International Society of the Learning Sciences

Reasoning about Equations with Tape Diagrams: Insights from Math Teachers and College Students

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Abstract: Research has shown that tape diagrams are beneficial for algebra learning. However, it is unclear whether certain visual features of tape diagrams have implications for learning. We investigated, with undergraduate students and math teachers, whether tape diagrams with different visual features (color, presence of outer lines, and position of the constant) differentially support reasoning about equations and whether people have preferences for certain visual features. Variations in visual features did not affect students' or teachers' reasoning accuracy; but each group displayed systematic preferences for most visual features considered. Future research should examine the effects of these visual features on performance while solving equations.

Introduction

Diagrams are a valuable tool for helping students understand mathematical concepts and procedures (e.g., Chu et al., 2017; Murata, 2008). Diagrams are thought to support students in solving problems, understanding structural relationships, and building conceptual understanding (e.g., Larkin & Simon, 1987). As such, curricular standards, such as the Common Core State Standards for Mathematics (CCSSM, 2010), designate diagrams as important learning tools. The CCSSM specifically recommend a type of diagram called tape diagrams (henceforth TDs). In TDs, quantities are depicted in horizontal bars, similar to strips of tape.

One domain in which TDs may be especially useful is algebra. Algebra presents challenges to many students, in part due to an overemphasis in algebra instruction on procedural skills, rather than conceptual understanding (Richland et al., 2012). Past work has suggested that TDs may support students in generating and solving algebraic equations (Booth & Koedinger, 2012) and in transitioning from informal to formal strategies (Nagashima, Bartel et al., 2020). TDs thus appear to hold promise for supporting algebraic thinking.

A close inspection of the literature on TDs indicates that visual features of TDs vary substantially across studies, raising the questions of whether particular visual features influence performance, and whether teachers or learners have preferences for specific visual features. Both questions are important when designing instructional materials, because small variations in design features can lead to large variations in learning (e.g., Barbieri et al., 2019).

One relevant study sought to analyze the affordances of TDs for student learning using a qualitative, teacher-centered approach termed Pedagogical Affordance Analysis (PAA) (Nagashima, Yang, et al., 2020). This study identified an optimal design for TDs based on substantial input from teachers. In a later classroom experiment, such TDs enhanced students' conceptual knowledge when used in early algebra learning activities (Nagashima, Bartel, et al., 2020). However, it remains unknown whether specific visual features have differential effects on student performance, or whether students' preferences align with the preferences of teachers. Understanding students' and teachers' preferences may inform researchers and practitioners who design instructional materials. Further, it is possible that a misalignment between the preferences of students and teachers could lead to misconceptions or student disengagement.

In the present investigation, we address three corresponding research questions: (1) What visual features of TDs help undergraduate students and math teachers accurately symbolize equations? (2) What visual features of TDs do students and math teachers prefer? (3) Do students and math teachers believe that TDs are useful, and do these judgments relate to their performance when working with TDs?

Method

Participants

We collected data with two samples of participants: undergraduate students ($N = 50$) and math teachers in the US ($N = 163$). Undergraduates were recruited through an introductory psychology participant pool at a

Midwestern university; teachers were recruited via postings on social media. Per self-report, the student sample was 56% female and 36% male; 8% declined to report gender. The student sample was 78% White and 20% Asian; 2% declined to report race or ethnicity. Students had varying levels of math knowledge, with 18% of students having never taken Calculus, 68% of students having taken one or two semesters of Calculus, and 10% having taken beyond two semesters of Calculus; 2% declined to report. The teacher sample was 49% female and 49% male; 2% declined to report gender. The teacher sample was 50% White, 15% Hispanic/Latinx, 14% Black/African American, 8% Asian, 5% Native American, 2% Native Hawaiian/Pacific Islander, and 4% multiple races/ethnicities; 2% declined to report race or ethnicity. On average, teachers had 9.4 years of experience (*SD* = 7.1). 121 (74%) of the teachers reported having seen TDs in the past, and 113 (69%) reported using diagrams similar to TDs in their teaching. Of the teachers, 48% taught grades 1-5, 31% taught grades 6-8, 11% taught grades 9-12, and 8% taught multiple grades, with 2% teaching other grades or declining to report.

