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A new commutator method for averaging lemmas

Pierre-Emmanuel Jabin*, Hsin-Yi Lin†, Eitan Tadmor‡

Abstract

This document corresponds to the talk that the first author gave at the Laurent Schwartz seminar on March 10th 2020. It introduces, in a simplified setting, a novel commutator method to obtain averaging lemma estimates. Averaging lemmas are a type regularizing effect on averages in velocity of solutions to kinetic equations. We introduce a new bilinear approach that naturally leads to velocity averages in $L^2([0,T],H^s_x)$. The new method outperforms classical averaging lemma results when the right-hand side of the kinetic equation has enough integrability. It also allows a perturbative approach to averaging lemmas which provides, for the first time, explicit regularity results for non-homogeneous velocity fluxes.

1 Introduction

The purpose of those notes is to present the **commutator method** for kinetic equations introduced in [26] and preview some of the upcoming results in [27]. In general we would like to consider kinetic equations written in the general form

$$\varepsilon \partial_t f + \mathrm{div}_x(a(x,v)\, f) = (-\Delta_v)^{\alpha/2} g, \tag{1}$$

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where $\varepsilon > 0$, $\alpha \geq 0$, and $a : \mathbb{R}^n_v \to \mathbb{R}^n$, $g : \mathbb{R}_t \times \mathbb{R}^n_v \times \mathbb{R}^n_x \to \mathbb{R}$ are given functions. ε can correspond to various macroscopic scales that are for instance introduced when hydrodynamics limits are considered.

For simplicity however in this note, we will consider the simpler setting where $\alpha = 0$ and $\mathrm{div}_x a(x, v) = 0$, leading to

$$\varepsilon \partial_t f + a(x, v) \cdot \nabla_x f = g, \quad \mathrm{div}_x a(x, v) = 0, \tag{2}$$

where we will have $f \in L^\infty([0,\ T],\ L^p(\mathbb{R}^{2n}))$ and $g \in L^1([0,\ T],\ L^{p'}(\mathbb{R}^{2n}))$ over some fixed time interval $[0,\ T]$ and where $1/p + 1/p' = 1$ as usual.

In many homogeneous setting, one simply has $a(v) = v$ but several applications require a more complex relation between the velocity v and the advection term $a(v)$. One may mention the kinetic formulation of scalar conservation laws, and kinetic models in relativistic or quantum settings (for example [17], [19]). Heterogeneous media or environment may however impose also an explicit dependence in x in the flux $a(x, v)$. This is a key distinction in the context of averaging lemmas as no quantitative regularizing effect had been proved so far in the heterogeneous case.

Here for simplicity again, we will limit ourselves to $a(v) = v$ in the homogeneous case or to a perturbation of this in the heterogeneous setting: $a(x, v) - v$ small enough in some appropriate smooth norm.

Averaging lemmas state that, by taking average in microscopic v variable, the velocity average of f

$$\rho_\phi(t, x) := \int f(t, x, v)\phi(v)\, dv,$$

has better regularity than f and g in x variable for any $\phi \in L^\infty_c$, the space of bounded and compactly supported functions. A typical example in the case $a(v) = v$ is the L^2 case: If both f and g belong to $L^2_{t,x,v}$ then ρ_ϕ belongs to $H^{1/2}_{t,x}$ gaining $1/2$ of a derivative.

Averaging lemmas have proved themselves as useful estimates in kinetic theory. First of all by providing compactness on moments, they are a critical tool to obtain existence of solutions with classical examples for the Vlasov-Maxwell system [10], or renormalized solutions to the Boltzmann equation [11]. Compactness of moments is also helpful when deriving hydrodynamic limits such as for the Boltzmann equation [20], or for the semiconductor Boltzmann-Poisson system [32].

In those examples, how much regularity is gained by averaging lemmas is not as important as the fact that compactness is obtained. Quantifying precisely the gain of regularity is however essential for applications to scalar conservation laws in particular or in other cases where kinetic formulations exist. The first such result was derived for scalar conservation laws in [30], isentropic gas dynamics in [31], but also so-called line-energy Ginzburg-Landau model as in [28]. We refer for example to [13, 21, 33] for more about the optimality of the regularity provided by averaging lemmas in that context.

Classical averaging lemmas were first introduced in [1] and [18] under L^2 setting. The classical approach involves taking Fourier transform in t and x of the kinetic equation and studying decay in the Fourier variable, which is connected with non degeneracy assumptions on the flux $a(v)$. Those non-degeneracy conditions are typically formulated in terms of the behavior in ε of $|\{|\tau + a(v)\cdot\xi| < \varepsilon\}$ (see for example [37]).

In combination with interpolation arguments, it was later extended to general L^p, $1 < p < \infty$ by [12] and later [6]. Of course wavelets can also be used instead of Fourier as in [14] or [38].

Because they are based on orthogonality, such approaches yield the best results when both f and g belong to L^2. Of course no uniform regularity can be obtained if $f,\ g \in L^1_{x,v}$ as was recognized in [18] (though equi-integrability in v is still enough for compactness, see [22, 16]). This will lead to a first key difference with our present bilinear method which is able to compensate a low integrability for $g \in L^q$ with $q < 2$ by a higher integrability in $f \in L^p$ provided $1/p + 1/q \leq 1$.

We want to close this very elementary introduction to the topic by citing the recent contributions in [3, 4, 5] which rely on the dispersive property of the kinetic equation in Fourier variables (x and v) in a spirit similar to the present approach.

