

CONNECTING ARROW'S THEOREM, VOTING THEORY, AND THE TRAVELING SALESPERSON PROBLEM

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ABSTRACT. Problems with majority voting over pairs as represented by Arrow's Theorem and those of finding the lengths of closed paths as captured by the Traveling Salesperson Problem (TSP) appear to have nothing in common. In fact, they are connected. As shown, pairwise voting and a version of the TSP share the same domain where each system can be simplified by restricting it to complementary regions to eliminate extraneous terms. Central for doing so is the Borda Count, where it is shown that its outcome most accurately reflects the voter preferences.

1. INTRODUCTION

Among the many challenges posed by discrete mathematics and the Social Sciences are aspects of voting theory as characterized by Arrow's Theorem and the properties of closed paths on a graph as typified by the Traveling Salesperson Problem (TSP). Surprisingly, both topics can be analyzed with essentially the same approach. After introducing the commonality with examples, it is shown how each situation can be simplified by emphasizing different regions of the associated geometry.

1.1. Voting theory and Arrow's Theorem. Central to Arrow's Theorem [1] is his IIA condition, which requires a profile's conclusion to be completely determined by the rankings of its associated paired comparisons. A $N = 70$ person majority vote example over the $n = 3$ alternatives $\{A_j\}_{j=1}^3$ is where

- (1) 25 prefer $A_1 \succ A_2 \succ A_3$, 23 prefer $A_2 \succ A_3 \succ A_1$, 22 prefer $A_3 \succ A_1 \succ A_2$.

This leads to the cyclic pairwise majority vote outcomes of

- (2) $A_1 \succ A_2$ by 47:23, $A_2 \succ A_3$ by 48:22, and $A_3 \succ A_1$ by 45:25,

which violate Arrow's objective of obtaining a transitive conclusion. This difficulty reflects the long-standing objective in voting theory, which is to replace cyclic outcomes with

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¹Even stronger, a decision method satisfies "Independence of irrelevant alternatives," IIA, iff it can be expressed in terms of independent paired comparison methods [10, 11]. This means it is not a negative feature if a decision method, such as the plurality vote, fails to satisfy IIA; it only means that the method cannot be so expressed. Conversely, any method that cannot be expressed in this manner does not satisfy IIA. Thus, Arrow's Theorem is a result about the problems that accompany paired comparisons.

appropriate transitive rankings, or at least with outcomes that identify a “best choice.” Prominent approaches were developed by the mathematicians Dodgson [2] in 1876 and Kemeny [3] in 1959.

A way to compare how an alternative fares in a paired comparison is to compute how its tally differs from the average score of $\frac{\text{number of voters}}{2} = \frac{N}{2}$. (In what follows, N is the number of voters, n is the number of alternatives.) So

$$(3) \quad d_{i,j} = \frac{1}{2}[A_i\text{'s tally} - A_j\text{'s tally}] = A_i\text{'s tally} - \frac{N}{2} = -d_{j,i}.$$

For the Eq. 2 values, where $N = 70$ and $n = 3$,

$$(4) \quad d_{1,2} = 47 - \frac{70}{2} = 12 = -d_{2,1}, \quad d_{2,3} = 13 = -d_{3,2}, \quad \text{and} \quad d_{1,3} = -10 = -d_{3,1}.$$

1.2. TSP. To introduce the notation for the TSP system discussed here, it takes 30 minutes to walk from home, H , to campus, C , while returning uphill takes 40. As the average is 35 minutes, returning home requires 5 minutes above the average denoted by $C \xrightarrow{5} H$; traveling in the opposite direction takes 5 fewer minutes or $H \xrightarrow{-5} C$. More generally, if $d_{i,j}$ represents the “difference from the average” cost of going from A_i to A_j , then

$$(5) \quad A_i \xrightarrow{d_{i,j}} A_j \text{ and } A_j \xrightarrow{-d_{i,j}} A_i \text{ are equivalent,}$$

or, as true with Eq. 3,

$$(6) \quad d_{i,j} = -d_{j,i}.$$

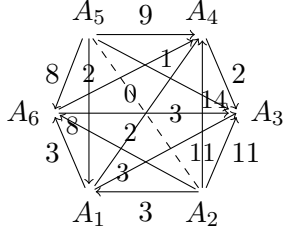


Figure 1. $\mathcal{G}_A^6$

Figure 1 is a typical TSP example, where the graph catalogues all “differences from averages” information about the alternatives $\{A_j\}_{j=1}^6$ and its $\binom{6}{2} = 15$ pairs. (Subscripts “A” and “S” refer, respectively, to whether the graph has an asymmetric or a symmetric structure.) Only positive cost directions are displayed because (Eqs. 5, 6) traveling counter to an arrow is a negative, or below average cost. A standard TSP objective is to discover the longest and shortest Hamiltonian circuits. To appreciate what will be developed, before reading more, let me ask the reader to find the longest such path in Fig. 1. Recall, a Hamiltonian circuit is a closed path that starts and ends at a selected vertex A_j and passes through each of the other vertices once. To illustrate, the Fig. 1 Hamiltonian path

$$A_1 \xrightarrow{-3} A_2 \xrightarrow{11} A_4 \xrightarrow{-1} A_6 \xrightarrow{3} A_3 \xrightarrow{-14} A_5 \xrightarrow{2} A_1$$

has length -2 , and its reversal has length 2 . In fact, thanks to Eqs. [5](#), [6](#), the tasks of finding the longest and shortest Hamiltonian paths coincide; the reversal of one is the other.

1.3. A common space. The voting and TSP questions involve the $\binom{n}{2}$ pairs defined by n alternatives where costs/differences between vertices of a $\{i, j\}$ pair are measured by $d_{i,j} = -d_{j,i}$ values.^{[2](#)} Thus, both settings share the $\mathbb{R}_A^{(n)}$ domain, where $\mathbf{d}_A^n \in \mathbb{R}_A^{(n)}$ has the form

$$(7) \quad \mathbf{d}_A^n = (d_{1,2}, d_{1,3}, \dots, d_{1,n}; d_{2,3}, \dots, d_{2,n}; d_{3,4}, \dots, d_{n-1,n}); d_{i,j} = -d_{j,i}.$$

Semicolons indicate changes in the first subindex. The $\mathbb{R}_A^{(n)}$ structure is developed next.

To start, recall that if a triplet $\{A_i, A_j, A_k\}$ defines a transitive ranking, it can be expressed with these $d_{s,u}$ values; e.g., should $d_{i,j} > 0$ and $d_{j,k} > 0$, then transitivity requires that $d_{i,k} > 0$. These inequalities are borrowed from the structure of points on a line where $p_i > p_j$ and $p_j > p_k$ require $p_i > p_k$. But these points also satisfy the stronger algebraic relationship $(p_i - p_j) + (p_j - p_k) = (p_i - p_k)$. The following mimics this equality.

Definition 1. [\[12\]](#) Vector $\mathbf{d}_A^n \in \mathbb{R}_A^{(n)}$ is strongly transitive iff each triplet $\{i, j, k\}$ satisfies

$$(8) \quad d_{i,j} + d_{j,k} = d_{i,k}.$$

.

