

## A DETERMINING FORM FOR THE 2D RAYLEIGH-BÉNARD PROBLEM

YU CAO, MICHAEL S. JOLLY<sup>†</sup>, AND EDRISS S. TITI

**ABSTRACT.** We construct a determining form for the 2D Rayleigh-Bénard (RB) system in a strip with solid horizontal boundaries, in the cases of no-slip and stress-free boundary conditions. The determining form is an ODE in a Banach space of trajectories whose steady states comprise the long-time dynamics of the RB system. In fact, solutions on the global attractor of the RB system can be further identified through the zeros of a scalar equation to which the ODE reduces for each initial trajectory. The twist in this work is that the trajectories are for the velocity field only, which in turn determines the corresponding trajectories of the temperature.

### 1. INTRODUCTION

It was shown in [19] that the long-time dynamics of the 2D Rayleigh-Bénard (RB) problem is entirely contained in the *global attractor*  $\mathcal{A}$ , which is a compact finite-dimensional subset of an infinite-dimensional Hilbert space  $H$ . An inertial manifold, if it exists, is a finite-dimensional invariant smooth manifold that contains the global attractor and attracts all the orbits at an exponential rate (see, e.g., [20]). The system obtained by restriction to an inertial manifold is called an *inertial form*. It is a finite-dimensional system of ODEs which reproduces the dynamics of the original system. While the existence of the inertial manifolds has been established for a considerable number of dissipative systems (see, e.g., [11, 21, 26, 27] and references therein), it has been an open problem since the 1980s for the 2D Navier-Stokes equations (NSE), and hence for the 2D RB problem as well.

The 2D NSE and 2D RB problem do enjoy a finite number of *determining parameters* (see, e.g., [9, 14, 18, 25]). For instance, in the case of determining Fourier modes, if two complete trajectories in the global attractor coincide upon projection  $P_m$  on a sufficiently large number  $m$  of low Fourier modes, then they must be the same (see, e.g., [9, 14, 18, 25]). Thus it is natural to expect the existence of a lifting map  $W : P_m \mathcal{A} \rightarrow \mathcal{A}$ . This property inspired the notion of a *determining form*, introduced in [15]. A determining form is an ODE in an infinite-dimensional Banach space of trajectories that captures the dynamics of the original system in a certain way. Rather than being a dimension reduction, as is the case for the inertial form, the determining form trades the infinite-dimensionality of physical space for that of time; the elements in its phase space are trajectories. It is an ODE in that it is represented by a globally Lipschitz vector field.

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<sup>†</sup>Corresponding author.



There are currently two approaches to constructing a determining form. The key step in either case is to extend the domain of the lifting map  $W$  to a Banach space  $X$  of projected trajectories. The determining form constructed here is based on the nudging approach to continuous data assimilation (see [1, 2]). It is given by

$$(1.1) \quad \frac{dv}{ds} = -\|v - I_h W(v)\|_X^2 (v - I_h u^*)$$

where  $u^*$  is some steady state of the original system, and  $\|\cdot\|_X$  is a sup norm on a Banach space of trajectories that evolve in the finite-dimensional range of some interpolant operator  $I_h$ . Note that the evolutionary variable is now  $s \in \mathbb{R}$ , not time. The trajectories in the global attractor of the original system are precisely the steady states ( $s$ -independent solutions) of (1.1). To show that (1.1) is an ODE in the true sense boils down to proving that the mapping  $W$  is globally Lipschitz on a ball in  $X$ , big enough to accommodate  $I_h \mathcal{A}$ . In addition to the 2D NSE (see [16]), this recipe has been carried out for the damped-driven nonlinear Schrödinger, damped-driven Korteweg–de Vries, and surface quasigeostrophic equations (see [3, 4, 22–24]), each with particular treatment and subtle twists in the analysis. This general procedure is developed in detail in Section 3.

In this paper we construct a determining form for the Rayleigh–Bénard problem. The novelty here is that the phase space  $X$  corresponds to projections of the velocity field alone. Still, both velocity and temperature of all trajectories in the global attractor of the 2D RB problem are identified through steady states of the determining form. This is the first such construction where the trajectories are in a subset of the system state variables. This was suggested in the context of data assimilation by [12, 13] where it was proved that coarse velocity data alone is sufficient to synchronize with a reference solution of the RB problem. The key difficulty in establishing the crucial Lipschitz property of the lifting map  $W$  is in getting *a priori* estimates that are independent of the nudging parameter. Doing this with nudging only in the velocity component adds an extra challenge.

We treat both no-slip and stress-free boundary conditions for the velocity field. Different analysis is needed for each case. In the stress-free case, the problem is equivalent to a periodic boundary condition problem in an extended domain with particular symmetries, which allows us to eliminate one of the nonlinear terms in the estimates. On the other hand, we do not in this case have the Poincaré inequality for (the first component of “velocity”)  $w$ , which is worked around by combining estimates of several norms. We observe that similar techniques are used in [7] to obtain sharper bounds on the size of the global attractor  $\mathcal{A}$  in the case of stress-free boundary conditions than previously known.

This paper rests on several pioneering works of Ciprian Foias. The very notion of a determining form was originally his idea, first realized in [15, 16] and spawned by the concepts of determining modes, he developed with Prodi [14], and inertial forms, with Constantin, Nicolaenko, Sell and Temam in [10, 11, 20]. The foundation of the analysis here is the global attractor, which he established and investigated for the Rayleigh–Bénard system with Manley and Temam [19] (see also [10, 27] and references therein). We have benefitted immeasurably from Ciprian’s insights, not just for this work, but for broader matters of science and life itself.

This paper is organized as follows. In Section 2, we lay out the functional setting for the RB problem subject to each set of boundary conditions. In Section 3, we recall the notion of interpolant operators and define the determining map. The main results (Theorem 3.9) are then presented at the end of the section. We give the proofs of Proposition 3.4, 3.5 and 3.7 in Section 4, 5 and 6, respectively.

## 2. NOTATION AND PRELIMINARIES

Under a similar change of variables as in [19], the 2D RB problem in an infinite strip  $\{(x_1, x_2) : 0 < x_2 < l\}$  with solid boundaries at  $x_2 = 0$  and  $x_2 = l$ , can be written as

$$(2.1a) \quad \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = g \theta \mathbf{e}_2,$$

$$(2.1b) \quad \frac{\partial \theta}{\partial t} - \kappa \Delta \theta + (u \cdot \nabla) \theta = \frac{u \cdot \mathbf{e}_2}{l},$$

$$(2.1c) \quad \nabla \cdot u = 0,$$

$$(2.1d) \quad u(0; x) = u_0(x), \quad \theta(0; x) = \theta_0(x),$$

where  $g$  denotes the gravitational acceleration. Unlike [19], we retain the dimension of the velocity  $u$  while the temperature fluctuation  $\theta$  is dimensionless. In this paper, we consider the following two sets of boundary conditions of physical interest.

No-slip:

in the  $x_2$ -variable:  $u, \theta = 0$  at  $x_2 = 0$  and  $x_2 = l$ ,

in the  $x_1$ -variable:  $u, \theta, p$  are of periodic  $L$ .

Stress-free:

in the  $x_2$ -variable:  $\frac{\partial u_1}{\partial x_2}, u_2, \theta = 0$  at  $x_2 = 0$  and  $x_2 = l$ ,

in the  $x_1$ -variable:  $u, \theta, p$  are of periodic  $L$ .

**2.1. Function spaces.** We will use the same notation indiscriminately for both scalar and vector Lebesgue and Sobolev spaces, which should not be a source of confusion.

We denote

$$(u, v) := \int_{\Omega} u \cdot v, \quad |u| := (u, u)^{1/2}, \quad \text{for } u, v \in L^2(\Omega),$$

$$((u, v)) := \int_{\Omega} \nabla u \cdot \nabla v, \quad \|u\| := ((u, u))^{1/2}, \quad \text{for } \nabla u, \nabla v \in L^2(\Omega),$$

for a domain  $\Omega$  that will be specified for each case of boundary conditions.

**2.1.1. No-slip BCs.** We define function spaces corresponding to the no-slip boundary conditions as in [12]. Let  $\Omega = \Omega_0 := (0, L) \times (0, l)$  and  $\mathcal{F}$  is the set of  $C^\infty(\Omega)$  functions, which are finite sums of the form  $\sum p_i(x_1) \phi_i(x_2)$  where each  $p_i$  is a trigonometric polynomial with period  $L$  and each  $\phi_i$  is compactly supported on the interval  $(0, l)$ .

Denote the space of smooth vector-valued functions which incorporates the divergence-free condition by

$$\mathcal{V} := \{u \in \mathcal{F} \times \mathcal{F} : \nabla \cdot u = 0\},$$

and the closures of  $\mathcal{V}$  and  $\mathcal{F}$  in  $L^2(\Omega)$  by  $H_0$  and  $H_1$ , respectively, which are endowed with the usual inner products and associated norms

$$(2.3) \quad \begin{aligned} (u, v)_{H_0} &:= (u, v), \quad (\psi, \phi)_{H_1} := (\psi, \phi), \quad \|u\|_{H_0} := (u, u)^{1/2}, \quad \|\psi\|_{H_1} := (\psi, \psi)^{1/2}. \end{aligned}$$

The closures of  $\mathcal{V}$  and  $\mathcal{F}$  in  $H^1(\Omega)$  will be denoted by  $V_0$  and  $V_1$ , respectively, endowed with the inner products and associated norms

$$((u, v))_{V_0} := ((u, v)), \quad ((\psi, \phi))_{V_1} := ((\psi, \phi)), \quad \|u\|_{V_0} := \|u\|, \quad \|\phi\|_{V_1} := \|\phi\|.$$

**2.1.2. Stress-free BCs.** Following [13], we consider the equivalent formulation of the 2D RB problem (2.1) subject to the fully periodic boundary conditions on the extended domain  $\Omega = (0, L) \times (-l, l)$  with the following special spatial symmetries: for  $(x_1, x_2) \in \Omega$ ,

$$\begin{aligned} u_1(x_1, x_2) &= u_1(x_1, -x_2), & u_2(x_1, x_2) &= -u_2(x_1, -x_2), \\ p(x_1, x_2) &= p(x_1, -x_2), & \theta(x_1, x_2) &= -\theta(x_1, -x_2). \end{aligned}$$

Observe that for  $(x_1, x_2) \in \Omega$  with  $x_2 = -l, 0, l$ , and for smooth enough functions one has

$$\frac{\partial u_1}{\partial x_2}, u_2, \theta = 0,$$

that is, one recovers the original corresponding physical boundary conditions when restricted to the physical domain  $\Omega_0$ .

We define function spaces corresponding to the “stress-free” boundary conditions, i.e., the periodic BCs with the above symmetries, as in [13], where

$\mathcal{F}_1$  is the set of trigonometric polynomials in  $(x_1, x_2)$ , with period  $L$  in the  $x_1$ -variable, that are even, with period  $2l$ , in the  $x_2$ -variable,

and

$\mathcal{F}_2$  is the set of trigonometric polynomials in  $(x_1, x_2)$ , with period  $L$  in the  $x_1$ -variable, that are odd, with period  $2l$ , in the  $x_2$ -variable.

The symmetries of the two velocity components lead us to take in the stress-free case

$$\mathcal{V} := \{u \in \mathcal{F}_1 \times \mathcal{F}_2 : \nabla \cdot u = 0\}.$$

The space  $H_0$  will again be the closure of  $\mathcal{V}$  in  $L^2(\Omega)$ , but  $H_1$  shall be that of  $\mathcal{F}_2$  in  $L^2(\Omega)$ , with inner products and norms as in (2.3).

Similarly, we denote the closures of  $\mathcal{V}$  and  $\mathcal{F}_2$  in  $H_{\text{per}}^1(\Omega)$  by  $V_0$  and  $V_1$ , respectively, but with the inner products

$$((u, v))_{V_0} := \frac{1}{|\Omega|} (u, v) + ((u, v)), \quad ((\psi, \phi))_{V_1} := ((\psi, \phi)),$$

and associated norms

$$\|u\|_{V_0} := \left( \frac{1}{|\Omega|} |u|^2 + \|u\|^2 \right)^{1/2}, \quad \|\phi\|_{V_1} := \|\phi\|.$$

## 2.2. The linear operators $A_i$ .

2.2.1. *No-slip BCs.* Let  $A_i : D(A_i) \rightarrow H_i$  ( $i = 0, 1$ ) be the unbounded linear operators defined by

$$(A_i u, v)_{H_i} = ((u, v))_{V_i}, \quad i = 0, 1, \quad \forall u, v \in D(A_i),$$

where  $D(A_0) = V_0 \cap H^2(\Omega)$  and  $D(A_1) = V_1 \cap H^2(\Omega)$ .

For each  $i = 0, 1$ , the operator  $A_i$  is self-adjoint and  $A_i^{-1}$  is a compact, positive-definite, self-adjoint linear operator in  $H_i$ . There exists a complete orthonormal set of eigenfunctions  $(\zeta_{i,j})_{j=1}^{\infty}$  in  $H_i$  such that  $A_i \zeta_{i,j} = \lambda_{i,j} \zeta_{i,j}$  where

$$0 < \lambda_{i,1} \leq \lambda_{i,2} \leq \cdots \leq \lambda_{i,m} \leq \cdots,$$

Observe that we have the following Poincaré inequalities:

$$(2.4) \quad |\phi|^2 \leq \lambda_1^{-1} \|\phi\|^2, \quad \forall \phi \in V_i,$$

$$(2.5) \quad \|\phi\|^2 \leq \lambda_1^{-1} |A_i \phi|^2, \quad \forall \phi \in D(A_i),$$

where  $\lambda_1 := \lambda_{1,1} = \lambda_{2,1}$ .

**Remark 2.1.** We observe that in this case  $|A_0 \phi|$  is equivalent to  $\|\phi\|_{H^2}$  for every  $\phi \in D(A_0)$ .

2.2.2. *Stress-free BCs.* Let  $A_i : D(A_i) \rightarrow H_i$  ( $i = 0, 1$ ) be the unbounded linear operators defined by  $A_i = -\Delta$ , where  $D(A_0) = V_0 \cap H^2(\Omega)$  and  $D(A_1) = V_1 \cap H^2(\Omega)$ .

**Remark 2.2.** The operator  $A_0$  is a nonnegative operator and possesses a sequence of eigenvalues with

$$0 = \lambda_{0,1} < \lambda_{0,2} \leq \cdots \leq \lambda_{0,m} \leq \cdots,$$

associated with an orthonormal basis  $\{\zeta_{0,m}\}_{m \in \mathbb{N}}$  of  $H_0$ . The operator  $A_1$  is a positive self-adjoint operator and possesses a sequence of eigenvalues with

$$0 < \lambda_{1,1} \leq \lambda_{1,2} \leq \cdots \leq \lambda_{1,m} \leq \cdots,$$

associated with an orthonormal basis  $\{\zeta_{1,m}\}_{m \in \mathbb{N}}$  of  $H_1$ . Observe that we have the Poincaré inequality for temperature:

$$(2.6) \quad |\theta|^2 \leq \lambda_1^{-1} \|\theta\|^2, \quad \forall \theta \in V_1,$$

$$(2.7) \quad \|\theta\|^2 \leq \lambda_1^{-1} |A_1 \theta|^2, \quad \forall \theta \in D(A_1),$$

where  $\lambda_1 = \lambda_{1,1}$ .

**Remark 2.3.** In the stress-free case, we do not have the Poincaré inequality for functions in  $V_0$ , but we have

$$(2.8) \quad |u|^2 \leq |\Omega| \|u\|_{V_0}^2, \quad \forall u \in V_0$$

by the definition of the norm  $\|\cdot\|_{V_0}$ .