In section 2 we present the simple result illustrating the method in the homogeneous case $a(v) = v$ while the next section focuses on the more complex inhomogeneous setting.

The results in this document correspond to the talk on the commutator method given by the first author at the Laurent Schwartz seminar on March 10th 2020, and announced in [26]. Later, an energy method was independently introduced in [2] recovering Th. 1. The perspective in [2] is somewhat different, considering only the homogeneous and stationary case but being more precise on hypoelliptic effects while restricting f and g to dual L^p exponents (vs. the larger range in that regard in Th. 2 below).

2 The homogeneous case $a(v) = v$

This section is devoted to the regularizing effect of Eq. (2) in the case where $a(v) = v$ does not depend on x. Because there are no derivatives in the right-hand side (this is the special case of (1)), it is a straightforward example of application of our method that still shows its interest with respect to classical approaches.

Specifically, we can prove the following theorem

Theorem 1. *Let $\varepsilon \leq 1$. If $f \in L^\infty\left([0,T],\ (L^2 \cap L^p)(\mathbb{R}^n_x \times \mathbb{R}^n_v)\right)$ solves (2) with $a(v) = v$ for some $g \in L^1\left([0,T],\ L^{p'}(\mathbb{R}^n_x \times \mathbb{R}^n_v)\right)$, where $1 < p < \infty$, then for all $\phi \in H^{3/2}(\mathbb{R}^n_v)$, $\rho_\phi \in L^2\left([0,T], H^{1/2}(\mathbb{R}^n_x)\right)$, and more precisely*

$$\|f\|^2_{L^2_t H^{1/2}_x H^{-3/2}_v} \leq C\left(\|f\|^2_{L^\infty_t L^p_{x,v}} + \|g\|^2_{L^1_t L^{p'}_{x,v}}\right),$$

where C is independent of ε.

There are obvious connections between this result and other dispersive approaches. First of all, by Wigner transform, this theorem with $p = 2$ connects to the local smoothing effect for the Schrödinger equation.

As we explain in details below, the gain of regularity is obtained through the use of an appropriate commutator, which is also roughly similar to the one found in hypoelliptic estimates for Fokker-Planck type of operators (see for example [25, 7]). Our method is also reminiscent of the multiplier method in [24] which was used to prove moment and trace lemmas for kinetic equations.

2.1 Proof of Theorem 1

It is useful to introduce the basic idea of our method in a general setting, and then narrow it down to our case shortly. Assume

$$\varepsilon \partial_t f + Bf = g,$$

where B is a skew-adjoint operator, $\varepsilon \leq 1$ and g are given. For any time-independent operator Q, we can consider

$$\varepsilon \partial_t \int f\,\overline{Qf}\,dx\,dv = \int \bar{f}[B,Q]\,f\,dx\,dv + \int g\,\overline{Qf}\,dx\,dv + \int f\,\overline{Qg}\,dx\,dv,$$

where $\overline{Qf} = \bar{Q}\,\bar{f}$ with $\bar{Q}$ the dual of Q on $L^2(\mathbb{R}^n, \mathbb{C})$ and $[B, Q]$ is the usual commutator

$$[B, Q] = B\,Q - Q\,B.$$

by the fundamental theorem of calculus we have

$$\begin{aligned} Re \int_0^T [B, Q]\, f \bar{f}\, dx\, dv\, dt \leq \sup_{t=0,T} \left| \int f\, \overline{Qf}\, dx\, dv \right| + \left| \int g\, \overline{Qf}\, dx\, dv\, dt \right| \\ + \left| \int f\, \overline{Qg}\, dx\, dv\, dt \right|. \end{aligned} \tag{3}$$

The idea is to find Q, bounded in some L^p spaces, such that the commutator of B and Q, $[B, Q]$, is positive-definite and gain extra derivatives. Hence by applying these conditions on (3) we would get the desired bound on f.

This type of method is well-known and was used for example by taking B to be of Schrödinger type, where the commutator appear naturally from the Hamilton vector field. Roughly speaking it involves constructing a proper symbol, which corresponds to Q, such that the Poisson bracket $[B,\, Q]$ implies a spacetime bound on f by Gårding's inequality. See for example [8], [15], [29] and [34].

In the context of averaging lemmas for kinetic equations, we fix B to be the kinetic transport operator,

$$B\, f = a(v) \cdot \nabla_x f. \tag{4}$$

We choose for Q the following bounded multiplier,

$$\mathcal{F}_{\xi,\zeta}(Qf) := m(\xi, \zeta) \mathcal{F}_{\xi,\zeta}(f),$$

where m is bounded and $\mathcal{F}_{\xi,\zeta}$ denotes the Fourier transform in both x, with dual variable ξ, and v with dual variable ζ. There exists a tempered distribution $K(x, v)$ such that $Qf = K \star_{x,v} f$ with $\mathcal{F}_{\xi,\zeta}(K) = m$. In this case and since $a(v) = v$ here, the commutator estimate becomes

$$\begin{aligned} &\int [a(v) \cdot \nabla_x, K \star_{x,v}] f \bar{f}\, dx\, dv \\ &= \int (v - w) \cdot \nabla_x K(x - y, v - w) f(y, w)\, dy\, dw f(x, v)\, dx\, dv. \end{aligned}$$

i.e. it is the quadratic form with the multiplier $\xi \cdot \nabla_\zeta m$.

