Aspects of non-associative gauge theory

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Abstract A smooth loop is the direct non-associative generalization of Lie group. In this paper, we review the theory of smooth loops and smooth loop bundles. This is then used to define a non-associative analog of the Chern-Simons functional.

1 Introduction

One of highly successful areas at the intersection of differential geometry, analysis, and mathematical physics is gauge theory. As it is well-known, this is the study of connections on bundles with particular Lie groups as the structure groups. In [5], the author initiated a theory of smooth loops, which are non-associative analogs of Lie groups, and began the development of gauge theory based on loops, i.e. a non-associative gauge theory. The purpose of this note is to review the theory of smooth loops and loop bundles, and to provide a more rigorous construction of a non-associative Chern-Simons functional on 3-manifolds. In particular, the affine space of connections in a standard gauge theory is replaced by an affine space $\mathcal{T}$ of torsions, modelled on 1-forms with values in a loop algebra (the tangent space to a loop at identity). We define a 1-form on $\mathcal{T}$, show that it is a closed form, and show that it is the exterior derivative of a function on $\mathcal{T}$, which we define to be the Chern-Simons functional. Finally, we show how this functional is affected by gauge transformations.

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2 Smooth Loops

For a detailed discussion of concepts related to smooth loops, the reader is referred to [5]. The reader can also refer to [6, 7, 9, 11, 12] for a discussion of these concepts.

Definition 1 A *loop* $\mathbb{L}$ is a set with a binary operation $p \cdot q$ with identity 1, and compatible left and right quotients $p\backslash q$ and p/q, respectively.

In particular, existence of quotients is equivalent to saying that for any $q \in \mathbb{L}$, the left and right product maps L_q and R_q are invertible maps. Restricting to the smooth category, we obtain the definition of a smooth loop.

Definition 2 A *smooth loop* is a smooth manifold $\mathbb{L}$ with a loop structure such that the left and right product maps are diffeomorphisms of $\mathbb{L}$.

Definition 3 A *pseudoautomorphism* of a smooth loop $\mathbb{L}$ is a diffeomorphism $h : \mathbb{L} \longrightarrow \mathbb{L}$ for which there exists another diffeomorphism $h' : \mathbb{L} \longrightarrow \mathbb{L}$, known as the partial pseudoautomorphism corresponding to h, such that for any $p, q \in \mathbb{L}$,

$$h(pq) = h'(p)\, h(q)\,. \tag{1}$$

In particular, $h' = R^{-1}_{h(1)} \circ h$. The element $h(1) \in \mathbb{L}$ is the *companion* of h'. As shown in [5], given h and h', we have the following properties

$$h(pq) = h'(p)\, h(q) \qquad h(q\backslash p) = h'(q) \backslash h(p) \qquad h'(p/q) = h(p)/h(q)\,. \tag{2}$$

It is then easy to see that the sets of pseudoautomorhisms and partial pseudoautomorphisms are both groups. Denote the former by Ψ and the latter by Ψ'. We also see that the *automorphism* group of $\mathbb{L}$ is the subgroup $H \subset \Psi$ which is the stabilizer of $1 \in \mathbb{L}$. We will use $\mathbb{L}$ to denote $\mathbb{L}$ with the action of Ψ and $\mathbb{L}'$ to denote $\mathbb{L}$ with the action of Ψ', if a distinction between the G-sets is needed.

Let $r \in \mathbb{L}$, then we may define a modified product $\circ_r$ on $\mathbb{L}$ via $p \circ_r q = (p \cdot qr)/r$, so that $\mathbb{L}$ equipped with product $\circ_r$ will be denoted by $(\mathbb{L}, \circ_r)$, the corresponding quotient will be denoted by $/_r$. We have the following properties [5].

Lemma 1 *Let $h \in \Psi$. Then, for any $p, q, r \in \mathbb{L}$,*

$$h'(p \circ_r q) = h'(p) \circ_{h(r)} h'(q) \qquad h'(p/_r q) = h'(p)\, /_{h(r)} h'(q)\,. \tag{3}$$

Consider the tangent space $\mathfrak{l} := T_1\mathbb{L}$ at $1 \in \mathbb{L}$. By analogy with Lie groups, for any $\xi \in \mathfrak{l}$, define the *fundamental vector field* $\rho(\xi)$ by pushing forward ξ by right translation, so that for any $q \in \mathbb{L}$, $\rho(\xi)_q = (R_q)_* \xi$.

