CONTINUITY OF A CLASS OF FBI TRANSFORMS ON SOBOLEV SPACES

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Abstract. We show that a subclass of the generalized FBI transforms that were introduced in the work [11] are bounded on Sobolev spaces.

1. Introduction

The classical FBI transform is a nonlinear transform which has the form

$$\mathcal{F}u(x,\xi)=\int_{\mathbb{R}^m} e^{i.\xi.(x-y)-|\xi||x-y|^2}u(y)\,dy,\ x,\xi\in\mathbb{R}^m \tag{1.1}$$

where u is a continuous function of compact support in $\mathbb{R}^m$ or a distribution of compact support in which case the integral is understood in a duality sense. This transform characterizes microlocal analyticity [22], microlocal smoothness [23], and microlocal Gevrey regularity [12]. It has been used extensively to study the local and microlocal regularity of solutions of linear and nonlinear partial differential equations. Among the numerous works where (1.1) or a variant have been used, we mention [1], [2], [3], [4], [5], [8], [6], [7], [12], [14], [17], [18], [19], [21], and [22].

In [22] (see also [15]) a more general FBI transform was considered where the phase function behaved much like the quadratic phase $i\xi.(x-y)-|\xi||x-y|^2$ in that the real part of the Hessian was required to be negative definite.

The work [11] introduced a class of FBI transforms where the real part of the Hessian of the phase function was allowed to degenerate. Examples of such transforms include, for each $k=2,\ldots$

$$\mathcal{F}_k u(x,\xi)=\int_{\mathbb{R}^m} e^{i\xi.(x-y)-|\xi||x-y|^{2k}}u(y)dy,\ x,\xi\in\mathbb{R}^m, \tag{1.2}$$

Note that when $k>1$, these transforms have a Hessian that degenerates at the origin. The more general FBI transforms of [11] were shown to characterize microlocal analyticity and microlocal smoothness. They also characterize microlocal regularity of Gevrey functions (see [9] and [17]).

This article establishes the boundedness of the transforms (1.2) on Sobolev spaces. The case when $k=1$ was treated in the work [10]. We mention that the transform (1.2) for $k=2$ was applied in the works [11] and [17] to prove microlocal CR regularity.

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2. Statements of results

For each $k = 1, 2, 3, \dots$, we consider the generalized FBI transform $\mathcal{F}_k(u, x, \xi)$ defined for $u \in \mathcal{E}'(\mathbb{R}^m)$ by

$$\mathcal{F}_k(u, x, \xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-y)-|\xi||x-y|^{2k}} u(y)dy$$

where the integral is understood to be in the duality sense when u is a distribution.

Theorem 2.1. *Let $\Omega' \subset\subset \Omega \subseteq \mathbb{R}^m$ be open sets, Ω bounded. Then for any $u \in \mathcal{E}'(\Omega')$,*

(a) $||u||^2_{H^t}$

$$\leq C\left(\int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u, x, \xi)|^2 |\xi|^{\frac{m}{2}} (1 + |\xi|^2)^{t + \frac{m-mk}{4k}} \, d\xi dx + ||u||^2_{H^{t-\frac{1}{4}}} \right);$$

(b) Conversely,

$$\int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u, x, \xi)|^2 |\xi|^{\frac{m}{2}} (1 + |\xi|^2)^{t + \frac{m-mk}{4k}} \, d\xi dx \leq C||u||^2_{H^t},$$

where in both (a) and (b), the constant C is independent of u.

We recall that a function $a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ is said to belong to the symbol class $S^k_{\rho,\delta}$ if for every pair of multi-indices α, β and every compact subset $K \subset \Omega$,

$$|\partial^\beta_x \partial^\alpha_\xi a(x, \xi)| \leq (1 + |\xi|)^{k - \rho|\alpha| + \delta|\beta|}.$$

Given a symbol $a(x, \xi) \in S^k_{\rho,\delta}$, the corresponding pseudodifferential operator $A(x, D) \in \Psi^k_{\rho,\delta}$ is defined by

$$A(x, D)u(x) = \int_{\mathbb{R}^m} \int_\Omega e^{i(x-y)\cdot\xi} a(x, \xi) u(y) \, dy d\xi, \quad u \in \mathcal{E}'(\Omega).$$