**Remark 2.4.** By the elliptic regularity of the operator  $A_0 + I$  (see [13, Remark 2.3]), we have in the stress-free case the equivalency

$$(2.9) \quad \widetilde{c}_E^2 \left( \frac{1}{|\Omega|} \|u\|_{L^2} + \|A_0 u\|_{L^2} \right) \leq \|u\|_{H^2} \leq c_E^2 \left( \frac{1}{|\Omega|} \|u\|_{L^2} + \|A_0 u\|_{L^2} \right), \quad \forall u \in D(A_0).$$

**2.3. The bilinear maps  $B_i$ .** Denote the dual space of  $V_i$  by  $V'_i$  ( $i = 0, 1$ ). Define the bilinear map  $B_0 : V_0 \times V_0 \rightarrow V'_0$  (and the trilinear map  $b_0 : V_0 \times V_0 \times V'_0 \rightarrow \mathbb{R}$ ) by the continuous extension of

$$b_0(u, v, w) := \langle B_0(u, v), w \rangle_{V'_0} = ((u \cdot \nabla)v, w), \quad u, v, w \in \mathcal{V}.$$

**2.3.1. No-slip BCs.** Define the scalar analogue  $B_1 : V_0 \times V_1 \rightarrow V'_1$  (and the trilinear map  $b_1 : V_0 \times V_1 \times V'_1 \rightarrow \mathbb{R}$ ) by the continuous extension of

$$b_1(u, \theta, \phi) := \langle B_1(u, \theta), \phi \rangle_{V'_1} = ((u \cdot \nabla)\theta, \phi), \quad u \in \mathcal{V}, \quad \theta, \phi \in \mathcal{F}.$$

The bilinear maps  $B_i$  (and the trilinear maps  $b_i$ ),  $i = 0, 1$ , have the orthogonality property:

$$(2.10) \quad b_0(u, v, v) = 0, \quad b_1(u, \theta, \theta) = 0, \quad u, v \in V_0, \quad \theta \in V_1.$$

**2.3.2. Stress-free BCs.** Define the scalar analogue  $B_1 : V_0 \times V_1 \rightarrow V'_1$  (and the trilinear map  $b_1 : V_0 \times V_1 \times V'_1 \rightarrow \mathbb{R}$ ) by the continuous extension of

$$b_1(u, \theta, \phi) := \langle B_1(u, \theta), \phi \rangle_{V'_1} = ((u \cdot \nabla)\theta, \phi), \quad u \in \mathcal{V}, \quad \theta, \phi \in \mathcal{F}_2.$$

The bilinear maps  $B_i$  (and the trilinear maps  $b_i$ ),  $i = 0, 1$ , have the same orthogonality property (2.10) as in the no-slip case. Furthermore, we have for each  $u \in D(A_0)$ ,

$$(2.11) \quad b_0(u, u, A_0 u) = 0,$$

which is not true in general in the no-slip case.

**2.4. Functional setting and bounds for the global attractor.** Following [19], we have the functional form of the RB problem (2.1):

$$(2.12a) \quad \frac{du}{dt} + \nu A_0 u + B_0(u, u) = \mathbb{P}_\sigma(g\theta e_2),$$

$$(2.12b) \quad \frac{d\theta}{dt} + \kappa A_1 \theta + B_1(u, \theta) = \frac{u \cdot e_2}{l},$$

$$(2.12c) \quad u(0; x) = u_0(x), \quad \theta(0; x) = \theta_0(x),$$

where  $\mathbb{P}_\sigma$  denotes the Helmholtz-Leray projector from  $L^2(\Omega)$  onto  $H_0$ .

**2.4.1. No-slip BCs.** It is shown in [19] that the RB system (2.1) with no-slip boundary conditions has a global attractor

$$(2.13) \quad \mathcal{A} = \{(u_0, \theta_0) \in H_0 \times H_1 : \exists \text{ a unique solution } (u, \theta)(t; u_0, \theta_0) \text{ of (2.1) for all } t \in \mathbb{R} \text{ and } \sup_t (\|u(t)\|_{V_0} + \|\theta(t)\|_{V_1}) < \infty\}.$$

Alternatively,  $\mathcal{A}$  is the maximal bounded invariant subset of  $V_0 \times V_1$  under the dynamics of (2.12). Moreover, there exists some (dimensional) constants  $J_i > 0$ ,  $i = 1, 2$ , such that

$$(2.14) \quad \sup_{t \in \mathbb{R}} \|u(t)\|_{V_0} \leq J_1, \quad \sup_{t \in \mathbb{R}} \|u(t)\|_{H^2} \leq J_2, \quad \forall (u, \theta) \in \mathcal{A}.$$

Henceforth, lowercase letters  $c_L, c_A, c_i, \dots$  will denote universal dimensionless positive constants; uppercase letters  $C, J_i, K, K_i, \dots$  will denote positive dimensional constants that depend on the physical parameters.

**2.4.2. Stress-free BCs.** The case of stress-free boundary conditions is studied further in [7]. With the stress-free boundary conditions, the RB system has steady states with arbitrarily large  $L^2$ -norms:

$$u(x) = (c, 0), \quad \theta(x) = 0, \quad c \in \mathbb{R},$$

which means that the system is not dissipative. However, since (see also [7])

$$\frac{d}{dt} \int_{\Omega} u(x, t) \, dx = 0,$$

we may assume in the stress-free case that the velocity field has a fixed average:

$$(2.15) \quad \int_{\Omega} u(x, t) \, dx = a, \quad \forall t \in \mathbb{R},$$

where  $a \in \mathbb{R}$  is fixed. Observe that the spatial average is conserved and the system is dissipative within each invariant affine space of fixed average  $a$ . It is shown in [7] that the RB system has a global attractor  $\mathcal{A} = \mathcal{A}_a$ , in each affine subspace of  $V_0 \times V_1$  where the spatial average (2.15) of velocity is fixed. Moreover, there exist some (dimensional) constants  $J_i = J_i(a) > 0$ ,  $i = 1, 2$ , such that (2.14) holds. In this case of stress-free boundary conditions, the dependence of  $J_i$ ,  $i = 1, 2$ , is shown in [7] to be algebraic in the physical parameters  $\nu, \kappa, l$  and  $L$ .

### 3. DETERMINING FORM AND MAIN RESULTS

In order to define the determining form, we need the notion of interpolant operators.

**3.1. Interpolant operators.** We recall a general class of interpolant operators introduced in [1, 2] for dealing with various determining parameters such as modes, nodes, volume elements, etc. These operators are finite-rank operators (bounded, linear and with finite-dimensional range) and are required to satisfy an approximation of identity type condition.

A finite-rank operator  $I_h : H^1(\Omega) \rightarrow H^1(\Omega)$  is a *Type I interpolant operator* if it satisfies

$$(3.1) \quad |\varphi - I_h(\varphi)| \leq c_0 h \|\varphi\|_{H^1}, \quad \forall \varphi \in H^1;$$

$$(3.2) \quad \|\varphi - I_h(\varphi)\|_{H^1} \leq \tilde{c}_0 \|\varphi\|_{H^1}, \quad \forall \varphi \in H^1.$$

A finite-rank operator  $I_h : H^2(\Omega) \rightarrow H^1(\Omega)$  is a *Type II interpolant operator* if it satisfies

$$(3.3) \quad |\varphi - I_h(\varphi)| \leq c_1 h \|\varphi\|_{H^1} + c_2 h^2 \|\varphi\|_{H^2}, \quad \forall \varphi \in H^2;$$

$$(3.4) \quad \|\varphi - I_h(\varphi)\|_{H^1} \leq \tilde{c}_1 \|\varphi\|_{H^1} + \tilde{c}_2 h \|\varphi\|_{H^2}, \quad \forall \varphi \in H^2.$$

In this paper, we construct a determining form for the RB system using Type II interpolants. The same can be done under slightly weaker assumptions on  $h$  for Type I interpolants (see [6]).

**Remark 3.1.** The orthogonal projection onto low Fourier modes, those with wave numbers  $k$  such that  $|k| \leq 1/h$ , is one example of a Type I interpolant. Another is finite volume elements. In addition, an example of a Type II interpolant is an interpolant operator that is based on nodal values satisfying (3.3) and (3.4). See, e.g., [1] for more details.

**Remark 3.2.** In the stress-free case, by definition, we have  $\|\varphi\|_{H^1} = \|\varphi\|_{V_0}$ , for  $\varphi \in V_0$ . Moreover, by (2.9) in Remark 2.4, replacing the absolute constants when necessary, we can replace  $\|\varphi\|_{H^2}$  by  $|A_0\varphi|$  in (3.3) and (3.4), for  $\varphi \in D(A_0)$ .

We need to modify the interpolant operator  $I_h$  so that it has a range of functions that are divergence-free and satisfy the boundary conditions. Motivated by [8, Proposition 2.1], we define the modified Type II interpolant operator  $\tilde{I}_h : H^2 \rightarrow V_0$  as

$$(3.5) \quad \tilde{I}_h := P_r I_h, \quad P_r \phi = \sum_{i=1}^r (\phi, \zeta_{0,i}) \zeta_{0,i}, \quad h^2 \sim \frac{1}{\lambda_{0,r}},$$

where we recall that  $\{\zeta_{0,i}\}$  are the eigenfunctions of the operator  $A_0$  in Section 2.2. The phase space  $(X, \|\cdot\|_X)$  of our determining form is then defined as

$$(3.6) \quad X := C_b(\mathbb{R}; \tilde{I}_h H^2), \quad \|v\|_X := \frac{\sup_{t \in \mathbb{R}} \|v(t)\|_{V_0}}{\nu \lambda_1^{1/2}}.$$

**Remark 3.3.** Based on the proof in [8, Proposition 2.1], we observe that  $\tilde{I}_h$  satisfies conditions (3.3) and (3.4) with modified constants  $c_i, \tilde{c}_i$ ,  $i = 1, 2$ . Furthermore, in the no-slip case, by the Poincaré inequality, modifying the constants  $c_i$  when necessary, we have

$$(3.7) \quad |\varphi - \tilde{I}_h(\varphi)| \leq c_1 h \|\varphi\|_{V_0} + c_2 h^2 |A_0\varphi|, \quad \forall \varphi \in D(A_0).$$

We also have (3.7) for the stress-free case by Remark 3.2.

**3.2. Auxiliary system and determining map.** Consider the following auxiliary system:

$$(3.8a) \quad \frac{dw}{dt} + \nu A_0 w + B_0(w, w) = \mathbb{P}_\sigma(g\eta e_2) - \mu\nu\lambda_1(\tilde{I}_h w - v),$$

$$(3.8b) \quad \frac{d\eta}{dt} + \kappa A_1 \eta + B_1(w, \eta) = \frac{w \cdot e_2}{l},$$

where  $v \in B_X(0, \rho) := \{\xi \in X : \|\xi\|_X < \rho\}$  with  $\rho > 0$  and  $\tilde{I}_h$  is a (modified) Type II interpolant operator. Note that the nudging term in (3.8) appears only in the momentum equation.

**Proposition 3.4** (Solutions to the auxiliary system). *Let  $\rho$  be a positive real number. Let  $\mu > 0$  be sufficiently large and  $h > 0$  sufficiently small (see conditions in Section 4). Then for each  $v \in B_X(0, \rho)$ , system (3.8) has a unique bounded solution  $(w(t), \eta(t))$  that exists for all  $t \in \mathbb{R}$  such that*

$$(3.9) \quad (w, \eta) \in C_b(\mathbb{R}, V_0 \times V_1) \cap L^2_{\text{loc}}(\mathbb{R}, D(A_0) \times D(A_1)), \quad \left( \frac{dw}{dt}, \frac{d\eta}{dt} \right) \in L^2_{\text{loc}}(\mathbb{R}, H_0 \times H_1).$$

The proof of Proposition 3.4 is given in Section 4. Note that this proposition provides a map, called the *determining map*,

$$\widetilde{W} : B_X(0, \rho) \rightarrow C_b(\mathbb{R}; V_0 \times V_1) \cap L^2_{\text{loc}}(\mathbb{R}; D(A_0) \times D(A_1)), \quad \widetilde{W}(v) := (w, \eta).$$

The projection of  $\widetilde{W}$  to the first component  $w$  induces a map  $W : B_X(0, \rho) \rightarrow Y$  with

$$Y := C_b(\mathbb{R}; V_0) \cap L^2_{\text{loc}}(\mathbb{R}; D(A_0)), \quad W(v) = w,$$

$$\|w\|_Y := \frac{\sup_{t \in \mathbb{R}} \|w(t)\|_{V_0}}{\nu \lambda_1^{1/2}} + \left( \frac{1}{\nu \lambda_1} \sup_{t \in \mathbb{R}} \int_t^{t + \frac{1}{\nu \lambda_1}} |A_0 w(\tau)|^2 d\tau \right)^{1/2}.$$

The induced map  $W$  will be used in the definition of the determining form. We denote  $Z := C_b(\mathbb{R}; V_1) \cap L^2_{\text{loc}}(\mathbb{R}; D(A_1))$  and

$$\|\eta\|_Z := \sup_{t \in \mathbb{R}} \|\eta(t)\|_{V_1} + \left( \nu \sup_{t \in \mathbb{R}} \int_t^{t + \frac{1}{\nu \lambda_1}} |A_1 \eta(\tau)|^2 d\tau \right)^{1/2}.$$

**Proposition 3.5.** *The maps  $\widetilde{W} : (B_X(0, \rho), \|\cdot\|_X) \rightarrow (Y \times Z, \|\cdot\|_Y + \|\cdot\|_Z)$  and  $W : (B_X(0, \rho), \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$  are Lipschitz.*

The proof of Proposition 3.5 is given in Section 5.

**Remark 3.6.** It is proved in [4] that the determining map  $\widetilde{W}$  is in fact Fréchet differentiable in the case of the 2D NSE.

**3.3. Determining form and long-time dynamics of the RB system.** Let  $(u^*, \theta^*)$  be a steady state of the RB problem (2.12); for instance, we may take  $(u^*, \theta^*) = (0, 0)$ , or in the case of stress-free boundary conditions,  $(u^*, \theta^*) = (c, 0)$ , where  $c$  is consistent with the fixed spatial average. Under the assumptions of Proposition 3.4, we will prove (in Theorem 3.9 (i)) that the differential equation

$$(3.10) \quad \frac{dv}{ds} = F(v) := -\|v - \widetilde{I}_h W(v)\|_X^2 (v - \widetilde{I}_h u^*), \quad v(0) = v_0 \in B_X(0, \rho),$$

is an *ODE* in the sense that the vector field  $F$  is globally Lipschitz in the ball  $B_X(0, \rho)$ , where  $\rho > 0$  is to be determined. The ODE (3.10) is called a *determining form* of the RB problem.

The connection between the long-time dynamics, i.e. the global attractor, of the RB problem (2.12) and the determining form will be made through the following result:

**Proposition 3.7.** *Let  $(u(t), \theta(t))$ ,  $t \in \mathbb{R}$ , be a solution of the RB problem (2.12) that lies in the global attractor  $\mathcal{A}$ . Suppose  $\mu, h$  satisfy the assumptions in Proposition 3.4. Suppose  $(w, \eta)$  is a solution to the system*

$$(3.11a) \quad \frac{dw}{dt} + \nu A_0 w + B_0(w, w) = \mathbb{P}_\sigma(g\eta e_2) - \mu\nu\lambda_1(\tilde{I}_h w - \tilde{I}_h u),$$

$$(3.11b) \quad \frac{d\eta}{dt} + \kappa A_1 \eta + B_1(w, \eta) = \frac{w \cdot e_2}{l},$$

and satisfies

$$(w, \eta) \in C_b(\mathbb{R}, V_0 \times V_1) \cap L^2_{\text{loc}}(\mathbb{R}, D(A_0) \times D(A_1)), \quad \left( \frac{dw}{dt}, \frac{d\eta}{dt} \right) \in L^2_{\text{loc}}(\mathbb{R}, H_0 \times H_1).$$

Then  $(w(t), \eta(t)) = (u(t), \theta(t))$  for all  $t \in \mathbb{R}$ .

The proof of Proposition 3.7 is given in Section 6.

**3.4. Main theorem.** In order to state the main theorem, we first prove the following result:

**Proposition 3.8.** *Let  $\tilde{I}_h$  be a (modified) Type II interpolant operator as in (3.5), with  $h < L$ . For every  $(u, \theta) \in \mathcal{A}$ , we have*

$$(3.12) \quad \|\tilde{I}_h u\|_X \leq R := ((\tilde{c}_1 + 1)J_1 + \tilde{c}_2 L J_2)/(\nu\lambda_1^{1/2}).$$

*Proof.* Let  $(u, \theta) \in \mathcal{A}$ . By (3.4), Remark 3.3, and the bound (2.14), we have

$$\begin{aligned} \|\tilde{I}_h u\|_{V_0} &\leq \|\tilde{I}_h u - u\|_{V_0} + \|u\|_{V_0} \\ &\leq \tilde{c}_1 \|u\|_{H^1} + \tilde{c}_2 h \|u\|_{H^2} + \|u\|_{V_0} \leq (\tilde{c}_1 + 1)J_1 + \tilde{c}_2 L J_2, \end{aligned}$$

which completes the proof by (3.6), the definition of the norm  $\|\cdot\|_X$ .  $\square$

The main results regarding the determining form are summarized in the following theorem:

**Theorem 3.9.** *Suppose the assumptions in Proposition 3.4 hold for  $\rho = 4R$ , where  $R > 0$  satisfies (3.12). Suppose also that  $h < L$  as in Proposition 3.8. Then the following hold.*

- (i) *The vector field  $F : B_X(0, \rho) \rightarrow X$  in the determining form (3.10) is Lipschitz. Hence the determining form (3.10) is an ODE in  $X$  which has short-time existence and uniqueness of solutions for every initial data  $v_0 \in B_X(0, \rho)$ .*
- (ii) *The ball  $B_X(\tilde{I}_h u^*, 3R) \subset B_X(0, \rho)$  is forward invariant in the evolution variable  $s$  under the dynamics of the determining form, which implies that (3.10) has a unique global solution for every initial data  $v_0 \in B_X(\tilde{I}_h u^*, 3R)$ .*
- (iii) *Every solution of (3.10) with initial data  $v_0 \in B_X(\tilde{I}_h u^*, 3R)$  converges to a steady state of (3.10) as  $s \rightarrow \infty$ .*
- (iv) *All the steady states of the determining form (3.10) that are contained in  $B_X(0, \rho)$  have the form  $v(t) = \tilde{I}_h u(t)$  for all  $t \in \mathbb{R}$ , where  $(u(\cdot), \theta(\cdot))$  is a trajectory in the global attractor  $\mathcal{A}$  of the RB problem (2.12) for a uniquely determined temperature  $\theta(\cdot)$ .*

We should emphasize that (3.10) governs an evolution of “trajectories” that are with range in a finite-dimensional space which correspond to velocity only. Yet it determines full trajectories of both the velocity and temperature on the global attractor of the RB system through the determining map  $\tilde{W}$ .