Definition 4 ([5]) The Maurer-Cartan form θ is an $\mathfrak{l}$-valued 1-form on $\mathbb{L}$, such that $\theta(\rho(\xi)) = \xi$. Equivalently, for any vector field X, $\theta(X)|_p = \left(R_p^{-1}\right)_* X_p \in \mathfrak{l}$.

This allows us to define brackets on $\mathfrak{l}$. For each $p \in \mathbb{L}$ define the bracket $[\cdot,\cdot]^{(p)}$ given for any $\xi, \eta \in \mathfrak{l}$ by $[\xi, \eta]^{(p)} = -\theta([\rho(\xi), \rho(\eta)])|_p$. We will denote $\mathfrak{l}$

equipped with the bracket $[\cdot,\cdot]^{(p)}$ by $\mathfrak{l}^{(p)}$. Define the *bracket function* $b : \mathbb{L} \longrightarrow \mathfrak{l} \otimes \Lambda^2\mathfrak{l}^*$ to be the map that takes $p \mapsto [\cdot,\cdot]^{(p)} \in \mathfrak{l} \otimes \Lambda^2\mathfrak{l}^*$, so that $b(\theta,\theta)$ is an $\mathfrak{l}$-valued 2-form on $\mathbb{L}$, i.e. $b(\theta,\theta) \in \Omega^2(\mathfrak{l})$.

Theorem 1 ([5, Theorem 3.10]) *The form* θ *satisfies* $d\theta = \frac{1}{2}db(\theta,\theta)$.

With respect to the action of Ψ, the bracket satisfies the following property.

Lemma 2 *If* $h \in \Psi(\mathbb{L})$ *and* $q \in \mathbb{L}$, *then, for any* $\xi,\eta,\gamma \in \mathfrak{l}$, $h'_*[\xi,\eta]^{(q)} = [h'_*\xi, h'_*\eta]^{h(q)}$.

We will assume that Ψ is a finite-dimensional Lie group, and suppose the Lie algebras of Ψ and $H_s = \mathrm{Aut}(\mathbb{L},\circ_s)$ are $\mathfrak{p}$ and $\mathfrak{h}_s$, respectively. In particular, $\mathfrak{h}_s$ is a Lie subalgebra of $\mathfrak{p}$. Also, we will assume that Ψ acts transitively on $\mathbb{L}$. The action of Ψ on $\mathbb{L}$ induces an action of the Lie algebra $\mathfrak{p}$ on $\mathfrak{l}$, which we will denote by $\cdot$.

Definition 5 Define the map $\varphi : \mathbb{L} \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ such that for each $s \in \mathbb{L}$ and $\gamma \in \mathfrak{p}$,

$$\varphi_s(\gamma) = \frac{d}{dt}(\exp(t\gamma)(s))/s\Big|_{t=0} \in \mathfrak{l}. \tag{4}$$

Lemma 3 *The map* φ *as in (4) is equivariant with respect to corresponding actions of* $\Psi(\mathbb{L})$, *in particular for* $h \in \Psi$, $s \in \mathbb{L}$, $\gamma \in \mathfrak{p}$, *we have*

$$\varphi_{h(s)}((\mathrm{Ad}_h)_*\gamma) = (h')_*\varphi_s(\gamma). \tag{5}$$

Moreover, the image of φ_s *is* $\mathfrak{l}^{(s)}$ *and the kernel is* $\mathfrak{h}_s$, *and hence,* $\mathfrak{p} \cong \mathfrak{h}_s \oplus \mathfrak{l}^{(s)}$.

Lemma 4 ([5]) *Suppose* $\xi \in \mathfrak{p}$ *and* $\eta,\gamma \in \mathfrak{l}$, *then*

$$\xi \cdot [\eta,\gamma]^{(s)} = [\xi\cdot\eta,\gamma]^{(s)} + [\eta,\xi\cdot\gamma]^{(s)} + a_s(\eta,\gamma,\varphi_s(\xi)) \tag{6a}$$

$$\xi\cdot\varphi_s(\eta) = \eta\cdot\varphi_s(\xi) + \varphi_s\left([\xi,\eta]_{\mathfrak{p}}\right) + [\varphi_s(\xi),\varphi_s(\eta)]^{(s)}. \tag{6b}$$

Similarly as for Lie groups, we may define a Killing form $K^{(s)}$ on $\mathfrak{l}^{(s)}$. For $\xi,\eta \in \mathfrak{l}$, we have

$$K^{(s)}(\xi,\eta) = \mathrm{Tr}\left(\mathrm{ad}_\xi^{(s)} \circ \mathrm{ad}_\eta^{(s)}\right), \tag{7}$$

where $\circ$ is just composition of linear maps on $\mathfrak{l}$ and $\mathrm{ad}_\xi^{(s)}(\cdot) = [\xi,\cdot]^{(s)}$. Clearly $K^{(s)}$ is a symmetric bilinear form on $\mathfrak{l}$. In [5] it is shown that for $h \in \Psi$, and $\xi,\eta \in \mathfrak{l}$ it satisfies $K^{(h(s))}(h'_*\xi, h'_*\eta) = K^{(s)}(\xi,\eta)$.

