

TOTAL MASS AND LIMITS OF QUASI-LOCAL MASS AT FUTURE NULL INFINITY

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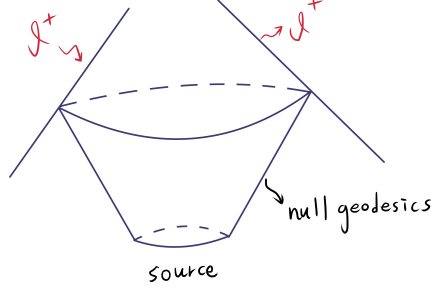
ABSTRACT. An idealized observer of an astronomical event is situated at future null infinity, where light rays emitted from the source approach. Mathematically, null infinity corresponds to the portion of the spacetime boundary defined by equivalence classes of null geodesics. But what can we observe at future null infinity? In the note, we start by reviewing the description of null infinity in terms of the Bondi-Sachs coordinate system. In particular, we reformulate and extend the invariance of Bondi mass and the mass loss formula in terms of a modified mass aspect 2-form. At the end, we discuss new results about the limit of quasilocal mass of unit spheres at null infinity in a Bondi-Sachs coordinate system. This is based on joint work with Po-Ning Chen, Jordan Keller, Ye-Kai Wang, and Shing-Tung Yau.

1. INTRODUCTION

This note is about mathematical general relativity and we shall discuss some recent applications of the theory of quasilocal mass [43, 44] to the study of gravitational radiation. We consider an isolated gravitating system that corresponds to an asymptotically flat spacetime, where gravity is weak at infinity. The future null infinity, denoted as $\mathscr{I}^+$, consists of equivalent classes of future null geodesics [1].

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^[1]This is similar to the definition of ideal boundary of a Cartan-Hadamard manifold as equivalent classes of geodesic rays. In physics literature, null infinity arises as a conformal compactification of the physical spacetime introduced by Penrose, see [33, 34, 21]



The mathematical (nonlinear) theory of gravitational radiation started around 1960's with the pioneering works of Bondi, Trautman, Sachs, van der Burg, Metzner, Penrose, Newman, Geroch etc [1, 39, 40, 2, 33, 42, 35, 31, 32, 21]. The spacetime near $\mathcal{I}^+$ is described in terms of the Bondi-Sachs coordinates (u, r, θ, ϕ) which are chosen in the following way. Level sets of u are null hypersurfaces generated by null geodesics, θ and ϕ are extended by constancy along the integral curves of the gradient vector field of u , and r corresponds to the “area distance”^[2]

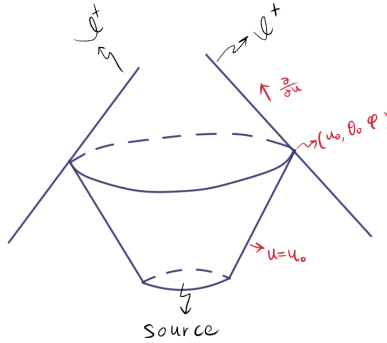
More precisely, one starts with (u, θ, ϕ) coordinates that satisfy the above conditions. For any other coordinate variable $\bar{r}$, in terms of the coordinates $(u, \bar{r}, \theta, \phi)$, the above conditions imply the metric is of the form

$$-\bar{U}\bar{V}du^2 - 2\bar{U}dud\bar{r} + \bar{g}_{ab}(dx^a + \bar{W}^a du)(dx^b + \bar{W}^b du), \quad (1)$$

where $a, b = 2, 3$ and $x^2 = \theta, x^3 = \phi$. Let σ_{ab} be a standard round metric on the unit 2-sphere (S^2, x^a) . We define a new coordinate variable r by

$$r = (\det \bar{g}_{ab} / \det \sigma_{ab})^{1/4}$$

and denote $h_{ab} = r^{-2}\bar{g}_{ab}$.



^[2] $s = 2 \ln r$ corresponds to the parameter of the inverse mean curvature (lapse) flow, which was an important tool in the study of the null Penrose inequality, see [36], as well as the Riemannian Penrose inequality, see [24].