We consider for example the specific formula

$$m(\xi,\zeta)=\frac{\xi}{|\xi|}\cdot\frac{\zeta}{(1+|\zeta|^2)^{1/2}},$$

and the corresponding kernel can be expressed as

$$K_0=R\cdot\nabla_v G_1^n,$$

where R is the Riesz kernel and G_1^n is the Bessel potential of order 1 in dimension n. With this choice by Plancherel identity,

$$\begin{aligned}\int [v\cdot\nabla_x,\, K\star_{x,v}]\, f\,\bar f\, dx\, dv\, dt &= \int \xi\cdot\nabla_\zeta m\,|\widehat f|^2\, d\xi\, d\zeta\, dt\\ &= \int\int\left[\frac{1}{(1+|\zeta|^2)^{1/2}}-\frac{\left|\frac{\xi}{|\xi|}\cdot\zeta\right|^2}{(1+|\zeta|^2)^{3/2}}\right]|\xi|\,|\widehat f|^2\, d\zeta\, d\xi\, dt\\ &\geq \int\int\frac{|\xi|}{(1+|\zeta|^2)^{3/2}}\,|\widehat f|^2\, d\zeta\, d\xi\, dt = \|f\|^2_{L^2\left([0,T],H^{1/2}(\mathbb{R}^n_x,H^{-3/2}(\mathbb{R}^n_v)\right)}.\end{aligned}$$

From classical Fourier theory (see for example [35]), K is bounded on L^p spaces for all $1<p<\infty$. Hence the right hand side of (3) is bounded as long as f is in $L^\infty\left([0,T],\ L^2\cap L^p(\mathbb{R}^n_x\times\mathbb{R}^n_v)\right)$ and g belongs to the space $L^1\left([0,\ T],\ L^{p'}(\mathbb{R}^n_x\times\mathbb{R}^n_v)\right)$, where we recall that $\frac{1}{p}+\frac{1}{p'}=1$.

2.2 A more complete result from [26]

We briefly present here the main result from [26] which deals with the more general Eq. (1), hence potentially with $\alpha>0$, and with more general integrability assumptions on f and g which are not necessarily anymore in dual spaces.

First of all some sort of non-degeneracy condition is needed on $a(v)$ as there is no gain of regularity if a is only constant for example. We assume $a(v)\in Lip(\mathbb{R}^n)$ with

$$a(v)\text{ one-to-one, and } J_{a^{-1}}\in L^\gamma, \tag{5}$$

where $J_{a^{-1}}=det(Da^{-1})$. This is not the standard non-degeneracy assumption and we refer to [26] for a discussion of that point which goes beyond the limited scope of these notes.

The method is considerably more intricate and requires several embeddings, leading to the introduction of the various coefficients defined below,

$$d_1 = \max\left\{n\left(\frac{1}{p_2}+\frac{1}{q_2}-\ell\right),0\right\}, \quad d_2 = \max\left\{n\left(\frac{2}{p_2}-\ell\right),0\right\},\ \ell = \frac{\gamma-2}{\gamma-1}, \tag{6}$$

$$d_3 = \max\left\{n\left(\frac{1}{p_1}+\frac{1}{q_1}-1\right),0\right\}, \quad d_4 = \max\left\{n\left(\frac{2}{p_1}-1\right),0\right\}. \tag{7}$$

The main result of [26] then reads

Theorem 2. *Given $\alpha \geq 0$, $T > 0$ and $0 < \varepsilon \leq 1$. Let $a \in Lip(\mathbb{R}^n)$ satisfy (5) with $\gamma \geq 2$. Let $f \in L^\infty([0,T], L^{p_1}_{loc}(\mathbb{R}^n_x, L^{p_2}_{loc}(\mathbb{R}^n_v)))$ solve (1) for some $g \in L^1([0,T], L^{q_1}_{loc}(\mathbb{R}^n_x, L^{q_2}_{loc}(\mathbb{R}^n_v)))$, with p_1, p_2, q_1, $q_2 \in [1,\infty]$. Then for all compactly supported $\phi \in W^{\alpha,\infty}(\mathbb{R}^n_v)$, one has that $\rho_\phi(t,x) \in L^2([0,T], H^s_{loc}(\mathbb{R}^n_x))$ for all $s < S$, with*

$$\|\rho_\phi\|^2_{L^2_t H^s_x} \leq C\left(\|f\|^2_{L^\infty_t L^{p_1}_{loc,x} L^{p_2}_{loc,v}} + \|g\|^2_{L^1_t L^{q_1}_{loc,x} L^{q_2}_{loc,v}}\right),$$

where C is independent of ε and $S = \frac{1}{2}\{(1-d_2)\theta - d_4\}$, with

$$\theta = \left[\min\left\{\frac{1-(d_3-d_4)}{\alpha+1+(d_1-d_2)}, 1\right\}\right],$$

where the d_i are defined in (6) and (7) for $i = 1,2,3,4$.

Note that because the same type of commutator estimates are at play, this method always produces averaging results ρ_ϕ in some Sobolev space H^s. We also note that the method performs in the same manner irrespective of the value of the small parameter ε. This may prove an important feature for use for example in hydrodynamic limits (see [23, 32, 36]).

It is not necessarily obvious to determine when Th 2 improves on existing results due to the interaction between the many parameters $p_1, p_2, q_1, q_2, \alpha$. In general, Th 2 will be all the more advantageous that we are close to the setting of Th 1 but we refer to [26] for a more precise discussion.

The proof of Th 2 is considerably more intricate than the simple proof for Th 1 that we presented. For this reason we only emphasize here some of the main point.