If $u \in \mathcal{E}'(\Omega)$, one says the point $(x_0, \xi^0) \in \Omega \times \mathbb{R}^m \setminus 0$ is not in the H^s wavefront set of u (denoted $(x_0, \xi^0) \notin \mathrm{WF}_s(u)$) if for some $\varphi(x) \in C^\infty_0(\Omega)$, $\varphi(x_0) \neq 0$, and an open cone $\Gamma \subset \mathbb{R}^m$ with vertex at the origin and containing ξ^0,

$$\int_\Gamma |\widehat{\varphi u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty.$$

It is well known that $(x_0, \xi^0) \notin \mathrm{WF}_s(u)$ if and only if whenever $P(x, D)$ is an elliptic pseudodifferential operator of order zero whose support is in a conic neighborhood of (x_0, ξ^0), $P(x, D)u \in H^s$. The following theorem is a microlocal version of Theorem 2.1.

Theorem 2.2. *Let $(x_0, \xi^0) \in \mathbb{R}^m \times \mathbb{R}^m \setminus \{0\}$ and $p(x, \xi) \in S^0_{1,0}$, with support in a conic neighborhood $\Omega_1 \times \Gamma$ of (x_0, ξ^0), $\Omega_1 \subset\subset \Omega$. Then for any $u \in \mathcal{E}'(\Omega')$ ($\Omega_1 \subset\subset \Omega' \subset\subset \Omega$), there exist constants $C_1, C_2 > 0$ independent of u such that:*

(a) $||P(x, D)u||^2_{H^t}$

$$\leq C_1\left(\int_{\mathbb{R}^m} \int_\Omega |\mathcal{F}_k(u, x, \xi)|^2 |p(x, \xi)|^2 |\xi|^{\frac{m}{2}} (1 + |\xi|^2)^{t + \frac{m-mk}{4k}} \, dx d\xi + ||u||^2_{H^{t-\frac{1}{4}}} \right);$$

$$(b)\ \int_{\mathbb{R}^m}\int_{\Omega}|\mathcal{F}_k(u,x,\xi)|^2|p(x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^{t+\frac{m-mk}{4k}}dxd\xi$$
$$\leq C_2\bigg(||P(x,D)u||^2_{H^t}+||u||^2_{H^{t-\frac{1}{4}}}\bigg).$$

3. Proofs of Theorems 2.1 & 2.2

Proof of Theorem 2.1. Observe that

$$|\mathcal{F}_k(u,x,\xi)|^2=\int_{\Omega}\int_{\Omega}e^{i\xi.(s-y)-|\xi|(|x-y|^{2k}+|x-s|^{2k})}u(y)\overline{u(s)}dyds,$$

which leads to

$$\begin{aligned}&\int_{\Omega}\int_{\mathbb{R}^m}|\mathcal{F}_k(u,x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi dx\\&=\int_{\Omega}\int_{\mathbb{R}^m}\int_{\Omega}\int_{\Omega}e^{i\xi.(s-y)-|\xi|(|x-y|^{2k}+|x-s|^{2k})}|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi dx\\&=\int_{\mathbb{R}^m}\int_{\Omega}\int_{\Omega}e^{i\xi.(s-y)}q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi,\end{aligned}$$

where

$$q(y,s,\xi)=\int_{\Omega}e^{-|\xi|(|x-y|^{2k}+|x-s|^{2k})}\,dx.$$

Let $Q(y,s,\xi)=q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s$. We will show that $q(y,s,\xi)\in S^{\frac{-m}{2k}}_{1,\frac{1}{2k}}$ and that for any $\Omega'\subseteq\Omega''\subset\subset\Omega$, there exist $c,b>0$ such that

$$Q(y,y,\xi)\geq c|\xi|^{\frac{m}{2}-\frac{m}{2k}}(1+|\xi|^2)^s\text{ for }y\in\Omega'',|\xi|\geq b.$$

We have

$$\begin{aligned}Q(y,y,\xi)&=\bigg(\int_{\Omega}e^{-2|\xi||x-y|^{2k}}dx\bigg)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\\&=\bigg(\int_{\Omega\setminus y}e^{-2|\xi||t|^{2k}}dt\bigg)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s.\end{aligned}$$