In terms of the Bondi-Sachs coordinate system (u, r, x^2, x^3) , the spacetime metric takes the form

$$g_{\alpha\beta}dx^\alpha dx^\beta = -UVdu^2 - 2Ududr + r^2h_{ab}(dx^a + W^a du)(dx^b + W^b du). \quad (2)$$

The index conventions here are $\alpha, \beta = 0, 1, 2, 3$, $a, b = 2, 3$, and $u = x^0, r = x^1$. See [20, 29] for more details of the construction of the coordinate system. The metric coefficients U, V, h_{ab}, W^a of (2) depend on u, r, θ, ϕ , but the assumption on r implies that $\det h_{ab}$ is independent of u and r . These gauge conditions thus reduce the number of metric coefficients of a Bondi-Sachs coordinate system to six (there are only two independent components in h_{ab}). On the other hand, the boundary conditions $U \rightarrow 1, V \rightarrow 1, W^a \rightarrow 0, h_{ab} \rightarrow \sigma_{ab}$ are imposed as $r \rightarrow \infty$ (such boundary conditions may not be satisfied in a radiative spacetime). The special gauge choice of the Bondi-Sachs coordinates implies a hierarchy among the vacuum Einstein equations, see [29, 22].

Assuming the outgoing radiation condition [35, 42, 29, 45, 41], the boundary condition and the vacuum Einstein equation imply that as $r \rightarrow \infty$, all metric coefficients can be expanded in inverse integral powers of r .^[3] In particular,

$$\begin{aligned} U &= 1 + O(r^{-2}), \\ V &= 1 - \frac{2m}{r} + O(r^{-2}), \\ W^a &= O(r^{-2}), \\ h_{ab} &= \sigma_{ab} + \frac{C_{ab}}{r} + O(r^{-2}) \end{aligned} \quad ,$$

where $m = m(u, x^a)$ is the mass aspect and $C_{ab} = C_{ab}(u, x^a)$ is the shear tensor of this Bondi-Sachs coordinate system. The Bondi-Sachs energy-momentum 4-vector associated with a u -slice is then

$$e(u) = \frac{1}{4\pi} \int_{S_\infty^2} m(u, x^a) dv_\sigma, \quad p_i(u) = \frac{1}{4\pi} \int_{S_\infty^2} m(u, x^a) Y_i dv_\sigma, \quad i = 1, 2, 3$$

where $\{Y_i = Y_i(x^a), i = 1, 2, 3\}$ is an orthonormal basis of the (-2) eigenspace of $\Delta = \Delta_\sigma$ (these are the usual $\ell = 1$ spherical harmonics) and dv_σ is the

^[3] The outgoing radiation condition assumes the traceless part of the r^{-2} term in the expansion of h_{ab} is zero. The presence of this traceless term will lead to a logarithmic term in the expansions of W^a and V . Spacetimes with metrics which admit an expansion in terms of $r^{-j} \log^i r$ are called “polyhomogeneous” and are studied in [20]. They do not obey the outgoing radiation condition or the peeling theorem [41], but they do appear as perturbations of the Minkowski spacetime by the work of Christodoulou-Klainerman [17].

area form of the metric σ . The positivity of the Bondi mass^[4]

$$\sqrt{e^2 - \sum_i p_i^2} \quad (3)$$

was proved by Schoen-Yau [37] and Horowitz-Perry [23] under the dominant energy condition and a global assumption on horizon, see also [19] and [25]. In Section 4 there is a discussion about the positivity of the Bondi mass and this global assumption. The supplementary equations imply the following equation satisfied by the mass aspect along $\mathcal{I}^+$:

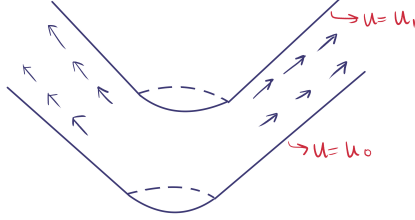
$$\partial_u m = \frac{1}{4} \nabla^a \nabla^b (\partial_u C_{ab}) - \frac{1}{8} |\partial_u C|_\sigma^2, \quad (4)$$

where $|\partial_u C|_\sigma^2 = \sigma^{ac} \sigma^{bd} \partial_u C_{ab} \partial_u C_{cd}$. Integrating over S_∞^2 with the metric σ yields the well-known Bondi mass loss formula:

$$\frac{d}{du} e(u) = -\frac{1}{32\pi} \int_{S_\infty^2} |\partial_u C|_\sigma^2 dv_\sigma \leq 0. \quad (5)$$

