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A characteristic-index inequality for closed embeddings of locally compact groups

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Abstract. The characteristic index of a locally compact connected group G is the non-negative integer d for which we have a homeomorphism $G \cong K \times \mathbb{R}^d$ with K maximal compact in G. We prove that the characteristic indices of closed connected subgroups are dominated by those of the ambient groups.
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Introduction

The *characteristic index* (here denoted by $\mathrm{ci}(G)$) of a locally compact connected group G was introduced in [13] (Theorem 13 therein) in the course of analyzing the structure of such groups: one can always find a closed submanifold $\mathbb{R}^d \subseteq G$ for some $d \in \mathbb{Z}_{\geq 0}$ such that, for a maximal compact subgroup $K \leq G$ the multiplication

$$K \times \mathbb{R}^d \ni (k, r) \mapsto k \cdot r \in G$$

is a homeomorphism. Now simply define the characteristic index by

$$\mathrm{ci}(G) := d;$$

a measure, in other words, of "how non-compact" G is. It is not difficult to see that this definition is appropriate: given homeomorphisms

$$K \times \mathbb{R}^d \cong G \cong K' \times \mathbb{R}^{d'}$$

as above, for maximal compact subgroups K and K', one can

- assume that $K = K'$ because all maximal compact subgroups are mutually conjugate [15, §4.13, first Theorem];

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- and that G is Lie by substituting $K/N \le G/N$ for $K \le G$ for a compact normal subgroup $N \trianglelefteq G$ with G/N Lie [15, §4.6, Theorem];
- hence affording a dimension count:

$$d = \dim G - \dim K = d'.$$

The notion of dimension in use here is the standard one in manifold theory [14, Theorem 1.2], coincident, in the cases of interest here, with various other notions of topological dimension available in the literature (e.g. the *small* and *large inductive dimension* of [6, Definitions 1.1.1 and 1.6.1] and the *covering dimension* of [6, Definition 1.6.7]).

The characteristic index was useful recently in [5] for the purpose of studying colimits in the category of locally compact groups. That analysis required an understanding of how characteristic indices behave under locally-compact-group morphisms, as we now recall.

By a *closed embedding* of topological groups we mean a continuous group morphism $f: H \to G$ that identifies H homeomorphically with a closed subgroup of G. On the other hand, a continuous group morphism $f: H \to G$ has *dense image* if $\overline{\operatorname{im} f} = G$. With this in place, the pertinent observations are that characteristic indices

(a) they can only *decrease* along dense-image morphisms $f: H \to G$ [5, Theorem 2.3] (a slight generalization of [13, Lemma 4.10], the analogue for quotients by closed normal subgroups)

(b) while on the other hand, they can only *increase* along closed embeddings $H \to G$ with H Lie and semisimple [5, Proposition 2.4].

This latter result, in particular, while sufficient as auxiliary material for [5], leaves open the natural question of whether or not $\operatorname{ci}(H) \le \operatorname{ci}(G)$ in full generality for any closed embedding $H \le G$ of connected locally compact groups. The aim of the present note is to prove that this is indeed the case:

Main Theorem. For a closed embedding $H \le G$ of connected locally compact groups the characteristic index of G dominates that of H:

$$\operatorname{ci}(H) \le \operatorname{ci}(G). \tag{0-1}$$

In part, the reason why this appears not to be as straightforward as one might hope is the tension between the two phenomena (a) and (b) above: as Example 2.3 makes clear, it is possible to

- start with a (semisimple, say) group H;
- enlarge its characteristic index by taking a product with a Euclidean group $N \cong \mathbb{R}^d$;
- and then bring the characteristic index back down (as in (a)) through a dense embedding $HN \le \overline{HN} = G$.

The example in question shows that this last step can shrink $\operatorname{ci}(G)$ all the way back down to $\operatorname{ci}(H)$, but the point of (0-1) is that so long as $H \le G$ is closed there can be no *further* characteristic-index shrinkage.