Suppose now $K^{(s)}$ is nondegenerate and $\mathfrak{p}$-invariant, so that the action of $\mathfrak{p}$ is skew-adjoint with respect to $K^{(s)}$. Moreover suppose $\mathfrak{p}$ is semisimple itself, so that it has a nondegenerate, invariant Killing form $K_{\mathfrak{p}}$. We will use $\langle\cdot,\cdot\rangle^{(s)}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ to denote the inner products using $K^{(s)}$ and $K_{\mathfrak{p}}$, respectively. Then, given the map $\varphi_s : \mathfrak{p} \longrightarrow \mathfrak{l}^{(s)}$, we can define its adjoint with respect to these two bilinear maps.

Definition 6 Define the map $\varphi_s^t : \mathfrak{l}^{(s)} \longrightarrow \mathfrak{p}$ such that for any $\xi \in \mathfrak{l}^{(s)}$ and $\eta \in \mathfrak{p}$,

$$\left\langle\varphi_s^t(\xi),\eta\right\rangle_{\mathfrak{p}} = \langle\xi,\varphi_s(\eta)\rangle^{(s)}. \tag{8}$$

Since $\mathfrak{h}_s \cong \ker \varphi_s$, we have $\mathfrak{p} \cong \mathfrak{h}_s \oplus \operatorname{Im} \varphi_s^t$, so that $\mathfrak{h}_s^{\perp} = \operatorname{Im} \varphi_s^t$.

Lemma 5 ([5, Lemma 3.43]) *Suppose Ψ acts transitively on $\mathbb{L}$, $\mathfrak{l}$ is an irreducible representation of $\mathfrak{h}$, and suppose the base field of $\mathfrak{p}$ is $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then, there exists a $\lambda \in \mathbb{F}$ such that for any $s \in \mathbb{L}$, $\varphi_s \varphi_s^t = \lambda \operatorname{id}_{\mathfrak{l}}$ and $\varphi_s^t \varphi_s = \lambda \pi_{\mathfrak{h}_s^{\perp}}$.*

Thus, given our prior assumption of the transitivity of the action of Ψ, the maps φ_s and φ_s^t are isomorphisms between $\mathfrak{l}$ and $\mathfrak{h}_s^{\perp}$. If $s \in \mathbb{L}$ is fixed, and there is no ambiguity, we will use the following notation. Given $\xi \in \mathfrak{p}$, $\hat{\xi} = \varphi_s(\xi) \in \mathfrak{l}$ and given $\eta \in \mathfrak{l}$, $\check{\eta} = \frac{1}{\lambda_s} \varphi_s^t(\eta) \in \mathfrak{h}_s^{\perp}$. We can also use φ_s^t to define a new bracket $[\cdot,\cdot]_{\varphi_s}$ on $\mathfrak{l}$, such that for $\xi, \eta \in \mathfrak{l}$,

$$[\xi, \eta]_{\varphi_s} = \varphi_s \left([\check{\xi}, \check{\eta}]_{\mathfrak{p}} \right). \tag{9}$$

Lemma 6 ([5, Lemma 3.50]) *Let $s \in \mathbb{L}$, then under the assumptions of Lemma 5, the bracket $[\cdot,\cdot]_{\varphi_s}$ satisfies the following properties. Suppose $\xi, \eta, \gamma \in \mathfrak{l}$, then*

1. $\left\langle [\xi, \eta]_{\varphi_s}, \gamma \right\rangle^{(s)} = -\left\langle \eta, [\xi, \gamma]_{\varphi_s} \right\rangle^{(s)}$.
2. For any $h \in \Psi$, $[\xi, \eta]_{\varphi_{h(s)}} = (h')_* \left[(h')_*^{-1} \xi, (h')_*^{-1} \eta \right]_{\varphi_s}$.

3 Loop bundles

Let M be a smooth, finite-dimensional manifold with a Ψ-principal bundle $\pi : \mathcal{P} \longrightarrow M$.