Since $0\in\Omega\setminus y$ for each $y\in\Omega''$, by compactness, there exists $\delta>0$ such that the ball $\mathbb{B}_\delta(0)\subseteq\Omega\setminus y$ for every $y\in\Omega''$. Hence $\mathbb{B}_1(0)\subseteq\frac{1}{\delta}(\Omega\setminus y)$ for every $y\in\Omega''$ and therefore, $\mathbb{B}_1(0)\subseteq|\xi|^{\frac{1}{2k}}(\Omega\setminus y)$ for every $y\in\Omega''$ and all $\xi\in\mathbb{R}^m$ satisfying $|\xi|\geq\frac{1}{\delta^{2k}}$. It follows that for any $y\in\Omega'',\xi\in\mathbb{R}^m,|\xi|\geq\frac{1}{\delta^{2k}}$,

$$\begin{aligned}\int_{\Omega\setminus y}e^{-2|\xi||t|^{2k}}dt&=\frac{1}{|\xi|^{\frac{m}{2k}}}\int_{|\xi|^{\frac{1}{2k}}(\Omega\setminus y)}e^{-2|v|^2}dv\\&\geq\frac{1}{|\xi|^{\frac{m}{2k}}}\int_{\mathbb{B}_1(0)}e^{-2|v|^{2k}}dv\\&=\frac{c}{|\xi|^{\frac{m}{2k}}},\ c>0,\end{aligned}$$

and so for such y and ξ,

$$Q(y,y,\xi)\geq c|\xi|^{\frac{m}{2}-\frac{m}{2k}}(1+|\xi|^2)^s.$$

Set $b=\frac{1}{\delta^{2k}}$, and let α,β,γ be multi-indices. We will show that there is a constant $C_{\alpha,\beta,\gamma}>0$ such that

$$\left|\partial_\xi^\alpha\partial_y^\beta\partial_s^\gamma q(y,s,\xi)\right|\le C_{\alpha,\beta,\gamma}(1+|\xi|)^{\frac{-m}{2k}-|\alpha|+\frac{1}{2k}(|\beta|+|\gamma|)}$$

for y and s in compact subsets and $|\xi|\ge b$.
Consider first $\partial_\xi^\alpha q(y,s,\xi)$:
Let $h(x,y,s)=|x-y|^{2k}+|x-s|^{2k}$, $F(r)=e^{-rh(x,y,s)}$ and $g(\xi)=|\xi|$. Then $e^{-|\xi|h(x,y,s)}=F(g(\xi))$. To estimate $\partial_\xi^\alpha F\circ g(\xi)$, we will use the multivariate version of the formula of Faá di Bruno which says that (see [13])

$$\partial_\xi^\alpha F\circ g(\xi)=\sum_{1\le\lambda\le|\alpha|}D^\lambda F(g(\xi))\sum_{s=1}^{|\alpha|}\sum_{p_s(\alpha,\lambda)}\alpha!\prod_{j=1}^{s}\frac{(D^{l_j}g)^{k_j}}{k_j!(l_j!)^{|k_j|}},$$

where

$$p_s(\alpha,\lambda)=\{(k_1,\dots,k_s;l_1,\dots,l_s):|k_j|>0,0<l_1\dots<l_s,\sum_{i=1}^{s}k_i=\lambda,\sum_{i=1}^{s}|k_i|l_i=\alpha\}.$$

Here, for two multi-indices $\nu=(\nu_1,\dots,\nu_d)$ and $\mu=(\mu_1,\dots,\mu_d)$, the linear order $\nu<\mu$ means one of the following holds:
(i) $|\nu|<|\mu|$;
(ii) $|\nu|=|\mu|$, and $\nu_1<\mu_1$, or
(iii) $|\nu|=|\mu|$, $\nu_1=\mu_1,\dots,\nu_k=\mu_k$, and $\nu_{k+1}<\mu_{k+1}$ for some $1\le k<d$.
We write $\nu\le\mu$ if $\nu_j\le\mu_j$ for every $1\le j\le d$.
Fix $(k_1,\dots,k_s;l_1,\dots,l_s)\in p_s(\alpha,\lambda)$. Then

$$D^\lambda F(r)=(-1)^\lambda h(x,y,s)^\lambda e^{-rh(x,y,s)},$$

and since $g(\xi)$ is homogeneous of degree 1, the factor $\prod_{j=1}^{s}\frac{(D^{l_j}g)^{k_j}}{k_j!(l_j!)^{|k_j|}}$ is homogeneous of degree $\sum_{j=1}^{s}(1-|l_j|)k_j=\lambda-|\alpha|$. It follows that $\partial_\xi^\alpha q(y,s,\xi)$ is a finite sum of constant multiples of terms of the type

