Snowmass White Paper: String Perturbation Theory

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The purpose of this White Paper is to review recent progress towards elucidating and evaluating string amplitudes, relating them to quantum field theory amplitudes, applying their predictions to string dualities, exploring their connection with gravitational physics, and deepening our understanding of their mathematical structure. We also present a selection of targets for future research.

INTRODUCTION

String theory unifies gravity and Yang-Mills theory in an ultraviolet finite quantum theory that is shedding light on a wide range of subjects including particle physics, gauge/gravity duality, black holes, gravitational waves, the early universe, and pure mathematics. Several comprehensive textbooks on string theory are available [1–8].

String theory may be approached from a number of complementary angles. String perturbation theory is a topological expansion in powers of the string coupling g_s obtained by summing over random fluctuating surfaces. The low-energy expansion in powers of the string scale α' is given by supergravity plus higher order string corrections to supergravity encoded in local effective interactions. Supergravity is valid for low energy but holds for all values of g_s while string perturbation theory is valid for small g_s but holds for all energies. The gauge/gravity correspondence formulates string theory in certain hyperbolic space-times via correlators in Yang-Mills theory. At the intersection of these approaches, effective interactions are calculable in string perturbation theory or in the gauge/gravity correspondence, and may be able to predict physical effects observable at low energy.

Superstring perturbation theory has grown into a rich subject whose study has revealed deep connections with quantum field theory amplitudes, D-branes, gauge/gravity duality, gravitational waves, black holes, algebraic geometry, and modular forms. At the heart of superstring perturbation theory are superstring amplitudes, which are primarily considered for massless gravitons, gauge bosons and their respective supersymmetry partners. Following the discovery of Type I, Type II, and Heterotic superstrings in [9–11], respectively, much of the formalism of superstring perturbation theory dates back to the 1980's (see [12, 13] and references therein for publications prior to 1988). Extensive lecture notes and video lectures on more recent developments may be found in [14–17].

PERTURBATIVE STRING AMPLITUDES

The developments of the past 20 years have significantly advanced the explicit computation of a number of string amplitudes, and sharpened our understanding of the formalism. This progress was driven by a confluence of ideas from different formulations of the superstring, including the Ramond-Neveu-Schwarz (RNS) formalism [18–20] and the pure-spinor formalism [21–23]. The mathematical structure of the RNS formulation, including the role of supermoduli, was revisited and further clarified in [24–26]. For the pure-spinor formalism, recent work advanced our understanding of its origin [27] and its interrelation with the RNS formalism [28, 29].

In both formalisms, at one loop and beyond, the chiral splitting procedure [13, 30], which splits the integrands of closed superstring amplitudes into left and right chiral integrands at fixed loop momenta, has become a valuable tool in evaluating closed string amplitudes. The structure of their monodromies constrains the dependence of chiral amplitudes on loop momenta and moduli and thereby significantly simplifies the construction of the amplitudes. Together with a specification of the D-brane setup, chiral amplitudes also encode the moduli-space integrand for open-string amplitudes.

Tree-level and one-loop amplitudes for four massless supergravity states in the Type I, Type II and Heterotic strings were evaluated in the papers that introduced these theories [9–11]. The following selection of more recent benchmark results grew out of progress driven by the development of novel tools for higher loop orders and larger numbers of external gauge and supergravity states.

Tree-level amplitudes are now known in compact and manifestly supersymmetric form for any number of massless external legs [31]. The construction was guided by BRST cohomology techniques and the use of multiparticle superfields in the pure-spinor formalism [32].

Following the early evaluation of the one-loop four-graviton amplitude in [33] and its supersymmetrization in [22], pure-spinor techniques together with chiral splitting and the multiparticle formalism led to explicit one-loop results up to seven points (i.e. up to seven external states) [34–36]. In the RNS formalism, the spin-structure sums can be performed at all multiplicities in a controlled function space [37, 38].

The two-loop measure on supermoduli space in the RNS formulation was constructed in [18, 39–41] using the super period matrix, and derived using algebraic geometry in [42]. The two-loop amplitude for four massless NS bosons was computed for Type II and Heterotic strings in the RNS formulation in [20, 43] and was extended to include external fermions in the pure-spinor formulation [44, 45]. Five-point two-loop amplitudes were constructed from an amalgam of pure-spinor methods and chiral splitting [46] in both the Type II and Heterotic strings, and their parity-even bosonic components were confirmed by a first-principles computation in the RNS formalism [47]. Two-loop contributions to the cosmological constant in orbifold compactifications were evaluated in [48, 49].