**Remark 3.10.** It is easy to see, as in [17], that the solution to (3.10) is always a convex combination of the initial condition and the chosen steady state. Indeed, since  $u^*$  is independent of  $s$ , we have

$$\frac{d}{ds}(v - \tilde{I}_h u^*) = -\|v - \tilde{I}_h W(v)\|_X^2(v - \tilde{I}_h u^*) .$$

and hence

$$(3.13) \quad v(s; t) = \beta(s)v_0(t) + (1 - \beta(s))\tilde{I}_h u^* \quad s \geq 0, t \in \mathbb{R} ,$$

where

$$(3.14) \quad \beta(s) = \exp\left(-\int_0^s \|v(\tau) - \tilde{I}_h W(v(\tau))\|_X^2 d\tau\right) .$$

So in fact  $\beta$  satisfies a scalar ODE, which for the RB problem written in the form (2.12) with  $(u^*, \theta^*) = (0, 0)$ , amounts to

$$(3.15) \quad v = \beta v_0, \quad \frac{d\beta}{ds} = -\beta \|\beta v_0 - \tilde{I}_h W(\beta v_0)\|_X^2, \quad \beta(0) = 1.$$

The dynamics of (3.15) are completely understood (see [17]). As  $s \rightarrow \infty$ , along the straight line through  $v_0$  and 0 in  $X$ , either  $v(s) \rightarrow 0$ , or  $v(s) \rightarrow \tilde{I}_h u$ , where  $(u, \theta)$  is the first trajectory in  $\mathcal{A}$ , with  $\tilde{I}_h u$  between  $v_0$  and 0. Thus the solutions in the global attractor can be identified as the zeros of the scalar function on the right-hand side of equation (3.15).

*Proof of Theorem 3.9.* Part (i). Define  $q : B_X(0, \rho) \rightarrow \mathbb{R}$  with  $q(v) := \|v - \tilde{I}_h W(v)\|_X$ . Let  $v_1, v_2 \in B_X(0, \rho)$ . By the triangle inequality and the definition of the vector field  $F$ ,

$$\begin{aligned} \|F(v_1) - F(v_2)\|_X &= \| [q^2(v_1) - q^2(v_2)](v_1 - \tilde{I}_h u^*) + q^2(v_2)(v_1 - v_2) \|_X \\ &\leq |q^2(v_1) - q^2(v_2)| \cdot \|v_1 - \tilde{I}_h u^*\|_X + |q^2(v_2)| \cdot \|v_1 - v_2\|_X. \end{aligned}$$

Hence, to show that  $F$  is Lipschitz (in the ball  $B_X(0, \rho)$ ), it suffices to show that the map  $q$  is Lipschitz. Note that

$$\begin{aligned} |q(v_1) - q(v_2)| &= |\|v_1 - \tilde{I}_h W(v_1)\|_X - \|v_2 - \tilde{I}_h W(v_2)\|_X| \\ &\leq \|v_1 - \tilde{I}_h W(v_1) - [v_2 - \tilde{I}_h W(v_2)]\|_X \\ &\leq \|v_1 - v_2\|_X + \|\tilde{I}_h W(v_1) - \tilde{I}_h W(v_2)\|_X. \end{aligned}$$

It suffices to show that

$$(3.16) \quad \|\tilde{I}_h W(v_1) - \tilde{I}_h W(v_2)\|_X \leq c\|v_1 - v_2\|_X.$$

Observe the following diagram:

$$B_X(0, \rho) \subset (X, \|\cdot\|_X) \xrightarrow{W} (Y, \|\cdot\|_Y) \xrightarrow{\tilde{I}_h} (X, \|\cdot\|_X).$$

To prove (3.16), it suffices to show that

$$(3.17) \quad \|w_1 - w_2\|_Y \leq c\|v_1 - v_2\|_X,$$

$$(3.18) \quad \|\tilde{I}_h w_1 - \tilde{I}_h w_2\|_X \leq c\|w_1 - w_2\|_Y,$$

where  $w_i := W(v_i)$  with  $i = 1, 2$ .

Proposition 3.5 implies that  $W$  is Lipschitz and hence we have (3.17). Inequality (3.18) follows from Remark 3.3 for the linear operator  $\tilde{I}_h$  and the definitions of the norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ . The proof of (i) is done.

By Proposition 3.8 and the triangle inequality<sup>1</sup>,

$$B_X(\tilde{I}_h(u^*), 3R) \subset B_X(0, \rho),$$

which implies short-time existence of a solution of the determining form (3.10). Thus, (ii) follows from the observation that

$$\|v(s; \cdot) - \tilde{I}_h(u^*)\|_X = \beta(s)\|v_0(\cdot) - \tilde{I}_h(u^*)\|_X, \quad s \geq 0,$$

where  $\beta$  is as in (3.14). Alternatively, (ii) follows from the dissipativity property of (3.10): for every fixed  $t \in \mathbb{R}$ ,

$$\frac{d}{ds} \|v(s; t) - \tilde{I}_h(u^*)\|_{V_0}^2 = -2\|v - \tilde{I}_h W(v)\|_X^2 \cdot \|v(s; t) - \tilde{I}_h(u^*)\|_{V_0}^2.$$

This property implies that the ball  $B_X(\tilde{I}_h(u^*), 3R)$  is forward invariant for all  $s \geq 0$ , which proves both (ii) and (iii).

To prove (iv) we observe that the steady states of equation (3.10) in the ball  $B_X(0, \rho)$  are either  $v = \tilde{I}_h(u^*)$  or  $v \in B_X(0, \rho)$  such that  $\|v - \tilde{I}_h W(v)\|_X = 0$ . In the first case  $(u^*, \theta^*) \in \mathcal{A}$  since  $(u^*, \theta^*)$  is a steady state of the RB system (2.12). In the second case we have  $v(t) = \tilde{I}_h W(v)(t)$  for all  $t \in \mathbb{R}$ . Let  $(w, \eta) = \tilde{W}(v)$ . It then follows from (3.8) that  $(w, \eta)$  is a bounded solution (thus a trajectory in the global attractor  $\mathcal{A}$  by (2.13)) to the RB system (2.12).

Conversely, since  $\rho = 4R$ , it follows from Proposition 3.8 that

$$\tilde{I}_h(\mathcal{A}) \subset B_X(\tilde{I}_h u^*, 3R) \subset B_X(0, \rho).$$

Thus, for every trajectory  $(u(\cdot), \theta(\cdot)) \subset \mathcal{A}$  it follows from the auxiliary system (3.8) and Proposition 3.7 that  $u(t) = W(\tilde{I}_h u)(t)$  for all  $t \in \mathbb{R}$ . In particular,  $\tilde{I}_h u = \tilde{I}_h W(\tilde{I}_h u)$ , which implies that  $\tilde{I}_h u$  is a steady state of equation (3.10) in  $B_X(0, \rho)$ .  $\square$

#### 4. PROOF OF PROPOSITION 3.4

Let  $\mu, h > 0$  and assume that  $\|v\|_X \leq \rho$ . For the case of no-slip boundary conditions, we assume that the following hold:

$$(4.1) \quad \mu\lambda_1^{1/2}c_1h \leq \frac{1}{4}, \quad \mu\lambda_1^22c_2^2h^4 \leq \frac{1}{8}, \quad \mu\nu\lambda_1(c_1h\lambda_1^{-1/2} + c_2h^2) \leq \frac{\nu}{8},$$

$$(4.2) \quad \mu\nu^2\lambda_1^2C_1 > \frac{5g^2K}{2\rho^2},$$

<sup>1</sup>Note that  $\|v\|_X \leq \|v - \tilde{I}_h u^*\|_X + \|\tilde{I}_h u^*\|_X \leq 3R + R = 4R$ .

$$(4.3) \quad \frac{1}{4}\mu\nu - 16K_1C_1^2\rho^4 > 0,$$

$$(4.4) \quad \frac{1}{2}\mu\nu\lambda_1 - \frac{g^2}{\kappa(\nu\lambda_1)^2} - \frac{\lambda_1\nu}{4}(K_2 \log K_2) - \frac{2c_L^2\nu^2}{\kappa}\rho^2 - \frac{2\nu^2}{l^2\kappa} \geq \frac{\kappa\lambda_1}{2},$$

where the constants  $K, C_1, K_1, K_2$  are defined in (4.16), (4.23), (4.21) and (5.14); they are all independent of  $\mu$  and  $h$ .

For the case of stress-free boundary conditions, we assume that the following hold:

$$(4.5) \quad \frac{1}{4}\mu\nu\lambda_1 - \left( \frac{2g^2}{|\Omega|\kappa\epsilon_2\lambda_1} + \frac{2g^2}{\kappa\epsilon_2} + \frac{\tilde{K}_1^2\epsilon_2}{\kappa l^2} \right) \geq \frac{\kappa\lambda_1}{2},$$

$$(4.6) \quad \frac{1}{8}\mu\lambda_1 - \frac{|\Omega|^{-1}}{4} \geq 0,$$

$$(4.7) \quad \frac{1}{4}\mu\nu\lambda_1 - K_{16} \geq \frac{\kappa\lambda_1}{4},$$

$$(4.8) \quad c_1h|\Omega|^{-1/2} \leq \frac{1}{8}, \quad 2c_2^2h^4\mu\lambda_1|\Omega|^{-1} \leq \frac{1}{8},$$

$$(4.9) \quad \mu\nu\lambda_1(c_1^2h^2 + c_2h^2) \leq \frac{\nu}{2},$$

where the constants  $\epsilon_2, \tilde{K}_1, K_{16}$ , being independent of  $\mu$  and  $h$ , are defined in (4.40), (4.46) and (5.41).

The uniqueness of bounded solutions follows from Proposition 3.5. In this section, we prove the existence of strong solutions.

**Remark 4.1.** Assumptions (4.4) and (4.7) are not needed for the proof of existence; they are used to prove the uniqueness of bounded solution.

**Step 1.** Let  $k$  be a fixed positive integer. For  $n \geq r$ , where  $r \in \mathbb{N}$  is fixed in (3.5), we consider a Galerkin approximation for system (3.8):

$$(4.10) \quad \begin{aligned} \frac{dw_n}{dt} + \nu A_0 w_n + P_{0,n} B_0(w_n, w_n) &= P_{0,n} \mathbb{P}_\sigma(g\eta_n \mathbf{e}_2) - \mu\nu\lambda_1 P_{0,n}(\tilde{I}_h w_n - v), \\ \frac{d\eta_n}{dt} + \kappa A_1 \eta_n + P_{1,n} B_1(w_n, \eta_n) &= P_{1,n} \left( \frac{w_n \cdot \mathbf{e}_2}{l} \right), \end{aligned}$$

with initial data

$$(4.11) \quad w_n(-k(\nu\lambda_1)^{-1}) = 0, \quad \eta_n(-k(\nu\lambda_1)^{-1}) = 0,$$

where  $P_{i,n}$  is the orthogonal projection onto  $H_{i,n} = \text{span}\{\zeta_{i,1}, \dots, \zeta_{i,n}\}$ . This is a finite system of ODEs with a quadratic polynomial nonlinearity. Hence, there exists  $T_n > -k(\nu\lambda_1)^{-1}$ , so that there exists a solution  $(w_n, \eta_n)$  to the initial value problem on the interval  $[-k(\nu\lambda_1)^{-1}, T_n]$ .

Thanks to the initial conditions (4.11), following the approach used to prove the existence and uniqueness of strong solutions for the Navier-Stokes equations and the RB system (see, e.g., [10, 27]), one can show by energy estimates that there exists  $T_* > -k(\nu\lambda_1)^{-1}$ , independent of  $n$ , such that solutions of (4.10) exist on  $[-k(\nu\lambda_1)^{-1}, T_*]$  and satisfy uniform bounds, in the relevant strong norms, which are independent of  $n$ . Therefore, by the Aubin-Lions compactness theorem, there

exists a subsequence  $\{(w_{n(j),k}, \eta_{n(j),k})\}_{j=1}^\infty$  which converges to a unique strong solution  $(w^{(k)}, \eta^{(k)})$  to system (3.8) on a common interval  $[-k(\nu\lambda_1)^{-1}, T_*]$  with initial data  $w^{(k)}(-k(\nu\lambda_1)^{-1}) = 0$  and  $\eta^{(k)}(-k(\nu\lambda_1)^{-1}) = 0$ . Let  $[-k(\nu\lambda_1)^{-1}, T_{**})$  be the maximum forward interval of existence for  $(w^{(k)}, \eta^{(k)})$ . Note that  $T_{**} \geq T_*$  and that from the above mentioned energy type estimates we have

$$(w^{(k)}, \eta^{(k)}) \in C([-k(\nu\lambda_1)^{-1}, T_{**}), V_0 \times V_1) \cap L_{\text{loc}}^2([-k(\nu\lambda_1)^{-1}, T_{**}), D(A_0) \times D(A_1)).$$

**Step 2.** Assume that  $T_{**} < \infty$ . In Section 4.1 and Section 4.2, for the no-slip and stress-free cases respectively, we show on the maximum interval of existence  $[-k(\nu\lambda_1)^{-1}, T_{**})$  for  $(w^{(k)}, \eta^{(k)})$  uniform (in time  $t$ ) bounds on the following quantities (omitting the superscript  $k$  for simplicity)

$$(4.12) \quad |\eta|^2, |w|^2, \|w\|^2, \int_t^{\min(t+T, T_{**})} |A_0 w(\tau)|^2 d\tau,$$

$$(4.13) \quad \|\eta\|^2, \int_t^{\min(t+T, T_{**})} |A_1 \eta(\tau)|^2 d\tau,$$

where  $T := (\nu\lambda_1)^{-1}$ .

**Remark 4.2.** All the bounds for (4.12) will be *independent* of  $k$  and  $T_{**}$ . On the other hand, bounds for (4.13) in this step may depend on  $k$ ; we will however, improve in the next step the bounds so that they will be independent of  $k$  and  $T_{**}$ .

For the no-slip case, the bounds (4.16), (4.24), (4.28), (4.35) and (4.37) in Section 4.1 imply that the solution  $(w^{(k)}, \eta^{(k)})$  cannot blow up in the space

$$C([-k(\nu\lambda_1)^{-1}, T_{**}), V_0 \times V_1) \cap L_{\text{loc}}^2([-k(\nu\lambda_1)^{-1}, T_{**}), D(A_0) \times D(A_1)),$$

and thus we may extend it beyond  $T_{**}$ , which contradicts the maximality of  $T_{**}$ . Therefore, we must have  $T_{**} = \infty$ .

The same argument works for the stress-free case by considering the bounds (4.54), (4.56), (4.58), (4.64) and (4.66) in Section 4.2.

**Step 3.** For  $(w^{(k)}, \eta^{(k)})$ , we show uniform bounds on the interval  $\mathcal{I}_k := [-k(\nu\lambda_1)^{-1} + (\nu\lambda_1)^{-1}, \infty)$ , for all the quantities in (4.12) and (4.13). These bounds will all be independent of  $k$ . Note that we need the extra time unit  $(\nu\lambda_1)^{-1}$  in  $\mathcal{I}_k$  due to the use of Lemma 4.3.

By Remark 4.2, the uniform bounds for (4.12) in Step 2, i.e.,

- (i) no-slip: (4.16), (4.24), (4.28);
- (ii) stress-free: (4.54), (4.56), (4.58),

are all valid on the interval  $[-k(\nu\lambda_1)^{-1}, \infty)$  and particularly on  $\mathcal{I}_k$ ; they are independent of  $k$ .

For the no-slip case, in subsection 4.1.4, letting  $\alpha_k = T = (\nu\lambda_1)^{-1}$  and  $t_1 = T_{**} = \infty$ , by (4.34), we have a uniform bound on the interval  $\mathcal{I}_k$  for  $\|\eta\|^2$ , where  $C_3$  in (4.34) is now independent of  $k$ . It follows that the uniform bound (4.36) is also valid for  $t \in \mathcal{I}_k$ .

The similar argument works for the stress-free case by considering (4.63) and (4.65) in subsection 4.2.3.