This condition was introduced for profiles in [\[9\]](#); the decision theory version used here comes from [\[12\]](#). The subspace of strongly transitive vectors, ST_A^n , is described next.

Theorem 1. [\[12\]](#) The set of strongly transitive $\mathbf{d}_{A,st}^n \in \mathbb{R}_A^{(n)}$, denoted by ST_A^n [\[3\]](#) is a $(n-1)$ -dimensional linear subspace of $\mathbb{R}_A^{(n)}$.

While details are in [\[12\]](#), proving that ST_A^n is a linear subspace is a standard exercise. The assertion about its dimension follows from the fact that any $d_{j,k}$ can be expressed as $d_{j,k} = d_{j,1} + d_{1,k}$ (Eq. [8](#)), so all $d_{j,k}$ values for $\mathbf{d}_{A,st}^n \in \text{ST}_A^n$ can be determined from the $(n-1)$ terms $\{d_{1,s}\}_{s=2}^n$. $\square$

The dimensions of $\mathbb{R}_A^{(n)}$ and ST_A^n dictate that $\mathbb{C}_A^n$, the orthogonal complement of ST_A^n , has dimension $\binom{n-1}{2}$. To motivate the form of these orthogonal vectors, expressing Eq. [8](#) as $x + y = z$, or $x + y - z = 0$, identifies $(1, 1, -1)$ as a normal vector. Thus the $x = y = -z = 1$ values define a normal vector $d_{i,j} = d_{j,k} = d_{k,i} = 1$, which is represented by the cyclic $A_i \succ A_j, A_j \succ A_k, A_k \succ A_i$ where the differences between values is the same constant.

Theorem 2. [\[12\]](#) The $\binom{n-1}{2}$ dimensional linear subspace $\mathbb{C}_A^n$ consists of cycles where one basis, which consists of three-cycles, is $\{\mathbf{d}_{1,j,k}^n\}_{1 < j < k \leq n}$. The only non-zero terms of $\mathbf{d}_{1,j,k}^n$ are $d_{1,j} = d_{j,k} = d_{k,1} = 1$.

²Rather than “difference from average,” $d_{i,j}$ can be anything; e.g., a natural choice is the difference between A_i and A_j values.

³In [\[9, 14\]](#), ST_A^n is called the “transitivity plane.”

According to these theorems, $\mathbb{R}_A^{(n)}$ nicely separates into linear subspaces of strongly transitive vectors, $\mathbb{ST}_A^n$, and cyclic vectors, $\mathbb{C}_A^n$, where $\mathbb{C}_A^n$ has a basis consisting of a special type of three-cycles. This means that $\mathbf{d}_A^n \in \mathbb{R}_A^{(n)}$ has a unique decomposition

$$(9) \quad \mathbf{d}_A^n = \mathbf{d}_{A,st}^n + \mathbf{d}_{A,cyclic}^n$$

where $\mathbf{d}_{A,st}^n$ and $\mathbf{d}_{A,cyclic}^n$ are, respectively, the orthogonal projection of $\mathbf{d}_A^n$ to $\mathbb{ST}_A^n$ and $\mathbb{C}_A^n$.

An immediate consequence of Eq. 9 is that a transitive vector, which is not strongly transitive, is a hybrid.

Corollary 1. *For $n \geq 3$, if $\mathbf{d}_A^n$ is transitive but not strongly transitive, then there are unique non-zero vectors $\mathbf{d}_{A,st}^n \in \mathbb{ST}_A^n$ and $\mathbf{d}_{A,cyclic}^n \in \mathbb{C}_A^n$ so that $\mathbf{d}_A^n = \mathbf{d}_{A,st}^n + \mathbf{d}_{A,cyclic}^n$.*

What limits the use of transitive vectors is that they fail to define a linear subspace. For instance, $d_{1,2} = d_{2,3} = 2$, $d_{1,3} = 1$ and $d_{1,2}^* = d_{2,3}^* = -1$, $d_{1,3}^* = -2$ are transitive triplets, but their sums (e.g., $d_{i,j} + d_{i,j}^* = d_{i,j}^+$) define the cyclic $d_{1,2}^+ = d_{2,3}^+ = 1$, $d_{1,3}^+ = -1$. According to Cor. 1, it is reasonable to treat a transitive vector as a strongly transitive choice that is lightly contaminated with cyclic terms. The following theorem partly explains the “lightly contaminated” modifier by showing that rankings of transitive $\mathbf{d}_A^n$ and its strongly transitive component $\mathbf{d}_{A,st}^n$ can differ, but not radically. This holds even for a non-transitive $\mathbf{d}_A^n$ that has, at least, a Condorcet winner and loser. (A Condorcet winner is a candidate who beats each of the other candidates in majority vote comparisons. A Condorcet loser loses all pairwise majority votes. They can exist even without transitivity; e.g., for $n = 5$, A_1 could be the Condorcet winner, A_5 the Condorcet loser, and A_2, A_3, A_4 define a cycle.)

Theorem 3. *If $\mathbf{d}_A^n$ has a unique Condorcet winner A_1 and a unique Condorcet loser A_n , then A_1 is strictly ranked above A_n in $\mathbf{d}_{A,st}^n$.*

The converse is to determine what happens by adding cyclic terms to a $\mathbf{d}_{A,st}^n$. If the resulting $\mathbf{d}_A^n$ is wildly cyclic, not much can be stated. But if A_1 and A_n are, respectively, the top and bottom ranked candidates of the $\mathbf{d}_{A,st}^n$ and if $\mathbf{d}_A^n$ remains transitive, the question is whether A_1 must be ranked above A_n in $\mathbf{d}_A^n = \mathbf{d}_{A,st}^n + \mathbf{d}_{A,cyclic}^n$. Proofs of this kind of results can be messy. But to illustrate Thm. 2, the basic idea is developed in Sect. 5 for $n = 3, 4$. The details of these proofs are similar to earlier relationships that were developed about the Kemeny and paired voting rankings relative to the Borda ranking [9, 14].

2. VOTING THEORY

As a central objective in voting and decision theory is to obtain transitive outcomes, the cyclic components of $\mathbf{d}_A^n$ introduce problems. A resolution is obvious; drop the troubling $\mathbf{d}_{A,cyclic}^n$ cyclic term and retain only $\mathbf{d}_{A,st}^n$. Dodgson’s method partly does so by replacing $\mathbf{d}_A^n$ with a vector that may have cyclic terms, but at least it has a Condorcet winner. (A_j is a Condorcet winner iff $d_{j,k} > 0$ for all $k \neq j$. That is, A_j is “better than” all other alternatives in paired comparisons.) Dodgson’s method [2], then, projects $\mathbf{d}_A^n \in \mathbb{R}_A^{(n)}$ to the nearest $\mathbb{R}_A^{(n)}$ subset where all vectors have a Condorcet winner. For $n \geq 4$, these regions

have cyclic components; e.g., for $n = 4$, the dimension of ST_A^4 is only three, while the Condorcet subset has the full $\binom{4}{2} = 6$ dimension. For instance, A_1 is the Condorcet winner in the eight $\mathbb{R}_A^6$ orthants where $d_{1,2} > 0, d_{1,3} > 0, d_{1,4} > 0$. Without imposing restrictions on the signs of $d_{2,3}, d_{2,4}, d_{3,4}$, cyclic behavior is allowed among $\{A_2, A_3, A_4\}$.