1. Preliminaries

The topological groups considered here are all Hausdorff. In fact, being T_0-separated ([15, §1.1, Definition, condition 4)]) will do: being Hausdorff follows [15, §1.16] (in the context of topological *groups*, not for arbitrary spaces), along with complete regularity [15, §1.18].

We record the following observation on the behavior of characteristic indices under passage to quotients, which aggregates a couple of results in the literature (on quotients by connected / discrete subgroups respectively).

Lemma 1.1. *Let G be a connected locally compact group and $N \trianglelefteq G$ a closed normal subgroup with identity connected component N_0. We then have*

$$\mathrm{ci}(G) = \mathrm{ci}(N_0) + \mathrm{rank}(N/N_0) + \mathrm{ci}(G/N).$$

where N/N_0 is finitely-generated abelian and its rank is the largest r for which there is an embedding $\mathbb{Z}^r \le N/N_0$.

Proof. We first handle the quotient by N_0:

$$\mathrm{ci}(G) = \mathrm{ci}(N_0) + \mathrm{ci}(G/N_0) \tag{1-1}$$

by [13, Lemma 4.10] (and [15, §4.6, Theorem], which ensures that the hypothesis of that lemma is met by connected locally compact groups).

Substituting $N/N_0 \trianglelefteq G/N_0$, we can now assume that the normal subgroup (N/N_0, in this case) is discrete. But then it will be finitely-generated abelian (by [5, Lemma 2.1], for instance), and hence [5, Proposition 0.3] applies to prove

$$\mathrm{ci}(G/N_0) = \mathrm{rank}(N/N_0) + \mathrm{ci}(G/N). \tag{1-2}$$

Combining (1-1) and (1-2) gives the desired result. ■

2. Dense embeddings

Altering the notational lettering momentarily in order to avoid confusion later, consider a dense embedding $S \le G$ of connected Lie groups (not closed, in general: $\overline{S} = G$). According to [5, Theorem 2.3] we have $\mathrm{ci}(S) \ge \mathrm{ci}(G)$. It will be handy below to have a more careful estimate of the difference between the two characteristic indices. Recall that the *radical* of a Lie group is its largest connected, normal, solvable subgroup [18, §1.3].

Lemma 2.1. *Let G be a connected Lie group and further consider*

- *a dense embedding $S \le \overline{S} = G$ with S connected;*
- *a maximal compact subgroup $K \le G$ with radical $A \le K$;*
- *the connected component $A_1 := (A \cap S)_0$.*

Given a decomposition

$$A_1 \cong \mathbb{T}^m \times \mathbb{R}^n \tag{2-1}$$

in the intrinsic topology on the Lie group S *where* $\mathbb{T}^m := \mathbb{R}^m/\mathbb{Z}^m$ *is the standard torus, we have*

$$\mathrm{ci}(S) - \mathrm{ci}(G) = n.$$

Proof. Dense embeddings of Lie groups are analyzed in enough detail in the proof of [7, Theorem 1] for us to be able to repurpose that argument.

Consider, as in that proof, the universal cover

$$\widetilde{G} \to G = \widetilde{G}/D$$

(where $D < \widetilde{G}$ is discrete and central, isomorphic to the fundamental group of G). Generally, tildes will adorn the connected components of preimages through this cover: we have $\widetilde{S}$, $\widetilde{A}$, $\widetilde{A_1}$, etc. $\widetilde{S}$. Note that

- $\widetilde{S} \le \widetilde{G}$ is closed because it is normal in a simply-connected Lie group [3, p.127].
- $\widetilde{A_1} \le \widetilde{G}$ is a closed Euclidean group, and having chosen a (closed, Euclidean) supplement for it in $\widetilde{A}$ in the sense that

  $$\widetilde{A} \cong \widetilde{A_1} \times \widetilde{A_2}$$

  the product

  $$\widetilde{S} \times \widetilde{A_2} \ni (s,a) \mapsto sa \in \widetilde{S}\widetilde{A_2} = \widetilde{G}$$

  is an isomorphism of analytic manifolds (not of groups, necessarily, because $\widetilde{S}$ and $\widetilde{A_2}$ need not commute). This, again, emerges as part of the proof of [7, Theorem 1].
- per the discussion immediately preceding [7, Theorem 1], we may as well assume that the discrete abelian group D is a closed subgroup of $\widetilde{K}$ whence the torsion-free component D_{free} in a decomposition

  $$D \cong D_{tors} \times D_{free} := (\text{torsion}) \times \mathbb{Z}^{\mathrm{rank}(D)} \tag{2-2}$$

  embeds in $\widetilde{A}$.