In particular,

$$e(u_1) \leq e(u_0) \text{ if } u_1 \geq u_0. \quad (6)$$



This formula indeed corresponds to energy loss, see [25] for a monotonicity formula for the quantity $e - \sqrt{\sum_i p_i^2}$.

2. INVARIANCE UNDER THE BMS GROUP

Rescaling the spacetime metric (2) by r^{-2} as $r \rightarrow \infty$, the limit of $r^{-2} g_{\alpha\beta} dx^\alpha dx^\beta$ approaches $\sigma_{ab} dx^a dx^b$, or the null metric on $\mathcal{I}^+$.^[5] Therefore, $\mathcal{I}^+$ can be

^[4] This was sometimes called the Bondi-Trautman (or Trautman-Bondi) mass. However, comparison with Trautman's work [39, 40] and Bondi's work [1, 2] shows that Trautman worked with a linear approximation, while Bondi used the full nonlinear theory of General Relativity. The setting of this article is fully nonlinear.

^[5] This is a special case of conformal compactification. In general, the metric on the unphysical spacetime is of the form $\Omega^2 g_{\alpha\beta} dx^\alpha dx^\beta$ and $\Omega = 0$ corresponds to $\mathcal{I}^+$, see [33, 34, 21].

view as a null three-manifold:

$$\mathcal{I}^+ = I \times (S^2, \sigma_{ab})$$

with $u \in I$, $x^a \in S^2$.

Each spacetime Bondi-Sachs coordinate system (u, r, x^a) induces such a limiting coordinate system (u, x^a) on $\mathcal{I}^+$, together with the mass aspect $m(u, x^a)$ and the shear $C_{ab}(u, x^a)$. Such a Bondi-Sachs coordinate system is by no means unique and the BMS group, which corresponds to the diffeomorphism group that preserves the gauge and boundary conditions, acts on the set of Bondi-Sachs coordinate systems.

A BMS group element induces a diffeomorphism $\mathbf{g}$ on $\mathcal{I}^+$ that is of the following form:

$$\mathbf{g} : (u, x^a) \mapsto (\bar{u}, \bar{x}^A), a = 2, 3, A = 2, 3 \quad (7)$$

such that

$$\begin{cases} \bar{x}^A &= g^A(x^a) \\ \bar{u} &= K(x^a)(u + f(x^a)) \end{cases} \quad (8)$$

where $g : (S^2, \sigma) \rightarrow (S^2, \bar{\sigma})$ is a conformal isometry, i.e. $g^* \bar{\sigma} = K^2 \sigma$ where $K = (\alpha_0 + \alpha_i Y_i)^{-1}$ and (α_0, α_i) is a future timelike unit vector.

Here is how the Poincaré group sits in the BMS group:

- (1) $f(x^a)$ is any smooth function on S^2 that is called a “supertranslation”. $f(x^a) = \sum a_i Y_i$ corresponds to an actual translation in the Poincaré group.
- (2) $K = (\alpha_0 + \sum \alpha_i Y_i)^{-1}$ corresponds to boosts in $O(3, 1)$.
- (3) Choices of $Y_i, i = 1, 2, 3$ correspond to $O(3) \subset O(3, 1)$.

The invariance/equivariance of the Bondi-Sachs energy-momentum is best described in terms of the modified mass aspect which is defined as:

$$\hat{m} = m - \frac{1}{4} \nabla^a \nabla^b C_{ab}, \quad (9)$$

where ∇ is the covariant derivative with respect to the metric σ .