- The use of renormalization. Denote the Fourier transform of f only in x by $\widetilde{f}$, vs $\widehat{f}$ for the whole Fourier transform in x and v. Fix a smooth convolution kernel $\Phi(v)$ with $supp(\Phi) \subseteq B(0,1)$. A key step in the proof is to consider $F_{s_1} = \widetilde{f} \star_v \Phi_{|\xi|^{-s_1}}$, where $\Phi_{|\xi|^{-s_1}}(v) = |\xi|^{ns_1}\Phi(v|\xi|^{s_1})$ for some exponent $s_1 \geq 0$. This function satisfies the equation

$$\varepsilon \partial_t F_{s_1} + ia(v)\cdot \xi F_{s_1} = (-\Delta_v)^{\alpha/2}\, \widetilde{g} \star_v \Phi_{|\xi|^{-s_1}} + Com, \tag{8}$$

  where the commutator term from the renormalization as introduced in [9] is given by

$$Com(v) = i \int (a(v) - a(w)) \cdot \xi \widetilde{f}(w)\, \Phi_{|\xi|^{-s_1}}(v-w)\, dw.$$

  The critical point in this step is to regularize in v the right-hand side so that we can bound F_{s_1} and $(-\Delta_v)^{\alpha/2}\, \widetilde{g} \star_v \Phi_{|\xi|^{-s_1}}$ in appropriately dual spaces as required by our commutator method. The bounds F_{s_1} or $(-\Delta_v)^{\alpha/2}\, \widetilde{g} \star_v \Phi_{|\xi|^{-s_1}}$ in those spaces will of course explicitly involve the convolution scale $|\xi|^{-s_1}$ leading to a careful optimization in s_1.

  The cost of this necessary regularizing step is the commutator Com that we can nevertheless show is appropriately small as $|\xi| \to \infty$.

- Pointwise estimate in $|\xi|$. The main estimate is obtained by applying our commutator method pointwise in $|\xi|$, that is we only integrate in v and in the directions over the sphere in ξ with given norm. Because we need that commutator to explicitly provide regularity, we first perform a change of variable $v \mapsto v' = a(v)$ and rewrite (8) as

$$\varepsilon \partial_t h + iv' \cdot \xi h = k \tag{9}$$

  for the corresponding right-hand side k.

The commutator method applied to Eq. (9) for a given fixed radius $|\xi|$ provides an L^2 bound on averages in velocity of h with explicit decay in $|\xi|$. This finally allows to recover the actual regularity of averages ρ_ϕ of f.

3 The inhomogeneous case with spatially dependent flux

We present in this subsection a preview of the upcoming [27] which develops the commutator method to tackle fluxes a with explicit spatial depen-

dence. For simplicity again, we consider an explicitly perturbative setting with $a(x,v) = v + b(x,v)$ and b small and we still assume that there is no derivative in the right-hand side, *i.e.* we again consider (2) or (1) with $\alpha = 0$.

As an example of possible result, we prove here

Theorem 3. *Assume that we have the perturbative condition*

$$\delta = \|(1+|\xi|+|\zeta|)\,(\widehat{a}-\widehat{v})\|_{L^1(\mathbb{R}^{2n})} < 1/2,$$

that $\varepsilon \leq 1$, and that f solves (2) with the following bounds

$$f \in L^\infty\left([0,T],\ (L^2\cap L^p)(\mathbb{R}^n_x\times\mathbb{R}^n_v)\right),\quad g\in L^1\left([0,T],\ L^{p'}(\mathbb{R}^n_x\times\mathbb{R}^n_v)\right),$$

where $1<p<\infty$. Then for all $\phi\in C^\infty(\mathbb{R}^n_v)$ and any $s<1/2-\delta$, one has that $\rho_\phi \in L^2\left([0,T],H^s(\mathbb{R}^n_x)\right)$ and more precisely

$$\|\rho_\phi\|^2_{L^2_t H^s_x} \leq C_s\left(\|f\|^2_{L^\infty_t L^p_{x,v}} + \|g\|^2_{L^1_t L^{p'}_{x,v}}\right),$$

where C_s is independent of ε.

Remark 1. *Of course $C_s\to\infty$ as $s\to 1/2-\delta$ and in fact $C_s\sim C(1/2-\delta-s)$. In general, we could also have a more general formulation for the assumption on a, which can be replaced by conditions like*

$$\|\nabla_x a\|_{C^k} \leq \eta \inf_{x,v} |det\,\nabla_v a|,$$

as will be shown in [27]. We emphasize that the approach is still perturbative in that case as shown by the smallness condition which ensures that $a(x,v)$ is still close to an x-independent flux. We state here the condition $\|(1+|\xi|+|\zeta|)\,\widehat{a-v}\|_{L^1(\mathbb{R}^{2n})}=\delta$ in Fourier as it appears naturally in the proof. Of course by interpolation this could be replaced by any condition $\|(1+|\xi|^k+|\zeta|^k)\,\widehat{a-v}\|_{L^1(\mathbb{R}^{2n})}=\delta'$ for some $k>0$ and enough smoothness on a at the cost of a worst δ'.

Theorem 3 reads in a very similar manner to Theorem 1 or Theorem 2. In particular we still gain $1/2$ derivative on the averages ρ_ϕ. Most applications however require some v derivatives in the right-hand side ($\alpha>0$ in (1)) and or some lack of integrability: $g\in L^1_t L^q_{x,v}$ with $1/p+1/q>1$. This creates an obvious need for an extension to Theorem 3 that would be comparable to how Theorem 2 generalizes Theorem 1. Such an extension is the focus of [27].