Through a double application of Lemma 1.1 we have

$$\begin{aligned}
\mathrm{ci}(G) &= \mathrm{ci}(\widetilde{G}) - \mathrm{rank}(D)\\
&= \mathrm{ci}(\widetilde{S}) + \dim \widetilde{A_2} - \mathrm{rank}(D)\\
&= \mathrm{ci}(S) + \mathrm{rank}(D \cap \widetilde{S}) + \dim \widetilde{A_2} - \mathrm{rank}(D),
\end{aligned}$$

so the goal is to argue that

$$\mathrm{rank}(D) = \mathrm{rank}(D \cap \widetilde{S}) + \dim \widetilde{A_2} + n \tag{2-3}$$

for n as in (2-1). A first observation is that since

- we are assuming that the free abelian summand in (2-2) is a subgroup of $\widetilde{A}$;
- and quotienting by D turns the Euclidean group $\widetilde{A}$ into a torus of the same dimension,

we have

$$\mathrm{rank}(D) = \dim \widetilde{A} = \dim \widetilde{A_1} + \dim \widetilde{A_2},$$

and hence the target equation (2-3) becomes

$$\dim \widetilde{A_1} = \mathrm{rank}(D \cap \widetilde{S}) + n.$$

To prove this last equality, notice that on the one hand the left-hand side $\dim \widetilde{A_1} = \dim A_1$ is exactly the $m+n$ of (2-1), while on the other, given the notation (2-2), we have

$$\mathrm{rank}(D \cap \widetilde{S}) = \mathrm{rank}(D_{free} \cap \widetilde{A}) = \mathrm{rank}(D_{free} \cap \widetilde{A_1}).$$

This is nothing but $\dim(A_1) = m$, and we are done. ■

Lemma 2.2. *Let $H \le G$ be a closed embedding of connected Lie groups with $G = \overline{HN}$ for a normal subgroup*

$$\mathbb{R}^d \cong N \trianglelefteq G.$$

We then have the inequality (0-1): $\mathrm{ci}(H) \le \mathrm{ci}(G)$.

Proof. In passing from $H \le \overline{HN}$ to

$$H/H \cap N \le \overline{HN}/H \cap N$$

an application of Lemma 1.1 (or a double application, rather) shows that the two characteristic indices decrease by the same amount, so we may as well assume that $H \cap N$ is trivial.

We apply Lemma 2.1 to the dense inclusion

$$S := HN \le G,$$

retaining the notation therein (for the groups K, A, A_1, etc.). Consider a decomposition (2-1) for $A_1 \le S = HN$. Since A is a torus and $H \le G$ is closed, the torus component $\mathbb{T}^m$ of that decomposition must contain the connected component $(H \cap A)_0$. Passing to Lie algebras, it follows in particular that the Euclidean component

$$\mathbb{R}^n \le \mathfrak{a}_1 := Lie(A_1)$$

of (the Lie-algebra version of) (2-1) intersects

$$\mathfrak{h} := Lie(H) \subset Lie(HN) \cong \mathbb{R}^d \rtimes \mathfrak{h}$$

trivially. It is a simple matter to prove, then, that the projection

$$\mathbb{R}^n \subset \mathfrak{a}_1 \le \mathfrak{h} \subset \mathbb{R}^d \rtimes \mathfrak{h} \to \mathbb{R}^d$$