Definition 7 Let $s : \mathcal{P} \longrightarrow \mathbb{L}$ be an equivariant map. In particular, the equivalence class $\lfloor p, s_p \rfloor_{\Psi}$ defines a section of the bundle $Q = \mathcal{P} \times_{\Psi} \mathbb{L}$. We will refer to s as *the defining map* (or *section*).

We will define several associated bundles related to $\mathcal{P}$. As it is well-known, sections of associated bundles are equivalent to equivariant maps. With this in mind, we also give properties of equivariant maps that correspond to sections of these bundles. Let $h \in \Psi$ and, as before, denote by h' the partial action of h.

Bundle	**Equivariant map**	**Equivariance property**
$\mathcal{P}$	$k : \mathcal{P} \longrightarrow \Psi$	$k_{ph} = h^{-1} k_p$
$Q' = \mathcal{P} \times_{\Psi'} \mathbb{L}'$	$q : \mathcal{P} \longrightarrow \mathbb{L}'$	$q_{ph} = (h')^{-1} q_p$
$Q = \mathcal{P} \times_{\Psi} \mathbb{L}$	$r : \mathcal{P} \longrightarrow \mathbb{L}$	$r_{ph} = h^{-1}(r_p)$
$\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$	$\eta : \mathcal{P} \longrightarrow \mathfrak{l}$	$\eta_{ph} = (h')_*^{-1} \eta_p$
$\mathfrak{p}_{\mathcal{P}} = \mathcal{P} \times_{(\mathrm{Ad}_{\xi})_*} \mathfrak{p}$	$\xi : \mathcal{P} \longrightarrow \mathfrak{p}$	$\xi_{ph} = \left(\mathrm{Ad}_h^{-1}\right)_* \xi_p$
$\mathrm{Ad}(\mathcal{P}) = \mathcal{P} \times_{\mathrm{Ad}_{\Psi}} \Psi$	$u : \mathcal{P} \longrightarrow \Psi$	$u_{ph} = h^{-1} u_p h$

(10)

Given equivariant maps $q, r : \mathcal{P} \longrightarrow \mathbb{L}'$, define an equivariant product using s, given for any $p \in \mathcal{P}$ by

$$q \circ_s r|_p = q_p \circ_{s_p} r_p. \tag{11}$$

Due to Lemma 1, the corresponding map $q \circ_s r : \mathcal{P} \longrightarrow \mathbb{L}'$ is equivariant, and hence $\circ_s$ induces a fiberwise product on sections of Q. Analogously, we define fiberwise quotients of sections of Q. Similarly, we define an equivariant bracket $[\cdot,\cdot]^{(s)}$ and the equivariant map φ_s. Other related objects such as the Killing form $K^{(s)}$ and the adjoint φ_s^t to φ_s are then similarly also equivariant.

Suppose the principal Ψ-bundle $\mathcal{P}$ has a principal Ehresmann connection given by the decomposition $T\mathcal{P} = \mathcal{HP} \oplus \mathcal{VP}$ and the corresponding vertical $\mathfrak{p}$-valued connection 1-form ω. Given an equivariant map $f : \mathcal{P} \longrightarrow S$, define

$$d^{\omega} f := f_* \circ \mathrm{proj}_{\mathcal{H}} : T\mathcal{P} \longrightarrow \mathcal{HP} \longrightarrow TS. \tag{12}$$

This is then a horizontal map since it vanishes on any vertical vectors. The map $d^{\omega} f$ is moreover still equivariant, and hence induces a covariant derivative on sections of the associated bundle $\mathcal{P} \times_{\Psi} S$. If S is a vector space, then this reduces to the usual definition of the exterior covariant derivative of a vector bundle-valued function and $d^{\omega} f$ is a vector-bundle-valued 1-form. Note that due to our initial assumption that $K^{(s)}$ is $\mathfrak{p}$-invariant, we also see that d^{ω} is metric-compatible with respect to $\langle\cdot,\cdot\rangle^{(s)}$.

Following [5], let us define the torsion of the defining map s with respect to the connection ω.