3.1 A first estimate with smoothness in velocity

Making explicit the perturbative nature of the flux, we rewrite our inhomogeneous kinetic equation

$$\partial_t f + (v + b(x,v)) \cdot \nabla_x f = g, \tag{10}$$

where we recall that $\mathrm{div}_x b = 0$ from $\mathrm{div}_x a = 0$.

Our first intermediary result uses a slightly different multiplier than before, namely

$$m(\xi,\zeta) = \frac{\xi}{|\xi|} \cdot \frac{\zeta}{(1+|\zeta|^2)^{\theta/2}}, \tag{11}$$

for $\theta < 1$ instead of $\theta = 1$ as in the proof of Theorem 1. This turns out to be necessary to control the contribution from $b(x,v)\cdot\nabla_x$ solely by the gain in derivatives from $v\cdot\nabla_x$. But unfortunately it also means that this first result can only apply to solutions f that are somewhat already smooth in v as per

Proposition 1. *Assume* $(1+|\xi|+|\zeta|)\,\widehat{b} \in L^1$ *and denote* $A = \|(1+|\xi|+|\zeta|)\,\widehat{b}\|_{L^1}$. *Assume moreover that* f *solves* (10) *then one has that*

$$\begin{aligned}(1-\theta-2A)\,\|f\|^2_{L^2_t H^{1/2}_x H^{-\theta/2}_v} &\le \|f^0\|^2_{L^2_x H^{(1-\theta)/2}_v} + \|f(t=T)\|^2_{L^2_x H^{(1-\theta)/2}_v} \\ &+ \|f\|^2_{L^2_{t,x} H^{(1-\theta)/2}_v} + Re\int_0^t \int m(\xi,\zeta)\,\widehat{f}^*\,\widehat{g}\,d\xi\,d\zeta\,dt.\end{aligned}$$

Proof. We first apply the Fourier transform in x and v to get

$$\partial_t \widehat{f} - \xi\cdot\nabla_\zeta \widehat{f} + i\,\xi\,(\widehat{b}(.,.)) \star_{\xi,\zeta} \widehat{f} = \widehat{g}, \tag{12}$$

where $\xi\,(\widehat{b}(.,.)) \star_{\xi,\zeta} \widehat{f} = (\widehat{b}(.,.)) \star_{\xi,\zeta} (\xi\,\widehat{f})$ from the divergence free condition on b.

From the definition (11), we observe that

$$\xi\cdot\nabla_\zeta m = \frac{|\xi|\,(1+|\zeta|^2) - \theta\,|\xi|^{-1}(\xi\cdot\zeta)^2}{(1+|\zeta|^2)^{\theta/2+1}} \ge (1-\theta)\,\frac{|\xi|}{(1+|\zeta|^2)^{\theta/2}}.$$

Now we calculate

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int m(\xi,\zeta)\,|\widehat{f}|^2\,d\xi\,d\zeta &\ge (1-\theta)\int \frac{|\xi|}{(1+|\zeta|^2)^{\theta/2}}\,|\widehat{f}|^2\,d\xi\,d\zeta \\ &+ Im\int_{\mathbb{R}^{2n}} m(\xi,\zeta)\,\xi\cdot(\widehat{b}(.,.)) \star_{\xi,\zeta} \widehat{f}\,\widehat{f}^*(\xi,\zeta)\,d\xi\,d\zeta + Re\int m(\xi,\zeta)\,\widehat{f}^*\,\widehat{g}.\end{aligned} \tag{13}$$

All terms are already in their proper form except for the second term in the right-hand side, for which we use the following

Lemma 1. *For any real valued Φ, we have that*

$$\begin{aligned} Im \int_{\mathbb{R}^{2n}} \Phi(\xi,\zeta)\, \xi \cdot (\widehat{b}(.,.)) \star_{\xi,\zeta} \widehat{f}\, \widehat{f}^*(\xi,\zeta)\, d\xi\, d\zeta \\ = \frac{1}{2} Im \int_{\mathbb{R}^{4n}} (\Phi(\xi,\zeta) - \Phi(\xi',\zeta'))\, \xi \cdot \widehat{b}(\xi-\xi',\zeta-\zeta') \\ \widehat{f}(\xi',\zeta')\, \widehat{f}^*(\xi,\zeta)\, d\xi\, d\xi'\, d\zeta\, d\zeta'. \end{aligned}$$

Assuming at the time being that Lemma 1 holds, we may combine it with (13) to we deduce that

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2} \int m(\xi,\zeta)\, |\widehat{f}|^2\, d\xi\, d\zeta \geq (1-\theta) \int \frac{|\xi|}{(1+|\zeta|^2)^{\theta/2}}\, |\widehat{f}|^2\, d\xi\, d\zeta \\ &+ Im \int_{\mathbb{R}^{4n}} \frac{m(\xi,\zeta) - m(\xi',\zeta')}{2}\, \xi \cdot \widehat{b}(\xi-\xi',\zeta-\zeta')\, \widehat{f}(\xi',\zeta')\, \widehat{f}^*(\xi,\zeta)\, d\xi\, d\xi'\, d\zeta\, d\zeta' \\ &+ Re \int m(\xi,\zeta)\, \widehat{f}^*\, \widehat{g}. \end{aligned} \tag{14}$$