(linear but not, in general, a Lie-algebra morphism) is on-to-one, whence $n \le d$. We now have

$$\begin{aligned}\mathrm{ci}(G) &= \mathrm{ci}(HN) - n\\ &= \mathrm{ci}(\mathbb{R}^d \rtimes H) - n\\ &= d + \mathrm{ci}(H) - n\\ &\ge \mathrm{ci}(H),\end{aligned}$$

where the first equality uses Lemma 2.1 (as indicated, with $S = HN \cong \mathbb{R}^d \rtimes H$), the third is an application of Lemma 1.1, and the last inequality is the above remark that $n \le d$. ■

To elucidate the phenomenon that underpins Lemma 2.2, some comments and examples are perhaps in order. Assume, as done at the start of the proof of Lemma 2.2, that H intersects N trivially. Abstractly, with its intrinsic topology (rather than the subspace topology inherited from G), HN is then isomorphic to the semidirect product $\mathbb{R}^d \rtimes H$ with respect to the adjoint action of

$$H \le G \text{ on } \mathbb{R}^d \cong N \le G.$$

On the one hand, according to Lemma 1.1 we have

$$\mathrm{ci}(HN) = \mathrm{ci}(\mathbb{R}^d \rtimes H) = d + \mathrm{ci}(H). \tag{2-4}$$

On the other hand though, by [5, Theorem 2.3], in passing to $G = \overline{HN}$ we have to then adjust the characteristic index *down* from that value because the embedding $HN \le G$ is dense:

$$\mathrm{ci}(HN) \ge \mathrm{ci}(G).$$

The point of Lemma 2.2, though, is that because $H \le G$ was closed, this latter discrepancy resulting from the dense embedding cannot be larger than the d we originally supplemented H with in (2-4). That dense embeddings can (in this regard) be pathological enough to achieve this upper bound can be illustrated with an example adapted from [7, Appendix] (used there for different but related purposes).

Example 2.3. We want a dense (connected-)Lie-group embedding $\mathbb{R}^d \rtimes H \le G$ with H and $\mathbb{R}^d$ both closed in G and such that

$$\mathrm{ci}(G) = \mathrm{ci}(H). \tag{2-5}$$

It will be enough to do this for $d = 1$, as the d^{th} Cartesian power of *that* example will then handle the general case.

Consider, as on [7, p.118], the universal cover

$$1 \to \mathbb{Z} = \langle \sigma \rangle \longrightarrow \widetilde{SL(2,\mathbb{R})} \longrightarrow SL(2,\mathbb{R}) \to 1.$$

The relevant objects are

- $H := \widetilde{SL(2,\mathbb{R})}$;
- acting trivially on $N := \mathbb{R}$;
- and $G := H \times N \times \mathbb{R}/D$ with

$$D := \{(\sigma^{m+n},\ m+n,\ m+n\gamma) \in G \mid m,n \in \mathbb{Z}\}$$

  for an irrational $\gamma \in \mathbb{R}$.

As noted in loc.cit., HD/H and ND/D (easily seen to be isomorphic to H and N respectively) are both closed in G, while their product is not. It follows, for dimension reasons, that we must have $\overline{HN} = G$.

As to characteristic indices, note first that $\mathrm{ci}(H) = 3$, since in fact $H = \widetilde{SL(2,\mathbb{R})}$ is homeomorphic to $\mathbb{R}^3$. On the other hand, G is obtained from $H \times \mathbb{R}^2$ by quotienting out a discrete (closed, central) subgroup $D \cong \mathbb{Z}^2$, meaning that by Lemma 1.1 its characteristic index is

$$\mathrm{ci}(G) = \mathrm{ci}(H) + \mathrm{ci}(\mathbb{R}^2) - \mathrm{rank}(\mathbb{Z}^2) = \mathrm{ci}(H);$$

(2-5), in other words.

3. Characteristic indices of maximal subgroups

In the discussion below diagrams of the form

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd} \tag{3-1}$$

indicate (locally trivial) fibrations [20, §2] with total space E, fiber F and base B. We chain several of these together (as will become apparent) to indicate that the fibers themselves are total spaces of further fibrations.

