of the principal $SO\left(7\right)$-bundle P of orthonormal frames to a principal G_2-subbundle, and hence each such reduction corresponds to a section σ of an $SO\left(7\right)/G_2$-bundle N and equivalently, an $SO\left(7\right)$-equivariant map $s : P \longrightarrow SO\left(7\right)/G_2 \cong S^7/\mathbb{Z}_2$. This is the picture used in [17].

We may also use the G_2-structure φ and the metric to define the *octonion bundle* $\mathbb{O}M \cong \Lambda^0 \oplus TM$ on M as a rank 8 real vector bundle equipped with an octonion product of sections given by

$$A \circ_\varphi B = \left(ab - g\left(\alpha, \beta\right), a\beta + b\alpha + \alpha \times_\varphi \beta \right) \tag{5}$$

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for any sections $A = (a, \alpha)$ and $B = (b, \beta)$. We set the metric $g = g_\varphi$, since we are fixing the metric, even though the G_2-structure may change. Here we define $\times_\varphi$ by $g(\alpha \times_\varphi \beta, \gamma) = \varphi(\alpha, \beta, \gamma)$ and given $A \in \Gamma(\mathbb{O}M)$, we write $A = (\operatorname{Re} A, \operatorname{Im} A)$. The metric on TM is extended to $\mathbb{O}M$ to give the octonion inner product $\langle A, B \rangle = ab + g(\alpha, \beta)$, which is Hermitian with respect to the octonion product. In the formula (1), the pair (a, α) can now be interpreted as a unit octonion section.

The *intrinsic torsion* of a G_2-structure is defined by $\nabla\varphi$, where ∇ is the Levi-Civita connection for the metric g that is defined by φ. Following [14], we have

$$\nabla_a \varphi_{bcd} = 2T_a{}^e \psi_{ebcd} \text{ and } \nabla_a \psi_{bcde} = -8T_{a[b}\varphi_{cde]} \tag{6}$$

where T_{ab} is the *full torsion tensor*, note that an additional factor of 2 is for convenience, and $\psi = *\varphi$ is the 4-form that is the Hodge dual of φ with respect to the metric g. The G_2-structure is known as *torsion-free* if $T = 0$, and in that case ∇ has holonomy contained in G_2. Conversely, if ∇ has holonomy contained in G_2, then there exists a torsion-free G_2-structure within the metric class. Let $V = (a, \alpha)$ be a unit octonion section, then define $\sigma_V(\varphi) = \sigma_{(a,\alpha)}(\varphi)$, as in (1). It has been shown in [10] that the torsion of the G_2-structure $\varphi_V = \sigma_V(\varphi)$ is given by

$$T^{(V)} = VTV^{-1} - (\nabla V) V^{-1} \tag{7}$$

where T is the torsion of φ, interpreted as a 1-form with values in the bundle of imaginary octonions $\operatorname{Im}\mathbb{O}M$. If we now define an octonion covariant derivative D on sections of $\mathbb{O}M$ via

$$DV = \nabla V - VT, \tag{8}$$

the expression (7) simply becomes

$$T^{(V)} = -(DV) V^{-1}. \tag{9}$$

As shown in [10], the derivative D has other nice properties - it is metric-compatible, and satisfies a partial product rule with respect to octonion product on $\mathbb{O}M$, that is, $D(UV) = (\nabla U) V + U(DV)$. Now given (9), the divergence of $T^{(V)}$ can be expressed as

$$\operatorname{div} T^{(V)} = -(\Delta_D V) V^{-1} - |DV|^2. \tag{10}$$

3 Energy functional

Given that the torsion varies across G_2-structures within the same metric class, an obvious question is how to pick a representative of the class with the "best" torsion. A reasonable way to try and characterize the best torsion is to look for critical points of a functional. Therefore, given the set $\mathcal{F}_g$ of all G_2-structures that are compatible with a given metric g, and assuming M is compact, define the functional $\mathcal{E} : \mathcal{F}_g \longrightarrow \mathbb{R}$ by

$$\mathcal{E}(\varphi) = \int_M \left| T^{(\varphi)} \right|^2 \operatorname{vol}, \tag{11}$$

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where $T^{(\varphi)}$ is the torsion of a G_2-structure φ. This is the functional used by Dwivedi, Gianniotis, and Karigiannis in [8].