Definition 8 The *torsion* $T^{(s,\omega)}$ of the defining map s with respect to ω is a horizontal $\mathfrak{l}$-valued 1-form on $\mathcal{P}$ given by $T^{(s,\omega)} = (s^*\theta) \circ \mathrm{proj}_{\mathcal{H}}$, where θ is Maurer-Cartan form of $\mathbb{L}$. Equivalently, at $p \in \mathcal{P}$, we have

$$T^{(s,\omega)}\Big|_p = \left(R_{s_p}^{-1}\right)_* d^{\omega} s|_p \,. \tag{13}$$

Thus, $T^{(s,\omega)}$ is the horizontal component of $\theta_s = s^*\theta$. We also easily see that it is Ψ-equivariant. Thus, $T^{(s,\omega)}$ is a *basic* (i.e. horizontal and equivariant) $\mathfrak{l}$-valued 1-form on $\mathcal{P}$, and thus defines a 1-form on M with values in the associated vector bundle $\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$. We have the following properties.

Theorem 2 *Suppose* $s : \mathcal{P} \longrightarrow \mathbb{L}$ *, then*

$$d^{\omega}\varphi_s = \mathrm{id}_{\mathfrak{p}} \cdot T^{(s,\omega)} - \left[\varphi_s, T^{(s,\omega)}\right]^{(s)} \tag{14a}$$

$$d^{\omega}\varphi_s^t = \varphi_s^t\left(\check{T} \cdot \mathrm{id}_{\mathfrak{l}}\right) - \left[\check{T}, \varphi_s^t\right]_{\mathfrak{p}}, \tag{14b}$$

where $\mathrm{id}_{\mathfrak{p}}$ *and* $\mathrm{id}_{\mathfrak{l}}$ *are the identity maps of* $\mathfrak{p}$ *and* $\mathfrak{l}$, *respectively, and* $\cdot$ *denotes the action of the Lie algebra* $\mathfrak{p}$ *on* $\mathfrak{l}$.

Proof Equation (14a) follows from [5, Theorem 4.11]. However to obtain (14b), suppose ξ is an $\mathfrak{l}$-valued map and η is a $\mathfrak{p}$-valued map. Then,

$$\left\langle \left(d^{\omega}\varphi_s^t\right)(\xi), \eta\right\rangle^{(s)} = \langle \xi, \left(d^{\omega}\varphi_s\right)\eta\rangle^{(s)} .$$

Using (14a) and (6b), we obtain (14b). □

Recall that the curvature $F^{(\omega)} \in \Omega^2(\mathcal{P}, \mathfrak{p})$ of the connection ω on $\mathcal{P}$ is given by

$$F^{(\omega)} = d\omega \circ \mathrm{proj}_{\mathcal{H}} = d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{p}}, \tag{15}$$

where wedge product is implied. Given the defining map s, define $\hat{F}^{(s,\omega)} \in \Omega^2(\mathcal{P}, \mathfrak{l})$ to be the projection of the curvature $F^{(\omega)}$ to $\mathfrak{l}$ with respect to s, such that for any $X_p, Y_p \in T_p\mathcal{P}$,

$$\hat{F}^{(s,\omega)} = \varphi_s\left(F^{(\omega)}\right). \tag{16}$$

Theorem 3 ([5, Theorem 4.19])
$\hat{F}^{(s,\omega)}$ and $T^{(s,\omega)}$ satisfy the following structure equation

$$\hat{F}^{(s,\omega)} = d^{\omega}T^{(s,\omega)} - \frac{1}{2}\left[T^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)}, \tag{17}$$

where a wedge product between the 1-forms $T^{(s,\omega)}$ is implied.

In the case of an octonion bundle over a 7-dimensional manifold, this relationship between the torsion and a curvature component has been shown in [2]. Using φ_s and φ_s^t, let us define an adapted covariant derivative as a map from the space of $\mathfrak{l}$-valued equivariant functions $\Omega^0_{basic}(\mathcal{P}, \mathfrak{l})$ to the space of horizontal $\mathfrak{l}$-valued equivariant (i.e. basic) 1-forms $\Omega^1_{basic}(\mathcal{P}, \mathfrak{l})$:

$$d^{\omega}_{\varphi_s} = \frac{1}{\lambda}\varphi_s \circ d^{\omega} \circ \varphi_s^t : \Omega^0_{basic}(\mathcal{P}, \mathfrak{l}) \longrightarrow \Omega^1_{basic}(\mathcal{P}, \mathfrak{l}). \tag{18}$$

We can see that this covariant derivative is metric-compatible as long as d^{ω} is.