The fibrations we are concerned with here will be at least locally trivial with everything in sight a Hausdorff, metrizable topological manifold, such as, say,

$$\begin{tikzcd} M \arrow[r] & G \arrow[d] \\ & M/G \end{tikzcd} \tag{3-2}$$

for any closed subgroup $M \le G$ of a Lie group (that this is indeed a fibration follows, for instance, from [20, §7.5]). They are in particular *Serre fibrations* in the sense of [1, Chapter VII, Definition 6.2], i.e. the 'espaces fibrés' of [19, Chapitre II, §2, Définition] (as mentioned in [19, Chapitre II, §2, Exemples]), so the results of this latter source apply.

In general, for a manifold X and coefficients R (an abelian group, or ring, or system of local coefficients [9, Appendix 3.H], etc.; whatever is appropriate), we write

$$\mathrm{dimh}_R(X) := \max\{m \in \mathbb{Z}_{\ge 0} \mid H_m(X,R)\}.$$

This is a version of, say, the *(co)homological dimension* of [17, §35].

Proposition 3.1. *Consider a topological manifold M fitting into a chain*

$$\begin{tikzcd} \mathbb{R}^d \arrow[r] & M_{n-1} \arrow[d] & \cdots \arrow[r] & M_1 \arrow[r] \arrow[d] & M \arrow[d] \\ & B_{n-1} & & B_1 & B_0 \end{tikzcd} \tag{3-3}$$

of manifold fibrations with all manifolds connected and all bases B_i compact. Then, d can be recovered as

$$d = \dim(M) - \mathrm{dimh}_{\mathbb{Z}/2}(M). \tag{3-4}$$

Proof. Consider a fibration (3-1) of connected topological manifolds, with B compact and such that

$$\mathrm{dimh}_{\mathbb{Z}/2}(F) = k \text{ and } H_k(F, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

It then follows from

- the fact that
$$\mathrm{dimh}_{\mathbb{Z}/2}(B) = \dim(B) \text{ and } H_{\dim(B)}(B, \mathbb{Z}/2) \cong \mathbb{Z}/2$$
([1, Chapter VI, Corollary 7.12]);

- together with the Serre spectral sequence
$$E^2_{p,q} := H_p(B, H_q(F, \mathbb{Z}/2)) \Rightarrow H_{p+q}(E, \mathbb{Z}/2)$$
attached to the fibration ([19, Chapitre II, §2, Théorème 2])

that

$$\mathrm{dimh}_{\mathbb{Z}/2}(E) = k + \dim(B) \text{ and } H_{k+\dim(B)}(E, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Applying this remark recursively, starting with the leftmost fibration

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

in (3-3) and proceeding rightward, we obtain

$$\mathrm{dimh}_{\mathbb{Z}/2}(M) = \dim(M) - d \text{ and } H_{\dim(M)-d}(M, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

As this is in fact an enhancement of the sought-after conclusion (3-4), we are done. ∎

Proposition 3.2. *For a maximal proper Lie subgroup $M \le G$ of a connected semisimple Lie group G we have*

$$\mathrm{ci}(M) \le \mathrm{ci}(G).$$

Proof. First, we reduce to the case of linear G by modding out a discrete central subgroup $D \trianglelefteq G$:

- the universal cover $\widetilde{G}$ has a linear quotient by a discrete central subgroup $D_1 \trianglelefteq \widetilde{G}$ [18, §1.4];
- whereas G is a quotient of $\widetilde{G}$ by some other discrete central subgroup $D_2 \trianglelefteq \widetilde{G}$;
- whereupon we can take
$$D := D_1D_2/D_2 \triangleleft G = \widetilde{G}/D_2 :$$
the quotient $G/D \cong \widetilde{G}/D_1D_2$ of the semisimple linear Lie group $\widetilde{G}/D_1$ will automatically be linear [8, Lemma 9].