Let $\hat{m}$ and $\hat{\bar{m}}$ be the modified mass aspects of the limiting Bond-Sachs coordinate systems (u, x^a) and $(\bar{u}, \bar{x}^A)$ on $\mathcal{I}^+$, respectively. Suppose (u, x^a) and $(\bar{u}, \bar{x}^A)$ are related by a BMS element (K, f) as in (8), then $\hat{m}$ and $\hat{\bar{m}}$ are related by

$$K^{-3}(\hat{m} - \frac{1}{4} \Delta(\Delta + 2)f) = \mathbf{g}^* \hat{\bar{m}}, \quad (10)$$

see [18, Section 6.9] for the special case of this formula when $K = 1$. In the next subsection, we show that this formula implies the invariance of the Bondi-Sachs energy-momentum. In addition, equation (4) becomes

$$\partial_u \hat{m} = -\frac{1}{8} |\partial_u C|_{\sigma}^2, \quad (11)$$

and the modified mass aspect is pointwise non-increasing (mass loss formula). Note that m and $\hat{m}$ define the same energy-momentum.

2.1. BMS invariance. In this subsection, let $\sigma = \sigma_{ab}$ be a round metric on a unit sphere S^2 and let Y_i be an orthonormal basis of the (-2) eigenspace of Δ_σ in the sense that

$$\nabla Y_i \cdot \nabla Y_j = \delta_{ij} - Y_i Y_j, i, j = 1, 2, 3, \quad (12)$$

where $\nabla = \nabla_\sigma$ is the gradient operator of σ .

Suppose $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a future timelike unit 4-vector, i.e. $\alpha_0 > 0$ and $\alpha_0^2 - \sum_i \alpha_i^2 = 1$ and denote $K = (\alpha_0 + \sum_j \alpha_j Y_j)^{-1}$. It is well-known that the conformal metric $\bar{\sigma} = K^2 \sigma$ is of constant Gauss curvature 1.

Lemma 1. *Let $\eta_{\alpha\beta}$ be the Minkowski metric. Let A_β^α be an element of $O(3, 1)$ such that $\eta_{\alpha\beta} = \eta_{\gamma\sigma} A_\alpha^\gamma A_\beta^\sigma$ (which also implies $\eta^{\gamma\sigma} = \eta^{\alpha\beta} A_\alpha^\gamma A_\beta^\sigma$) and $A_0^0 = \alpha_0, A_0^k = \alpha_k, k = 1, 2, 3$. Then*

$$\bar{Y}_i = K(A_i^0 + A_i^k Y_k), i = 1, 2, 3$$

form an orthonormal basis of the -2 eigenspace of the Laplace operator $\bar{\Delta}$ of $\bar{\sigma}$ in the sense that

$$\bar{\nabla} \bar{Y}_i \cdot \bar{\nabla} \bar{Y}_j = \delta_{ij} - \bar{Y}_i \bar{Y}_j, i, j = 1, 2, 3$$

Proof. This is a direct calculation using the definition $\bar{\sigma} = K^2 \sigma$, the formula $K^{-1} = \alpha_0 + \sum_j \alpha_j Y_j$, and the relation (12). $\square$

The following proposition shows how the energy-momentum transforms under a boost element of the BMS group.

Proposition 2. *Let μ be a function on S^2 and denote*

$$e = \int \mu dv_\sigma, \quad p_i = \int \mu Y_i dv_\sigma,$$

where dv_σ is the volume form of σ . Suppose $A_\beta^\alpha \in O(3, 1)$ satisfies $A_0^0 = \alpha_0, A_0^k = \alpha_k$ and let $\bar{Y}_i = (A_i^0 + A_i^k Y_k)K, i = 1, 2, 3$. Denote

$$\bar{e} = \int (K^{-3} \mu) dv_{\bar{\sigma}}, \quad \bar{p}_i = \int (K^{-3} \mu \bar{Y}_i) dv_{\bar{\sigma}},$$

where $dv_{\bar{\sigma}}$ is the volume form of $\bar{\sigma}$. Then

$$\bar{e} = A_0^0 e + A_0^k p_k, \quad \bar{p}_i = A_i^0 e + A_i^k p_k.$$

In particular, $\bar{e}^2 - \sum \bar{p}_i^2 = e^2 - \sum p_i^2$.