Lemma 7 *Suppose $\xi, \eta \in \Omega^0_{basic}(\mathcal{P}, \mathfrak{l})$, then d^{ω} is metric-compatible if and only if*

$$d\langle\xi, \eta\rangle_s = \left\langle d^{\omega}_{\varphi_s}\xi, \eta\right\rangle_s + \left\langle \xi, d^{\omega}_{\varphi_s}\eta\right\rangle_s.$$

Proof This can be shown by explicitly expanding $d^{\omega}_{\varphi_s}$ and noting that since $\varphi_s\varphi_s^t = \lambda\,\mathrm{id}_{\mathfrak{l}}$, $(d^{\omega}\varphi_s)\varphi_s^t = -\varphi_s\left(d\varphi_s^t\right)$. □

Theorem 4 *$\hat{F}^{(s,\omega)}$ satisfies the following Bianchi identity*

$$d^{\omega}_{\varphi_s}\hat{F}^{(s,\omega)} = F_{\mathfrak{h}_s}\dot{\wedge}T^{(s,\omega)}, \tag{19}$$

where $F_{\mathfrak{h}_s} = \pi_{\mathfrak{h}_s}F$ and $\dot{\wedge}$ denotes the action of $\mathfrak{p}$ on $\mathfrak{l}$ combined with the wedge product of p-forms.

Proof We can write $F = F_{\mathfrak{h}_s} + \frac{1}{\lambda}\varphi_s^t\left(\hat{F}\right)$, so applying $\varphi_s \circ d^{\omega}$, the left-hand side vanishes due to the standard Bianchi identity, and we are left with

$$d^{\omega}_{\varphi_s}\hat{F} = -\varphi_s\left(d^{\omega}F_{\mathfrak{h}_s}\right) = (d^{\omega}\varphi_s) \wedge F_{\mathfrak{h}_s}.$$

Using (14a), we obtain (19). □

3.1 Gauge theory

As discussed earlier, equivariant horizontal forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M. So let us now switch perspective, and work in terms of sections of bundles. Recall that the space of connections on $\mathcal{P}$ is an affine space modelled on $\Omega^1(\mathfrak{p}_{\mathcal{P}})$. Thus, any connection $\tilde{\omega} = \omega + A$ for some $A \in \Omega^1(\mathfrak{p}_{\mathcal{P}})$. Then,

$$T^{(s,\tilde{\omega})} = T^{(s,\omega)} + \varphi_s(A) \tag{20}$$

The space of possible torsions of s therefore comes from deformations by elements of $\varphi_s^t\left(\Omega^1(\mathcal{A})\right)$. So define the *torsion space* $\mathcal{T}_s \cong \Omega^1(\mathcal{A})$. Therefore, for any $\xi \in \Omega^1(\mathcal{A})$, the torsion and curvature of $\omega_\xi = \omega + \check{\xi}$ are given by

$$T^{\left(s,\omega_\xi\right)} = T^{(s,\omega)} + \xi \tag{21a}$$

$$\hat{F}^{\left(s,\omega_\xi\right)} = \hat{F}^{(s,\omega)} + d^{\omega}_{\varphi_s}\xi + \frac{1}{2}[\xi,\xi]_{\varphi_s}. \tag{21b}$$

Since our prior assumption of transitivity of the action of Ψ implies that φ_s is surjective, we can find a reference connection ω_0 for which $T^{(s,\omega_0)} = 0$. In particular, ω_0 will have curvature with values in $\mathfrak{h}_s$, and in particular $\hat{F}^{(s,\omega_0)} = 0$. The torsion will unchanged if we add to ω an $\mathfrak{h}_s$-valued 1-form, hence the equivalence $\mathcal{T}_s \cong \Omega^1(\mathcal{A})$ is independent of the choice of a particular ω_0.

Suppose h is a section of the associated bundle $\mathrm{Ad}(\mathcal{P})$, then it defines a gauge-transformation and the gauge transformed connection is $h^*\omega$. In particular, for the section $s \in \Gamma(Q)$, we have

$$d^{h^*\omega}s = (h_*)^{-1}d^{\omega}(h(s)). \tag{22}$$

Since the torsion is determined by the covariant derivative of s, transformations of the connection and the defining section s are very closely related. Indeed, as shown in [5], the corresponding transformation of torsion is given by

$$\begin{aligned} T^{(h(s),\omega)} &= \left(R_{h(s)}\right)_*^{-1} d^{\omega}(h(s)) \\ &= h'_* \circ (R_s)_*^{-1} \circ (h_*)^{-1} d^{\omega}(h(s)) \\ &= h'_* T^{(s,h^*\omega)}, \end{aligned} \tag{23}$$

which follows from Definition 8 and properties of h (2). Recall that we assumed that Ψ acts transitively on $\mathbb{L}$, so that, for a fixed connection ω, all the possible torsions are obtained by the action of Ψ on s, with the non-trivial transformations given by cosets of Ψ/H_s, where $H_s = \mathrm{Stab}(s)$. On the other hand, as (23) shows, transformations of s correspond to gauge transformations of the connection. Since,

$$d^{h^*\omega}s = d^{\omega}s + (h)_*^{-1}(d^{\omega}h)\cdot s, \tag{24}$$

we obtain

$$T^{(s,h^*\omega)} = T^{(s,\omega)} + \varphi_s\left((h_*)^{-1}(d^\omega h)\right). \tag{25}$$