**Step 4.** For each positive integer  $m$ , consider a (sub)sequence of solutions  $\{(w^{(k)}, \eta^{(k)})\}_{k=m+1}^\infty$ . By Step 3, this sequence satisfies all the uniform bounds on (4.12) and (4.13) (with  $T_{**} = \infty$ ) on the interval  $\mathcal{I}_{m+1} = [-m(\nu\lambda_1)^{-1}, \infty)$ , and in particular on  $[-m(\nu\lambda_1)^{-1}, m(\nu\lambda_1)^{-1}]$ . Thus,

$$(4.14) \quad \int_{-m(\nu\lambda_1)^{-1}}^{m(\nu\lambda_1)^{-1}} |A_0 w^{(k)}(\tau)|^2 d\tau < \infty, \quad \int_{-m(\nu\lambda_1)^{-1}}^{m(\nu\lambda_1)^{-1}} |A_1 \eta^{(k)}(\tau)|^2 d\tau < \infty,$$

where the bounds in (4.14) may depend on  $m$ , but are independent of  $k$ . In particular, (4.14) implies that

$$(4.15) \quad \int_{-m(\nu\lambda_1)^{-1}}^{m(\nu\lambda_1)^{-1}} \left| \frac{dw^{(k)}(\tau)}{d\tau} \right|^2 d\tau < \infty, \quad \int_{-m(\nu\lambda_1)^{-1}}^{m(\nu\lambda_1)^{-1}} \left| \frac{d\eta^{(k)}(\tau)}{d\tau} \right|^2 d\tau < \infty,$$

are bounded uniformly in  $k$ , with bounds that may depend on  $m$ .

Applying the Aubin-Lions compactness theorem using (4.14), (4.15), and the uniform, with respect to  $t$  and  $k$ , bounds on the quantities

$$|\eta^{(k)}|^2, |w^{(k)}|^2, \|w^{(k)}\|^2, \|\eta^{(k)}\|^2, \quad t \in [-m(\nu\lambda_1)^{-1}, m(\nu\lambda_1)^{-1}],$$

we obtain a subsequence  $\{(w^{(k_l, m)}, \eta^{(k_l, m)})\}_{l=1}^\infty$  that converges to a solution of system (3.8) on the closed interval  $[-m(\nu\lambda_1)^{-1}, m(\nu\lambda_1)^{-1}]$ .

We then apply the Cantor diagonal process to nested subsequences, relabeling when necessary, to get a subsequence  $\{(w^{(k_m, m)}, \eta^{(k_m, m)})\}_{m=1}^\infty$  that converges to a solution  $(w, \eta)$  on  $[-M(\nu\lambda_1)^{-1}, M(\nu\lambda_1)^{-1}]$  for all  $M \in \mathbb{N}$ . Note that  $(w, \eta)$  is defined on  $(-\infty, \infty)$ . Hence,  $(w, \eta)$  satisfies all the uniform bounds on (4.12) and (4.13) for  $t \in \mathbb{R}$  and thus (3.9). The proof of Proposition 3.4 is complete.

**4.1. No-slip BCs (bounds on  $[-k(\nu\lambda_1)^{-1}, T_{**}]$  with  $T_{**} < \infty$ ).** For simplicity, we will omit the superscript  $k$  in  $(w^{(k)}, \eta^{(k)})$  in this section and the next (stress-free BCs). All estimates are rigorous on the maximal interval  $[-k(\nu\lambda_1)^{-1}, T_{**}]$ .

**4.1.1. Bound for  $|\eta|$ .** By a similar argument as in [19, Lemma 2.1], we can show, by employing the maximum principle for the heat equation, that (see the Appendix)

$$(4.16) \quad |\eta(t)| \leq 2|\Omega| := K, \quad \forall t \in [-k(\nu\lambda_1)^{-1}, T_{**}].$$

**4.1.2. Bounds for  $|w|$  and  $\|w\|$ .** Taking the  $L^2$  inner product of the auxiliary equation (3.8a) with  $w$  and  $A_0 w$  respectively, we have

$$(4.17) \quad \frac{1}{2} \frac{d}{dt} |w|^2 + \nu \|w\|^2 = g(\eta e_2, w) - \mu \nu \lambda_1 (\tilde{I}_h w - v, w),$$

(4.18)

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu |A_0 w|^2 + (B_0(w, w), A_0 w) = g(\eta e_2, A_0 w) - \mu \nu \lambda_1 (\tilde{I}_h w - v, A_0 w),$$

where we use  $b_0(w, w, w) = 0$ . By the Cauchy-Schwarz, Young and Poincaré inequalities, we have

$$\begin{aligned}
 (4.19) \quad & -\mu\nu\lambda_1(\tilde{I}_h w - v, w) \leq \mu\nu\lambda_1 \left[ |(\tilde{I}_h w - w, w)| + |(v, w)| - (w, w) \right] \\
 & \leq \mu\nu\lambda_1 \left[ c_1 h \|w\| \cdot |w| + c_2 h^2 |A_0 w| \cdot |w| + |v| \cdot |w| - |w|^2 \right] \\
 & \qquad \qquad \qquad \text{(by Remark 3.3)} \\
 & \leq \mu\nu\lambda_1 \left[ c_1 h \lambda_1^{-1/2} \|w\|^2 + 2c_2^2 h^4 |A_0 w|^2 + 2|v|^2 - \frac{3}{4} |w|^2 \right] \\
 & \leq \frac{\nu}{4} \|w\|^2 + \frac{\nu}{8} \lambda_1^{-1} |A_0 w|^2 + 2\mu\nu\lambda_1 |v|^2 - \frac{3}{4} \mu\nu\lambda_1 |w|^2 \quad \text{(by (4.1))},
 \end{aligned}$$

and

$$\begin{aligned}
 (4.20) \quad & -\mu\nu\lambda_1(\tilde{I}_h w - v, A_0 w) \leq \mu\nu\lambda_1 \left[ |(\tilde{I}_h w - w, A_0 w)| + |(v, A_0 w)| - (w, A_0 w) \right] \\
 & = \mu\nu\lambda_1 \left[ |(\tilde{I}_h w - w, A_0 w)| + |((v, w))| - (w, A_0 w) \right] \quad \text{(since } v(t) \in V_0 \text{)} \\
 & \leq \mu\nu\lambda_1 \left[ c_1 h \|w\| \cdot |A_0 w| + c_2 h^2 |A_0 w|^2 + \|v\| \cdot \|w\| \right] - \mu\nu\lambda_1 \|w\|^2 \\
 & \qquad \qquad \qquad \text{(by Remark 3.3)} \\
 & \leq \mu\nu\lambda_1 \left[ c_1 h \lambda_1^{-1/2} |A_0 w|^2 + c_2 h^2 |A_0 w|^2 + \|v\|^2 - \frac{3}{4} \|w\|^2 \right] \\
 & \leq \frac{\nu}{8} |A_0 w|^2 + \mu\nu\lambda_1 \|v\|^2 - \frac{3}{4} \mu\nu\lambda_1 \|w\|^2 \quad \text{(by (4.1))}.
 \end{aligned}$$

For the nonlinear term, we have

$$\begin{aligned}
 (4.21) \quad & |(B_0(w, w), A_0 w)| \leq \|w\|_{L^4}^2 \|\nabla w\|_{L^4}^2 |A_0 w| \quad \text{(Hölder)} \\
 & \leq c_L^2 |w|^{1/2} \|w\| \cdot |A_0 w|^{3/2} \quad \text{(Ladyzhenskaya)} \\
 & \leq \frac{\nu}{8} |A_0 w|^2 + K_1 |w|^2 \|w\|^4, \quad K_1 := \frac{27c_L^8}{2\nu^3}.
 \end{aligned}$$

Combining (4.16)–(4.21), we get

$$\begin{aligned}
(4.22) \quad & \frac{1}{2} \frac{d}{dt} (|w|^2 + \lambda_1^{-1} \|w\|^2) + \nu (\|w\|^2 + \lambda_1^{-1} |A_0 w|^2) \\
& \leq g|\eta||w| + \frac{\nu}{4} \|w\|^2 + \frac{\nu}{8} \lambda_1^{-1} |A_0 w|^2 + 2\mu\nu\lambda_1 \|v\|^2 - \frac{3}{4} \mu\nu\lambda_1 \|w\|^2 \\
& \quad + \lambda_1^{-1} \left( g|\eta||A_0 w| + \frac{\nu}{8} |A_0 w|^2 + \mu\nu\lambda_1 \|v\|^2 - \frac{3}{4} \mu\nu\lambda_1 \|w\|^2 \right) \\
& \quad + \lambda_1^{-1} \left( \frac{\nu}{8} |A_0 w|^2 + K_1 |w|^2 \|w\|^4 \right) \\
& \leq \frac{g^2 K}{2\nu\lambda_1} + \frac{\nu\lambda_1}{2} |w|^2 + \frac{\nu}{4} \|w\|^2 + \frac{3\nu}{8} \lambda_1^{-1} |A_0 w|^2 + \frac{2g^2 K}{\nu\lambda_1} + \frac{\nu}{8} \lambda_1^{-1} |A_0 w|^2 \\
& \quad + 3\mu\nu\lambda_1 \|v\|_X^2 \nu^2 - \frac{3}{4} \mu\nu\lambda_1 (|w|^2 + \lambda_1^{-1} \|w\|^2) + \lambda_1^{-1} K_1 |w|^2 \|w\|^4.
\end{aligned}$$

Hence,

$$\begin{aligned}
(4.23) \quad & \frac{1}{2} \frac{d}{dt} (|w|^2 + \lambda_1^{-1} \|w\|^2) + \frac{1}{2} \mu\nu\lambda_1 (|w|^2 + \lambda_1^{-1} \|w\|^2) \\
& \quad + \left( \frac{1}{4} \mu\nu\lambda_1 - K_1 |w|^2 \|w\|^2 \right) \lambda_1^{-1} \|w\|^2 + \frac{\nu\lambda_1^{-1}}{2} |A_0 w|^2 \\
& \leq 3\mu\nu\lambda_1 \|v\|_X^2 \nu^2 + \frac{5g^2 K}{2\nu\lambda_1} \\
& \leq \mu\nu\lambda_1 C_1 \rho^2 \quad (\text{by (4.2)}), \quad C_1 := 4\nu^2.
\end{aligned}$$

We now show that

$$(4.24) \quad |w|^2 + \lambda_1^{-1} \|w\|^2 \leq 4C_1 \rho^2, \quad t \in [-k(\nu\lambda_1)^{-1}, T_{**}].$$

By continuity and the initial condition  $w(-k(\nu\lambda_1)^{-1}) = 0$ , there exists  $t_* \in [-k(\nu\lambda_1)^{-1}, T_{**}]$  such that

$$|w|^2 + \lambda_1^{-1} \|w\|^2 \leq 4C_1 \rho^2, \quad t \in [-k(\nu\lambda_1)^{-1}, t_*].$$

It then follows from (4.24) and (4.3) that

$$\frac{1}{4} \mu\nu\lambda_1 - K_1 |w|^2 \|w\|^2 \geq 0, \quad t \in [-k(\nu\lambda_1)^{-1}, t_*].$$

Let

$$\tilde{T} = \sup \{ \tau \in [-k(\nu\lambda_1)^{-1}, T_{**}] : |w(t)|^2 + \lambda_1^{-1} \|w(t)\|^2 \leq 4C_1 \rho^2 \text{ for all } t \in [-k(\nu\lambda_1)^{-1}, \tau] \}.$$

Notice that  $\tilde{T} \geq t_* > -k(\nu\lambda_1)^{-1}$ . We claim that  $\tilde{T} = T_{**}$ . If not, then  $\tilde{T} < T_{**}$ , and

$$(4.25) \quad |w(\tilde{T})|^2 + \lambda_1^{-1} \|w(\tilde{T})\|^2 = 4C_1 \rho^2,$$

$$\begin{aligned}
(4.26) \quad & \frac{1}{2} \frac{d}{dt} (|w|^2 + \lambda_1^{-1} \|w\|^2) + \frac{1}{2} \mu\nu\lambda_1 (|w|^2 + \lambda_1^{-1} \|w\|^2) \\
& \quad + \frac{\nu\lambda_1^{-1}}{2} |A_0 w|^2 \leq \mu\nu\lambda_1 C_1 \rho^2, \quad \forall t \in [-k(\nu\lambda_1)^{-1}, \tilde{T}].
\end{aligned}$$

Dropping the term  $\frac{\nu\lambda_1^{-1}}{2}|A_0w|^2$ , we have by the Gronwall inequality that

$$|w(\tilde{T})|^2 + \lambda_1^{-1}\|w(\tilde{T})\|^2 \leq 2C_1\rho^2(1 - e^{\mu\nu\lambda_1(-k(\nu\lambda_1)^{-1}-\tilde{T})}) < 2C_1\rho^2,$$

which contradicts (4.25).

4.1.3. *Bound for  $\int_t^{\min(t+T, T_{**})} |A_0w(\tau)|^2 d\tau$ .* Henceforth, we let  $T = (\nu\lambda_1)^{-1}$ .

Inequality (4.26) implies that

$$\frac{1}{2}\frac{d}{dt}(|w|^2 + \lambda_1^{-1}\|w\|^2) + \frac{\nu\lambda_1^{-1}}{2}|A_0w|^2 \leq \mu\nu\lambda_1 C_1\rho^2.$$

For any  $t \in [-k(\nu\lambda_1)^{-1}, T_{**})$ , integrating on both sides from  $t$  to  $\min(t+T, T_{**})$ , observing that  $\min(t+T, T_{**}) - t \leq T$ , and using the bound (4.24), we have

$$(4.27) \quad \nu \int_t^{\min(t+T, T_{**})} |A_0w(\tau)|^2 d\tau \leq 4C_1\rho^2\lambda_1 + T\mu\nu\lambda_1^2 C_1\rho^2.$$

Since  $T_{**} < \infty$ , it follows that

$$(4.28) \quad \nu \int_{-k(\nu\lambda_1)^{-1}}^{T_{**}} |A_0w(\tau)|^2 d\tau < \infty.$$

4.1.4. *Bound for  $\|\eta\|$ .* Taking the  $L^2$  inner product of the equation (3.8b) with  $\eta$ , and applying the Cauchy-Schwarz and Young inequalities, we have

$$(4.29) \quad \frac{1}{2}\frac{d}{dt}|\eta|^2 + \kappa\|\eta\|^2 \leq \frac{\kappa\lambda_1}{4}|\eta|^2 + \frac{1}{\kappa l^2\lambda_1}|w|^2.$$

Let  $\tilde{k} = k(\nu\lambda_1)^{-1}$  and  $\alpha_k = \frac{T_{**}+\tilde{k}}{2}$ . For any  $t \in [-\tilde{k}, -\tilde{k} + \alpha_k)$ , integrating (4.29) from  $t$  to  $t + \alpha_k$ , we have

$$(4.30) \quad \kappa \int_t^{t+\alpha_k} \|\eta(\tau)\|^2 d\tau \leq \frac{K^2}{2} + \alpha_k \left( \frac{\kappa\lambda_1 K^2}{4\rho^2} + \frac{4C_1}{\kappa l^2\lambda_1} \right) \rho^2 =: \beta_k.$$

By taking the  $L^2$  inner product of the equation (3.8b) with  $A_1\eta$ , we have

$$(4.31) \quad \frac{1}{2}\frac{d}{dt}\|\eta\|^2 + \kappa|A_1\eta|^2 + (B_1(w, \eta), A_1\eta) = \frac{(w \cdot e_2, A_1\eta)}{l} \leq \frac{\kappa}{4}|A_1\eta|^2 + \frac{1}{l^2\kappa}|w|^2.$$

Integrating by parts, we have (as in [12, (3.22)])

$$\begin{aligned} (4.32) \quad |(B_1(w, \eta), A_1\eta)| &\leq \|w\| \cdot \|\nabla\eta\|_{L^4}^2 \quad (\text{H\"older}) \\ &\leq c_L\|w\| \cdot \|\eta\| \cdot |A_1\eta| \quad (\text{Ladyzhenskaya}) \\ &\leq \frac{c_L^2}{\kappa}\|w\|^2\|\eta\|^2 + \frac{\kappa}{4}|A_1\eta|^2. \end{aligned}$$

Consequently,

$$(4.33) \quad \frac{d}{dt}\|\eta\|^2 + \kappa|A_1\eta|^2 \leq \frac{2c_L^2}{\kappa}\|w\|^2\|\eta\|^2 + \frac{2}{l^2\kappa}|w|^2 \leq \frac{8c_L^2 C_1 \lambda_1}{\kappa}\rho^2\|\eta\|^2 + \frac{8C_1}{l^2\kappa}\rho^2.$$

We now recall the following uniform Gronwall inequality from [19].