Design and Procedure

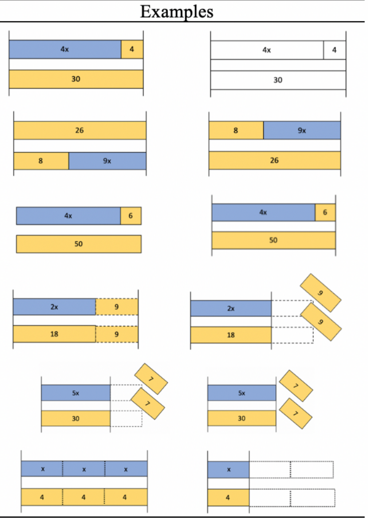
Each participant completed an online survey via a web-based platform. In the survey, participants were presented with three tasks: Symbolization, Preference Ratings, and Ratings of Perceived Usefulness.

Symbolization Task: Participants were presented with TDs and were asked to generate corresponding equations. For example, participants might be presented with the TD in the first row of Table 1 and be asked to generate the corresponding equation (the correct answer is $4x + 4 = 30$). The diagrams varied in visual features, including whether or not they used color, whether they included vertical lines, and whether the constant was on the top or bottom tape. Responses were coded as correct or incorrect. Undergraduates were asked to symbolize six one-operation and six two-operation equations; teachers were asked to symbolize four equations of each type.

Preference Ratings: Participants were then shown 12 pairs of TDs that varied by one feature and were asked to choose which TD best represented the equation (see Table 1). Following this task, participants responded to a set of general questions regarding their preferences (e.g., "What features did you like/dislike about the tape diagrams you just saw?"). Responses were coded for the specific visual features and for whether they noted *aesthetic* or *conceptual* reasons for liking or disliking a feature (see Table 2).

Table 1. Variations in visual features of diagrams.

Equation	Feature	Examples
$4x + 4 = 30$	Color	
$8 + 9x = 26$	Constant Location	
$4x + 6 = 50$	Line	
$2x + 9 – 9 = 27 - 9$	Dynamic: Subtraction	
$5x + 7 – 7 = 37 -7$	Phantom Dashed: Subtraction	
$3x = 12$	Full Dashed vs. Phantom Dashed: Division	



Ratings of Perceived Usefulness: Participants responded to questions about the perceived usefulness of TDs on a five-point Likert scale (Strongly Disagree to Strongly Agree) (e.g., "Tape diagrams helped me represent the equations"). They also responded to three questions regarding whether they thought younger students would find TDs helpful or not helpful (e.g., "Tape diagrams would help younger students represent equations").

Results

We first analyzed whether specific visual features influenced participants' abilities to generate equations to correspond with TDs. Success on the one-operator items was at ceiling, so we focus here on the two-operator items. Based on previous research (Nagashima, Yang et al., 2020), we selected the "baseline" TD as including color, including outer lines, and having the constant on the bottom. We then compared each of the variants to this baseline. Thus, the TDs were categorized into four groups: baseline, baseline without color, baseline without outer lines, and baseline with the constant in the top tape.

Undergraduates performed similarly at symbolizing all four types of TDs (M = .69, SD = .06), F(3, 196) = .54, p = .65. Likewise, teachers also performed similarly at symbolizing all four types of TDs, (M = .44, SD = .04), F(3, 486) = .83, p = .47. Thus, variations in visual features did not affect symbolization accuracy.

We next analyzed whether participants preferred specific visual features. We evaluated the same features used in the symbolization task, as well as additional features that represented intermediate steps in equation solving (Table 1). Descriptively, both students and teachers had general, group-level preferences for four of the six features: use of color, having the constant in the bottom tape, using full dashed lines to represent division, and using phantom dashed lines to represent the removal of break-away pieces (Figure 3). The only feature for which there was no clear preference in either group was representing subtraction with dynamic break-away pieces vs. a static representation.