Assuming for example that $|\zeta| \leq |\zeta'|$, it remains to estimate

$$\begin{aligned} |m(\xi,\zeta) - m(\xi',\zeta')| \leq & |\zeta|^{1-\theta} \left| \frac{\xi}{|\xi|} - \frac{\xi'}{|\xi'|} \right| \\ & + \left| \frac{\zeta}{(1+|\zeta|^2)^{\theta/2}} - \frac{\zeta'}{(1+|\zeta'|^2)^{\theta/2}} \right|. \end{aligned}$$

Hence, with similar calculations for the second term,

$$|m(\xi,\zeta) - m(\xi',\zeta')| \leq 2\, |\zeta|^{1-\theta}\, \frac{1+|\xi-\xi'|}{\max(1,|\xi|,|\xi'|)} + 2\, \frac{1+|\zeta-\zeta'|}{\max(1,|\zeta|^\theta,|\zeta'|^\theta)}.$$

Since $\|(1+|\xi|+|\zeta|)\, \widehat{b}\|_{L^1} = A$, by Cauchy-Schwartz, we have that

$$\begin{aligned} &\frac{1}{2}\, Im \int_{\mathbb{R}^{4d}} (m(\xi,\zeta) - m(\xi',\zeta'))\, \xi \cdot \widehat{b}(\xi-\xi',\zeta-\zeta')\, \widehat{f}(\xi',\zeta')\, \widehat{f}^*(\xi,\zeta)\, d\xi'\, d\zeta'\, d\xi\, d\zeta \\ &\leq A \int \left(|\zeta|^{1-\theta} + \frac{|\xi|}{\max(1,|\zeta|^\theta)} \right) |\widehat{f}(\xi,\zeta)|^2\, d\xi\, d\zeta. \end{aligned}$$

Inserting this into (14), we conclude as claimed that

$$(1-\theta-2\,A)\int_0^T\int\frac{|\xi|}{(1+|\zeta|^2)^{\theta/2}}\,|\widehat{f}|^2\,d\xi\,d\zeta\,dt\le\int(1+|\zeta|^{1-\theta})\,|\widehat{f^0}|^2\,d\xi\,d\zeta$$
$$+\int(1+|\zeta|^{1-\theta})\,|\widehat{f}(t=T)|^2\,d\xi\,d\zeta+A\,\|f\|^2_{L^2_{t,x}H_v^{(1-\theta)/2}}$$
$$+Re\int_0^t\int m(\xi,\zeta)\,f^*\,\widehat{g}\,d\xi\,d\zeta.$$

□

It remains to prove Lemma 1

Proof of Lemma 1. Simply write

$$Im\int_{\mathbb{R}^{2d}}\Phi(\xi,\zeta)\,\xi\cdot(\widehat{b}(.,.))\star_{\xi,\zeta}\,\widehat{f}\,\widehat{f}^*(\xi,\zeta)\,d\xi\,d\zeta$$
$$=Im\int_{\mathbb{R}^{4d}}\Phi(\xi,\zeta)\,\xi\cdot\widehat{b}(\xi-\xi',\zeta-\zeta'))\,\widehat{f}(\xi',\zeta')\,\widehat{f}^*(\xi,\zeta)\,d\xi\,d\xi'\,d\zeta\,d\zeta'$$
$$=Im\int_{\mathbb{R}^{4d}}\Phi(\xi',\zeta')\,\xi'\cdot\widehat{b}(\xi'-\xi,\zeta'-\zeta))\,\widehat{f}^*(\xi',\zeta')\,\widehat{f}(\xi,\zeta)\,d\xi\,d\xi'\,d\zeta\,d\zeta',$$

by swapping ξ and ξ' and ζ and ζ'. Taking then the complex conjugate

$$Im\int_{\mathbb{R}^{2d}}\Phi(\xi,\zeta)\,\xi\cdot\widehat{b}(.,.))\star_{\xi,\zeta}\,\widehat{f}\,\widehat{f}^*(\xi,\zeta)\,d\xi\,d\zeta$$
$$=-Im\int_{\mathbb{R}^{4d}}\Phi(\xi',\zeta')\,\xi'\cdot\widehat{b}^*(\xi'-\xi,\zeta'-\zeta))\,\widehat{f}(\xi',\zeta')\,\widehat{f}^*(\xi,\zeta)\,d\xi\,d\xi'\,d\zeta\,d\zeta'$$
$$=-Im\int_{\mathbb{R}^{4d}}\Phi(\xi',\zeta')\,\xi'\cdot\widehat{b}(\xi-\xi',\zeta-\zeta'))\,\widehat{f}(\xi',\zeta')\,\widehat{f}^*(\xi,\zeta)\,d\xi\,d\xi'\,d\zeta\,d\zeta',$$

since b is real valued and hence $\widehat{b}^*(\xi,\zeta)=\widehat{b}(-\xi,-\zeta)$.