Kemeny adopted the more ambitious goal [3] of replacing $\mathbf{d}_A^n$ with a transitive outcome. Similar to Dodgson, he created a projection mapping that sends $\mathbf{d}_A^n$ into the nearest $\mathbb{R}_A^{\binom{n}{2}}$ subset consisting of transitive outcomes.

Both methods provide electoral relief at a first level, but a deeper investigation reveals a host of other subtle difficulties that cast doubt on the reliability of these approaches. Important results in this direction were found in a series of papers by Ratliff [4, 5, 6]. For instance, the above “projection” descriptions makes it reasonable to expect that the Dodgson and Kemeny outcomes are related; perhaps the Dodgson winner always is the top-ranked Kemeny candidate. But Ratliff proved that, in general, such assertions are false. A small sample of his findings follows.

Theorem 4. [4, 5, 6] *For $n \geq 4$ candidates, select an integer k , where $1 \leq k \leq n$. There exist paired comparison examples where the Dodgson winner is the k^{th} ranked Kemeny winner.*

Dodgson's method can be generalized to select a committee of $k \geq 2$ candidates by using Dodgson's projection method to a subset of $\mathbb{R}_A^{\binom{n}{2}}$ where all vectors have k candidates where each is ranked above the $(n - k)$ remaining candidates. For integers k and s satisfying $1 \leq s < n - k$, $k \neq s$, there exist examples where with the Dodgson's generalized approach, no candidate in the Dodgson committee of s is in the Dodgson committee of k candidates.

As Ratliff proved, rather than being top-ranked, the Dodgson winner can end up being anywhere in a Kemeny outcome; it can even be bottom-ranked. Moreover, the Dodgson winner need not be in a Dodgson-Ratliff committee of two or three. These unexpected conclusions are consequences of the cyclic terms that remain even after the Dodgson and Kemeny projections. For instance, unless Kemeny's outcome is strongly transitive, it contains cyclic components (Cor. [1]) that can create other difficulties.

Stated differently, the Dodgson and Kemeny procedures remove only as many of the $\mathbf{d}_A^n$ cyclic components as needed to attain their objectives. Without a thorough cleansing, it is reasonable to expect other mysterious properties: this happens. This also holds for Arrow's Theorem; its negative conclusion is strictly a consequence of the $\mathbf{d}_{A,\text{cyclic}}^n$ component. Completely removing $\mathbf{d}_{A,\text{cyclic}}^n$ converts Arrow's assertion into a positive result [10, 11]. Indeed, the ultimate goal for decision and voting problems should be to eliminate *all cyclic components of a given $\mathbf{d}_A^n$* . Doing so is a common mathematical computation.

2.1. Orthogonal Projection. The standard way to eliminate the unwanted cyclic terms from a given $\mathbf{d}_A^n$ is with the orthogonal projection

$$(10) \quad P : \mathbb{R}_A^{\binom{n}{2}} \rightarrow \text{ST}_A^n.$$

The computations require finding a basis for ST_A^n (using Eq. [8](#)) and carrying out the associated vector analysis. The resulting approach follows; see [12](#) for details and examples.

Definition 2. For alternative A_j , the weighted sum is

$$(11) \quad \mathcal{S}_A(A_j) = \sum_{k=1}^n d_{j,k}; \quad j = 1, \dots, n.$$

With Eq. [4](#) values,

$$(12) \quad S_A(A_1) = 12 - 10 = 2, \quad S_A(A_2) = -12 + 13 = 1, \quad S_A(A_3) = -13 + 10 = -3.$$

Theorem 5. [12](#) For $\mathbf{d}_A^n \in \mathbb{R}_A^{\binom{n}{2}}$, the $d_{i,j}$ value in $\mathbf{d}_{A,st}^n$ is

$$(13) \quad d_{i,j} = \frac{1}{n} [\mathcal{S}_A(A_i) - \mathcal{S}_A(A_j)].$$

Of surprise, this projection is equivalent to the well known Borda Count. This Borda procedure tallies a n -candidate ballot by assigning $n - j$ points to the j^{th} ranked candidate. A candidate's Borda tally, $\mathcal{B}(A_j)$, is the sum of points assigned to A_j over all N ballots. Using the Eq. [1](#) example, $\mathcal{B}(A_1) = 25(3 - 1) + 23(3 - 3) + 22(3 - 2) = 72$, $\mathcal{B}(A_2) = 25(3 - 2) + 23(3 - 1) + 22(3 - 3) = 71$, and $\mathcal{B}(A_3) = 25(3 - 3) + 23(3 - 2) + 22(3 - 1) = 67$.

Theorem 6. [12](#) For any $n \geq 3$, the Eq. [10](#) orthogonal projection of $\mathbf{d}_A^n \in \mathbb{R}_A^{\binom{n}{2}}$ to ST_A^n , which is $\mathbf{d}_{A,st}^n$, is equivalent to the Borda Count. More precisely,

$$(14) \quad \mathcal{B}(A_j) = (n - 1) \frac{N}{2} + S_A(A_j), \quad j = 1, \dots, n.$$

and

$$(15) \quad \mathcal{B}(A_j) - \mathcal{B}(A_k) = S_A(A_j) - S_A(A_k) = n d_{j,k}.$$

Proof: As known (e.g., [8](#), [9](#), [10](#)), a way to compute $\mathcal{B}(A_j)$ is to sum A_j 's tallies over each of its $(n - 1)$ majority vote paired comparisons. (As an example, using the Eq. [2](#) tallies, in the $\{A_1, A_2\}$ and $\{A_1, A_3\}$ elections, A_1 receives, respectively, 47 and 25 votes; this $47 + 25 = 72$ value agrees with A_1 's above computed $\mathcal{B}(A_1) = 72$.) According to Eq. [3](#),

$$(16) \quad \mathcal{B}(A_j) = \sum_{k=1, k \neq j}^n [d_{j,k} + \frac{N}{2}] = (n - 1) \frac{N}{2} + S_A(A_j).$$

Thus, for voting theory, $S_A(A_j) - S_A(A_k) = \mathcal{B}(A_j) - \mathcal{B}(A_k)$. $\square$

Theorem [6](#) has an interesting consequence. To set the stage, consider all of those voting methods where the outcome is determined by assigning a score to each candidate; e.g., this includes almost all standard methods such as all positional methods, cumulative voting, Approval Voting, etc. As these scores satisfy strong transitivity, the outcome is in ST_A^n . A natural objective is to have an outcome that most accurately reflects the views (i.e., preferences) of the voters. That is, find the ST_A^n outcome that is closest to the data, which is the orthogonal projection of $\mathbf{d}_A^n$. This is equivalent to the Borda Count (Thm. [6](#)).

Combining comments from the above provide new explanations for earlier results. For instance, the following, which has the spirit of Thm. 3, was proved by using these tools.