**Lemma 4.3** (Uniform Gronwall). *Let  $g, h, y$  be three positive locally integrable functions on  $(t_0, t_1)$  which satisfy for all  $t$  with  $t_0 \leq t < t + \alpha < t_1$ ,*

$$\frac{dy}{dt} \leq gy + h, \quad \int_t^{t+\alpha} g(\tau) d\tau \leq a_1, \quad \int_t^{t+\alpha} h(\tau) d\tau \leq a_2, \quad \int_t^{t+\alpha} y(\tau) d\tau \leq a_3,$$

where  $a_1, a_2, a_3, \alpha$  are positive constants. Then

$$y(t + \alpha) \leq \left( \frac{a_3}{\alpha} + a_2 \right) e^{a_1}, \quad t_0 \leq t < t + \alpha < t_1.$$

Applying Lemma 4.3 to (4.33) with

$$\begin{aligned} t_0 &= -k(\nu\lambda_1)^{-1}, \quad t_1 = T_{**}, \quad \alpha = \alpha_k, \\ g(t) &= \frac{8c_L^2 C_1 \lambda_1 \rho^2}{\kappa}, \quad h(t) = \frac{8C_1 \rho^2}{l^2 \kappa}, \quad y(t) = \|\eta(t)\|^2, \\ a_1 &= \frac{8c_L^2 C_1 \lambda_1}{\kappa} \rho^2 \alpha, \quad a_2 = \frac{8C_1}{l^2 \kappa} \rho^2 \alpha, \quad a_3 = \frac{\beta_k}{\kappa} \alpha, \end{aligned}$$

we get

$$(4.34) \quad \sup_{t \in [-\tilde{k} + \alpha, T_{**})} \|\eta(t)\|^2 \leq \left( \frac{a_3}{\alpha} + a_2 \right) e^{a_1} =: C_3,$$

and thus

$$(4.35) \quad \sup_{t \in [-\tilde{k}, T_{**})} \|\eta(t)\|^2 \leq \left( \frac{a_3}{\alpha} + a_2 \right) e^{a_1} + \sup_{t \in [-\tilde{k}, -\tilde{k} + \alpha]} \|\eta(t)\|^2 < \infty.$$

4.1.5. *Bound for  $\int_t^{\min(t+T, T_{**})} |A_1 \eta(\tau)|^2 d\tau$ .* For any  $t \in [-\tilde{k} + \alpha_k, T_{**})$ , inserting the bound (4.34) in (4.33) and then integrating from  $t$  to  $\min(t + T, T_{**})$  on both sides, we have

$$(4.36) \quad \kappa \int_t^{\min(t+T, T_{**})} |A_1 \eta(\tau)|^2 d\tau \leq C_3 + \left( \frac{8c_L^2 C_1 \lambda_1 C_3}{\kappa} + \frac{8C_1}{l^2 \kappa} \right) \rho^2 T.$$

Since  $T_{**} < \infty$ , it follows that

$$(4.37) \quad \int_{-k(\nu\lambda_1)^{-1}}^{T_{**}} |A_1 \eta(\tau)|^2 d\tau < \infty.$$

**4.2. Stress-free BCs (bounds on  $[-k(\nu\lambda_1)^{-1}, T_{**}]$  with  $T_{**} < \infty$ ).** The argument using the maximum principle for showing the bound for  $|\eta|$  in Section 4.1 also works here. Taking advantage of the orthogonality property that  $b_0(w, w, A_0 w) = 0$  in the case of stress-free BCs, we combine the estimates of  $\|w\|_{V_0}$  and  $|\eta|$  together.

4.2.1. *Bounds for  $\|w\|_{V_0}$  and  $|\eta|$ .* Taking the  $L^2$  inner products of the auxiliary system (3.8) with  $w, A_0 w$  and  $\eta$  respectively, we have

$$(4.38) \quad \epsilon_1 \left( \frac{1}{2} \frac{d}{dt} |w|^2 + \nu \|w\|^2 \right) = \epsilon_1 \left( g(\eta e_2, w) - \mu \nu \lambda_1 (\tilde{I}_h w - v, w) \right), \quad \epsilon_1 := \frac{1}{|\Omega|},$$

$$(4.39) \quad \frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu |A_0 w|^2 = g(\eta e_2, A_0 w) - \mu \nu \lambda_1 (\tilde{I}_h w - v, A_0 w),$$

$$(4.40) \quad \epsilon_2 \left( \frac{1}{2} \frac{d}{dt} |\eta|^2 + \kappa \|\eta\|^2 \right) = \epsilon_2 \left( \frac{(w \cdot e_2, \eta)}{l} \right), \quad \epsilon_2 := (\nu \lambda_1)^2,$$

where we used  $b_0(w, w, w) = 0$ ,  $b_0(w, w, A_0 w) = 0$  and  $b_1(w, \eta, \eta) = 0$ . Note that equations (4.38)–(4.40) have the same dimension and no nonlinear term appears in the equations above.

Now we estimate the right-hand sides of the three equations above as follows:

(4.41)

$$\begin{aligned}
-\mu\nu\lambda_1\epsilon_1(\tilde{I}_h w - v, w) &\leq \mu\nu\lambda_1\epsilon_1 \left( |(\tilde{I}_h w - w, w)| + |(v, w)| - (w, w) \right) \\
&\leq \frac{\mu\nu\lambda_1}{|\Omega|} \left( c_1 h \|w\|_{V_0} \cdot |w| + c_2 h^2 |A_0 w|^2 \cdot |w| + |v|^2 + \frac{1}{4} |w|^2 - |w|^2 \right) \\
&\quad \text{(by Remark 3.3)} \\
&\leq \frac{\mu\nu\lambda_1}{|\Omega|} \left( c_1 h |\Omega|^{1/2} \|w\|_{V_0}^2 + 2c_2^2 h^4 |A_0 w| + \frac{1}{8} |w|^2 + |v|^2 + \frac{1}{4} |w|^2 - |w|^2 \right) \\
&\quad \text{(by (2.8))} \\
&\leq \frac{1}{8} \mu\nu\lambda_1 \|w\|_{V_0}^2 + \frac{\nu}{8} |A_0 w|^2 + \mu\nu\lambda_1\epsilon_1 |v|^2 - \frac{3}{4} \mu\nu\lambda_1\epsilon_1 |w|^2 + \frac{1}{8} \mu\nu\lambda_1\epsilon_1 |w|^2 \\
&\quad \text{(by (4.8))} \\
&\leq \frac{1}{4} \mu\nu\lambda_1 \|w\|_{V_0}^2 + \mu\nu\lambda_1\epsilon_1 |v|^2 - \frac{3}{4} \mu\nu\lambda_1\epsilon_1 |w|^2 + \frac{\nu}{8} |A_0 w|^2
\end{aligned}$$

(4.42)

$$\begin{aligned}
-\mu\nu\lambda_1(\tilde{I}_h w - v, A_0 w) &\leq \mu\nu\lambda_1 \left( |(\tilde{I}_h w - w, A_0 w)| + |(v, A_0 w)| - (w, A_0 w) \right) \\
&\leq \mu\nu\lambda_1 \left( c_1 h \|w\|_{V_0} \cdot |A_0 w| + c_2 h^2 |A_0 w|^2 + \|v\|^2 + \frac{1}{4} \|w\|^2 - \|w\|^2 \right) \\
&\quad \text{(by Remark 3.3)} \\
&\leq \mu\nu\lambda_1 \left( \frac{1}{4} \|w\|_{V_0}^2 + c_1^2 h^2 |A_0 w|^2 + c_2 h^2 |A_0 w|^2 + \|v\|^2 + \frac{1}{4} \|w\|^2 - \|w\|^2 \right) \\
&\leq \frac{1}{4} \mu\nu\lambda_1 \|w\|_{V_0}^2 + \frac{\nu}{8} |A_0 w|^2 + \mu\nu\lambda_1 \|v\|^2 - \frac{3}{4} \mu\nu\lambda_1 \|w\|^2 \quad \text{(by (4.9)).}
\end{aligned}$$

$$\begin{aligned}
(4.43) \quad \epsilon_1 |(g\eta e_2, w)| &\leq \frac{g}{|\Omega|} |\eta| \cdot |w| \leq \frac{g\lambda_1^{-1/2}}{|\Omega|} \|\eta\| \cdot |w| \\
&\leq \frac{\kappa\epsilon_2}{8} \|\eta\|^2 + \frac{2}{\kappa\epsilon_2} \frac{g^2\lambda_1^{-1}}{|\Omega|^2} |w|^2 \leq \frac{\kappa\epsilon_2}{8} \|\eta\|^2 + \frac{2}{\kappa\epsilon_2} \frac{g^2\lambda_1^{-1}}{|\Omega|} \|w\|_{V_0}^2,
\end{aligned}$$

(4.44)

$$|(g\eta e_2, A_0 w)| \leq g \|\eta e_2\| \cdot \|w\| \leq \frac{\kappa\epsilon_2}{8} \|\eta\|^2 + \frac{2g^2}{\kappa\epsilon_2} \|w\|^2 \leq \frac{\kappa\epsilon_2}{8} \|\eta\|^2 + \frac{2g^2}{\kappa\epsilon_2} \|w\|_{V_0}^2,$$

(4.45)

$$\frac{\epsilon_2}{l} |(w \cdot e_2, \eta)| \leq \frac{\epsilon_2}{l} |w \cdot e_2| \cdot |\eta| \leq \frac{\tilde{K}_1 \epsilon_2}{l} \|w\|_{V_0} \|\eta\| \leq \frac{\kappa\epsilon_2}{4} \|\eta\|^2 + \frac{\tilde{K}_1^2 \epsilon_2}{\kappa l^2} \|w\|_{V_0}^2,$$

where

$$(4.46) \quad \tilde{K}_1 := |\Omega|^{1/2} \lambda_1^{-1/2}.$$

Combining (4.38)–(4.45), we have

$$(4.47) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\epsilon_1 |w|^2 + \|w\|^2 + \epsilon_2 |\eta|^2) + \epsilon_1 \nu \|w\|^2 + \nu |A_0 w|^2 + \kappa \epsilon_2 \|\eta\|^2 \\ & \leq \frac{1}{2} \mu \nu \lambda_1 \|w\|_{V_0}^2 - \frac{3}{4} \mu \nu \lambda_1 (\epsilon_1 |w|^2 + \|w\|^2) \\ & \quad + \left( \frac{2g^2 \lambda_1^{-1}}{|\Omega| \kappa \epsilon_2} + \frac{2g^2}{\kappa \epsilon_2} + \frac{\tilde{K}_1^2 \epsilon_2}{\kappa l^2} \right) \|w\|_{V_0}^2 + \mu \nu \lambda_1 (\epsilon_1 |v|^2 + \|v\|^2) \\ & \quad + \frac{1}{2} \kappa \epsilon_2 \|\eta\|^2 + \frac{\nu}{2} |A_0 w|^2, \end{aligned}$$

and thus, after dropping nonnegative terms on the left,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|w\|_{V_0}^2 + \epsilon_2 |\eta|^2) \\ & \quad + \|w\|_{V_0}^2 \left( \frac{1}{4} \mu \nu \lambda_1 - \left( \frac{2g^2 \lambda_1^{-1}}{|\Omega| \kappa \epsilon_2} + \frac{2g^2}{\kappa \epsilon_2} + \frac{\tilde{K}_1^2 \epsilon_2}{\kappa l^2} \right) \right) \\ & \quad + \frac{\kappa \lambda_1}{2} \cdot \epsilon_2 |\eta|^2 \leq \mu \nu \lambda_1 \|v\|_X^2 \nu^2 \lambda_1, \end{aligned}$$

By (4.5), we have

$$\frac{d}{dt} (\|w\|_{V_0}^2 + \epsilon_2 |\eta|^2) + (\|w\|_{V_0}^2 + \epsilon_2 |\eta|^2) \cdot (\lambda_1 \kappa) \leq 2 \mu \nu \lambda_1 \|v\|_X^2 \nu^2 \lambda_1,$$

which implies by the Gronwall inequality that

$$(4.48) \quad \|w\|_{V_0}^2 + (\nu \lambda_1)^2 |\eta|^2 \leq \frac{2 \mu \nu \lambda_1}{\lambda_1 \kappa} \|v\|_X^2 \nu^2 \lambda_1,$$

and in particular

$$(4.49) \quad |\eta|^2 \leq \tilde{C}_0 \mu \|v\|_X^2, \quad \tilde{C}_0 := \frac{2 \nu \lambda_1 \nu^2 \lambda_1}{\lambda_1 \kappa (\nu \lambda_1)^2} = \frac{2 \nu}{\lambda_1 \kappa}.$$

We use (4.49) to improve the bound on  $\|w\|_{V_0}^2$ . Instead of (4.43) and (4.44), we now estimate as follows

$$(4.50) \quad \epsilon_1 |(g \eta e_2, w)| \leq g \epsilon_1 |\eta| |w| \leq \frac{g^2}{\nu} |\eta|^2 + \frac{\nu}{4} \epsilon_1^2 |w|^2,$$

$$(4.51) \quad |(g \eta e_2, A_0 w)| \leq g |\eta| |A_0 w| \leq \frac{g^2}{\nu} |\eta|^2 + \frac{\nu}{4} |A_0 w|^2.$$

Combining (4.38), (4.39), (4.41), (4.42), (4.50) and (4.51), we have

$$\begin{aligned}
 (4.52) \quad & \frac{1}{2} \frac{d}{dt} (\epsilon_1 |w|^2 + \|w\|^2) + \epsilon_1 \nu \|w\|^2 + \nu |A_0 w|^2 \\
 & \leq \left( \frac{1}{2} \mu \nu \lambda_1 - \frac{3}{4} \mu \nu \lambda_1 \right) \|w\|_{V_0}^2 + \mu \nu \lambda_1 \|v\|_{V_0}^2 + \frac{\nu}{4} |A_0 w|^2 \\
 & \quad + \frac{g^2}{\nu} |\eta|^2 + \frac{\nu}{4} \epsilon_1^2 |w|^2 + \frac{g^2}{\nu} |\eta|^2 + \frac{\nu}{4} |A_0 w|^2,
 \end{aligned}$$

which implies that

$$\frac{1}{2} \frac{d}{dt} \|w\|_{V_0}^2 + \|w\|_{V_0}^2 \left( \frac{1}{4} \mu \nu \lambda_1 - \frac{\epsilon_1 \nu}{4} \right) + \frac{\nu}{2} |A_0 w|^2 \leq \frac{2g^2}{\nu} |\eta|^2 + \mu \nu \lambda_1 \|v\|_{V_0}^2.$$

Therefore, by (4.6),

$$(4.53) \quad \frac{d}{dt} \|w\|_{V_0}^2 + \frac{1}{4} \mu \nu \lambda_1 \|w\|_{V_0}^2 + \nu |A_0 w|^2 \leq 2\mu \left( \frac{2g^2 \tilde{C}_0}{\nu} + \nu \lambda_1 \nu^2 \lambda_1 \right) \|v\|_X^2.$$

Dropping the term  $\nu |A_0 w|^2$  in (4.53) and using the Gronwall inequality, we conclude that

$$(4.54) \quad \|w\|_{V_0}^2 \leq \tilde{C}_1 \|v\|_X^2$$

where

$$(4.55) \quad \tilde{C}_1 := \frac{2\mu \left( \frac{2g^2 \tilde{C}_0}{\nu} + \nu \lambda_1 \nu^2 \lambda_1 \right)}{\frac{1}{4} \mu \nu \lambda_1} = \frac{32g^2}{\lambda_1 \kappa \nu \lambda_1} + 8\nu^2 \lambda_1.$$

Note that the constant  $\tilde{C}_1$  is independent of  $\mu$ .

By (4.40) and (4.45), we have

$$\frac{1}{2} \frac{d}{dt} |\eta|^2 + \kappa \|\eta\|^2 \leq \frac{\kappa}{4} \|\eta\|^2 + \frac{\tilde{K}_1^2}{\kappa l^2} \|w\|_{V_0}^2$$

and thus by (4.54) and the Poincaré inequality,

$$\frac{1}{2} \frac{d}{dt} |\eta|^2 + \frac{\kappa \lambda_1}{2} |\eta|^2 \leq \tilde{K}_2 \|v\|_X^2, \quad \tilde{K}_2 := \frac{\tilde{K}_1^2 \tilde{C}_1}{\kappa l^2}.$$

Consequently, by the Gronwall inequality again, we have

$$(4.56) \quad |\eta|^2 \leq \tilde{C}_2 \|v\|_X^2, \quad \tilde{C}_2 := \frac{2\tilde{K}_2}{\lambda_1 \kappa},$$

where  $\tilde{C}_2$  is also independent of  $\mu$ .