We now use the divergence free condition on b to obtain

$$Im\int_{\mathbb{R}^{2d}}\Phi(\xi,\zeta)\,\xi\cdot\widehat{b}(.,.))\star_{\xi,\zeta}\,\widehat{f}\,\widehat{f}^*(\xi,\zeta)\,d\xi\,d\zeta$$
$$=-Im\int_{\mathbb{R}^{4d}}\Phi(\xi',\zeta')\,\xi\cdot\widehat{b}(\xi-\xi',\zeta-\zeta'))\,\widehat{f}(\xi',\zeta')\,\widehat{f}^*(\xi,\zeta)\,d\xi\,d\xi'\,d\zeta\,d\zeta',$$

which, by taking the average, leads to

$$\begin{aligned} Im \int_{\mathbb{R}^{2d}} \Phi(\xi,\zeta)\, \xi \cdot \widehat{b}(.,.)) \star_{\xi,\zeta} \widehat{f}\, \widehat{f^*}(\xi,\zeta)\, d\xi\, d\zeta \\ = \frac{1}{2} Im \int_{\mathbb{R}^{4d}} (\Phi(\xi,\zeta) - \Phi(\xi',\zeta'))\, \xi \cdot \widehat{b}(\xi-\xi',\zeta-\zeta') \\ \widehat{f}(\xi',\zeta')\, \widehat{f^*}(\xi,\zeta)\, d\xi\, d\xi'\, d\zeta\, d\zeta'. \end{aligned}$$

□

3.2 Regularizing in velocity

As mentioned when stating Prop. 1, it does not immediately yield the answer that we are looking for since it requires some a priori regularity in velocity on f which is not readily available. Eq. 2 is obviously not invariant by convolution in velocity and any regularization in v comes with a corresponding loss of regularity in x.

We need to proceed carefully by making the trade-off between x and v as quantitative as possible. For this reason, we define f_r by

$$\widehat{f_r} = \widehat{f}\, \Phi(|\zeta| + |\xi|), \tag{15}$$

where we typically consider

$$\Phi(x) = \frac{1}{(1+|x|)^r}, \quad r \in (0,\ 1).$$

For a fixed value of $|\xi|$, $\widehat{f_r}$ decays faster than $\widehat{f}$ in ζ. This will allow us to control the extra derivatives needed in v through the gain of regularity in x.

More precisely, we can prove from Prop. 1

Proposition 2. *Assume that $f \in L^\infty_t L^2_{x,v}$ and $g \in L^1_t L^2_{x,v}$ satisfy (10). Then f_r solves*

$$\partial_t f_r + (v + b(x,v)) \cdot \nabla_x f_r = g_r + c,$$

where $\widehat{g_r} = \widehat{g}\, \Phi$ and $\|c\|_{L^\infty_t L^2_x H^r_v} \leq r(1+\widetilde{A})\|f\|_{L^\infty_t L^2_{x,v}}$ for $\widetilde{A} = \|(1+|\xi|^2+|\zeta|^2)\widehat{b}\|_{L^1}$.

Proof. The proof follows classical commutator estimates for the theory of renormalized solutions which we perform here in Fourier.

Multiplying Eq. (12) on $\widehat{f}$ by Φ, we observe that

$$\partial_t \widehat{f_r} - \xi \cdot \nabla_\zeta \widehat{f_r} + i\,\xi\,(\widehat{b}(.,.)) \star_{\xi,\zeta}\ \widehat{f_r} = \widehat{g_r} + \widehat{c},$$

where $\widehat{g_r} = \widehat{g}\,\Phi$ and

$$\widehat{c} = \widehat{c_1} + \widehat{c_2} = -\widehat{f}\,\xi \cdot \nabla_\zeta \Phi(|\zeta| + |\xi|)$$
$$+ i\,\xi \int \widehat{b}(\xi - \xi', \zeta - \zeta')\ (\Phi\,(|\zeta'| + |\xi'|) - \Phi\,(|\zeta| + |\xi|))\ \widehat{f}(\xi', \zeta')\,d\xi'\,d\zeta'.$$

Note that

$$\int (1 + |\zeta|)^{2\,r}\,|\widehat{c_1}|^2\,d\xi\,d\zeta \le \int |\widehat{f}|^2\,(1 + |\zeta|)^{2\,r}\ |\Phi'\,(|\zeta| + |\xi|)|^2\ |\xi|^2\,d\xi\,d\zeta.$$

By the definition of Φ, we hence have that

$$\int (1 + |\zeta|)^{2\,r}\,|\widehat{c_1}|^2\,d\xi\,d\zeta \le r^2 \int |\widehat{f}|^2\,\frac{|\xi|^2\,(1 + |\zeta|)^{2\,r}}{(1 + |\zeta| + |\xi|)^{2\,r+2}}\,d\xi\,d\zeta,$$

showing indeed that

$$\|c_1\|_{L^\infty_t L^2_x H^r_v} \le r\,\|f\|_{L^\infty_t L^2_{x,v}}. \tag{16}$$

We perform similar calculations for c_2, first with

$$|\widehat{c_2}| \le |\xi| \int |\widehat{b}(\xi - \xi', \zeta - \zeta')|\ |\Phi\,(|\zeta'| + |\xi'|) - \Phi\,(|\zeta| + |\xi|)|\ |\widehat{f}(\xi', \zeta')|\,d\xi'\,d\zeta'.$$

Next we observe that

$$|\Phi(x) - \Phi(y)| \le r\,|x - y|\,(1 + |x - y|)\ \inf\left((1 + |x|)^{-r-1},\ (1 + |y|)^{-r-1}\right),$$

implying that

$$|\Phi\,(|\zeta| + |\xi|) - \Phi\,(|\zeta'| + |\xi'|)| \le r\,\frac{|\zeta - \zeta'| + |\xi - \xi'|}{(1 + |\xi| + |\zeta|)^{r+1}}\ (1 + |\zeta - \zeta'| + |\xi - \xi'|)\,.$$