As we have seen in the previous section, given a G_2-structure φ, any other G_2-structure within the same metric class is given by $\sigma_V (\varphi)$ for a unit octonion section V. Therefore, the functional (11) is equivalent to the functional $\mathcal{E}_\mathbb{O}$: $\Gamma (S\mathbb{O}M) \longrightarrow \mathbb{R}$ given by

$$\mathcal{E}_\mathbb{O} (V) = \int_M \left| T^{(V)} \right|^2 \text{vol} = \int_M |DV|^2 \text{vol} \tag{12}$$

where we have also applied (9). Hence, in fact, the functional $\mathcal{E}_\varphi$ is equivalent to an energy functional with respect to the derivative D. This is the functional used in [10, 11].

On the other hand, following the approach in [17], recall that a principal H-subbundle of a principal G-bundle P may be characterized by an equivariant map $s : P \longrightarrow G/H$, or equivalently, as a section σ of the associated bundle $N = P \times_G (G/H) \cong P/H$. Assuming that G is semi-simple, so that it admits a bi-invariant metric, we may define a metric η on N, together with the corresponding Levi-Civita connection ∇^η. Moreover, given a metric g on the base manifold, we may induce a metric on $T^*M \otimes \sigma^*TN$, which is compatible with the splitting $TN = \mathcal{V}N \oplus \mathcal{H}N$ induced by ∇^η. Using this metric, we may then define an energy functional $\mathcal{E}_\Gamma : \Gamma (N) \longrightarrow \mathbb{R}$ on sections of N:

$$\mathcal{E}_\Gamma (\sigma) = \int_M |d\sigma|^2 \text{vol} . \tag{13}$$

Alternatively, suppose that moreover G is compact, so that P is compact. Then, let us define an energy functional on G-equivariant maps $s : P \longrightarrow G/H$:

$$\mathcal{E}_G (s) = \int_P |ds|^2 \text{vol}_P \tag{14}$$

where an induced metric on $T^*P \otimes s^*T (G/H)$ is used. It is then shown in [17], that for any section $\sigma \in \Gamma (N)$ and its corresponding G-equivariant map $s \in C_G^\infty (P, G/H)$, $\mathcal{E}_G (s) = c_1 \mathcal{E}_\Gamma (\sigma) + c_2$ where c_1 and c_2 are uniform constants.

Consider the orthogonal splitting $d\sigma = d^\mathcal{V}\sigma + d^\mathcal{H}\sigma$ into horizontal and vertical parts. Since the horizontal component of the metric is given by π^*g, where $\pi : N \longrightarrow M$ is the bundle projection map, we find that for any $X \in TM$,

$$\left| d^\mathcal{H}\sigma (X) \right|^2 = (\pi^*g) (d\sigma (X) , d\sigma (X)) = g ((\pi \circ \sigma)_* X, (\pi \circ \sigma)_* X) = g (X, X) .$$

Thus, the horizontal part of $d\sigma$ contributes only a constant term to (13), and it is thus sufficient to consider just the vertical component

$$\mathcal{E}_\Gamma^\mathcal{V} (\sigma) = \int_M \left| d^\mathcal{V}\sigma \right|^2 \text{vol} . \tag{15}$$

In the G_2 case, Loubeau and Sá Earp show in [17] that this functional is equivalent to (11).

Theorem 3.1 ([17]) *If M is 7-dimensional, P is the $SO\left(7\right)$-principal bundle of oriented orthonormal frames, and N is an associated $SO\left(7\right)/G_2$-bundle over M, then $\left|d^{\mathcal{V}}\sigma\right|^2 = \frac{8}{3}\left|T^{(\sigma)}\right|^2$ where $T^{(\sigma)}$ is the torsion tensor of the G_2-structure defined by the section σ.*

4 Gradient flow

Given the functionals defined in the previous section, we may consider critical points and negative gradient flows of the functionals. This is summarized below.