Proof. From the last lemma, we know that $\bar{Y}_i = K(A_i^0 + A_i^k Y_k), i = 1, 2, 3$ forms an orthonormal basis for the (-2) eigenspace of $\Delta_{\bar{\sigma}}$. Since $dv_{\bar{\sigma}} = K^2 dv_\sigma$, we compute

$$\bar{e} = \int (K^{-3} \mu) dv_{\bar{\sigma}} = \int (A_0^0 + A_0^k Y_k) \mu dv_\sigma = A_0^0 \int \mu dv_\sigma + A_0^k \int Y_k \mu dv_\sigma$$

and

$$\bar{p}_i = \int (K^{-3}\mu)\bar{Y}_i dv_{\bar{\sigma}} = \int (A_i^0 + A_i^k Y_k)\mu dv_{\sigma} = A_i^0 \int \mu dv_{\sigma} + A_i^k \int Y_k \mu dv_{\sigma}.$$

Write $p_0 = e, \bar{p}_0 = \bar{e}$, the formula becomes $\bar{p}_{\alpha} = A_{\alpha}^{\beta} p_{\beta}, \alpha, \beta = 0, 1, 2, 3$. Since we require $\eta_{\alpha\beta} = \eta_{\gamma\sigma} A_{\alpha}^{\gamma} A_{\beta}^{\sigma}$, the last formula is exactly how a co-vector transforms (a vector v^{α} transforms by $\bar{v}^{\beta} = A_{\alpha}^{\beta} v^{\alpha}$). $\square$

3. A MODIFIED MASS ASPECT 2-FORM

Definition 3. The modified mass aspect 2-form $\mathfrak{m}$ of a limiting Bondi-Sachs coordinate system (u, x^a, σ) of $\mathcal{I}^+$ is defined to be

$$\mathfrak{m} = \hat{m} dv_{\sigma}$$

where $\hat{m}$ is the modified mass aspect $\hat{m}$ defined in (9) and $dv_{\sigma} = \sqrt{\det \sigma} dx^2 \wedge dx^3$ is the volume form of the Riemannian metric σ .

In terms of the modified mass aspect 2-form, equation (10) becomes

$$K^{-1}(\mathfrak{m} - \frac{1}{4}\Delta(\Delta + 2)f dv_{\sigma}) = \mathfrak{g}^* \bar{\mathfrak{m}}, \quad (13)$$

where $\bar{\mathfrak{m}} = \hat{\bar{m}} dv_{\bar{\sigma}}$ is the modified mass aspect 2-form of the limiting Bondi-Sachs coordinate system $(\bar{u}, \bar{x}^A, \bar{\sigma})$.

We can then integrate both sides of (13) on any section Σ of $\mathcal{I}^+$ that is of the form $u = h(x^a)$ where h is any continuous function.

Proposition 4. *For any section Σ of $\mathcal{I}^+$, suppose $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ are the modified mass aspect 2-forms of two Bondi-Sachs coordinate systems which are related by a BMS group element that is a pure supertranslation, then the energy integrals are the same*

$$\int_{\Sigma} \mathfrak{m} = \int_{\Sigma} \bar{\mathfrak{m}}.$$

In general, the energy-momentum are related by the boost associated with K .

We note that in this formulation, Σ does not need to be the level set of any Bondi-Sachs coordinate u on $\mathcal{I}^+$.

The energy loss formula (6) can also be extended by means of the modified mass aspect two-form.

Definition 5. For any two sections Σ_1 and Σ_2 on $\mathcal{I}^+$, Σ_1 is said to be in the retarded future of Σ_2 if there exists a limiting Bondi-Sachs coordinate system (u, x^a) such that Σ_1 and Σ_2 are given by $u = h_1(x^a)$ and $u = h_2(x^a)$ respectively, and that $h_1(x^a) \geq h_2(x^a)$ for each $x^a \in S^2$.

One easily check that this notion is independent of the choice of the limiting Bondi-Sachs coordinate system because $K > 0$ and (8).