$$\int_\Omega h(x,y,s)^\lambda e^{-|\xi|h(x,y,s)}q_\lambda(\xi)dx,$$

where $q_\lambda(\xi)$ is homogeneous of degree $\lambda-|\alpha|$.
For a multi-index β, we next consider $\partial_y^\beta\partial_\xi^\alpha q(y,s,\xi)$:
From the form of $\partial_\xi^\alpha q(y,s,\xi)$ that we have seen, we only need to consider terms of the form

$$\int_\Omega\partial_y^\beta\left\{h(x,y,s)^\lambda e^{-|\xi|h(x,y,s)}\right\}q_\lambda(\xi)dx,$$

where $q_\lambda(\xi)$ is homogeneous of degree $\lambda-|\alpha|$ and $1\le\lambda\le|\alpha|$. We have

$$\int_\Omega\partial_y^\beta\left\{h(x,y,s)^\lambda e^{-|\xi|h(x,y,s)}\right\}q_\lambda(\xi)\,dx=\sum_{\delta\le\beta}\binom{\beta}{\delta}\int_\Omega\partial_y^{\beta-\delta}\left\{h(x,y,s)^\lambda\right\}\partial_y^\delta e^{-|\xi|h(x,y,s)}q_\lambda(\xi)\,dx.$$

In the latter sum, consider a term

$$\partial_y^{\beta-\delta}\Big\{h(x,y,s)^\lambda\Big\}\partial_y^\delta e^{-|\xi|h(x,y,s)}.$$

Once again we use Faá di Bruno's multivariable formula to compute

$$\partial_t^\delta e^{-|\xi||t|^{2k}} = \partial_t^\delta F(g(t)),$$

where $g(t)=|t|^{2k}$ and $F(r)=e^{-r|\xi|}$. We have

$$\partial_t^\delta e^{-|\xi||t|^{2k}} = \sum_{1\le|\lambda'|\le|\delta|}(D^{\lambda'}F)(g(t))\sum_{s=1}^{|\delta|}\sum_{p_s(\delta,\lambda')}\delta!\prod_{j=1}^{s}\frac{(D^{l_j}g)^{k_j}}{k_j!(l_j!)^{|k_j|}},$$

where $\sum_{i=1}^s k_i=\lambda'$, $\sum_{i=1}^s|k_i|l_i=\delta$, and $0<l_1<\ldots<l_s$.
Fix $\lambda', 1\le|\lambda'|\le|\delta|$ and $(k_1,\ldots,k_s;l_1,\ldots,l_s)\in p_s(\delta,\lambda')$. For each $1\le j\le s$, $D^{l_j}g(t)$ is either 0 or homogenous of degree $2k-|l_j|\ge 0$. Therefore,

$$\prod_{j=1}^{s}\frac{(D^{l_j}g)^{k_j}}{k_j!(l_j!)^{|k_j|}}$$

is either 0 or a homogeneous polynomial of degree $\sum_{j=1}^s k_j(2k-|l_j|)=2k\lambda'-|\delta|$.
Thus $\partial_y^\delta e^{-|\xi||x-y|^{2k}}$ is a constant linear combination of terms of the form

$$g_{\lambda'}(x-y)|\xi|^{\lambda'}e^{-|\xi||x-y|^{2k}},$$

where $g_{\lambda'}$ is either 0 or a homogeneous polynomial of degree $2k\lambda'-|\delta|$. It follows that

$$\begin{aligned}&\int_\Omega\partial_y^{\beta-\delta}\Big\{h(x,y,s)^\lambda\Big\}\partial_y^\delta e^{-|\xi|h(x,y,s)}q_\lambda(\xi)\,dx\\ &=\int_\Omega\partial_y^{\beta-\delta}\Big\{h(x,y,s)^\lambda\Big\}\Big(\partial_y^\delta e^{-|\xi||x-y|^{2k}}\Big)e^{-|\xi||x-s|^{2k}}q_\lambda(\xi)\,dx\end{aligned}$$

is a constant linear combination of terms of the form

$$\int_\Omega\partial_y^{\beta-\delta}\Big\{h(x,y,s)^\lambda\Big\}g_{\lambda'}(x-y)q_\lambda(\xi)|\xi|^{\lambda'}e^{-|\xi|h(x,y,s)}\,dx,$$