Space	Functional	Critical points	Negative gradient flow
$\mathcal{F}_g$	$\mathcal{E}\left(\varphi\right)$	$\operatorname{div} T^{(\varphi)} = 0$	$\frac{\partial\varphi_t}{\partial t} = 2\operatorname{div} T^{(\varphi_t)}\lrcorner\psi_t$
$\Gamma\left(S\mathbb{O}M\right)$	$\mathcal{E}_{\mathbb{O}}\left(V\right)$	$\Delta_D V + \left|DV\right|^2 V = 0$	$\frac{\partial V_t}{\partial t} = \Delta_D V_t + \left|DV_t\right|^2 V_t$
$\Gamma\left(N\right)$	$\mathcal{E}_\Gamma\left(\sigma\right)$	$\tau^{\mathcal{V}}\left(\sigma\right) = 0$	$\frac{\partial\sigma_t}{\partial t} = \tau^{\mathcal{V}}\left(\sigma_t\right)$
$C_G^\infty\left(P, G/H\right)$	$\mathcal{E}_G\left(s\right)$	$\tau^{\mathcal{H}}\left(s\right) = 0$	$\frac{\partial s_t}{\partial t} = \tau^{\mathcal{H}}\left(s_t\right)$

where $\tau^{\mathcal{V}}\left(\sigma\right) := \operatorname{Tr}_g\left(\nabla^\eta d^{\mathcal{V}}\sigma\right)$ is the *vertical tension field* of the functional $\mathcal{E}_\Gamma\left(\sigma\right)$ and $\tau^{\mathcal{H}}\left(s\right) := \operatorname{Tr}_g^{\mathcal{H}}\left(\nabla^\eta ds\right)$ is the *horizontal tension field* of the functional $\mathcal{E}_G\left(s\right)$. It is proved in [20, Theorem 1] that $\sigma \in \Gamma\left(N\right)$ is a harmonic *section*, i.e. a critical point of the functional (13), if and only if the corresponding G-equivariant map $s \in C_G^\infty\left(P, G/H\right)$ is a *horizontally harmonic map*, that is $\tau^{\mathcal{H}}\left(s\right) = 0$. In the expression for $\tau^{\mathcal{H}}\left(s\right)$, the trace is just over the horizontal distribution in TP. It should be emphasized that the reason that the critical points of $\mathcal{E}_G$ are not exactly harmonic maps is that we are varying over only the equivariant maps, rather than arbitrary maps. On the other hand, Wood does prove in [20, Theorem 3], that if G/H is a normal G-homogeneous manifold and the metric on P is constructed from any compatible metric on G, then σ is a harmonic section if and only s is a harmonic map, that is, $\tau\left(s\right) := \operatorname{Tr}_g\left(\nabla^\eta ds\right) = 0$. Crucially, these conditions are satisfied for $G = SO\left(7\right)$, $H = G_2$, and P the orthonormal frame bundle on M. Moreover, as shown in [17], given these conditions, a family $\sigma_t \in \Gamma\left(N\right)$ satisfies the harmonic section flow $\frac{\partial\sigma_t}{\partial t} = \tau^{\mathcal{V}}\left(\sigma_t\right)$ if and only if there is a corresponding family $s_t \in C_G^\infty\left(P, G/H\right)$ that satisfies the harmonic map flow $\frac{\partial s_t}{\partial t} = \tau\left(s_t\right)$. Also, Wood has shown in [19] that equivariance is preserved along the harmonic map flow, so that if the initial condition is equivariant, then the flow will continue to be equivariant. This shows a close relationship between harmonic map theory and the theory of harmonic sections, and hence the flow (3) of isometric G_2-structures.

On the other hand, one must be careful when applying harmonic map results. In particular, the energy $\mathcal{E}_G\left(s\right)$ contains a topological term that can never be arbitrarily small, and thus standard small initial energy long-time existence results [5] for harmonic maps cannot be applied. Similarly, while a constant map is always harmonic, an equivariant map $s : P \longrightarrow G/H$ can never be constant (if $H \neq G$). Thus existence of non-trivial harmonic equivariant maps and hence harmonic sections is not guaranteed, as expected.

Some results from the theory of harmonic maps do carry over, at least in the G_2-case. It was shown in [8, 11] that almost monotonicity and ε-regularity results similar to the harmonic map heat flow [5, 6, 18] hold for the flow (3).