Theorem 6. *For any two sections Σ_1 and Σ_2 on $\mathcal{I}^+$ such that Σ_1 is in the retarded future of Σ_2 , we have*

$$\int_{\Sigma_1} \mathbf{m} \leq \int_{\Sigma_2} \mathbf{m}.$$

Proof. Equation (11) implies that the mass aspect 2-form $\mathbf{m}$, as a 2-form on the three-manifold $\mathcal{I}^+$, verifies

$$d\mathbf{m} = -\frac{1}{8}|\partial_u C|_\sigma^2 du \wedge dv_\sigma, \quad (14)$$

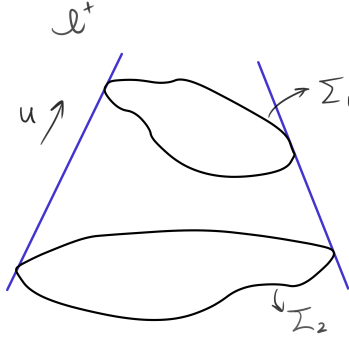
where d is the exterior derivative operator on $\mathcal{I}^+$ as a differentiable manifold.

Integrating both sides of (14) over the region between Σ_1 and Σ_2 and applying Stokes's theorem yield

$$\int_{\Sigma_1} \mathbf{m} - \int_{\Sigma_2} \mathbf{m} = -\frac{1}{8} \int_{u=h_2(x^a)}^{u=h_1(x^a)} \int_{(S^2, \sigma)} |\partial_u C|_\sigma^2 du \wedge dv_\sigma, \quad (15)$$

which is non-positive by the retarded future condition. $\square$

This theorem should be considered as an extension of the classical Bondi mass loss formula (6) which only applies to the case when Σ_1 and Σ_2 are both smooth and u level sets of a fixed Bondi-Sachs coordinate system.



4. WANG-YAU QUASILOCAL MASS

The quasilocal mass is attached to a 2-dimensional closed surface Σ which bounds a spacelike region in spacetime. Σ is assumed to be a topological 2-sphere, but with different intrinsic geometry and extrinsic geometry, we expect to read off the effect of gravitation in the spacetime vicinity of the surface. Suppose the surface is spacelike, i.e. the induced metric σ is Riemannian. An essential part of the extrinsic geometry is measured by the mean curvature vector field $\mathbf{H}$ of Σ . $\mathbf{H}$ is a normal vector field of the surface such that the null expansion along any null normal direction ℓ is given by the pairing of $\mathbf{H}$ and ℓ .

In [43], Wang-Yau proposed the following definition of quasilocal mass. To evaluate the quasilocal mass of a 2-surface Σ with the physical data $(\sigma, \mathbf{H})$, one first solves the optimal isometric embedding equation, see (16) below, which gives an embedding of Σ into the Minkowski spacetime with the image surface Σ_0 that has the same induced metric as Σ , i.e. σ . One then compares the extrinsic geometries of Σ and Σ_0 and evaluates the quasilocal mass from $\sigma, \mathbf{H}$ and $\mathbf{H}_0$.

Assuming the mean curvature vector $\mathbf{H}$ is spacelike, the physical surface Σ with physical data $(\sigma, \mathbf{H})$ gives $(\sigma, |\mathbf{H}|, \alpha_{\mathbf{H}})$ where $|\mathbf{H}| > 0$ is the Lorentz norm of $\mathbf{H}$ and $\alpha_{\mathbf{H}}$ is the connection one-form determined by $\mathbf{H}$. Given an isometric embedding $X : \Sigma \rightarrow \mathbb{R}^{3,1}$ of σ . Let Σ_0 be the image $X(\Sigma)$ and $(\sigma, |\mathbf{H}_0|, \alpha_{\mathbf{H}_0})$ be the corresponding data of Σ_0 .