We will define *loop gauge transformations* to be precisely those that act non-trivially on s. Infinitesimally this corresponds to taking $h = \exp(\check{\eta})$ for $\eta \in \Omega^0(\mathcal{A})$, so that

$$T^{(s,u^*\omega)} = T^{(s,\omega)} + d^\omega_{\varphi_s}\eta, \tag{26}$$

hence at $T^{(s,\omega)} \in \mathcal{T}_s$, the tangent vectors to $\mathcal{T}_s$ in the directions of loop gauge transformations correspond precisely to the image of $d^\omega_{\varphi_s}$. Although this is beyond the scope of this note, the L_2-norm of T may be considered as a functional on gauge orbits in $\mathcal{T}_s$. Critical points then become analogues of the Coulomb gauge condition in gauge theory [1, 2, 3, 4, 5, 8].

The above considerations allow us to consider analogues of various functionals defined in gauge theory [5]. The key difference of course is that $\hat{F}$ does not satisfy the standard Bianchi identity.

Let us now specialize to the case of M being a smooth compact 3-dimensional manifold. Following the standard theory, as in [10], let us define a 1-form ρ on $\mathcal{T}_s$, for $\chi \in \Omega^1(\mathcal{A})$, which is also interpreted as an element of $T_\omega\mathcal{T}$, by

$$\rho(\chi)|_\omega = \int_M \left\langle \hat{F}^{(s,\omega)}, \chi\right\rangle^{(s)}. \tag{27}$$

Theorem 5 *Suppose M is a smooth compact 3-dimensional manifold, then $\rho = d\vartheta$, where ϑ is a functional on $\mathcal{T}_s \cong \Omega^1(\mathcal{A})$ given by*

$$\vartheta(\xi) = \frac{1}{2}\int_M \left\langle d^\omega_{\varphi_s}\xi + \frac{1}{3}[\xi,\xi]_{\varphi_s}, \xi\right\rangle^{(s)} dt. \tag{28}$$

The critical points of ϑ correspond to $\omega_\xi = \omega_0 + \check{\xi}$ for which $\hat{F}^{(s,\omega_\xi)} = 0$.

Proof Consider $\omega_\xi = \omega + \check{\xi}$, then using Stokes' Theorem, to first order we get

$$\begin{aligned}\rho(\chi)|_{\omega_\xi} - \rho(\chi)|_\omega &= \int_M \left\langle d^\omega_{\varphi_s}\xi, \chi\right\rangle^{(s)} + O\left(|\xi|^2\right)\\ &= \int_M d\langle \xi, \chi\rangle^{(s)} + \int_M \left\langle \xi, d^\omega_{\varphi_s}\chi\right\rangle^{(s)} + O\left(|\xi|^2\right)\\ &= \int_M \left\langle \xi, d^\omega_{\varphi_s}\chi\right\rangle^{(s)} + O\left(|\xi|^2\right)\end{aligned}$$

Using the same argument as in [10], we see that $d\rho = 0$. Since $\mathcal{T}_s$ is a contractible space, by Poincare lemma, $\rho = d\vartheta$ for some function ϑ on $\mathcal{T}_s$. Consider now a path $\omega(t) = \omega_0 + t\check{\xi}$ from ω_0 to $\omega = \omega_0 + \check{\xi}$, where ω_0 is such that $T^{(s,\omega_0)} = 0$. Integrating it explicitly, and noting that since ρ is closed, this is path-independent, we get,

$$\begin{aligned}\vartheta(\xi)-\vartheta(0) &= \int_0^1 \rho_{\omega(t)}\left(\varphi_s\left(\dot{\omega}(t)\right)\right)dt \\ &= \int_0^1\int_M \left\langle \hat{F}^{(s,\omega(t))},\xi\right\rangle^{(s)} dt \\ &= \int_0^1\int_M \left\langle t d^{\omega}_{\varphi_s}\xi+\frac{1}{2}t^2\left[\xi,\xi\right]_{\varphi_s},\xi\right\rangle^{(s)} dt \\ &= \frac{1}{2}\int_M \left\langle d^{\omega}_{\varphi_s}\xi+\frac{1}{3}\left[\xi,\xi\right]_{\varphi_s},\xi\right\rangle^{(s)} dt.\end{aligned}$$