Let $p_{x_0,t_0}(x,t)$ be the backward heat kernel on M, that is, the solution of the backward heat equation for $0 \leq t \leq t_0$ that converges to a delta function at $(x,t) = (x_0,t_0)$. Then, given a time-dependent octonion section V_t or equivalently, a 3-form $\varphi_t = \sigma_{V(t)}(\varphi)$ for some fixed G_2-structure φ, define the $\mathcal{F}$-functional [11]

$$\mathcal{F}(x_0,t_0,t) = (t_0 - t) \int_M \left| T^{(V_t)}(x) \right|^2 p_{x_0,t_0}(x,t) \operatorname{vol}(x), \qquad (16)$$

where $T^{(V_t)} = -(DV_t)V_t^{-1}$ is the torsion of the G_2-structure φ_t. In [8], the analogous quantity is denoted by $\Theta_{(x_0,t_0)}(\varphi(t))$. It is then shown in both [8, Theorem 5.3] and [11, Proof of Corollary 7.2] that $\mathcal{F}$ satisfies an *almost* monotonicity formula along the flow (2). Suppose V_t is a solution of the flow (2) for $0 \leq t < t_0$ with initial energy $\mathcal{E}(0) = \mathcal{E}_0$. Then, there exists a constant $C > 0$, that only depends on the background geometry, such that for any t and τ satisfying $t_0 - 1 \leq \tau \leq t < t_0$, $\mathcal{F}$ satisfies the following relation

$$\mathcal{F}(x_0,t_0,t) \leq C\mathcal{F}(x_0,t_0,\tau) + C(t-\tau)\left(\mathcal{E}_0 + \mathcal{E}_0^{\frac{1}{2}}\right). \qquad (17)$$

In [8], the last term in (17) was $C(t-\tau)(\mathcal{E}_0 + 1)$, which of course follows from (17) for a different constant C. In both [8] and [11] similar versions of an ε-regularity result is proven for $\mathcal{F}$. We'll state it as in [11].

Theorem 4.1 ([8, Theorem 5.7] and [11, Theorem 7.1]) *Given $\mathcal{E}_0$, there exist $\varepsilon > 0$ and $\beta > 0$, both depending on M and β also depending on $\mathcal{E}_0$, such that if V is a solution of the flow (2) on $M \times [0,t_0)$ with energy bounded by $\mathcal{E}_0$, and if*

$$\mathcal{F}(x_0,t_0,t) \leq \varepsilon \qquad (18)$$

for $t \in [t_0 - \beta, t_0)$, then V extends smoothly to $U_{x_0} \times [0,t_0]$ for some neighborhood U_{x_0} of x_0 with $|DV| = \left|T^{(V)}\right|$ bounded uniformly.

Then, Theorem 4.1 was used in [8, 11] to show long-time existence of the isometric heat flow and convergence to a G_2-structure with $\operatorname{div} T = 0$ given sufficiently small initial pointwise torsion.

Given a G_2-structure 3-form φ, in [8] a concept of *entropy* was defined:

$$\lambda(\varphi,\sigma) = \max_{(x,t) \in M \times (0,\sigma)} \left\{ t \int_M \left| T^{(\varphi)}(y) \right|^2 p_{(x,t)}(y,0) \operatorname{vol}(y) \right\}. \qquad (19)$$

This mirrors similar entropy concepts defined for the mean curvature flow, Yang-Mills flow, and the harmonic map heat flow, in [7], [16], and [2], respectively. The quantity $\lambda(\varphi,\sigma)$ is shown in [8] to be invariant under the scaling $(\varphi,\sigma) \mapsto (c^3\varphi, c^2\sigma)$. While the same quantity could be defined for an octonion section

V, if considered as a function of V, λ would lose the scaling property for V. So in this case, using the 3-form has an advantage. Overall, one of the key results in [8] is long term existence and convergence of the flow (3) given sufficiently small entropy.

Theorem 4.2 ([8, Theorem 5.15]) *Let φ_0 be a G_2-structure on a compact 7-manifold M. For any $\delta, \sigma > 0$, there exists $\varepsilon > 0$, such that if $\lambda\left(\varphi_0, \sigma\right) < \varepsilon$, then the flow (3) with initial condition $\varphi\left(0\right) = \varphi_0$ exists for all time and converges smoothly to a G_2-structure φ_∞ that satisfies $\operatorname{div} T^{(\varphi_\infty)} = 0$ and $\left|T^{(\varphi_\infty)}\right| < \delta$.*

Although good progress has been made on properties of the flows (2) and (3), many questions still remain. For example, is it possible to prove long-time existence given small initial energy, rather than entropy or pointwise torsion? If we combine the equivariant harmonic map approach with the octonion approach, then everything could be reformulated in terms of equivariant maps from the orthonormal frame bundle P to S^7 equipped with the octonion product. It is likely that the additional algebraic structure could help achieve stronger results.