Theorem 7. [9, 14] *For $n \geq 3$, Kemeny's method ranks a Borda winner over the Borda loser. Conversely, the Borda Count ranks a Kemeny winner over the Kemeny bottom ranked candidate.*

For paired comparison majority votes, the Borda outcome ranks the Condorcet winner over the Condorcet bottom ranked candidate. Conversely, if the paired comparisons define a transitive ranking, the Borda winner is ranked over the Borda bottom ranked candidate.

The Kemeny outcome is a transitive vector, while the strongly transitive Borda ranking comes from the orthogonal projection of $\mathbf{d}_A^n$ to $\mathbb{ST}_A^n$. As Cor. 1 asserts, a transitive ranking is a strongly transitive ranking clouded by cyclic terms, which captures the flavor of the Kemeny outcome.

Theorem 6 provides an explanation why the Borda Count has so many positive properties; examples with decision methods are in [12]. The following completes the introductory comments about Arrow's Theorem.

Theorem 8. [8, 9] *If $\mathbf{d}_A^n \in \mathbb{ST}_A^n$, then the Borda Count and majority vote paired comparisons both satisfy Arrow's Theorem. The Borda Count is the only positional voting method that satisfies Arrow's conditions.*

The last assertion holds because the Borda Count is the only positional method where its outcomes are determined by the outcomes of majority votes over pairs. The outcome for all other positional methods need not be related, in any manner, to the paired comparison outcomes [7].

3. TURNING TO THE TSP

According to Eq. 9, $\mathbf{d}_A^n \in \mathbb{R}^{\binom{n}{2}}$ can be uniquely expressed as $\mathbf{d}_A^n = \mathbf{d}_{A,st}^n + \mathbf{d}_{A,cyclic}^n$, where $\mathbf{d}_{A,st}^n \in \mathbb{ST}_A^n$ and $\mathbf{d}_{A,cyclic}^n \in \mathbb{C}_A^n$. As described above, voting and decision theories seek transitive outcomes, which means that the cyclic $\mathbf{d}_{A,cyclic}^n$ component imposes obstacles. The natural resolution is to eliminate $\mathbf{d}_{A,cyclic}^n$ and retain $\mathbf{d}_{A,st}^n$ by projecting $\mathbf{d}_A^n$ into $\mathbb{ST}_A^n$.

In contrast, TSP and other closed path concerns involve *cycles*, so the linear behavior now is what creates barriers. This change in the objective converts the strongly transitive $\mathbf{d}_{A,st}^n$ from being the desired component into a troublemaker. Thus, to analyze TSP issues, mimic what was done for voting by orthogonally projecting $\mathbf{d}_A^n$ into $\mathbb{C}_A^n$ to eliminate the $\mathbf{d}_{A,st}^n$ term and retain $\mathbf{d}_{A,cyclic}^n$.

What simplifies finding $\mathbf{d}_{A,cyclic}^n$ is that $\mathbf{d}_{A,cyclic}^n = \mathbf{d}_A^n - \mathbf{d}_{A,st}^n$, and $\mathbf{d}_{A,st}^n$, which is equivalent to the Borda Count (Thm. 6), is easily computed (Thm. 5). The decomposition of the Fig. 1 graph is in Fig. 2 where the $\mathcal{G}_{A,cpi}^6$ and $\mathcal{G}_{A,cyclic}^6$ entries represent, respectively, $\mathbf{d}_{A,st}^6$ and $\mathbf{d}_{A,cyclic}^6$.

The computation of $\mathcal{G}_{A,cpi}^6$ follows from Def. 2 and Thm. 5. For instance, $S_A(A_1) = [-3+3+2-2+3] = 3$ while $S_A(A_2) = [3+8+0+11+11] = 33$, so $d_{1,2} = \frac{1}{6}[S_A(A_1) - S_A(A_2)] = -5$

is the $\mathbf{d}_{A,st}^6$ entry, which is the length of the $A_1 \rightarrow A_2$ leg in $\mathcal{G}_{cpi}^6$. The cpi subscript of $\mathcal{G}_{A,cpi}^6$, which means “closed path independent,” is described next.

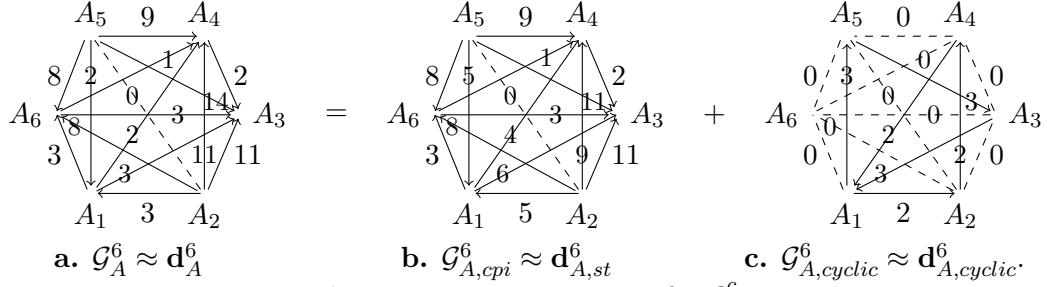


Figure 2. Decomposition of a $\mathcal{G}_A^6$

Thanks to the linearity of the decomposition, the length of any closed path (not just Hamiltonian circuits) in $\mathcal{G}_A^6$ equals the sum of the path's lengths in $\mathcal{G}_{A,cpi}^6$ and $\mathcal{G}_{A,cyclic}^6$. As an example, the -4 length of the $\mathcal{G}_A^6$ closed path $A_1 \xrightarrow{-3} A_2 \xrightarrow{8} A_6 \xrightarrow{3} A_3 \xrightarrow{-14} A_5 \xrightarrow{2} A_1$ equals the sum of its path's lengths of $A_1 \xrightarrow{-5} A_2 \xrightarrow{8} A_6 \xrightarrow{3} A_3 \xrightarrow{-11} A_5 \xrightarrow{5} A_1$ in $\mathcal{G}_{A,cpi}^6$ and $A_1 \xrightarrow{2} A_2 \xrightarrow{0} A_6 \xrightarrow{0} A_3 \xrightarrow{-3} A_5 \xrightarrow{-3} A_1$ in $\mathcal{G}_{A,cyclic}^6$. As this path's length in $\mathcal{G}_{A,cpi}^6$ is zero, its -4 length in $\mathcal{G}_{A,cyclic}^6$ equals the $\mathcal{G}_A^6$ path length. That this closed path in $\mathcal{G}_{A,cpi}^6$ has length zero is not a lucky coincidence. Instead, *all closed paths* in $\mathcal{G}_{A,cpi}^n$ have length zero, so these paths are “closed path independent” when computing $\mathcal{G}_A^6$ path lengths.

Theorem 9. *For $n \geq 3$, a closed path in $\mathcal{G}_{A,cpi}^n$ has length zero. The length of a closed path in $\mathcal{G}_A^n$ equals its length in $\mathcal{G}_{A,cyclic}^n$.*

The length of any $\mathcal{G}_A^n$ path starting at V_i and ending at V_j equals its length in $\mathcal{G}_{A,cyclic}^n$ plus the $V_i \rightarrow V_j$ length in $\mathcal{G}_{A,cpi}^n$.