Setting $\vartheta(0)=0$, and noting that $\xi=T^{(s,\xi)}$ and $d^{\omega}_{\varphi_s}\xi=\hat{F}^{(s,\xi)}-[\xi,\xi]_{\varphi_s}$, we recover

$$\vartheta(\xi)=\frac{1}{2}\int_M\left(\left\langle T,\hat{F}\right\rangle^{(s)}-\frac{1}{6\lambda^2}\left\langle T,[T,T]_{\varphi_s}\right\rangle^{(s)}\right), \tag{29}$$

which (up to a factor of $\frac{1}{2}$), is the Loop Chern-Simons Functional defined in [5]. In particular, we see that $d\vartheta|_{\omega}=0$ if and only if $\hat{F}^{(s,\omega)}=0$, that is, connections for which this holds are critical points of the functional ϑ. □

Unlike in the case of the standard Chern-Simons Functional, ρ does not necessarily vanish along orbits of the non-associative gauge action. As we see from (26), vectors tangent to the orbits are given by $d^{\omega}_{\varphi_s}\eta$ for some $\eta\in\Omega^0(\mathcal{A})$. Using (19), we find

$$\rho\left(d^{\omega}_{\varphi_s}\eta\right)\Big|_{\omega}=\int_M\left\langle F^{\omega}_{\mathfrak{h}_s}\dot{\wedge}T^{(s,\omega)},\eta\right\rangle^{(s)}. \tag{30}$$

Now let us consider how ϑ is affected by gauge transformations. Consider a path $t\in[0,1]$ connecting $T^{(s,\omega)}$ to $T^{(s,u^*\omega)}$. In particular, this is equivalent to a path $\xi(t)\in\Omega^1(\mathcal{A})$ such that $\xi(0)=0$ and $\xi(1)=\varphi_s(u^*\omega-\omega)$. Then, define $\omega(t)=\omega+\check{\xi}(t)$, so that

$$\begin{aligned}\vartheta(\xi(1))-\vartheta(0) &= \int_0^1\rho_{\omega(t)}\left(\varphi_s\left(\dot{\omega}(t)\right)\right)dt \\ &= \int_0^1\int_M\left\langle\hat{F}^{(s,\omega(t))},\dot{\xi}(t)\right\rangle^{(s)} dt.\end{aligned} \tag{31}$$

As in the standard gauge theory [10], we may extend $\mathcal{P}$, and all the associated bundles, to a bundle over $\tilde{M}=M\times[0,1]$. In a local trivialization, let us define the connection $A=A_0dt+A_idx^i$ on $\tilde{M}$ with $A_0=0$ and $(A_i)_{(p,t)}=\omega_i(t)_p$. Then, we see that the curvature F_A of this connection is given by $(F_A)_{0i}=\dot{A}_i(t)$ and $(F_A)_{ij}=\left(F^{(\omega)}\right)_{ij}$. Hence

$$\hat{F}_A=\dot{\xi}_i(t)\,dt\wedge dx^i+\left(\hat{F}^{(s,\omega)}\right)_{ij}dx^i\wedge dx^j=-\dot{\xi}(t)\wedge dt+\hat{F}^{(s,\omega)}, \tag{32}$$

so that $\left\langle\hat{F}_A,\hat{F}_A\right\rangle=-2\left\langle\hat{F}^{(s,\omega(t))},\dot{\xi}(t)\right\rangle^{(s)}\wedge dt$, and thus (31) becomes

$$\vartheta\left(\xi\left(1\right)\right)-\vartheta\left(0\right)=-2\int_{\tilde{M}}\left\langle \hat{F}_{A},\hat{F}_{A}\right\rangle . \qquad (33)$$

This shows that there is a relation between Chern-Simons and a Chern-Weil-like functionals, similar to standard gauge theory. However, the 4-form $\left\langle \hat{F}_{A},\hat{F}_{A}\right\rangle$ on a 4-manifold is not necessarily independent of the choice of connection, so it is not a topological invariant. On the other hand, the above discussion shows that in this particular case, it is independent of the path $\omega\left(t\right)$, so it is important to understand if there is an invariant theory that is related to this non-associative gauge theory.

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