At three loops, a measure on supermoduli space in the RNS formulation was proposed in [50], further analyzed in [51], and used to discuss the vanishing of two- and three-point amplitudes in [52, 53]. The leading low-energy term in the three-loop four-point amplitude was calculated in the non-minimal pure-spinor formalism in [54]. Moreover, a recent proposal for the four-point amplitude at all energies [55] grew out of input from ambi-twistor strings [56–58]. Beyond three loops, recent results include proposed non-renormalization theorems [59, 60], and the existence of a super-period matrix on supermoduli space with Ramond punctures [51, 61].

The superstring amplitudes described above are given by integrals over the moduli spaces of ordinary compact Riemann surfaces and vertex insertion points on each surface of remarkably simple integrands. These integral representations require an analytic continuation in the external momenta. At one-loop order, the analytic continuation was proven to exist, shown to reproduce all expected physical singularities in terms of a double dispersion relation, and used to compute the decay rate of a large class of massive string states in [62–64].

As will be reviewed in later sections, the remarkable simplicity of the known string amplitudes greatly facilitates deriving gravity and gauge amplitudes from the $\alpha' \to 0$ limit of the corresponding string amplitudes, verifying quantitative predictions of dualities, and extracting number-theoretic structures from the low-energy expansion. However, it is not clear whether this remarkable simplicity persists to higher loop order in the RNS formulation due to the increased complexity of supermoduli space [24, 65, 66], and in the pure-spinor formulation due to divergences associated with the composite b-ghost [23].

Future targets: An immediate goal in the RNS formulation is to lift some of the obstacles to higher loop and higher multiplicity amplitudes imposed by the complexity of supermoduli space. Alternatively, one may seek to extract low-energy effective interactions for comparison with S-duality predictions directly from the more involved supermoduli presentation of the amplitudes. A parallel goal is to systematize and simplify the sums over spin structures, possibly by making use of the novel perspective on the GSO projection developed in [67, 68], and thereby to provide a useful mathematical delimitation for the space of functions to which chiral amplitudes belong at arbitrary loop order.

The manifest supersymmetry of the pure-spinor formulation simplifies the evaluation of numerous superstring amplitudes, but higher multiplicities and loop orders are marred by divergences in the $\lambda, \bar{\lambda}$-ghost integrals and the difficulties encountered in evaluating b-ghost correlators. An important goal is to identify the physical origin of these divergences and to resolve them, possibly building upon ideas in [69, 70].

Another goal is to elucidate further the relation between the RNS formulation and the pure-spinor formulation. This would lead to a deeper understanding of the superstring, and might allow more general backgrounds for the two formulations. It is not yet known how to describe the RNS formulation in backgrounds with finite non-zero Ramond-Ramond flux, or how to describe the pure-spinor formulation in backgrounds which are not ten-dimensional supergravity solutions. Progress towards understanding this relation between the two formulations has been made in [28, 29],

Perhaps the most daring goal is to obtain higher loop and higher multiplicity amplitudes by indirect methods such as used in bootstrapping conformal field theory or gauge and gravity amplitudes.

CONNECTIONS WITH FIELD THEORY

Over the past few decades, considerable progress has been made in evaluating and simplifying on-shell scattering amplitudes in gauge and gravity theories (see corresponding White Papers including [71]) thereby opening up a wide frontier of results with increasing number of loops and external legs. Supersymmetric theories lend themselves to the greatest degree of simplification and have been studied most extensively in this context. Some of their structures carry over to non-supersymmetric theories. We refer to the textbooks [72, 73] and the recent reviews [74, 75] on modern techniques for quantum field theory (QFT) amplitudes.

Several of these modern techniques were inspired by developments in string theory, such as the relation between supergravity amplitudes and their Yang-Mills counterparts. The intimate relation between open- and closed-string amplitudes, illustrated at tree-level by the Kawai-Lewellen-Tye (KLT) relations [76], and at loop level by chiral splitting [13, 30], may be viewed as a natural framework for the double-copy structure of supergravity amplitudes due to Bern, Carrasco and Johansson (BCJ) [77].