Let T be a future timelike unit Killing field of $\mathbb{R}^{3,1}$ and define $\tau = -\langle X, T \rangle$. Define a function ρ and a 1-form j_a on Σ :

$$\rho = \frac{\sqrt{|\mathbf{H}_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|\mathbf{H}|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}}$$

$$j_a = \rho \nabla_a \tau - \nabla_a \left(\sinh^{-1} \left(\frac{\rho \Delta\tau}{|\mathbf{H}_0| |\mathbf{H}|} \right) \right) - (\alpha_{\mathbf{H}_0})_a + (\alpha_{\mathbf{H}})_a,$$

where ∇_a is the covariant derivative with respect to the metric σ , $|\nabla\tau|^2 = \nabla^a \tau \nabla_a \tau$ and $\Delta\tau = \nabla^a \nabla_a \tau$. ρ is the quasilocal mass density and j_a is the quasilocal momentum density. A full set of quasilocal conserved quantities was defined in [8, 9] using ρ and j_a .

The optimal isometric embedding equation for (X, T) is

$$\begin{cases} \langle dX, dX \rangle &= \sigma \\ \nabla^a j_a &= 0. \end{cases} \quad (16)$$

The first equation is the isometric embedding equation into the Minkowski spacetime and the second one is the Euler-Lagrange equation of the quasilocal energy [43, 44] in the space of isometric embeddings. The quasi-local mass is defined to be

$$E(\Sigma, X, T) = \frac{1}{8\pi} \int_{\Sigma} \rho.$$

Σ_0 is essentially the “unique” surface in the Minkowski spacetime that best matches the physical surface Σ . If the original surface Σ happens to be a surface in the Minkowski spacetime, the above procedure identifies $\Sigma_0 = \Sigma$ up to a global isometry. In [7], we prove such a minimizing and uniqueness property for a solution of the optimal isometric embedding equation.

A prototype form of the quasilocal mass (see Brown-York [4], Liu-Yau [27], Booth-Mann [3], Kijowski [26], etc) is

$$\frac{1}{8\pi} \int_{\Sigma} |\mathbf{H}_0| - \frac{1}{8\pi} \int_{\Sigma} |\mathbf{H}|.$$

The positivity is proved by Shi-Tam [38] and Liu-Yau [28]. However, for a surface in the Minkowski spacetime, the above expression may not be zero [30]. The optimal isometric embedding equation gives the necessary correction, so the definition of Wang-Yau is *positive* in general, and zero for surfaces in the Minkowski spacetime, see [44]. In general, the optimal isometric embedding equation is difficult to solve. However, in a perturbative configuration, when a family of surfaces limit to a surface in the Minkowski spacetime, the optimal isometric embedding equation is solvable, subject to the positivity of the limiting mass [7].

5. LARGE SPHERE LIMIT AT NULL INFINITY

In [6], we evaluate the large sphere limit of quasilocal mass at $\mathcal{I}^+$ which recovers the Bondi-Sachs energy momentum. At a retarded time $u = u_0$, we consider the family of large spheres Σ_r parametrized by the Bondi-Sachs coordinate r . The positivity of the Bondi mass guarantees the unique solvability of the optimal isometric embedding system (16) with a solution (X_r, T_r) . Suppose X_r and T_r admit expansions:

$$T_r = T^{(0)} + \sum_{k=1}^{\infty} T^{(-k)} r^{-k}$$

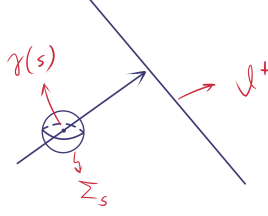
$$X_r = rX^{(1)} + X^{(0)} + \sum_{k=1}^{\infty} X^{(-k)} r^{-k},$$

then $T^{(0)}$ is shown to be proportional to the Bondi-Sachs energy-momentum in [6, Theorem 2] and $T^{(0)}$ being future timelike makes $T^{(-k)}$ and $X^{(-k+1)}$ solvable inductively for $k = 1, 2, \dots$.

In [11], we also evaluate the large sphere limit of quasilocal mass on an asymptotically hyperbolic initial data set. This in particular gives a new proof of the positivity of the Bondi mass.