The proof of the first assertion follows immediately from the strong transitivity of the $\mathcal{G}_{A,cpi}^n$ components. To check that $\mathcal{G}_{A,cpi}^6$ has this property, select any triplet from Fig. 2b—perhaps $\{A_1, A_5, A_3\}$; the goal is to show that $d_{1,5} + d_{5,3} = d_{1,3}$. From $\mathcal{G}_{A,cpi}^6$, this requires $d_{1,5} + d_{5,3} = -5 + 11$ to equal $d_{1,3} = 6$, which it does.

Next, a standard induction exercise shows that if $\mathbf{d}_{A,st}^n$ is strongly transitive, then *any path* from A_j to A_k has the same length as the direct $A_j \rightarrow A_k$ path. Reversing this last arc creates a closed path with length zero. The rest of the theorem follows immediately.

The last assertion of Thm. 9 requires the “ -1 ” length of $\mathcal{G}_A^6$ path,

$$A_1 \xrightarrow{-3} A_2 \xrightarrow{11} A_4 \xrightarrow{2} A_3 \xrightarrow{-11} A_2 \xrightarrow{0} A_5,$$

which skips A_6 but visits A_2 twice, to equal this path's length in $\mathcal{G}_{A,cyclic}^6$ plus -5 . The -5 comes from the $A_1 \xrightarrow{-5} A_5$ arc in $\mathcal{G}_{A,cpi}^6$ (Fig. 2b). This path's length in $\mathcal{G}_{A,cyclic}^6$ is

$$A_1 \xrightarrow{2} A_2 \xrightarrow{2} A_4 \xrightarrow{0} A_3 \xrightarrow{0} A_2 \xrightarrow{0} A_5$$

or 4, and, as Thm. 9 promises, $-1 = 4 - 5$.

3.1. Finding the longest and shortest Hamiltonian circuits. According to Thm. [9](#), the longest/shortest Hamiltonian path in $\mathcal{G}_A^6$ (Fig. 2a) can be found by ignoring $\mathcal{G}_A^6$ and, instead, analyzing the reduced $\mathcal{G}_{A,cyclic}^6$ with its two three-cycles (Fig. 3) $A_1 \xrightarrow{3} A_5 \xrightarrow{3} A_3 \xrightarrow{3} A_1$ and $A_1 \xrightarrow{2} A_2 \xrightarrow{2} A_4 \xrightarrow{2} A_1$. The obvious strategy is to use these cycles as fully as possible. To avoid premature closure by returning to a previously visited vertex, use at most two arcs of each three-cycle.

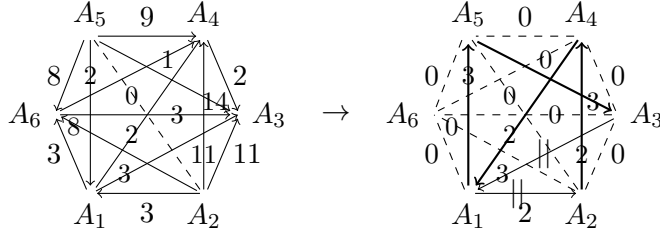


Figure 3. Finding $\mathcal{G}_A^6$'s longest Hamiltonian circuit via $\mathcal{G}_{A,cyclic}^6$.

Consequently, starting at A_1 , use only the edges $A_1 \xrightarrow{3} A_5 \xrightarrow{3} A_3$ from the first three-cycle; this is why $A_3 \xrightarrow{3} A_1$ arc is crossed out in Fig. 3. As only one arc can leave A_1 , this eliminates the crossed-out $A_1 \xrightarrow{2} A_2$ arc. Thus, the Hamiltonian circuit depends upon the four bold $\mathcal{G}_{A,cyclic}^6$ arcs in Fig. 3. Finding the longest Hamiltonian circuit now is immediate—use zero-length arcs to ensure all vertices are visited and to connect the two basic ones. An answer is the Hamiltonian path

$$(A_1 \xrightarrow{3} A_5 \xrightarrow{3} A_3) \xrightarrow{0} A_6 \xrightarrow{0} (A_2 \xrightarrow{2} A_4 \xrightarrow{2} A_1)$$

of length 10. The reversal of this closed path has length -10 . It is clear from the graph's structure that these are the longest and shortest $\mathcal{G}_{A,cyclic}^6$ Hamiltonian circuits, so (Thm. 3) they define the longest and shortest $\mathcal{G}_A^6$ Hamiltonian circuits with the same lengths.

To appreciate what is going on, notice that what complicates computing the Fig. 2a path length $A_5 \xrightarrow{14} A_3 \xrightarrow{-3} A_1 \xrightarrow{-2} A_5$ of $14 - 3 - 2 = 9$ are the subtractions/cancellations. To see what they are, let u , v , and w be the canceled terms, respectively, for arcs $\widehat{A_5 A_3}$, $\widehat{A_3 A_1}$, and $\widehat{A_1 A_5}$. That is, after cancellations, the path length computation would be $(14 - u) + (-3 - v) + (-2 - w) = 9$ where $u + v + w = 0$, which satisfies cpi from Eq. [8](#). That is, the cancellations in computing closed path lengths are linear expressions that define a ST_A^n graph. The optimal choice of cancellations for a $\mathcal{G}_A^n$ is the ST_A^n graph that most closely approximates $\mathcal{G}_A^n$, which is its orthogonal projection into ST_A^n , or $\mathcal{G}_{A,cpi}^n$. Indeed, with Fig. 2b, the optimal cancelation is from $\mathcal{G}_{A,cpi}^6$ where $u + v + w = 11 - 6 - 5 = 0$. Removing these cancellations leaves $\mathcal{G}_{A,cyclic}^6$ with the path length $A_5 \xrightarrow{3} A_3 \xrightarrow{3} A_1 \xrightarrow{3} A_5$ of length 9 where no modifications are needed in the computations.

3.2. The symmetric case. For the standard symmetric TSP, the distance between vertices is the same in each direction. Thus the $d_{i,j} = -d_{j,i}$ asymmetric condition is replaced

with the $d_{i,j} = d_{j,i}$ requirement. This requires a different decomposition of the new $\mathbb{R}_S^{(n)}$ where, with the orthogonal projection, leads to a different type of inherent symmetries. Otherwise, everything remains much the same. Of importance, the decomposition provides a best possible simplification for closed paths. Details for the general symmetric and asymmetric cases are given elsewhere [13], but a flavor of what happens is described next.

A $n = 6$ example is given in Fig. 4; recall, the S subscript in $\mathcal{G}_S^6$ refers to “symmetric.” While a typical goal is to find the shortest Hamiltonian circuit, everything extends to the analysis of any path, whether closed or not.

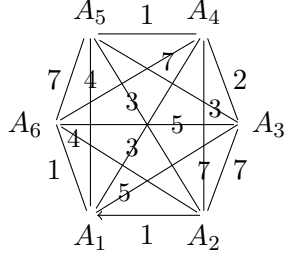


Figure 4. A symmetric $\mathcal{G}_S^6 \in \mathbb{G}_S^6$

Central to the discussion is the average Hamiltonian path length in $\mathcal{G}_S^n$ denoted by $T(\mathcal{G}_S^n)$.