where $g_{\lambda'}$ is either 0 or homogeneous of degree $2k\lambda'-|\delta|, 1\le\lambda'\le|\delta|, q_\lambda(\xi)$ homogeneous of degree $\lambda-|\alpha|, 1\le\lambda\le|\alpha|$. The same argument shows that for any multi-index γ, $\partial_\xi^\alpha\partial_y^\beta\partial_s^\gamma q(y,s,\xi)$ is a constant linear combination of terms of the form

$$\int_\Omega\partial_s^{\gamma-\delta'}\partial_y^{\beta-\delta}\Big\{h(x,y,s)^\lambda\Big\}g_{\lambda''}(x-s)g_{\lambda'}(x-y)q_\lambda(\xi)|\xi|^{\lambda'+\lambda''}e^{-|\xi|h(x,y,s)}\,dx,$$

where $g_{\lambda''}$ is either 0 or a homogeneous polynomial of degree $2k\lambda''-|\delta'|\ge 0,\ \ 1\le\lambda''\le|\delta'|,|\delta'|\le|\gamma|,|\delta|\le|\beta|,1\le\lambda''\le|\delta'|$, and $g_{\lambda'}$, and q_λ are as before. Since $h(x,y,s)^\lambda$ is a polynomial of degree $2k\lambda$, we may assume that

$$|\gamma|-|\delta'|+|\beta|-|\delta|\le 2k\lambda.$$

Clearly, for some constant $C>0$,

$$\left|\partial_s^{\gamma-\delta'}\partial_y^{\beta-\delta}\left\{h(x,y,s)^{\lambda}\right\}\right| \leq Ch(x,y,s)^{\lambda-\frac{|\gamma|}{2k}-\frac{|\beta|}{2k}+\frac{|\delta'|}{2k}+\frac{|\delta|}{2k}},$$

and

$$\left|g_{\lambda''}(x-s)g_{\lambda'}(x-y)\right| \leq Ch^{\lambda'-\frac{|\delta|}{2k}+\lambda''-\frac{|\delta'|}{2k}}.$$

Thus

$$\begin{aligned}&\left|\partial_s^{\gamma-\delta'}\partial_y^{\beta-\delta}\left\{h(x,y,s)^{\lambda}\right\}g_{\lambda''}(x-s)g_{\lambda'}(x-y)q_{\lambda}(\xi)|\xi|^{\lambda'+\lambda''}e^{-|\xi|h(x,y,s)}\right|\\ &\leq C_1h(x,y,s)^{\lambda+\lambda'+\lambda''-\frac{|\beta|}{2k}-\frac{|\gamma|}{2k}}|\xi|^{\lambda+\lambda'+\lambda''}|\xi|^{-|\alpha|}e^{-|\xi|h(x,y,s)}.\end{aligned}$$

We claim that we may assume $\lambda+\lambda'+\lambda''-\frac{|\beta|}{2k}-\frac{|\gamma|}{2k}\geq 0$. Indeed, this follows from the fact that unless

$$|\gamma|-|\delta'|+|\beta|-|\delta|\leq 2k\delta,\ 2k\lambda'\geq|\delta| \text{ and } 2k\lambda''\geq|\delta'|,$$

the product

$$\partial_s^{\gamma-\delta'}\partial_y^{\beta-\delta}\left\{h(x,y,s)^{\lambda}\right\}g_{\lambda''}(x-s)g_{\lambda'}(x-y)$$

would be zero. Thus

$$\begin{aligned}&\left|h(x,y,s)^{\lambda+\lambda'+\lambda''-\frac{|\beta|}{2k}-\frac{|\gamma|}{2k}}|\xi|^{\lambda+\lambda'+\lambda''}|\xi|^{-|\alpha|}e^{-|\xi|h(x,y,s)}\right|\\ &=\left(h(x,y,s)|\xi|\right)^{\lambda+\lambda'+\lambda''-\frac{|\beta|}{2k}-\frac{|\gamma|}{2k}}e^{-\frac{|\xi|}{2}h(x,y,s)}|\xi|^{\frac{|\beta|+|\gamma|}{2k}-|\alpha|}e^{-|\frac{|\xi|}{2}|h(x,y,s)}\\ &\leq C_2|\xi|^{\frac{|\beta|+|\gamma|}{2k}-|\alpha|}e^{-\frac{|\xi|}{2}h(x,y,s)}\end{aligned}$$