The BCJ-formulation of the gravitational double copy relies on the so-called color-kinematics duality of its Yang-Mills building blocks [78]. This duality consists of a formal exchange symmetry of "color" degrees of freedom and kinematic variables and can in many cases be derived from properties of open-string amplitudes, as one can see from two complementary perspectives.

First, the manifestly gauge-invariant incarnation of the color-kinematics duality at tree-level in the form of BCJ relations among partial amplitudes [78] stems from monodromy properties of open-string worldsheets [79, 80], see [81–85] for loop-level generalizations.

Second, the color-kinematics duality can be manifested at the level of cubic-vertex diagrams where kinematic factors obey the same Jacobi identities as color factors. Suitable representations of open-string amplitudes lead to solutions of kinematic Jacobi identities at tree-level [86], one loop [87, 88] and two loops [46] through their respective point-particle limits [89]. A more general viewpoint on kinematic Jacobi identities is offered by a residue theorem for moduli spaces of Riemann surfaces with marked points [90] and from nested b-ghost action on vertex operators of the pure-spinor formalism in Siegel gauge [91].

Parametrizations of gauge-theory amplitudes subject to kinematic Jacobi identities have shaped the state-of-the-art for multiloop supergravity amplitudes with different amounts of space-time supersymmetry [92–94]. Their ultraviolet properties and the associated non-renormalization theorems for gravitational operators can often be anticipated from low-energy expansions of string amplitudes [95–98]. In particular, studies of the ultraviolet divergences in supergravity at five-loop order [99] were preceded by a corresponding analysis based on string theory [100].

Similar derivations of the gravitational double copy and the color-kinematics duality of gauge theories at low loop orders have been given within the Cachazo-He-Yuan formalism (CHY) [101, 102] and ambi-twistor string theories [56–58]. These models are based on worldsheet degrees of freedom similar to those of standard Type II strings but their moduli-space integrals are localized via scattering equations, and dimensionful parameters such as α' are usually absent.

The development of ambi-twistor models and their amplitude prescriptions has strongly benefitted from the availability of superstring prototypes. In particular, as exploited in the recent three-loop proposal for Type-II four-point amplitudes [55], the chiral correlation functions of superstrings and ambi-twistor strings can be freely exchanged when expressed in terms of logarithmic forms [103–105].

Just as string theory elegantly explains far-reaching properties of field theory amplitudes, the evaluation of string amplitudes also benefits from field theory structures. For instance, the open-superstring tree-level amplitudes themselves admit a double-copy construction [31] akin to the KLT relations for gravitational amplitudes [106].

Given that the field theory version of the KLT formula usually involves pairs of physical amplitudes, one may interpret the disk integrals contributing to open superstrings as scalar amplitudes. Hence, the open superstring is said to be a double copy of super-Yang-Mills with "Z-theory", a putative ultraviolet-soft theory of scalars with bi-adjoint ϕ^3 theory and Goldstone bosons in its low-energy limit [107–109].

Similar double-copy constructions apply to bosonic and Heterotic strings [110, 111]: In their massless tree amplitudes, the super-Yang–Mills building blocks of the open superstring [31] are replaced by the tree-level amplitudes of massive gauge theories dubbed $(DF)^2+$YM and $(DF)^2+$YM$+\phi^3$ [111] which have been constructed in the context of conformal-supergravity amplitudes [112].

Future targets: It is an open question whether the double-copy structure of supergravity and the color-kinematics duality of gauge theories persist to all orders in perturbation theory. String theory may offer a suitable framework to

investigate the systematics at higher loops – either by providing a natural formulation that applies for any number of loops and legs or by pinpointing fundamental obstructions.

For instance, the five-loop four-point function [113] and one-loop six-point function [87, 114] of maximal supergravity in general dimensions $D \leq 10$ have so far resisted a double-copy representation in terms of cubic-vertex diagrams. While workarounds using the generalized double copy [115] and linearized Feynman propagators [116, 117] have been found, it would be interesting to check the viability of kinematic Jacobi identities and the traditional form of the double copy [77] for these examples from a string-theory perspective.

Another important line of follow-up research concerns double-copy structures of loop-level string amplitudes. Loop-level KLT relations for closed-string amplitudes are unknown at the time of writing, and a field theory-type double copy for open-superstring amplitudes has been pioneered at one loop [36, 118]. The latter calls for a connection with the interpretation of disk integrals as Z-theory amplitudes and a systematic understanding in the light of cohomology bases in the chiral-splitting setting.