Definition 3. For a graph $\mathcal{G}_S^n$, let $S_S(A_j)$ be the sum of arc lengths attached to vertex A_j , $j = 1, \dots, n$. Let $T(\mathcal{G}_S^n) = \frac{1}{n-1} \sum_{j=1}^n S_S(A_j)$.

So $S_S(A_j)$ (Def. 3) and $S_A(A_j)$ (Eq. 2) are the same. That $T(\mathcal{G}_S^n)$ is the average Hamiltonian path length follows from the fact that $\frac{1}{n-1} S_S(A_j)$ is the average length of the $(n-1)$ arcs attached to A_j . Illustrating with Fig. 4, $S_S(A_1) = 1 + 5 + 3 + 4 + 1 = 14$, $S_S(A_2) = 22$, $S_S(A_3) = 22$, $S_S(A_4) = 20$, $S_S(A_5) = 18$, $S_S(A_6) = 24$, so $T(\mathcal{G}_S^6) = \frac{1}{5}(120) = 24$. When considering non-Hamiltonian closed paths, restrict the $S_S(A_j)$ values and the summation defining T to the relevant arcs and vertices.

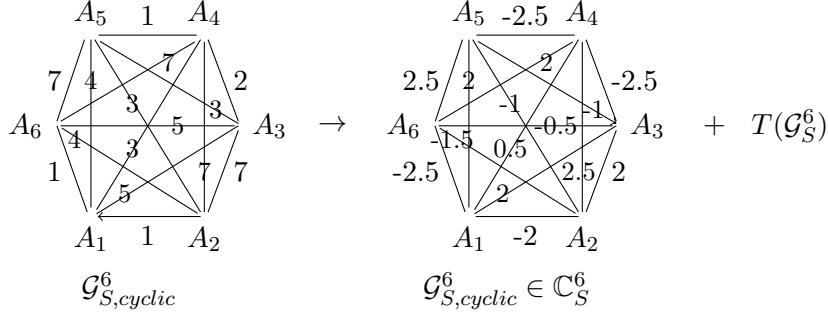


Figure 5. Decomposition of a $\mathcal{G}_S^6 \in \mathbb{G}_S^6$

The scheme replaces entries of $\mathcal{G}_S^n$ with values that, after removing irrelevant terms (which are similar to $\mathcal{G}_{A, cpi}^n$), can be viewed as differences from the average arc length, as in Fig. 5. Thus negative values represent “smaller than average” costs. The theorem is that

the length of a Hamiltonian circuit in $\mathcal{G}_S^n$ is the length of its path in $\mathcal{G}_{S,cycle}^n$ plus $T(\mathcal{G}_S^n)$. Standard modifications handle paths that are not closed and/or incomplete graphs.

Each $\mathcal{G}_{S,cyclic}^6$ vertex has an arc with negative (i.e., below average) length; this assertion holds more generally for any $\mathcal{G}_{S,cyclic}^n$. By observation and using arcs with negative values as often as possible, an Hamiltonian path in $\mathcal{G}_{S,cyclic}^6$ with shortest length of -11 is

$$(17) \quad A_1 \xrightarrow{-2.5} A_6 \xrightarrow{-0.5} A_3 \xrightarrow{-2.5} A_4 \xrightarrow{-2.5} A_5 \xrightarrow{-1} A_2 \xrightarrow{-2} A_1.$$

This defines the shortest Hamiltonian path in $\mathcal{G}_S^6$ with length $24 - 11 = 13$.

In general, the shortest $\mathcal{G}_S^n$ Hamiltonian path is bounded below by $T(\mathcal{G}_S^n)$ plus the sum of the n smallest $\mathcal{G}_{S,cyclic}^n$ arc lengths. With $\mathcal{G}_S^6$, this is $24 + (-2.5 - 2.5 - 2.5 - 2 - 1.5 - 1) = 12$. This shortest Hamiltonian path length is larger than, but close to, its lower bound.

Although the $\mathcal{G}_S^6$ entries can be random numbers, the advantage of the $\mathcal{G}_{S,cyclic}^6$ entries is that each entry has a meaning with respect to a path's length. Thus, a way to find the shortest Hamiltonian path is to rank the arcs according to length, where "smaller is better." Start with the first n arcs to determine whether they form a path. If not, then iteratively add arcs to see whether it creates a Hamiltonian path. To illustrate, the $\mathcal{G}_{S,cyclic}^6$ arcs with negative values are

$$(18) \quad \begin{array}{c|ccc|ccc} \text{Length} & \text{Arc} & \text{Arc} & \text{Arc} & \text{Length} & \text{Arc} & \text{Arc} \\ \hline -2.5 & \widehat{A_1 A_6}, & \widehat{A_3 A_4}, & \widehat{A_4 A_5} & -2 & \widehat{A_1 A_2} & \\ -1.5 & \widehat{A_2 A_6} & & & -1 & \widehat{A_3 A_5}, & \widehat{A_2 A_5} \\ -0.5 & \widehat{A_3 A_6} & & & & & \end{array}$$

This array emphasizes using $\widehat{A_1 A_6}$, $\widehat{A_3 A_4}$, $\widehat{A_4 A_5}$, where $\widehat{A_3 A_6}$ forms a transition between the first and the second arc, and $\widehat{A_5 A_2}$ includes the missing A_2 , while $\widehat{A_2 A_1}$ completes the journey. This is the Eq. [17](#) tour.

$\mathbb{C}_A^n$: TSP and graphs

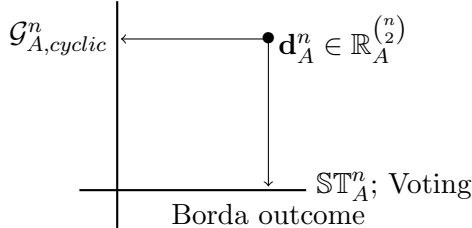


Figure 6. Resolutions via projections

4. CONCLUSION

Figure 6 summarizes much of what was discussed. Namely, both the voting problem and TSP can be described in terms of a point $\mathbf{d}_A^n \in \mathbb{R}_A^{(n)}$. This space has an orthogonal decomposition into the strongly transitive linear subspace $\mathbb{ST}_A^n$ and the linear subspace $\mathbb{C}_A^n$

consisting of three-cycles. The $\mathbf{d}_{A,st}^n$ component of $\mathbf{d}_A^n$ in the $\mathbb{ST}_A^n$ direction resolves several concerns from voting theory, but frustrates the analysis of paths in TSP. Conversely, $\mathbf{d}_A^n$'s component in the $\mathbb{C}_A^n$ direction, $\mathbf{d}_{A,cyclic}^n$, determines all closed path properties for the TSP, but erects barriers and paradoxes for voting theory. A resolution is to project $\mathbf{d}_A^n$ into the appropriate subspace that has positive properties for the problem being considered.