4.2.2. *Bound for  $\int_t^{\min(t+T, T_{**})} |A_0 w(\tau)|^2 d\tau$ .* For any  $t \in [-k(\nu \lambda_1)^{-1}, T_{**})$ , dropping the term  $\frac{1}{4} \mu \nu \lambda_1 \|w\|_{V_0}^2$  in (4.53) and integrating, then using the bound (4.54), we have

$$(4.57) \quad \nu \int_t^{\min(t+T, T_{**})} |A_0 w(\tau)|^2 d\tau \leq \tilde{C}_1 \|v\|_X^2 + 2\mu T \left( \frac{2g^2 \tilde{C}_0}{\nu} + \nu \lambda_1 \nu^2 \lambda_1 \right) \|v\|_X^2.$$

Since  $T_{**} < \infty$ , it follows that

$$(4.58) \quad \nu \int_{-k(\nu\lambda_1)^{-1}}^{T_{**}} |A_0 w(\tau)|^2 d\tau < \infty.$$

4.2.3. *Bound for  $\|\eta\|$ .* Proceeding as in the no-slip case (Section 4.1.4) but using the bounds (4.54) and (4.56) for  $|w|$  and  $|\eta|$  in (4.29) instead, we get for any  $t \in [-\tilde{k}, -\tilde{k} + \alpha_k]$ ,  $\tilde{k} = k(\nu\lambda_1)^{-1}$  and  $\alpha_k = \frac{T_{**} + \tilde{k}}{2}$ ,

$$(4.59) \quad \kappa \int_t^{t+\alpha_k} \|\eta(\tau)\|^2 d\tau \leq \frac{\tilde{C}_2}{2} + \alpha_k \left( \frac{\kappa\lambda_1\tilde{C}_2}{4} + \frac{\tilde{C}_1}{\kappa l^2\lambda_1} \right) \|v\|_X^2 =: \tilde{\beta}_k$$

Similarly as in (4.31), we have

$$(4.60) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta\|^2 + \kappa |A_1 \eta|^2 + (B_1(w, \eta), A_1 \eta) \\ \leq \frac{\kappa}{4} |A_1 \eta|^2 + \frac{1}{l^2\kappa} |w|^2 \leq \frac{\kappa}{4} |A_1 \eta|^2 + \frac{|\Omega|}{l^2\kappa} \|w\|_{V_0}^2 \\ \leq \frac{\kappa}{4} |A_1 \eta|^2 + \frac{|\Omega|\tilde{C}_1}{l^2\kappa} \|v\|_X^2. \end{aligned}$$

For the nonlinear term, we have

$$(4.61) \quad \begin{aligned} |(B_1(w, \eta), A_1 \eta)| &\leq |A_1 \eta| \cdot \|w\|_{L^4} \|\nabla \eta\|_{L^4} \quad (\text{Hölder}) \\ &\leq c_L |A_1 \eta| \cdot |w|^{1/2} \|w\|_{V_0}^{1/2} \|\eta\|^{1/2} |A_1 \eta|^{1/2} \quad (\text{Ladyzhenskaya}) \\ &\leq c_L |\Omega|^{1/2} \|w\|_{V_0} |A_1 \eta|^{3/2} \|\eta\|^{1/2} \quad (\text{by (2.8)}) \\ &\leq c_L |\Omega|^{1/2} \tilde{C}_1^{1/2} \|v\|_X |A_1 \eta|^{3/2} \|\eta\|^{1/2} \\ &\leq \frac{\kappa}{4} |A_1 \eta|^2 + \tilde{K}_3 \|v\|_X^4 \|\eta\|^2 \quad (\text{Young}) \quad \tilde{K}_3 := \frac{27}{4\kappa^3} c_L^4 |\Omega|^2 \tilde{C}_1^2. \end{aligned}$$

By (4.60) and (4.61), we have

$$(4.62) \quad \frac{d}{dt} \|\eta\|^2 + \kappa |A_1 \eta|^2 \leq 2\tilde{K}_3 \|v\|_X^4 \|\eta\|^2 + \tilde{K}_4 \|v\|_X^2, \quad \tilde{K}_4 := \frac{2|\Omega|\tilde{C}_1}{l^2\kappa}.$$

Proceeding as in Section 4.1.4, using Lemma 4.3 with

$$\begin{aligned} t_0 &= -k(\nu\lambda_1)^{-1}, \quad t_1 = T_{**}, \quad \alpha = \alpha_k, \\ g(t) &= 2K_3\rho^4, \quad h(t) = K_4\rho^2, \quad y(t) = \|\eta(t)\|^2, \\ a_1 &:= 2\tilde{K}_3\alpha\rho^4, \quad a_2 := \tilde{K}_4\rho^2\alpha, \quad a_3 := \frac{\tilde{\beta}_k}{\kappa}\alpha, \end{aligned}$$

we get

$$(4.63) \quad \sup_{t \in [-\tilde{k} + \alpha, T_{**}]} \|\eta(t)\|^2 \leq \tilde{C}_4 \|v\|_X^2, \quad \tilde{C}_4 := (\tilde{C}_3 + \tilde{K}_4 T) e^{2\tilde{K}_3 T \rho^4},$$

and as in Section (4.1),

$$(4.64) \quad \sup_{t \in [-\tilde{k}, T_{**}]} \|\eta(t)\|^2 < \infty.$$

4.2.4. *Bound for  $\int_t^{\min(t+T, T_{**})} |A_1\eta(\tau)|^2 d\tau$ .* Similarly as in Section 4.1.5, combining (4.62) and (4.63), we get for any  $t \in [-\tilde{k} + \alpha_k, T_{**})$ ,

$$(4.65) \quad \kappa \int_t^{\min(t+T, T_{**})} |A_1\eta(\tau)|^2 d\tau \leq \tilde{C}_4 \|v\|_X^2 + T \left( 2\tilde{K}_3 \|v\|_X^4 \tilde{C}_4 \|v\|_X^2 + \tilde{K}_4 \|v\|_X^2 \right).$$

Also,

$$(4.66) \quad \kappa \int_{-k(\nu\lambda_1)^{-1}}^{T_{**}} |A_1\eta(\tau)|^2 d\tau < \infty.$$

## 5. LIPCHITZ PROPERTY OF THE MAP $\widetilde{W}$

We assume in this section that  $\|v_i\|_X \leq \rho$ ,  $i = 1, 2$ . Let  $\varphi = w_1 - w_2$ ,  $\psi = \eta_1 - \eta_2$  and  $\gamma = v_1 - v_2$  where  $(w_i, \eta_i) = \widetilde{W}(v_i)$ . We establish in this section the Lipchitz property of the map  $\widetilde{W}$  for each set of boundary conditions.

By the auxiliary system (3.8), we have

$$(5.1) \quad \frac{d\varphi}{dt} + \nu A_0\varphi + B_0(w_2, \varphi) + B_0(\varphi, w_1) = \mathbb{P}_\sigma(g\psi e_2) - \mu\nu\lambda_1(\tilde{I}_h\varphi - \gamma),$$

$$(5.2) \quad \frac{d\psi}{dt} + \kappa A_1\psi + B_1(w_2, \psi) + B_1(\varphi, \eta_1) = \frac{\varphi \cdot e_2}{l}.$$

### 5.1. No-slip BCs.

5.1.1. *Bound for  $\|\varphi\|^2$  and  $|\psi|^2$  by  $\|\gamma\|_X^2$ .* Taking the  $L^2$  inner product of (5.1)–(5.2) with  $A_0\varphi$  and  $\psi$  respectively, we have

$$(5.3) \quad \frac{1}{2} \frac{d}{dt} \|\varphi\|^2 + \nu |A_0\varphi|^2 + (B_0(w_2, \varphi), A_0\varphi) + (B_0(\varphi, w_1), A_0\varphi) \\ = (g\psi e_2, A_0\varphi) - \mu\nu\lambda_1(\tilde{I}_h\varphi - \gamma, A_0\varphi),$$

$$(5.4) \quad \frac{1}{2} \frac{d}{dt} |\psi|^2 + \kappa \|\psi\|^2 + (B_1(\varphi, \eta_1), \psi) = \frac{1}{l} (\varphi \cdot e_2, \psi).$$

Proceeding as for (4.20), we find

$$(5.5) \quad -\mu\nu\lambda_1(\tilde{I}_h\varphi - \gamma, A_0\varphi) \leq \frac{\nu}{8} |A_0\varphi|^2 + \frac{1}{2} \mu\nu\lambda_1 \|\gamma\|_X^2 - \frac{1}{2} \mu\nu\lambda_1 \|\varphi\|^2.$$

By the Cauchy-Schwarz, Young and Poincaré inequalities, we have

$$(5.6) \quad (g\psi e_2, A_0\varphi) \leq g \|\psi\| \cdot \|\varphi\| \leq \frac{\kappa(\nu\lambda_1)^2}{4} \|\psi\|^2 + \frac{g^2}{\kappa(\nu\lambda_1)^2} \|\varphi\|^2,$$

$$(5.7) \quad \frac{1}{l} (\varphi \cdot e_2, \psi) \leq \frac{1}{l} |\varphi| \cdot |\psi| \leq \frac{1}{l\lambda_1} \|\varphi\| \cdot \|\psi\| \leq \frac{\kappa}{8} \|\psi\|^2 + \frac{2}{l^2\lambda_1^2\kappa} \|\varphi\|^2.$$

For the two nonlinear terms involving  $B_0$ , we have (see [28])

$$(5.8) \quad \begin{aligned} |(B_0(w_2, \varphi), A_0\varphi)| &\leq c_T \|w_2\| \cdot \|\varphi\| \left( \log \frac{e|A_0\varphi|}{\lambda_1^{1/2}\|\varphi\|} \right)^{1/2} |A_0\varphi| \\ &\leq \frac{c_T^2}{\nu} \|w_2\|^2 \|\varphi\|^2 \log \frac{e|A_0\varphi|}{\lambda_1^{1/2}\|\varphi\|} + \frac{\nu}{4} |A_0\varphi|^2 \end{aligned}$$

and by the Brézis-Gallouet inequality (see [5, 28])

$$(5.9) \quad \begin{aligned} |(B_0(\varphi, w_1), A_0\varphi)| &\leq c_B \|w_1\| \cdot \|\varphi\| \left( \log \frac{e|A_0\varphi|}{\lambda_1^{1/2}\|\varphi\|} \right)^{1/2} |A_0\varphi| \\ &\leq \frac{c_B^2}{\nu} \|w_1\|^2 \|\varphi\|^2 \log \frac{e|A_0\varphi|}{\lambda_1^{1/2}\|\varphi\|} + \frac{\nu}{4} |A_0\varphi|^2 \end{aligned}$$

For the nonlinear term involving  $B_1$ , we have

$$(5.10) \quad \begin{aligned} |(B_1(\varphi, \eta_1), \psi)| &\leq \|\varphi\|_{L^4} \|\psi\|_{L^4} \|\eta_1\| \leq c_L |\varphi|^{1/2} \|\varphi\|^{1/2} |\psi|^{1/2} \|\psi\|^{1/2} \|\eta_1\| \\ &\leq \frac{c_L}{\sqrt{\lambda_1}} \|\varphi\| \|\psi\| \|\eta_1\| \leq \frac{\kappa}{8} \|\psi\|^2 + \frac{2c_L^2}{\kappa\lambda_1} \|\varphi\|^2 \|\eta_1\|^2. \end{aligned}$$

Combining the estimates above, we have for  $\|\varphi\|$ ,

$$(5.11) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varphi\|^2 + \|\varphi\|^2 &\left[ \frac{1}{2} \mu\nu\lambda_1 - \frac{g^2}{\kappa(\nu\lambda_1)^2} \right. \\ &+ \frac{\nu|A_0\varphi|^2}{4\|\varphi\|^2} - (c_T^2 \|w_2\|^2 + c_B^2 \|w_1\|^2) \left( \log \frac{e|A_0\varphi|}{\lambda_1^{1/2}\|\varphi\|} \right) \nu^{-1} \Big] \\ &+ \nu|A_0\varphi|^2 \left[ 1 - \frac{1}{8} - \frac{1}{2} - \frac{1}{4} \right] - \frac{\kappa(\nu\lambda_1)^2}{4} \|\psi\|^2 \\ &\leq \frac{\mu\nu\lambda_1}{2} \|\gamma\|^2. \end{aligned}$$

But the second line of (5.11) can be estimated by

$$(5.12) \quad \begin{aligned} \frac{\nu|A_0\varphi|^2}{4\|\varphi\|^2} - (c_T^2 \|w_2\|^2 + c_B^2 \|w_1\|^2) &\left( \log \frac{e|A_0\varphi|}{\lambda_1^{1/2}\|\varphi\|} \right) \nu^{-1} \\ &\geq \frac{\nu|A_0\varphi|^2}{4\|\varphi\|^2} - \frac{c_T^2 \|w_2\|^2 + c_B^2 \|w_1\|^2}{\nu} \left( 1 + 2 \log \frac{|A_0\varphi|}{\lambda_1^{1/2}\|\varphi\|} \right) \\ &= \frac{\lambda_1\nu}{4} \left[ \frac{|A_0\varphi|^2}{\lambda_1\|\varphi\|^2} - \frac{c_T^2 \|w_2\|^2 + c_B^2 \|w_1\|^2}{\nu^2\lambda_1/4} \left( 1 + \log \frac{|A_0\varphi|^2}{\lambda_1\|\varphi\|^2} \right) \right] \\ &\geq \frac{\lambda_1\nu}{4} (-\epsilon \log \epsilon) \end{aligned}$$

where we used the elementary relation (see [16, p.371])

$$(5.13) \quad \chi - \epsilon(1 + \log \chi) \geq -\epsilon \log \epsilon, \quad \forall \chi \geq 1,$$

with

$$(5.14) \quad \epsilon := \frac{c_T^2 \|w_2\|^2 + c_B^2 \|w_1\|^2}{\nu^2 \lambda_1 / 4} \leq \frac{4(c_T^2 + c_B^2) \rho^2}{\nu^2 \lambda_1} =: K_2.$$

Hence,

$$(5.15) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varphi\|^2 + \|\varphi\|^2 & \left[ \frac{1}{2} \mu \nu \lambda_1 - \frac{g^2}{\kappa(\nu \lambda_1)^2} - \frac{\lambda_1 \nu}{4} (K_2 \log K_2) \right] \\ & + \frac{\nu}{8} |A_0 \varphi|^2 - \frac{\kappa(\nu \lambda_1)^2}{4} \|\psi\|^2 \\ & \leq \frac{\mu \nu \lambda_1}{2} \|\gamma\|^2. \end{aligned}$$

Combining (5.4), (5.7) and (5.10), we have

$$(5.16) \quad \frac{1}{2} \frac{d}{dt} |\psi|^2 - \|\varphi\|^2 \left[ \frac{2c_L^2}{\kappa \lambda_1^2} \|\eta_1\|^2 + \frac{2}{l^2 \lambda_1^2 \kappa} \right] + \left[ \kappa \|\psi\|^2 - \frac{\kappa}{8} \|\psi\|^2 - \frac{\kappa}{8} \|\psi\|^2 \right] \leq 0.$$

Combining the differential inequalities (5.15) and (5.16) for  $\|\varphi\|^2$  and  $|\psi|^2$ , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\varphi\|^2 + (\nu \lambda_1)^2 |\psi|^2 \right) & + \frac{\kappa(\nu \lambda_1)^2}{2} \|\psi\|^2 + \frac{\nu}{8} |A_0 \varphi|^2 \\ & + \|\varphi\|^2 \left[ \frac{1}{2} \mu \nu \lambda_1 - \frac{g^2}{\kappa(\nu \lambda_1)^2} - \frac{\lambda_1 \nu}{4} (K_2 \log K_2) - \frac{2c_L^2 (\nu \lambda_1)^2}{\kappa \lambda_1^2} \rho^2 - \frac{2(\nu \lambda_1)^2}{l^2 \lambda_1^2 \kappa} \right] \\ & \leq \frac{1}{2} \mu \nu \lambda_1 \|\gamma\|_X^2 \nu^2 \lambda_1. \end{aligned}$$

Consequently, by (4.4) and the Poincaré inequality,

$$(5.17) \quad \begin{aligned} \frac{d}{dt} \left( \|\varphi\|^2 + (\nu \lambda_1)^2 |\psi|^2 \right) & + \kappa(\nu \lambda_1)^2 \|\psi\|^2 + \frac{\nu}{4} |A_0 \varphi|^2 + \kappa \lambda_1 \|\varphi\|^2 \\ & \leq \frac{d}{dt} \left( \|\varphi\|^2 + (\nu \lambda_1)^2 |\psi|^2 \right) + \kappa \lambda_1 \left( \|\varphi\|^2 + (\nu \lambda_1)^2 |\psi|^2 \right) + \frac{\nu}{4} |A_0 \varphi|^2 \\ & \leq \mu \nu \lambda_1 \|\gamma\|_X^2 \nu^2 \lambda_1. \end{aligned}$$

Dropping the terms  $\frac{\nu}{4} |A_0 \varphi|^2$  in the second inequality, using the Gronwall inequality and the fact that  $\|w_j\|$ ,  $|\eta_j|$  are bounded, we obtain

$$(5.18) \quad \|\varphi\|^2 + (\nu \lambda_1)^2 |\psi|^2 \leq \frac{\mu \nu \lambda_1}{\kappa} \|\gamma\|_X^2 \nu^2.$$