6. UNIT SPHERE LIMIT AT NULL INFINITY

Both the positivity of Bondi mass (3) and the mass loss formula (5) are global statements about $\mathcal{I}^+$ that require the information in all direction of (θ, ϕ) on S_{∞}^2 . In [10, 13, 14, 15, 16], we study the limit of quasilocal mass along a single direction (θ, ϕ) and obtain quasilocal quantities at $\mathcal{I}^+$. Consider a null geodesic γ with affine parameter s that approaches $\mathcal{I}^+$. Around each point $\gamma(s)$, consider a geodesic 2-sphere Σ_s of unit radius. The geometry of Σ_s approaches the geometry of a standard unit round sphere of $\mathbb{R}^3$.



In the limit $s \rightarrow \infty$, we obtain the limit of the quasilocal mass $\lim_{s \rightarrow \infty} E(\Sigma_s)$ which is of the order of $\frac{1}{s^2}$ with $E(\Sigma_s) \geq 0$. In [10], we study the case of linear gravitational perturbation of the Schwarzschild black hole à la Chandrasekhar [5]. The linearized vacuum Einstein equation is solved by separation of variables and solutions of linearized waves are obtained. The optimal isometric embedding system can be solved and the quasilocal mass can be evaluated by solving equations of the following forms on the standard 2-sphere S^2 :

$$\begin{aligned} \Delta(\Delta + 2)\tau &= \text{physical data}, \\ (\Delta + 2)N &= \text{physical data}, \end{aligned} \tag{17}$$

where τ and N are functions on the standard 2-sphere and Δ is the Laplace operator. All distinctive features of the linearized waves such as frequency and mode parameters are recovered.

In [13], we study the case of the Vaidya spacetime. The metric of the Vaidya spacetime takes the form:

$$-(1 - \frac{2m(u)}{r})du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The quasilocal mass of a unit sphere approaching null infinity is computed in [13]:

$$E(\Sigma_s) = -\frac{1}{16\pi s^2} \int_{S^2} (\partial_u m) f^2 + l.o.t. \geq 0$$

In particular, the positivity of quasilocal mass corresponds to the mass loss formula in the Vaidya case.

What happens in the general case? One may expect that the limit of quasilocal mass in the direction of (θ_0, ϕ_0) should recover the mass aspect function $m(\cdot, \theta_0, \phi_0)$. But notice that the mass aspect function is not point-wise positive, only the integrated Bondi mass is positive. Besides, the positivity of Bondi mass requires a global assumption on horizon [37, 23]. For example, on a Schwarzschild spacetime with negative mass parameter (thus there is a naked singularity that is not shielded by a horizon), the Bondi mass is negative, but the quasilocal mass is still positive. On the other hand,

the Vaidya spacetime is non-vacuum, indeed both the dominant energy condition and the mass loss formula correspond to $\partial_u m \leq 0$.

In order to understand such a quasilocal mass at the purely gravitational level, we compute the case for the null infinity of a general spacetime in a Bondi-Sachs coordinate system [15, 16]. The $\frac{1}{s^2}$ term of the quasilocal mass of a unit sphere approaching null infinity is (up to a constant factor)

$$\int_{B^3} \left[\frac{1}{8} |\partial_u C|^2 + \det(h_0^{(-1)} - h^{(-1)}) \right] + \frac{1}{4} \int_{S^2} [(tr_\Sigma k^{(-1)})^2 - \tau^{(-1)} \Delta(\Delta + 2)\tau^{(-1)}], \quad (18)$$

in which $h^{(-1)}$ and $k^{(-1)}$ are part of the physical data and $h_0^{(-1)}$ and $\tau^{(-1)}$ depend on the solution of the optimal isometric embedding system. A priori, the expression may depend on the mass aspect and the shear tensor. However, some rather amazing cancellations show that the answer depends only on the shear tensor:

Theorem 7. [15, 16] *The limit of the quasilocal mass of unit-size sphere at $\mathcal{I}^+$ is a positive quasilocal quantity that depends only on the shear tensor.*

This quantity should be considered as a quasilocal measurement of radiation. It is very interesting to compare (18) with the global loss formula (15). The first term $\frac{1}{8} \int_{B^3} |\partial_u C|^2$ in (18) should be considered as the principal term of radiation, which also appears in (15), and other terms in (18) should be considered as correction terms due to the quasilocal nature of the quantity.

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