References

[1] L. Bagaglini. The Energy Functional of G_2-Structures compatible with a Background Metric. *Journal of Geometric Analysis*, 2019. doi:10.1007/s12220-019-00264-6.

[2] J. Boling, C. Kelleher, and J. Streets. Entropy, stability and harmonic map flow. *Trans. Amer. Math. Soc.*, 369(8):5769–5808, 2017. doi:10.1090/tran/6949.

[3] R. L. Bryant. Metrics with exceptional holonomy. *Ann. of Math. (2)*, 126(3):525–576, 1987. doi:10.2307/1971360.

[4] R. L. Bryant. Some remarks on G_2-structures. In *Proceedings of Gökova Geometry-Topology Conference 2005*, pages 75–109. Gökova Geometry/Topology Conference (GGT), Gökova, 2006, math/0305124.

[5] Y. M. Chen and W. Y. Ding. Blow-up and global existence for heat flows of harmonic maps. *Invent. Math.*, 99(3):567–578, 1990. doi:10.1007/BF01234431.

[6] Y. M. Chen and M. Struwe. Existence and partial regularity results for the heat flow for harmonic maps. *Math. Z.*, 201(1):83–103, 1989. doi:10.1007/BF01161997.

[7] T. H. Colding and W. P. Minicozzi, II. Generic mean curvature flow I: generic singularities. *Ann. of Math. (2)*, 175(2):755–833, 2012. doi:10.4007/annals.2012.175.2.7.

[8] S. Dwivedi, P. Gianniotis, and S. Karigiannis. A gradient flow of isometric G_2 structures. *The Journal of Geometric Analysis*, 2019, 1904.10068. doi:10.1007/s12220-019-00327-8.

[9] S. Grigorian. Short-time behaviour of a modified Laplacian coflow of G2-structures. *Adv. Math.*, 248:378–415, 2013, 1209.4347. doi:10.1016/j.aim.2013.08.013.

[10] S. Grigorian. G_2-structures and octonion bundles. *Adv. Math.*, 308:142–207, 2017, 1510.04226. doi:10.1016/j.aim.2016.12.003.

[11] S. Grigorian. Estimates and monotonicity for a heat flow of isometric G_2-structures. *Calc. Var. Partial Differential Equations*, 58(5):Art. 175, 37, 2019. doi:10.1007/s00526-019-1630-0.

[12] N. J. Hitchin. The geometry of three-forms in six dimensions. *J. Differential Geom.*, 55(3):547–576, 2000, math/0010054. URL http://projecteuclid.org/euclid.jdg/1090341263.

[13] S. Karigiannis. Deformations of G_2 and $Spin(7)$ Structures on Manifolds. *Canadian Journal of Mathematics*, 57:1012, 2005, math/0301218. doi:10.4153/CJM-2005-039-x.

[14] S. Karigiannis. Flows of G_2-Structures, I. *Q. J. Math.*, 60(4):487–522, 2009, math/0702077. doi:10.1093/qmath/han020.

[15] S. Karigiannis, B. McKay, and M.-P. Tsui. Soliton solutions for the Laplacian coflow of some G_2-structures with symmetry. *Differential Geom. Appl.*, 30(4):318–333, 2012, 1108.2192. doi:10.1016/j.difgeo.2012.05.003.

[16] C. Kelleher and J. Streets. Entropy, stability, and Yang-Mills flow. *Commun. Contemp. Math.*, 18(2):1550032, 51, 2016. doi:10.1142/S0219199715500327.

[17] E. Loubeau and H. N. Sá Earp. Harmonic flow of geometric structures. 2019, 1907.06072.

[18] M. Struwe. On the evolution of harmonic maps in higher dimensions. *J. Differential Geom.*, 28(3):485–502, 1988. URL `http://projecteuclid.org/euclid.jdg/1214442475`.

[19] C. M. Wood. An existence theorem for harmonic sections. *Manuscripta Math.*, 68(1):69–75, 1990. doi:10.1007/BF02568751.

[20] C. M. Wood. Harmonic sections and equivariant harmonic maps. *Manuscripta Math.*, 94(1):1–13, 1997. doi:10.1007/BF02677834.