for some $C_2>0$, where we have used the fact that for any $d\geq 0$, the function t^de^{-t} is bounded on $[0,\infty)$.
It follows that for some constants $C'>0, C>0$,

$$\begin{aligned}\left|\partial_\xi^{\alpha}\partial_y^{\beta}\partial_s^{\gamma}q(y,s,\xi)\right| &\leq C'|\xi|^{\frac{|\beta|+|\gamma|}{2k}-|\alpha|}\int_{\Omega}e^{-\frac{|\xi|}{2}h(x,y,s)}\,dx\\ &\leq C|\xi|^{\frac{|\beta|+|\gamma|}{2k}-|\alpha|-\frac{m}{2k}}.\end{aligned}$$

We have shown that $q(y,s,\xi)\in S^{-\frac{m}{2k}}_{1,\frac{1}{2k}}$. Let $\varphi(\xi)\in C_0^{\infty}(\mathbb{R}^m), \varphi(\xi)\equiv 1$ for $|\xi|\leq\frac{b}{2}$ and $\varphi(\xi)\equiv 0$ for $|\xi|\geq b$. We write

$$\int_{\Omega}\int_{\mathbb{R}^m}|\mathcal{F}_k(u,x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi dx = A_1+A_2,$$

where

$$A_1=\int_{\Omega}\int_{\mathbb{R}^m}|\mathcal{F}_k(u,x,\xi)|^2(1-\varphi(\xi))|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi dx,$$

and

$$A_2=\int_{\Omega}\int_{\mathbb{R}^m}|\mathcal{F}_k(u,x,\xi)|^2\varphi(\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi\,dx.$$

We have

$$\begin{aligned}A_1 &= \int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi\\ &= \langle Tu,u\rangle,\end{aligned}$$

where

$$Tu(s)=\int_{\mathbb{R}^m}\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\,dyd\xi$$

is a pseudodifferential operator in the class $\Psi^{2s+\frac{mk-m}{2k}}_{1,\frac{1}{2k}}$. By the boundedness of pseudodifferential operators in this class ([H]), there exists $C_1>0$ such that

$$A_1\le C_1||u||^2_{H^{s+\frac{mk-m}{4k}}}.$$

The integral A_2 is of the form

$$A_2=\langle Su,u\rangle,$$

where S is a smoothing operator and hence for any $M>0$ there exists $C_M>0$ such that

$$A_2\le C_M||u||^2_{H^{-M}}.$$

It follows that for some $C>0$,

$$\int_\Omega\int_{\mathbb{R}^m}|\mathcal{F}_k(u,x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi dx\le C||u||^2_{H^{s+\frac{mk-m}{4k}}},$$

which establishes part (b) of Theorem 2.1.

To prove part (a), observe that the amplitude of the operator T is

$$B(y,s,\xi)=(1-\varphi(\xi))q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s,$$

and therefore, for any $\Omega'\subseteq\Omega''\subset\subset\Omega$, as we saw before , for some $C>0$,

$$\begin{aligned}B(y,y,\xi)&=(1-\varphi(\xi))q(y,y,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\\ &\ge C(1+|\xi|^2)^{s+\frac{mk-m}{4k}}\ \text{ for } y\in\Omega'',|\xi|\ge b.\end{aligned}$$

Hence by Garding's inequality, there exists $C>0$ such that

$$||u||^2_{H^{s+\frac{mk-m}{4k}}}\le A_1+C||u||^2_{H^{s+\frac{mk-m}{4k}-\frac{1}{4}}}.$$

Therefore, for some $C>0$,

$$||u||^2_{H^{s+\frac{mk-m}{4}}}\le C\Big(\int_\Omega\int_{\mathbb{R}^m}|\mathcal{F}_k(u,x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi dx+||u||^2_{H^{s+\frac{mk-m}{4}-\frac{1}{4}}}\Big),$$

which proves part (a) of Theorem 2.1. □

Proof of Theorem 2.2. We have

$$\int_{\mathbb{R}^m}\int_\Omega |\mathcal{F}_k(u,x,\xi)|^2|p(x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,dxd\xi$$
$$=\int_\Omega\int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)-|\xi|(|x-y|^{2k}+|x-s|^{2k})}|p(x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi dx$$
$$=\int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)}q_1(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi,$$

where

$$q_1(y,s,\xi)=\int_\Omega e^{-|\xi|(|x-y|^{2k}+|x-s|^{2k})}|p(x,\xi)|^2\,dx.$$

For any multi-indices α,β,γ,

$$\partial_s^\gamma\partial_y^\beta\partial_\xi^\alpha q_1(y,s,\xi)=\sum_{\delta\le\alpha}\binom{\alpha}{\delta}\int_\Omega\partial_s^\gamma\partial_y^\beta\partial_\xi^\delta\left(e^{-|\xi|h(x,y,s)}\right)\partial_\xi^{\alpha-\delta}|p(x,\xi)|^2\,dx.$$