5.1.2. *Bound for  $\int_t^{t+T} |A_0 \varphi|^2$  and  $\int_t^{t+T} \|\psi\|^2$  by  $\|\gamma\|_X^2$ .* The inequality (5.17) implies that

$$\frac{d}{dt} \left( \|\varphi\|^2 + (\nu \lambda_1)^2 |\psi|^2 \right) + \frac{\nu}{4} |A_0 \varphi|^2 + \kappa(\nu \lambda_1)^2 \|\psi\|^2 \leq \mu \nu \lambda_1 \|\gamma\|_X^2 \nu^2 \lambda_1.$$

Integrating from  $t$  to  $t+T$ ,  $T = (\nu \lambda_1)^{-1}$ , and using the bound (5.18), we have

$$(5.19) \quad \frac{\nu}{4} \int_t^{t+T} |A_0 \varphi(\tau)|^2 d\tau + \kappa(\nu \lambda_1) \int_t^{t+T} \|\psi(\tau)\|^2 d\tau \leq \mu \nu \lambda_1 \left( \lambda_1 T + \frac{1}{\kappa} \right) \|\gamma\|_X^2 \nu^2.$$

5.1.3. *Bounds for  $\|\psi\|^2$  and  $\int_t^{t+T} |A_1\psi|^2$  by  $\|\gamma\|_X^2$ .* Taking the  $L^2$  inner product of (5.2) with  $A_1\psi$ , we have

$$(5.20) \quad \frac{1}{2} \frac{d}{dt} \|\psi\|^2 + \kappa |A_1\psi|^2 + b_1(w_2, \psi, A_1\psi) + b_1(\varphi, \eta_1, A_1\psi) = \frac{1}{l} (\varphi \cdot \mathbf{e}_2, A_1\psi), .$$

Integrating by parts, we have

$$(5.21) \quad \begin{aligned} |b_1(w_2, \psi, A_1\psi)| &\leq \|w_2\| \cdot \|\nabla\psi\|_{L^4}^2 \quad (\text{Hölder}) \\ &\leq c_L \|w_2\| \cdot \|\psi\| \cdot |A_1\psi| \quad (\text{Ladyzhenskaya}) \\ &\leq \frac{\kappa}{8} |A_1\psi|^2 + \frac{2c_L^2}{\kappa} \|w_2\|^2 \|\psi\|^2. \end{aligned}$$

Similarly,

$$(5.22) \quad \begin{aligned} |b_1(\varphi, \eta_1, A_1\psi)| &\leq \|\varphi\| \cdot \|\nabla\eta_1\|_{L^4} \|\nabla\psi\|_{L^4} \quad (\text{Hölder}) \\ &\leq c_L \|\varphi\| \cdot \|\eta_1\|^{1/2} |A_1\eta_1|^{1/2} \|\psi\|^{1/2} |A_1\psi|^{1/2} \quad (\text{Ladyzhenskaya}) \\ &\leq \frac{1}{2\nu} \|\varphi\|^2 + \frac{c_L^2 \nu}{2} \|\eta_1\| \cdot |A_1\eta_1| \cdot \|\psi\| \cdot |A_1\psi| \quad (\text{Young}) \\ &\leq \frac{1}{2\nu} \|\varphi\|^2 + \frac{\kappa}{8} |A_1\psi|^2 + \frac{c_L^4 \nu^2}{2\kappa} \|\eta_1\|^2 |A_1\eta_1|^2 \|\psi\|^2. \end{aligned}$$

By Cauchy-Schwarz and Young inequalities,

$$(5.23) \quad \frac{1}{l} |(\varphi \cdot \mathbf{e}_2, A_1\psi)| \leq \frac{\kappa}{4} |A_1\psi|^2 + \frac{1}{\kappa l^2} |\varphi|^2.$$

Combining (5.20)–(5.23), we obtain

$$(5.24) \quad \frac{1}{2} \frac{d}{dt} \|\psi\|^2 + \frac{\kappa}{2} |A_1\psi|^2 \leq \left( \frac{2c_L^2}{\kappa} \|w_2\|^2 + \frac{c_L^4 \nu^2}{2\kappa} \|\eta_1\|^2 |A_1\eta_1|^2 \right) \|\psi\|^2 + \frac{1}{2\nu} \|\varphi\|^2 + \frac{1}{\kappa l^2} |\varphi|^2.$$

Let the function  $g$  and  $h$  in Lemma 4.3 be

$$(5.25) \quad g := 2 \left( \frac{2c_L^2}{\kappa} \|w_2\|^2 + \frac{c_L^4 \nu^2}{2\kappa} \|\eta_1\|^2 |A_1\eta_1|^2 \right), \quad h := \frac{1}{\nu} \|\varphi\|^2 + \frac{2}{\kappa l^2} |\varphi|^2.$$

By the bounds (4.24), (4.34) and (4.36), we have

$$(5.26) \quad \begin{aligned} \int_t^{t+T} g(s) ds &\leq \frac{4c_L^2}{\kappa} \cdot 4C_1 \rho^2 \lambda_1 T + \frac{c_L^4 \nu^2}{\kappa^2} C_3 \left[ C_3 + \left( \frac{8c_L^2 C_1 \lambda_1 C_3}{\kappa} + \frac{8C_1}{l^2 \kappa} \right) \rho^2 T \right] T \\ &=: a_1. \end{aligned}$$

By (5.18) and the Poincaré inequality, we have

$$(5.27) \quad \int_t^{t+T} h(s) ds \leq T \left( \frac{1}{2\nu} + \frac{1}{\kappa l^2 \lambda_1} \right) \frac{\mu \nu \lambda_1}{\kappa} \|\gamma\|_X^2 \nu^2 =: K_{11} \|\gamma\|_X^2 =: a_2.$$

By (5.19),

$$(5.28) \quad \int_t^{t+T} \|\psi(\tau)\|^2 d\tau \leq \frac{\mu \nu \lambda_1}{\kappa (\nu \lambda_1)} \left( \lambda_1 T + \frac{1}{\kappa} \right) \|\gamma\|_X^2 \nu^2 =: K_{12} \|\gamma\|_X^2 =: a_3.$$

Dropping the term  $\frac{\kappa}{2} |A_1\psi|^2$  in (5.24), applying Lemma 4.3 with (5.26), (5.27) and (5.28) we have

$$\sup_{t \in \mathbb{R}} \|\psi(t)\|^2 \leq e^{a_1} \left( K_{11} + \frac{K_{12}}{T} \right) \|\gamma\|_X^2 =: K_{13} \|\gamma\|_X^2.$$

Now, by integrating (5.24) from  $t$  to  $t+T$  and using (5.26) and (5.27), we get

$$\kappa \int_t^{t+T} |A_1\psi(\tau)|^2 d\tau \leq (K_{13} + a_1 K_{13} + K_{11}) \|\gamma\|_X^2.$$

## 5.2. Stress-free BCs.

5.2.1. *Bounds for  $|\psi|^2$ ,  $|\varphi|^2$  and  $\|\varphi\|^2$  by  $\|\gamma\|_X^2$ .* Taking the  $L^2$  inner product of (5.1)–(5.2) with  $\varphi$  and  $\psi$  respectively and taking the  $L^2$  inner product of (5.1) with  $A_0\varphi$  we have

$$(5.29) \quad \epsilon_1 \left( \frac{1}{2} \frac{d}{dt} |\varphi|^2 + \nu \|\varphi\|^2 + b_0(\varphi, w_1, \varphi) \right) = \epsilon_1 \left( g(\psi \mathbf{e}_2, \varphi) - \mu \nu \lambda_1 (\tilde{I}_h \varphi - \gamma, \varphi) \right)$$

$$(5.30) \quad \frac{1}{2} \frac{d}{dt} \|\varphi\|^2 + \nu |A_0\varphi|^2 + b_0(w_2, \varphi, A_0\varphi) + b_0(\varphi, w_1, A_0\varphi) \\ = g(\psi \mathbf{e}_2, A_0\varphi) - \mu \nu \lambda_1 (\tilde{I}_h \varphi - \gamma, A_0\varphi)$$

$$(5.31) \quad \epsilon_2 \left( \frac{1}{2} \frac{d}{dt} |\psi|^2 + \kappa \|\psi\|^2 + b_1(\varphi, \eta_1, \psi) \right) = \epsilon_2 \left( \frac{1}{l} (\varphi \cdot \mathbf{e}_2, \psi) \right),$$

where, as in (4.38), (4.40),  $\epsilon_1 = |\Omega|^{-1}$ ,  $\epsilon_2 = (\nu \lambda_1)^2$ .

For the linear terms, as in (4.41)–(4.45) we have

$$(5.32) \quad -\mu \nu \lambda_1 \epsilon_1 (\tilde{I}_h \varphi - \gamma, \varphi) \leq \frac{1}{4} \mu \nu \lambda_1 \|\varphi\|_{V_0}^2 + \mu \nu \lambda_1 \epsilon_1 |\gamma|^2 - \frac{3}{4} \mu \nu \lambda_1 \epsilon_1 |\varphi|^2 + \frac{\nu}{8} |A_0\varphi|^2,$$

$$(5.33) \quad -\mu \nu \lambda_1 (\tilde{I}_h \varphi - \gamma, A_0\varphi) \leq \frac{1}{4} \mu \nu \lambda_1 \|\varphi\|_{V_0}^2 + \frac{\nu}{8} |A_0\varphi|^2 + \mu \nu \lambda_1 \|\gamma\|^2 - \frac{3}{4} \mu \nu \lambda_1 \|\varphi\|^2,$$

$$(5.34) \quad \epsilon_1 |(g\psi \mathbf{e}_2, \varphi)| \leq \frac{\kappa \epsilon_2}{8} \|\psi\|^2 + \frac{2}{\kappa \epsilon_2} \frac{g^2 \lambda_1^{-1}}{|\Omega|} \|\varphi\|_{V_0}^2,$$

$$(5.35) \quad |(g\psi \mathbf{e}_2, A_0\varphi)| \leq \frac{\kappa \epsilon_2}{8} \|\psi\|^2 + \frac{2g^2}{\kappa \epsilon_2} \|\varphi\|_{V_0}^2,$$

$$(5.36) \quad \frac{\epsilon_2}{l} |(\varphi \cdot \mathbf{e}_2, \psi)| \leq \frac{\kappa \epsilon_2}{4} \|\psi\|^2 + \frac{\tilde{K}_1^2 \epsilon_2}{\kappa l^2} \|\varphi\|_{V_0}^2.$$

For the nonlinear terms, we have

$$\begin{aligned}
 (5.37) \quad \epsilon_1 |b_0(\varphi, w_1, \varphi)| &\leq \epsilon_1 \|w_1\| \cdot \|\varphi\|_{L^4}^2 \quad (\text{Hölder}) \\
 &\leq \epsilon_1 c_L \|w_1\| \cdot |\varphi| \cdot \|\varphi\|_{V_0} \quad (\text{Ladyzhenskaya}) \\
 &\leq \epsilon_1 c_L \tilde{C}_1^{1/2} \rho |\varphi| \cdot \|\varphi\|_{V_0} \quad (\text{by (4.54)}) \\
 &\leq \epsilon_1 c_L \tilde{C}_1^{1/2} \rho |\Omega|^{1/2} \|\varphi\|_{V_0}^2 \quad (\text{by (2.8)})
 \end{aligned}$$

$$\begin{aligned}
 (5.38) \quad \epsilon_2 |b_1(\varphi, \eta_1, \psi)| &\leq \epsilon_2 \|\varphi\|_{L^4} \|\eta_1\| \|\psi\|_{L^4} \quad (\text{Hölder}) \\
 &\leq \epsilon_2 c_L |\varphi|^{1/2} \|\varphi\|_{V_0}^{1/2} \|\eta_1\| |\psi|^{1/2} \|\psi\|^{1/2} \quad (\text{Ladyzhenskaya}) \\
 &\leq \epsilon_2 c_L |\Omega|^{1/4} \|\varphi\|_{V_0} \|\eta_1\| \lambda_1^{-1/4} \|\psi\| \quad (\text{by (2.8)}) \\
 &\leq \epsilon_2 c_L |\Omega|^{1/4} \lambda_1^{-1/4} \tilde{C}_4^{1/2} \rho \|\varphi\|_{V_0} \|\psi\| \quad (\text{by (4.63)}) \\
 &\leq \frac{\epsilon_2 \kappa}{4} \|\psi\|^2 + \frac{\epsilon_2 K_{13}}{\kappa} \|\varphi\|_{V_0}^2 \quad (\text{Young}) \quad K_{13} := c_L^2 |\Omega|^{1/2} \lambda_1^{-1/2} \tilde{C}_4 \rho^2
 \end{aligned}$$

$$\begin{aligned}
 (5.39) \quad |b_0(\varphi, w_1, A_0 \varphi)| &\leq \|\varphi\|_{L^\infty} \|w_1\| |A_0 \varphi| \quad (\text{Hölder}) \\
 &\leq c_A |\varphi|^{1/2} \|\varphi\|_{H^2}^{1/2} \|w_1\| |A_0 \varphi| \quad (\text{2D Agmon}) \\
 &\leq c_A c_E |\varphi|^{1/2} \left( \frac{1}{|\Omega|^{1/2}} |\varphi|^{1/2} + |A_0 \varphi|^{1/2} \right) \|w_1\| |A_0 \varphi| \quad (\text{by (2.9)}) \\
 &\leq \frac{c_A c_E}{|\Omega|^{1/2}} |\varphi| \cdot |A_0 \varphi| \tilde{C}_1^{1/2} \rho + c_A c_E |\varphi|^{1/2} |A_0 \varphi|^{3/2} \tilde{C}_1^{1/2} \rho \quad (\text{by (4.54)}) \\
 &\leq \frac{\nu}{4} |A_0 \varphi|^2 + \frac{K_{14}}{\nu} |\varphi|^2 \quad (\text{Young})
 \end{aligned}$$

where

$$K_{14} = 2\hat{K}_1^2 + \frac{54}{\nu^3} \hat{K}_2^4, \quad \hat{K}_1 = \frac{c_A c_E \tilde{C}_1^{1/2} \rho}{|\Omega|^{1/2}}, \quad \hat{K}_2 = c_A c_E \tilde{C}_1^{1/2} \rho;$$

$$\begin{aligned}
 (5.40) \quad |b_0(w_2, \varphi, A_0 \varphi)| &\leq \|w_2\|_{L^4} \|\nabla \varphi\|_{L^4} |A_0 \varphi| \quad (\text{Hölder}) \\
 &\leq c_L |w_2|^{1/2} \|w_2\|_{H^1}^{1/2} \|\nabla \varphi\|_{H^1}^{1/2} \|\nabla \varphi\|_{H^1}^{1/2} |A_0 \varphi| \quad (\text{Ladyzhenskaya}) \\
 &\leq c_L |\Omega|^{1/4} \|w_2\|_{V_0} \|\varphi\|^{1/2} \|\varphi\|_{H^2}^{1/2} |A_0 \varphi| \\
 &\leq c_L c_E |\Omega|^{1/4} \tilde{C}_1^{1/2} \rho \|\varphi\|^{1/2} \left( \frac{1}{|\Omega|^{1/2}} |\varphi|^{1/2} + |A_0 \varphi|^{1/2} \right) |A_0 \varphi| \quad (\text{by (2.9)}) \\
 &\leq c_L c_E |\Omega|^{1/4} \tilde{C}_1^{1/2} \rho \|\varphi\|^{1/2} \left( \frac{|\Omega|^{1/4}}{|\Omega|^{1/2}} \|\varphi\|_{V_0}^{1/2} + |A_0 \varphi|^{1/2} \right) |A_0 \varphi| \\
 &\leq \frac{\nu}{4} |A_0 \varphi|^2 + \frac{K_{15}}{\nu} \|\varphi\|_{V_0}^2 \quad (\text{Young})
 \end{aligned}$$

where

$$K_{15} = 2\hat{K}_3^2 + \frac{54}{\nu^3}\hat{K}_4^4, \quad \hat{K}_3 = c_E c_L \tilde{C}_1^{1/2} \rho, \quad \hat{K}_4 = c_E c_L |\Omega|^{1/4} \tilde{C}_1^{1/2} \rho.$$