To introduce the rest of the material in this concluding section, let me brag about my two granddaughters. Heili has a deep interest in neurobiology and gymnastics, while Tatjana is heavily involved in the social sciences and ballet. Who is older? It is impossible to tell from this information; comparing ages requires having relevant inputs. More generally, to realize any specified objective, for the data to be useful; they must, in some manner, be directed toward the stated purpose. This truism extends to voting theory and the TSP.

To see this with Arrow's result, return to the introductory voting example where the alternatives are three cities. Suppose the 25 voters with the $A_1 \succ A_2 \succ A_3$ ranking judge cities according the ease of finding parking, the 23 who prefer $A_2 \succ A_3 \succ A_1$ evaluate cities according to the popularity of local professional basketball teams, and the 22 preferring $A_3 \succ A_1 \succ A_2$ assess them according to available types of craft beer. Thus the $A_1 \succ A_2$ outcome by 47:23 is a conglomeration of attitudes about parking, basketball, and beer. Of importance, *nothing* is directed toward Arrow's explicitly stated objective of having a transitive ranking. This means (with these apple and orange comparisons) that rather than a surprise, non-transitive outcomes should be anticipated. Indeed, expect a transitive outcome *only if* the inputs contribute toward this transitivity target. This comment leads to the decomposition (Fig. 6); the $\mathbf{d}_{A,cyclic}^n$ component of $\mathbf{d}_A^n$ that runs counter to the stated objective is identified and dismissed. This leads to the approach described in the first paragraph of the proof of Thm. [5](#), which is called the "summation method" in [8](#).

Another way to handle this problem is to express the inputs, the preferences of voters, in terms of the transitivity objective. To do so, describe the $A_1 \succ A_2 \succ A_3$ ranking as $(A_1 \succ A_2, 1)$, $(A_1 \succ A_3, 2)$ and $(A_2 \succ A_3, 1)$, which restates this ranking in terms of pairs that now are in a strongly transitive format. Here $d_{1,2} = 1$ captures that A_1 is ranked one spot above A_2 , while $d_{1,3} = 2$ means that A_1 is ranked two spots above A_3 . Choosing the triplet (A_1, A_2, A_3) requires checking whether $d_{1,2} + d_{2,3} = 1 + 1$ equals $d_{1,3} = 2$, which it does. By using this strongly transitive approach, which is called "Intensity of IIA" or IIIA (introduced in [8](#)), the profile information for $\{A_1, A_3\}$ has 25 voters with $(A_1 \succ A_3, 2)$, 23 with $(A_1 \succ A_3, -1)$ and 22 with $(A_1 \succ A_3, -1)$. The IIIA tally for $\{A_1, A_3\}$ sums the products of the number of voters with each ranking times its intensity. Here the tally is $25(2) + 23(-1) + 22(-1) = 5$ leading to $A_1 \succ A_3$, rather than the above $A_3 \succ A_1$ (Eq. [2](#)) that forced a cycle. Even stronger, the IIIA tallies over the three pairs not only are transitive, they are strongly transitive.

With any number of alternatives, the IIIA outcome for the majority vote always is strongly transitive. It must be because $\mathbb{ST}_A^n$ is a linear subspace, and the outcome is a summation of strongly transitive profiles. This IIIA method is equivalent to the Borda Count [8](#); a conclusion that should be anticipated (particularly with Thm. [6](#)). As shown in [8](#), by replacing IIA with IIIA, Arrow's Theorem now has a positive conclusion. That is,

by using data that is consistent with the objective of transitivity, the problems of Arrow's Theorem disappear.

Similarly with TSP, each difference from the average cost between two vertices, as catalogued with $\mathcal{G}_A^6$ (Fig. 1), could be based on different attributes; e.g., the $d_{1,2}$ difference between A_1 and A_2 might reflect the topography while $d_{2,3}$ between A_2 and A_3 could be caused by traffic restrictions. As the objective concerns path lengths of closed curves, the goal must be to use inputs that contribute to the specified goal. Here the strongly transitive component $\mathbf{d}_{A,cpi}^n$ pushes the outcome in a linear fashion that conflicts with the goal of having circular paths. Dismissing this $\mathbf{d}_{A,cpi}^n$ term (as indicated in Fig. 6) leaves $\mathbf{d}_{A,cyclic}^n$ data, which are consistent with the global objective of finding closed path properties.

5. PROOFS

Beyond proving Thm. 3, an intent of this section is to demonstrate the above tools.

Proof of Thm. 3. Assume that the paired rankings defined by $\mathbf{d}^n$ have A_1 and A_n , respectively, the unique Condorcet winner and loser. As A_1 is the unique Condorcet winner, the associated $\mathbf{d}_A^n$ must have $d_{1,k} > 0$ for $k = 2, \dots, n$. Similarly, as A_n is the unique Condorcet loser, it must be that $d_{n,k} < 0$ for all $k = 1, \dots, n-1$. From Eq. 2, this means that $S_A(A_1) > 0$ and $S_A(A_n) < 0$. According to Eq. 15, A_1 is Borda ranked above A_n . As the Borda ranking is the $\mathbf{d}_{A,st}^n$ ranking, this proves the theorem. Notice, the Condorcet uniqueness conditions are unnecessary conveniences. $\square$

It remains to show for $n = 3, 4$ that if A_1 and A_n are, respectively, the top and bottom ranked alternative for $\mathbf{d}_{A,st}^n$, and if $\mathbf{d}_A^n = \mathbf{d}_{A,st}^n + \mathbf{d}_{A,cyclic}^n$ is transitive, then $A_1 \succ A_n$. The approach uses the fact, which follows from strong transitivity, that $d_{1,n}$ for $\mathbf{d}_{A,st}^n$ is an upper bound for all other $d_{s,u}$ values. This size of $d_{1,n}$ requires the cyclic perturbations to affect and reverse smaller $d_{u,s}$ values, which violates transitivity before they can impact on the $d_{1,n}$ term to reverse $A_1 \succ A_n$. Assume the $\mathbf{d}_{A,st}^n$ ranking is $A_1 \succ A_2 \succ \dots \succ A_{n-1} \succ A_n$.

The proof is immediate for $n = 3$. All cyclic $n = 3$ vectors are α multiples of $(1, -1; 1)$, so $\mathbf{d}_A^3 = \mathbf{d}_{A,st}^3 + \mathbf{d}_{A,cyclic}^3 = (d_{1,2} + \alpha, d_{1,3} - \alpha; d_{2,3} + \alpha)$. If $\alpha \geq 0$, then the $A_1 \succ A_2$ and $A_2 \succ A_3$ rankings remain unchanged. The $A_1 \succ A_3$ ranking persists as long as $d_{1,3} - \alpha > 0$. As soon as $\alpha > d_{1,3}$, the $A_1 \succ A_3$ ranking reverses to become $A_3 \succ A_1$, which converts the set of pairwise rankings from transitive to cyclic.