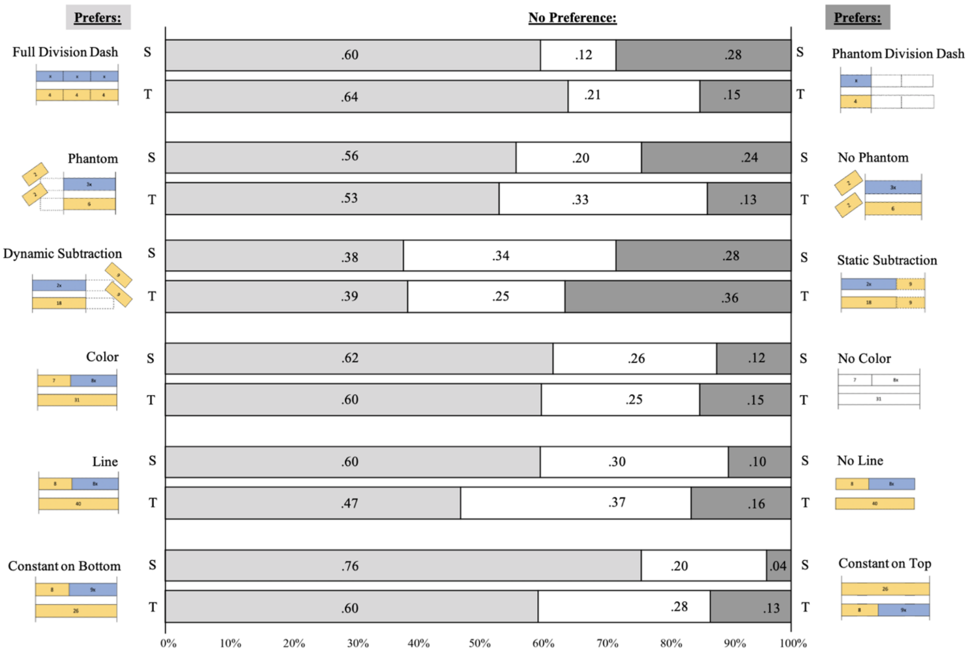


Figure 1. Preferences for TD visual features as a function of specific features and whether the participant was a student or teacher. "No preference" ratings indicate participants who were inconsistent in their ratings (e.g., stating that they preferred color on the first trial and no color on the second trial).

Table 2 presents data on participants' responses to the open-ended questions (e.g., "What features did you like about the tape diagrams you just saw? Be as specific as possible"). Descriptively, across groups, the majority of responses focused on either color/no color or line/no line. Students tended to provide aesthetic reasons for their preferences, while teachers tended to provide conceptual reasons. Teachers mentioned the representation of intermediate steps (e.g., dynamic subtraction) more frequently than students.

Table 2. Number of students (S) and teachers (T) who provided aesthetic and conceptual reasons for their preferences for specific types of TDs.

	Aesthetic			Conceptual		
	Like	Dislike	Example	Like	Dislike	Example
Color v. No Color	S: 20 T: 5	S: 6 T: 7	"I liked the different colors used…"	S: 8 T: 17	S: N/A T: 11	"I liked the colors to distinguish constants from expressions with x"
Line v. No Line	S: 8 T: 1	S: N/A T: 1	"I like the bars on the ends"	S: 4 T: 12	S: 1 T: 1	"I also liked the vertical lines to show clearly that the two values were equal"

Finally, we asked whether participants viewed TDs as useful for supporting equation solving. For each sample, we regressed the percent of equations that participants accurately symbolized on their self-reported usefulness of TDs score. For both samples, the more useful participants perceived the TDs to be, the higher their accuracy at symbolizing (students: $F(1,48) = 6.03$, $p = .017$, $R2 = .11$; teachers: $F(1,161) = 87.17$, $p < .001$, $R2 = 35$). Undergraduates rated TDs as more useful for younger students than for themselves, $t(49) = 4.42$, $p < .001$, whereas teachers rated TDs as equally useful for younger students and themselves , $t(163) = .24$, $p = .81$.

General discussion

Both students and teachers preferred TDs that used color, represented the constant in the bottom tape, used full dashed lines to represent division, and used phantom dashed lines to represent removing break-away pieces in subtraction. A smaller majority preferred outer lines. Participants based these preferences on aesthetic and conceptual grounds, with teachers' preferences being primarily conceptually based. Variations in visual features did not affect undergraduates' or teachers' success at generating equations to correspond to the TDs.

For teachers who wish to use TDs in instruction, this information about preferred features may be useful. Building on their pedagogical content knowledge, teachers may integrate the preferred TDs in lessons, for example, to illustrate solution procedures or to support self-explanation activities. Using TDs with preferred visual features may promote students' enjoyment of and engagement with the material.

In both samples, success at symbolizing TDs was associated with the perceived usefulness of the diagrams. Some participants may have understood the conventions upon which TDs are based better than others, and this third variable may have given rise to the observed association. Participants who viewed TDs as useful also tended to judge that they would be useful for younger learners. Future studies should investigate younger students' preferences for visual features and should examine whether visual features differentially support learning and symbolization in younger students.

In sum, this study revealed systematic preferences for visual features of tape diagrams in both teachers and undergraduates. These preferences can inform the design of curricular materials that involve such diagrams.

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Acknowledgments

This research was supported by NSF Award #1760922 and by the Institute of Education Sciences, U.S. Department of Education, through Award #R305B150003 to the University of Wisconsin-Madison. The opinions expressed are those of the authors and do not represent views of NSF or the U.S. Department of Education.