Combining (5.29)–(5.40), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\epsilon_1 |\varphi|^2 + \|\varphi\|^2 + \epsilon_2 |\psi|^2) + \epsilon_1 \nu \|\varphi\|^2 + \nu |A_0 \varphi|^2 + \epsilon_2 \kappa \|\psi\|^2 \\ & \leq \|\varphi\|_{V_0}^2 \left[ \frac{1}{2} \mu \nu \lambda_1 - \frac{3}{4} \mu \nu \lambda_1 + \frac{2g^2 \lambda_1^{-1}}{\kappa \epsilon_2 |\Omega|} + \frac{2g^2}{\kappa \epsilon_2} + \frac{\tilde{K}_1^2 \epsilon_2}{\kappa l^2} \right. \\ & \quad \left. + \epsilon_1 c_L \tilde{C}_1^{1/2} \rho |\Omega|^{1/2} + \frac{\epsilon_2 K_{13}}{\kappa} + \frac{K_{15}}{\nu} + \frac{K_{14} |\Omega|}{\nu} \right] \\ & \quad + \mu \nu \lambda_1 \|\gamma\|_{V_0}^2 + \frac{3}{4} \nu |A_0 \varphi|^2 + \frac{3}{4} \kappa \epsilon_2 \|\psi\|^2. \end{aligned}$$

It follows that

$$\frac{1}{2} \frac{d}{dt} (\|\varphi\|_{V_0}^2 + \epsilon_2 |\psi|^2) + \|\varphi\|_{V_0}^2 \left( \frac{1}{4} \mu \nu \lambda_1 - K_{16} \right) + \frac{1}{4} \epsilon_2 \kappa \|\psi\|^2 + \frac{\nu}{4} |A_0 \varphi|^2 \leq \mu \nu \lambda_1 \|\gamma\|_{V_0}^2$$

where

$$(5.41) \quad K_{16} := \frac{2g^2 \lambda_1^{-1}}{\kappa \epsilon_2 |\Omega|} + \frac{2g^2}{\kappa \epsilon_2} + \frac{\tilde{K}_1^2 \epsilon_2}{\kappa l^2} + \epsilon_1 c_L \tilde{C}_1^{1/2} \rho |\Omega|^{1/2} + \frac{\epsilon_2 K_{13}}{\kappa} + \frac{K_{15}}{\nu} + \frac{K_{14} |\Omega|}{\nu}.$$

By (4.7), we have

$$(5.42) \quad \begin{aligned} & \frac{d}{dt} (\|\varphi\|_{V_0}^2 + \epsilon_2 |\psi|^2) + (\|\varphi\|_{V_0}^2 + \epsilon_2 |\psi|^2) \frac{\kappa \lambda_1}{4} + \frac{1}{4} \epsilon_2 \kappa \|\psi\|^2 + \frac{\nu}{2} |A_0 \varphi|^2 \\ & \leq 2 \mu \nu \lambda_1 \|\gamma\|_X^2 \nu^2 \lambda_1. \end{aligned}$$

Dropping  $\frac{1}{4} \epsilon_2 \kappa \|\psi\|^2 + \frac{\nu}{2} |A_0 \varphi|^2$  on the left and using the Gronwall inequality, we conclude that

$$(5.43) \quad \|\varphi\|_{V_0}^2 + \epsilon_2 |\psi|^2 \leq \mu C_6 \|\gamma\|_X^2, \quad C_6 := \frac{8 \lambda_1 \nu^3}{\kappa},$$

and in particular,

$$(5.44) \quad \|\varphi\|_{V_0}^2 \leq \mu C_6 \|\gamma\|_X^2.$$

5.2.2. *Bound for  $\int_t^{t+T} \|\psi\|^2$  and  $\int_t^{t+T} |A_0 \varphi|^2$  by  $\|\gamma\|_X^2$ .* Using the inequality (5.42) and proceeding as in the no-slip case, we get

$$(5.45) \quad \epsilon_2 \kappa \int_t^{t+T} \|\psi(\tau)\|^2 d\tau + \nu \int_t^{t+T} |A_0 \varphi(\tau)|^2 d\tau \leq (8 \mu \nu \lambda_1 \nu^2 \lambda_1 T + 4 \mu C_6) \|\gamma\|_X^2.$$

5.2.3. *Bound for  $\|\psi\|^2$  and  $\int_t^{t+T} |A_1 \psi|^2$  by  $\|\gamma\|_X^2$ .* Proceeding as in the no-slip case, we get (5.24):

$$\frac{1}{2} \frac{d}{dt} \|\psi\|^2 + \frac{\kappa}{2} |A_1 \psi|^2 \leq \left( \frac{2c_L^2}{\kappa} \|w_2\|^2 + \frac{c_L^4 \nu^2}{2\kappa} \|\eta_1\|^2 |A_1 \eta_1|^2 \right) \|\psi\|^2 + \frac{1}{2\nu} \|\varphi\|^2 + \frac{1}{\kappa l^2} |\varphi|^2.$$

Using the bounds (4.54), (4.63) and (4.65), we have

$$(5.46) \quad \int_t^{t+T} g(s) ds \leq \frac{4c_L^2}{\kappa} \cdot \tilde{C}_1 \rho^2 T + \frac{c_L^4 \nu^2}{\kappa^2} \tilde{C}_4 \rho^2 \left[ \tilde{C}_4 \rho^2 + T \left( 2\tilde{K}_3 \rho^4 \tilde{C}_4 \rho^2 + \tilde{K}_4 \rho^2 \right) \right] =: a_1.$$

By (5.43) and (2.8), we have

$$(5.47) \quad \int_t^{t+T} h(s) ds \leq T \left( \frac{1}{\nu} + \frac{2|\Omega|}{\kappa l^2} \right) \mu C_6 \|\gamma\|_X^2 =: \tilde{K}_{11} \|\gamma\|_X^2.$$

Applying Lemma 4.3 with (5.46), (5.47) and (5.45) yields

$$\sup_{t \in \mathbb{R}} \|\psi(t)\|^2 \leq e^{a_1} \left[ \tilde{K}_{11} + \frac{1}{\kappa \epsilon_2 T} (8\mu\nu\lambda_1\nu^2\lambda_1 T + 4\mu C_6) \right] \|\gamma\|_X^2 =: \tilde{K}_{12} \|\gamma\|_X^2.$$

By integrating (5.24) from  $t$  to  $t+T$  and using (5.46) and (5.47), we get

$$\kappa \int_t^{t+T} |A_1 \psi(\tau)|^2 d\tau \leq (\tilde{K}_{12} + a_1 \tilde{K}_{12} + \tilde{K}_{11}) \|\gamma\|_X^2.$$

## 6. PROOF OF PROPOSITION 3.7

Let  $\delta = w - u$  and  $\xi = \eta - \theta$ . Taking the difference of the RB system (2.12) and the auxiliary equations (3.11), we have

$$\begin{aligned} \frac{d\delta}{dt} + \nu A_0 \delta + B_0(w, w) - B_0(u, u) &= \mathbb{P}_\sigma(g\xi e_2) - \mu\nu\lambda_1(\tilde{I}_h \delta), \\ \frac{d\xi}{dt} + \kappa A_1 \xi + B_1(w, \eta) - B_1(u, \theta) &= \frac{\delta \cdot e_2}{l}. \end{aligned}$$

Applying the (essentially) same calculation in Section 5, we conclude that

$$\|\delta(t)\|^2 = |\xi(t)|^2 = 0, \quad \forall t \in \mathbb{R},$$

which completes the proof.

## 7. APPENDIX

Let  $\mathcal{T}(t; x) = \eta(t; x) + (1 - \frac{x_2}{l})$  where  $x = (x_1, x_2) \in \Omega$ . Observe that for a given smooth enough  $w$  with  $\nabla \cdot w = 0$ ,  $\mathcal{T}$  satisfies, on  $[-k(\nu\lambda_1)^{-1}, T_{**}]$ ,

$$(7.1) \quad \frac{\partial \mathcal{T}}{\partial t} - \kappa \Delta \mathcal{T} + (w \cdot \nabla) \mathcal{T} = 0,$$

$$(7.2) \quad \mathcal{T}(-k(\nu\lambda_1)^{-1}; x_1, x_2) = 1 - \frac{x_2}{l}.$$

with boundary conditions

in the  $x_2$ -variable:  $\mathcal{T} = 0$  at  $x_2 = 0$  and  $x_2 = l$ ,

in the  $x_1$ -variable:  $\mathcal{T}$  is of periodic  $L$ .

Observe that  $0 \leq \mathcal{T}(-k(\nu\lambda_1)^{-1}; x) \leq 1$ , and thus

$$\mathcal{T}_-(k(\nu\lambda_1)^{-1}; x) = 0, \quad (\mathcal{T} - 1)_+(k(\nu\lambda_1)^{-1}; x) = 0,$$

where we denote for any real number  $M$ ,  $M_+ = \max(M, 0)$  and  $M_- = \max(-M, 0)$ .

Note that  $\tilde{\mathcal{T}} := \mathcal{T}_-$  satisfies (7.1) a.e. and also the boundary conditions. The chain rule and integration by parts yield

$$\begin{aligned} \int_{\Omega} ((w \cdot \nabla) \tilde{\mathcal{T}}) \tilde{\mathcal{T}} \, dx &= \sum_{i,j} \int_{\Omega} w_i (\partial_i \tilde{\mathcal{T}}_j) \tilde{\mathcal{T}}_j \, dx \\ &= \sum_{i,j} \int_{\Omega} w_i \partial_i \frac{(\tilde{\mathcal{T}}_j)^2}{2} \, dx \\ &= - \sum_j \int_{\Omega} (\nabla \cdot w) \frac{(\tilde{\mathcal{T}}_j)^2}{2} \, dx \\ &= 0, \end{aligned}$$

where the boundary term vanishes due to the boundary conditions. Hence, multiplying (7.1) by  $\mathcal{T}_-$  and integrating over  $\Omega$ , we obtain

$$\frac{1}{2} \frac{d}{dt} |\mathcal{T}_-(t)|^2 + \kappa |\nabla \mathcal{T}_-(t)|^2 = 0,$$

which implies that

$$|\mathcal{T}_-(t)|^2 \leq |\mathcal{T}_-(-k(\nu\lambda_1)^{-1})|^2 = 0 \quad \text{for } t \in [-k(\nu\lambda_1)^{-1}, T_{**}).$$

It follows that  $\mathcal{T}_-(t) = 0$  and thus  $\mathcal{T}(t) \geq 0$ .

We now show that  $\mathcal{T} \leq 1$ . Observe that

$$\frac{\partial}{\partial t} (\mathcal{T} - 1) - \kappa \Delta (\mathcal{T} - 1) + (w \cdot \nabla) (\mathcal{T} - 1) = 0.$$

Proceeding similarly as above, we obtain,

$$\frac{1}{2} \frac{d}{dt} |(\mathcal{T} - 1)_+|^2 + \kappa |\nabla (\mathcal{T} - 1)_+|^2 = 0,$$

which implies that

$$|(\mathcal{T} - 1)_+(t)|^2 \leq |(\mathcal{T} - 1)_+(-k(\nu\lambda_1)^{-1})|^2 = 0 \quad \text{for } t \in [-k(\nu\lambda_1)^{-1}, T_{**}),$$

and thus  $\mathcal{T}(t) \leq 1$ .

We conclude that

$$0 \leq \mathcal{T}(t; x) \leq 1, \quad \text{a.e. } x \in \Omega, \quad t \in [-k(\nu\lambda_1)^{-1}, T_{**}),$$

which implies that

$$|\eta(t; x)| \leq 1 + \sup_{x \in \Omega} |1 - \frac{x_2}{l}| \leq 2,$$

and thus

$$(7.3) \quad \|\eta(t)\|_{L^2(\Omega)} \leq 2|\Omega|, \quad \forall t \in [-k(\nu\lambda_1)^{-1}, T_{**}).$$

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## REFERENCES

[1] A. Azouani, E. Olson and E. S. Titi, *Continuous data assimilation using general interpolant observables*, J. Nonlinear Sci. **24** (2014), 277–304.

[2] A. Azouani and E. S. Titi, *Feedback control of nonlinear dissipative systems by finite determining parameters—a reaction-diffusion paradigm*, Evol. Equ. Control Theory **3** (2014), 579–594.

[3] L. Bai and M. Yang, *A determining form for a nonlocal system*, Advanced Nonlinear Studies **17** (2017), 705–713.

[4] A. Biswas, C. Foias, C. F. Mondaini and E. S. Titi, *Downscaling data assimilation algorithm with applications to statistical solutions of the Navier–Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **36** (2019), 295–326.

[5] H. Brézis and T. Gallouet, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal. **4** (1980), 677–681.

[6] Y. Cao, *A determining form and algebraic bounds on the global attractor for the 2D Rayleigh–Bénard problem*, Indiana University, 2020.

[7] Y. Cao, M. S. Jolly, E. S. Titi and J. P. Whitehead, *Algebraic bounds on the Rayleigh–Bénard attractor*, arXiv:1905.01399v2 [math.AP], 2019.

[8] E. Celik, E. Olson and E. S. Titi, *Spectral filtering of interpolant observables for a discrete-in-time downscaling data assimilation algorithm*, SIAM J. Appl. Dyn. Syst. **18** (2019), 1118–1142.

[9] B. Cockburn, D. A. Jones and E. S. Titi, *Determining degrees of freedom for nonlinear dissipative equations*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), 563–568.

[10] P. Constantin, and C. Foias, *Navier–Stokes Equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[11] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations*, Springer Science & Business Media, vol. 70, 2012.

[12] A. Farhat, M. S. Jolly and E. S. Titi, *Continuous data assimilation for the 2D Bénard convection through velocity measurements alone*, Phys. D **303** (2015), 59–66.

[13] A. Farhat, E. Lunasin and E. S. Titi, *Continuous data assimilation for a 2D Bénard convection system through horizontal velocity measurements alone*, J. Nonlinear Sci. **27** (2017), 1065–1087.

[14] C. Foias and G. Prodi, *Sur le comportement global des solutions non-stationnaires des équations de Navier–Stokes en dimension 2*, Rend. Sem. Mat. Univ. Padova **39** (1967), 1–34.

[15] C. Foias, M. S. Jolly, R. Kravchenko and E. S. Titi, *A determining form for the two-dimensional Navier–Stokes equations: the Fourier modes case*, J. Math. Phys. **53** (2012): 115623, 30.

[16] C. Ciprian, S. Michael, R. Kravchenko and E. S. Titi, *A unified approach to determining forms for the 2D Navier–Stokes equations—the general interpolants case*, Russian Mathematical Surveys **69** (2014): 359.

[17] C. Foias, M. S. Jolly, D. Lithio and E. S. Titi, *One-dimensional parametric determining form for the two-dimensional Navier–Stokes equations*, J. Nonlinear Sci. **27** (2017), 1513–1529.

[18] C. Foias, O. Manley, R. Rosa and R. Temam, *Navier–Stokes Equations and Turbulence*, Encyclopedia of Mathematics and its Applications, vol. 83, Cambridge University Press, Cambridge, 2001.

[19] C. Foias, O. Manley and R. Temam, *Attractors for the Bénard problem: existence and physical bounds on their fractal dimension*, Nonlinear Anal. **8** (1987), 939–967.

[20] C. Foias, G. R. Sell and R. Temam, *Inertial manifolds for nonlinear evolutionary equations*, J. Differential Equations **73** (1988), 309–353.

[21] C. Foias, B. Nicolaenko, G. R. Sell and R. Temam, *Inertial manifolds for the Kuramoto–Sivashinsky equation and an estimate of their lowest dimension*, J. Math. Pures Appl. **67** (1988), 197–226.

[22] M. S. Jolly, V. R. Martinez, R. Vincent, T. Sadigov and E. S. Titi, *Edriss, A determining form for the subcritical surface quasi-geostrophic equation*, J. Dynam. Differential Equations **31** (2019), 1457–1494.

[23] M. S. Jolly, T. Sadigov and E. S. Titi, *A determining form for the damped driven nonlinear Schrödinger equation—Fourier modes case*, J. Differential Equations **258** (2015), 2711–2744.

- [24] M. S. Jolly, Michael, T. Sadigov and E. S. Titi, *Determining form and data assimilation algorithm for weakly damped and driven Korteweg-de Vries equation—Fourier modes case*, Nonlinear Anal. Real World Appl. **36** (2017), 287–317.
- [25] D. A. Jones and E. S. Titi, *Upper bounds on the number of determining modes, nodes, and volume elements for the Navier-Stokes equations*, Indiana Univ. Math. J. **42** (1993), 875–887.
- [26] J. Mallet-Paret and G. R. Sell, George, *Inertial manifolds for reaction diffusion equations in higher space dimensions*, J. Amer. Math. Soc. **1** (1988), 805–866.
- [27] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, vol. 68. Second Ed., Springer-Verlag, New York, 1997.
- [28] E. S. Titi, *On a criterion for locating stable stationary solutions to the Navier-Stokes equations*, Nonlinear Anal. **11** (1987), 1085–1102.

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revised*



CAO

Department of Mathematics, Indiana University, Bloomington, IN 47405

*E-mail address:* cao20@iu.edu



CAO

Department of Mathematics, Indiana University, Bloomington, IN 47405

*E-mail address:* msjolly@indiana.edu

EDRISS S. TITI

Department of Mathematics, Texas A&M University, 3368 TAMU, College Station, TX 77843-3368, USA.

Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK.

*E-mail address:* Titi@damtp.cam.ac.uk and titi@math.tamu.edu