We saw in the proof of Theorem 2.1 that for some $C>0$,

$$\left|\partial_s^\gamma\partial_y^\beta\partial_\xi^\delta\left(e^{-|\xi|h(x,y,s)}\right)\right|\le C|\xi|^{-\delta+\frac{|\beta|+|\gamma|}{2k}}.$$

Since $|p(x,\xi)|^2\in\mathbb{S}^0_{1,0}$, for some $C'>0$,

$$\left|\partial_\xi^{\alpha-\delta}|p(x,\xi)|^2\right|\le C'|\xi|^{|\delta|-|\alpha|}$$

and hence

$$(1-\varphi(\xi))q_1(y,s,\xi)\in\mathbb{S}^{-\frac{m}{2k}}_{1,\frac{1}{2k}}.$$

Write

$$\int_{\mathbb{R}^m}\int_\Omega|\mathcal{F}_k(u,x,\xi)|^2|p(x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dxd\xi=A_1+A_2,$$

where

$$A_1=\int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q_1(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi,$$

and

$$A_2=\int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)}\varphi(\xi)q_1(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi.$$

We have $A_1=\langle T_1u,u\rangle$, where

$$T_1u(s)=\int_{\mathbb{R}^m}\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q_1(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\,dyd\xi.$$

We recall from the proof of Theorem 2.1 that

$$Tu(s)=\int_{\mathbb{R}^m}\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\,dyd\xi,$$

where $q(y,s,\xi)=\int_\Omega e^{-|\xi|(|x-y|^{2k}+|x-s|^{2k})}\,dx$. Write

$$P(x,D)u(x)=\int_{\mathbb{R}^m}\int_\Omega e^{i\xi.(s-y)}p(x,\xi)u(y)\,dyd\xi.$$

We observe that if $P^*(x,D)$ denotes the adjoint of $P(x,D)$, then the principal symbol of the composition $P^* \circ T \circ P$ is the same as that of T_1. Indeed, the principal symbol of T_1 is given by

$$(1-\varphi(\xi))q_1(y,y,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s = (1-\varphi(\xi))|p(x,\xi)|^2 q(y,y,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s,$$

while that of $P^* \circ T \circ P$ is

$$\overline{p(x,\xi)}(1-\varphi(\xi))q(y,y,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s p(x,\xi).$$

Therefore, the difference $E = T_1 - P^* \circ T \circ P$ is a pseudodifferential operator in the class $\Psi^{2s+\frac{mk-m}{2k}-\frac{1}{2}}_{1,\frac{1}{2k}}$.

It follows that

$$\begin{aligned} A_1 &= \langle T_1 u, u\rangle \\ &= \langle P^* \circ T \circ P(u), u\rangle + \langle Eu, u\rangle \\ &= \langle T(Pu), Pu\rangle + \langle Eu, u\rangle. \end{aligned}$$

By Garding's inequality, there are constants $C_1', C_2' > 0$ such that

$$\mathrm{Re}\left\{\langle T(Pu), Pu\rangle\right\} \geq C_1'||Pu||^2_{H^{s+\frac{mk-m}{4k}}} - C_2'||Pu||^2_{H^{s+\frac{mk-m}{4k}-\frac{1}{4}}}.$$

We also have, for some $C_3 > 0$,

$$|\langle Eu, u\rangle| \leq C_3||u||^2_{H^{s+\frac{mk-m}{4k}-\frac{1}{4}}}.$$

Hence for some $C_1, C_2 > 0$, since $P(x,D)$ is of order 0,

$$A_1 \geq C_1||Pu||^2_{H^{s+\frac{mk-m}{4k}}} - C_2||Pu||^2_{H^{s+\frac{mk-m}{4k}-\frac{1}{4}}}.$$

Since A_2 involves a smoothing operator, the proof of (a) is completed.
(b) follows from the continuity of T_1 and the fact that A_2 involves a smoothing operator. □

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