Denoting

$$L(\zeta - \zeta', \xi - \xi') = (|\zeta - \zeta'| + |\xi - \xi'|)\ (1 + |\zeta - \zeta'| + |\xi - \xi'|)\,,$$

we hence obtain that

$$|\widehat{c}_2| \leq r\,\frac{|\xi|}{(1+|\xi|+|\zeta|)^{r+1}}\int |\widehat{f}(\xi',\zeta')|\,|\widehat{b}(\xi-\xi',\zeta-\zeta')|\,L\,d\xi'\,d\zeta'$$
$$\leq \frac{r}{(1+|\zeta|)^r}\int |\widehat{f}(\xi',\zeta')|\,|\widehat{b}(\xi-\xi',\zeta-\zeta')|\,L\,d\xi'\,d\zeta'.$$

Therefore

$$\|(1+|\zeta|)^r\,\widehat{c}_2\|_{L^2_{\xi,\zeta}} \leq r\,\|\widehat{b}\,L\|_{L^1_{\xi,\zeta}}\,\|\widehat{f}\|_{L^2_{\xi,\zeta}},$$

and adding (16), by the definition of $\widetilde{A}$

$$\|c\|_{L^\infty_t L^2_x H^r_v} \leq \|c_1\|_{L^\infty_t L^2_x H^r_v} + \|c_2\|_{L^\infty_t L^2_x H^r_v} \leq r\,(1+\widetilde{A})\,\|f\|_{L^\infty_t\,L^2_{x,v}}, \tag{17}$$

concluding the proof. □

3.3 Proof of Theorem 3

The strategy is of course to apply Prop. 1 on the regularized f_r as defined by (15). Since by Prop. 2, f_r solves the kinetic equation (10) with $g_r + c$ as a right-hand side, we deduce from Prop. 1

$$(1-\theta-2A)\,\|f_r\|^2_{L^2_t H^{1/2}_x H^{-\theta/2}_v} \leq \|f^0_r\|^2_{L^2_x H^{(1-\theta)/2}_v} + \|f_r(t=T)\|^2_{L^2_x H^{(1-\theta)/2}_v}$$
$$+\,\|f_r\|^2_{L^2_{t,x}H^{(1-\theta)/2}} + Re\int_0^t\int m(\xi,\zeta)\,\widehat{f}^*_r\,(\widehat{g}_r+\widehat{c})\,d\xi\,d\zeta\,dt. \tag{18}$$

Provided that $r \geq (1-\theta)/2$ then, we trivially have that

$$\|f_r(t)\|_{L^2_x H^{(1-\theta)/2}_v} \leq \|f(t)\|_{L^2_{x,v}}.$$

The symbol $m(\xi,\zeta)/(1+|\zeta|)^{1-\theta}$ corresponds to the product of two Calderon-Zygmund operators and is thus bounded on every L^p space for $1<p<\infty$. We hence also have that

$$Re\int_0^t\int m(\xi,\zeta)\,\widehat{f}^*_r\,(\widehat{g}_r+\widehat{c})\,d\xi\,d\zeta\,dt \leq \|f_r\|_{L^\infty_t L^2_x H^{(1-\theta)/2}_v}\,\|c\|_{L^1_t L^2_x H^{(1-\theta)/2}_v}$$
$$+\|f_r\|_{L^\infty_t L^p_x W^{(1-\theta)/2,p}_v}\,\|g_r\|_{L^1_t L^{p'}_x W^{(1-\theta)/2,p'}_v}.$$

We have similarly that if $r \geq (1-\theta)/2$

$$\|f_r(t)\|_{L^1_t L^p_x W^{(1-\theta)/2,p}_v} \leq \|f\|_{L^1_t L^p_{x,v}}, \quad \|g_r(t)\|_{L^1_t L^{p'}_x W^{(1-\theta)/2,p'}_v} \leq \|g\|_{L^1_t L^{p'}_{x,v}}.$$

By the estimate on c in Prop. 2, one also has the bound

$$\|c\|_{L^1_t L^2_x H^{(1-\theta)/2}_v} \leq t\, r\, (1+\widetilde{A})\, \|f\|_{L^\infty_t L^2_{x,v}}.$$

As a consequence from (18)

$$(1-\theta-2A)\, \|f_r\|^2_{L^2_t H^{1/2}_x H^{-\theta/2}_v} \leq (C+t\, r)\, (1+\widetilde{A})\, \|f\|^2_{L^\infty_t L^2_{x,v} \cap L^p_{x,v}} + C\, \|g\|^2_{L^1_t L^{p'}_{x,v}}.$$

Finally by the definition of Φ,

$$\|f_r\|_{L^2_t H^{1/2}_x H^{-\theta/2}_v} = \|f\|_{L^2_t H^{1/2-r}_x H^{-\theta/2-r}_v},$$

so that

$$(1-\theta-2A)\, \|f\|^2_{L^2_t H^{1/2-r}_x H^{-\theta/2-r}_v} \leq (C+t\, r)\, (1+\widetilde{A})\, \|f\|^2_{L^\infty_t L^2_{x,v} \cap L^p_{x,v}} + C\, \|g\|_{L^1_t L^{p'}_{x,v}}.$$

This obviously leads to taking r as small as possible to obtain the largest possible regularity. Since we have to take $r \geq (1-\theta)/2$, we simply choose $r = (1-\theta)/2$, and conclude that

$$(1-\theta-2A)\, \|f\|^2_{L^2_t H^{\theta/2}_x H^{-1/2}_v} \leq (C+t)\, (1+\widetilde{A})\, \|f\|^2_{L^\infty_t L^2_{x,v} \cap L^p_{x,v}} + C\, \|g\|_{L^1_t L^{p'}_{x,v}}.$$

Optimizing in θ in terms of A finishes the proof by taking $s = \theta/2$.

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