By Lemma 1.1 we have
$$\mathrm{ci}(G) - \mathrm{ci}(G/D) = \mathrm{rank}(D)$$
and
$$\mathrm{ci}(M) - \mathrm{ci}(MD/D) = \mathrm{rank}(M \cap D),$$
so subtracting the two and using the obvious inequality $\mathrm{rank}(D) \ge \mathrm{rank}(M \cap D)$ we obtain
$$\mathrm{ci}(G) - \mathrm{ci}(M) \ge \mathrm{ci}(G/D) - \mathrm{ci}(MD/D).$$
In other words, if the conclusion holds for the embedding $MD/D \le G/D$ of linear groups then it holds in its original form. For that reason, we will henceforth assume that everything in sight is linear.

[16, Theorem 3.1] then applies, ensuring that either

- the radical R of M is compact;
- or the homogeneous space G/M is compact.

In the former case (M has compact radical R) we have
$$\mathrm{ci}(M) = \mathrm{ci}(M/R) = \mathrm{ci}(L)$$
for a Levi factor $L \le M$ [18, §1.3] and the problem reduces to the inclusion $L \le G$ where the smaller group L is also semisimple. The desired conclusion is now precisely [5, Proposition 2.4].

This leaves the case when G/M is compact. We then have, on the one hand, the fibration (3-2) with compact base and fiber $M \cong K_M \times \mathbb{R}^{\mathrm{ci}(M)}$ for a maximal compact subgroup $K_M \le M$, and on the other the analogous decomposition $G \cong K_G \times \mathbb{R}^{\mathrm{ci}(G)}$. The fact that the "non-compact piece" must have the same dimension
$$\mathrm{ci}(G) = \mathrm{ci}(M)$$
now follows from Proposition 3.1. ∎

4. The main result

In this section we prove the main theorem stated in the Introduction.

Remark 4.1. The case when G is abelian is not difficult to deduce from the existing literature.

First, one can pass to the quotient $G/\mathrm{comp}(G)$ by the largest compact subgroup ([11, Definition 5.4, Proposition 5.8]), thus assuming without loss of generality that G is *compact-free*. It then follows [11, Proposition 5.25] that G is a vector group $\mathbb{R}^d$ and H is a vector-space summand thereof [11, Theorem 5.19]; the conclusion follows from the fact that for a vector group $(\mathbb{R}^d, +)$ the characteristic index is d.

Before embarking on the proof, a remark on what will *not* work. Suppose that in (0-1) we have restricted attention to Lie groups (as we will). Since a connected Lie group is analytically isomorphic to a manifold of the form

$$G \cong K \times \mathbb{R}^{\mathrm{ci}(G)}$$

for a maximal compact subgroup $K \leq G$ (as follows, for instance, from [15, §4.13, first Theorem]), one might hope that the main theorem would follow from a general result to the effect that for any analytic closed embedding

$$M_1 \times \mathbb{R}^m \subseteq M_2 \times \mathbb{R}^n \tag{4-1}$$

for compact analytic manifolds M_i we have to have $m \leq n$. This is not the case:

Example 4.2. Consider the closed analytic embedding

$$\mathbb{R}^2 \ni (x,\ y) \mapsto (\varphi(x,y),\ x^2+y^2) \in \mathbb{S}^2 \times \mathbb{R},$$

where $\varphi : \mathbb{R}^2 \to \mathbb{S}^2$ is the inverse of the stereographic projection [14, Problem 1-7], mapping $\mathbb{R}^2$ isomorphically onto the complement of the north pole in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. This is

- analytic because its two components φ and $(x,y) \mapsto x^2+y^2$ are;
- one-to-one because φ already is;
- proper because $(x,y) \mapsto x^2+y^2$ is.

We thus have a closed analytic embedding of the form (4-1), with $m = 2 > 1 = n$.

In the proof of the main result we will, at one point, have to work with connected Lie subgroups $M \leq G$ of a semisimple Lie group that are *maximal* among proper connected Lie subgroups. These have been studied extensively by Mostow in [16], which deals mostly the case of linear G.

Although this is not stated explicitly in [16] (as far as I can tell), such maximal subgroups are always automatically closed, regardless of whether or not G is linear. This follows by assembling together a number of remarks.