For all $\alpha < 0$ values, $A_1 \succ A_3$. But the system becomes cyclic as soon as $-\alpha$ equals the second largest of $\{d_{1,2}, d_{2,3}\}$. For instance, suppose $d_{1,2} < d_{2,3}$. For $\alpha = -d_{1,2}$, the rankings are $A_1 \succ A_3, A_2 \succ A_3$, and $A_2 \sim A_1$ defining the transitive $A_1 \sim A_2 \succ A_3$. But once $\alpha = -d_{2,3}$, the rankings are the non-transitive $A_1 \succ A_3, A_3 \sim A_2, A_2 \succ A_1$. Stated differently, if $\mathbf{d}_A^3$ has a transitive ranking, then $A_1 \succ A_3$, which proves the assertion.

The theme of the proof for $n \geq 4$ is that, because of strong transitivity, $d_{1,n}$ from $\mathbf{d}_{A,st}^n$ is so large that before the cyclic components can reverse the $A_1 \succ A_n$ ranking, they change enough rankings of other pairs to define a cyclic outcome that violates the transitivity of $\mathbf{d}_A^n$. One approach follows:

- (1) Assume that the cyclic terms force $A_n \succ A_1$. It must be shown that the pairs do not define a transitive outcome.
- (2) The size of the perturbations of cyclic terms that accompany the $A_n \succ A_1$ assumption requires some other alternative, say A_k , to satisfy $A_k \succ A_n$. If $\mathbf{d}_A^n$ is transitive, as assumed, then $A_k \succ A_n \succ A_1$, and $A_k \succ A_1$.
- (3) The size of the cyclic perturbations that cause $A_k \succ A_1$ forces some other alternative, A_j , to satisfy $A_j \succ A_k$. This requires $A_j \succ A_k \succ A_n \succ A_1$, or $A_j \succ A_1$.
- (4) The next step is to show that this $A_j \succ A_1$ ranking drives the size of certain cyclic terms to be large enough so that $A_n \succ A_j$.
- (5) Step 4 is the sought contradiction. The $A_j \succ A_n$ ranking follows from the assumption of the transitivity of $\mathbf{d}_A^n$, and the conflicting $A_n \succ A_j$ is a direct result of computations based on the impacts of the cyclic terms. Thus the assumption that $\mathbf{d}_A^n$ is transitive is false, which proves the conclusion.

To illustrate this program with $n = 4$, all of the following $d_{i,j}$ values come from $\mathbf{d}_{A,st}^4$. Assume that $\mathbf{d}_A^4$ is transitive. Using the Thm. 2 basis for $\mathbb{C}_A^4$, where $\alpha_{j,k}$ is the coefficient for $\mathbf{d}_{1,j,k}^4$, a representation for $\mathbf{d}_A^4 = \mathbf{d}_{A,st}^4 + \mathbf{d}_{A,cyclic}^4$ is

$$(19) \quad \mathbf{d}_A^4 = (d_{1,2} + \alpha_{2,3} + \alpha_{2,4}, d_{1,3} - \alpha_{2,3} + \alpha_{3,4}, d_{1,4} - \alpha_{2,4} - \alpha_{3,4}; d_{2,3} + \alpha_{2,3}, d_{2,4} + \alpha_{2,4}, d_{3,4} + \alpha_{3,4}).$$

According to Eq. [19](#), if $A_4 \succ A_1$, then $d_{1,4} - \alpha_{2,4} - \alpha_{3,4} < 0$, or $0 < d_{1,4} < \alpha_{2,4} + \alpha_{3,4}$. This forces one or both of $\alpha_{3,4}$, $\alpha_{2,4}$ to be positive.

Step 2. Start with the assumption that $\alpha_{3,4} > 0$, which requires from Eq. [19](#) that $A_3 > A_4$. From the transitivity of $\mathbf{d}_A^4$, this requires $A_3 > A_1$, or $d_{1,3} - \alpha_{2,3} + \alpha_{3,4} < 0$, which leads to the inequality

$$(20) \quad 0 < d_{1,3} + \alpha_{3,4} < \alpha_{2,3}.$$

Step 3. Combining $\alpha_{2,3} > 0$ (Eq. [20](#)) with $\mathbf{d}_A^4$'s transitivity (and Eq. [19](#)), requires $A_2 > A_3 > A_4 > A_1$. The $A_2 > A_1$ ranking forces (Eq. [19](#)) $\alpha_{2,3} + \alpha_{2,4} < -d_{1,2} < 0$, which, with Eq. [20](#), becomes

$$(21) \quad \alpha_{2,4} < -d_{1,2} - \alpha_{2,3} < -d_{1,2} - d_{1,3} - \alpha_{3,4} < 0.$$

Step 4. As $\alpha_{2,4} < 0$ (Step 3), it follows from $A_4 > A_1$ and $d_{1,4} - \alpha_{2,4} - \alpha_{3,4} < 0$ that

$$(22) \quad \alpha_{3,4} > d_{1,4} \geq d_{j,k}.$$

The last inequality follows because $d_{1,4}$ is the largest $d_{j,k}$ value (strong transitivity).

Step 5. By using Eq. [21](#), the computation for the $\{A_2, A_4\}$ ranking depends on the sign of $d_{2,4} + \alpha_{2,4} < -d_{1,2} - d_{1,3} + [-\alpha_{3,4} + d_{2,4}]$. Because $-\alpha_{3,4} + d_{2,4} < 0$ (Eq. [22](#)), it follows that $d_{2,4} + \alpha_{2,4} < 0$, which is $A_4 \succ A_2$ and the desired contradiction.

The remaining case of $\alpha_{2,4} > 0$, $\alpha_{3,4} \leq 0$ is simpler because from $A_4 \succ A_1$ (Eq. [19](#))

$$(23) \quad \alpha_{2,4} > d_{1,4} \geq d_{j,k}.$$

Step 2. According to Eq. [19](#), the $\alpha_{2,4} > 0$ inequality and the transitivity of $\mathbf{d}_A^4$ mandates $A_2 > A_4 > A_1$, or from $A_2 \succ A_1$ that $d_{1,2} + \alpha_{2,3} + \alpha_{2,4} < 0$, or that $\alpha_{2,3} < -d_{1,2} - \alpha_{2,4} < 0$.

Step 3. Using this inequality and Eq. [23](#), the $\{A_2, A_3\}$ ranking equation is determined by $d_{2,3} + \alpha_{2,3} < -d_{1,2} + [d_{2,3} - \alpha_{2,4}] < 0$, or $A_3 > A_2 > A_4 > A_1$.

Step 4. According to the $A_3 > A_1$ ranking, $d_{1,3} - \alpha_{2,3} + \alpha_{3,4} < 0$ or (Step 2) $d_{1,3} + \alpha_{3,4} < \alpha_{2,3} < -d_{1,2} - \alpha_{2,4} < 0$. What follows uses the inequality $\alpha_{3,4} < -d_{1,3} - d_{1,2} - \alpha_{2,4}$.

Step 5. The $\{A_3, A_4\}$ ranking depends on the sign of $d_{3,4} + \alpha_{3,4} < -d_{1,2} - d_{1,3} + [-\alpha_{2,4} + d_{3,4}] < 0$. As the term in the brackets is negative, the value is negative, leading to the $A_4 > A_3$ contradiction. This completes the proof for $n = 4$.

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