- Let us focus for the moment on the linear-G case. As noted in loc.cit. (in the course of the proof of [16, Theorem 3.1]), the Lie algebra

$$\mathfrak{m} := Lie(M) \subset \mathfrak{g} := Lie(G) \tag{4-2}$$

  is maximal and hence *algebraic* in the sense of [2, Definition 1].

  To see this, recall that in general, a Lie algebra $\mathfrak{h}$ of matrices has an *algebraic hull* $\mathfrak{h}^*$ [2, Definition 2]: the smallest algebraic Lie algebra containing $\mathfrak{h}$. Now, Lie algebras have the same derived subalgebras as their algebraic hulls [2, Proposition 1 part 3))], and since $\mathfrak{g}$ is semisimple and hence coincides with its derived subalgebra [12, §5.2, Corollary], we cannot have

$$\mathfrak{m} \subsetneq \mathfrak{m}^* = \mathfrak{g}.$$

- But then the Lie group corresponding to $\mathfrak{m}$ is expressible by polynomial equations (e.g. as explained on [4, p.195]), so it will be closed.
- All of that is still in the context of a *linear* semisimple G. Generally, given a maximal connected Lie subgroup $M \subset G$, the Lie-algebra inclusion (4-2) will stay as-is upon quotienting by a discrete central subgroup $D \subset G$ with G/D linear (see the proof of Proposition 3.2 for more on D). But we have just argued that

$$MD/D \cong M/M \cap D \subset G/D$$

  is closed, and hence so is the connected component $M = (MD)_0$ of its preimage through $G \to G/D$.

Henceforth, whenever handling maximal Lie subgroups (always of *semisimple* Lie groups), they will be assumed connected and proper. Their automatic closure will also be taken for granted, per the above remarks.

Proof of the Main Theorem. There is no cost to assuming that G and H are Lie: G has a normal compact subgroup $K \trianglelefteq G$ with G/K Lie [15, §4.6, Theorem], and the passage from $H \leq G$ to

$$HK/K \cong H/H \cap K \leq G/K$$

changes nothing ($HK \leq G$ is still closed because K is compact, and the characteristic indices do not change because again, we are modding out compact normal subgroups).

The Lie-group version of the result, in turn, is now amenable to induction by $\dim(G) - \dim(H)$ (which quantity we refer to as the *dimension difference* of the inclusion). There is, of course, nothing to prove in the base case of dimension-difference 0.

So long as we can find an abelian, connected, proper and non-trivial normal subgroup $N \trianglelefteq G$ we can decompose the original inclusion as

$$H \leq \overline{HN} \leq G.$$

Each of thee two successive inclusions has strictly smaller dimension difference, so we can appeal to the induction hypothesis assuming those inclusions have been taken care of. Note furthermore that the right-hand embedding

$$\overline{HN} \leq G$$

further reduces to

$$\overline{HN}/N \le G/N$$

by Lemma 1.1. In this fashion, we can boil down the problem to two cases:

(a) G is of the form $\overline{HN}$ for a connected, normal, abelian group $N \trianglelefteq G$;

(b) G is semisimple.

Case (a) we defer until later (Lemma 2.2), noting here only that since N is abelian and connected it must be of the form

$$\text{torus} \times \mathbb{R}^n$$

(see for instance [10, Chapter II, Exercise C.2]), and the torus component can always be annihilated with no change to characteristic indices. For that reason, when we return to (a) in Lemma 2.2, we will be assuming that N is a Euclidean group (i.e. one of the form $(\mathbb{R}^n, +)$).

The rest of the present proof, then, focuses on the semisimple-G case ((b) above). The strategy will be to again shrink the dimension difference for as long as it is possible. Specifically, assuming $H \le G$ is not maximal among connected (proper) Lie subgroups, it can be embedded into such a maximal subgroup $M \le G$. We would then have to handle the two inclusions

$$H \le M \quad \text{and} \quad M \le G$$

separately, given our induction hypothesis. For the former, we can simply proceed as before: if M is not semisimple break up $H \le M$ into successive embeddings again resorting to induction, etc. As to the latter, we once more handle it separately as a special case in Proposition 3.2. ■

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