STRING FIELD THEORY

Although perturbative superstring theory can be formulated in terms of correlation functions of vertex operators using a two-dimensional worldsheet action, an alternative formulation is as a string field theory in ten space-time dimensions in which the wavefunction of the string field depends on both the zero modes and the excited modes of the worldsheet variables. In contrast to the field theories from the point-particle limit of superstrings discussed in the previous section, string field theory obtains exact-in-α' expressions for superstring amplitudes from Feynman-type rules for the string field. Recent lecture notes on string field theory include [119, 120].

For computing perturbative scattering amplitudes, string field theory is more complicated than using the worldsheet approach. However, there are several aspects of superstring theory such as tachyon condensation [121–123] and mass renormalization [124–126] which are easier to understand using superstring field theory. Moreover, it is expected that non-perturbative features of superstring theory such as duality symmetries and background independence will be easier to study using superstring field theory.

Just as there are different formulations of perturbative superstring theory with different worldsheet actions, there are also different formulations of superstring field theory. The best-understood formulation is covariant RNS superstring field theory, and by introducing spurious free fields which decouple from the rest of the action [127], it is now possible to describe both the Neveu-Schwarz and Ramond sectors of open and closed superstring field theory [128]. However, space-time supersymmetry is not manifest in this formulation, and "picture-changing operators" need to be introduced with complicated sewing rules [66]. Other formulations include light-cone superstring field theory [129] and WZW-like superstring field theory [130] which have RNS-like versions in which space-time supersymmetry is not manifest, but also have Green-Schwarz-like versions in which space-time supersymmetry is manifest. There have been several recent works on establishing the relation between these different formulations of superstring field theory [131–133].

Future targets: The relation of open string field theory with closed string field theory needs to be better understood and will probably lead to important new developments. Since open string field theory amplitudes should contain closed string states as poles, open string field theory should also be able to describe closed string fields [134]. It has been conjectured by many authors [135–138] that AdS/CFT duality can be understood as a duality between open string field theory for the CFT side and closed string field theory for the AdS side.

STRING DUALITIES

Five superstring theories in 10-dimensional flat Minkowski space-time admit a perturbative expansion in the string coupling, namely Type I, Type IIA, Type IIB, and Heterotic superstrings with gauge groups $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$. The low-energy approximation to the sector of massless states in each string theory is governed by the corresponding supergravity. There also exists an 11-dimensional supergravity theory with the maximal number of 32 supercharges.

String dualities are discrete transformations that relate these different theories [139, 140], often after suitable compactification. The conjectured existence of these dualities has led to posit that the five perturbative string theories are but different manifestations of a single unifying "M-theory". A schematic illustration is given in Figure 1. We refer to [141] for a modern textbook on supergravity; to [142] for a collection of essays on M-theory; to [143] for an early overview on dualities, and to [4, 6] for textbooks with extensive discussions of string dualities.
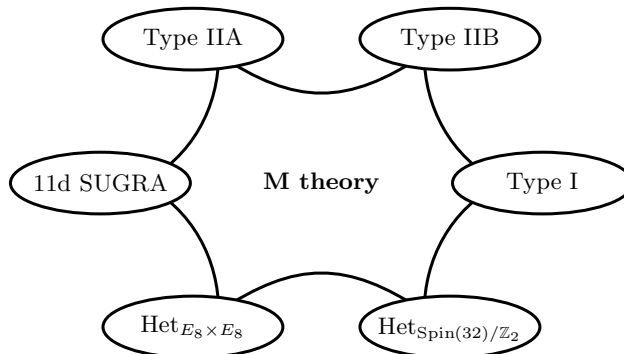
FIG. 1. *The web of string dualities, see for instance [4, 6].*

The large web of dualities between superstring theories includes perturbative as well as non-perturbative dualities. T-duality relates string theories compactified on different tori to one another, such as Type IIA to Type IIB, and Heterotic $E_8 \times E_8$ to $\mathrm{Spin}(32)/\mathbb{Z}_2$, and gives rise to arithmetic duality groups and associated automorphic forms (see [144–146] and references therein). Mirror symmetry similarly relates string theories compactified on different Calabi-Yau manifolds. For the vast literature on these subjects we refer to [147–151], and references therein. Compactification of Type II on a K3 Calabi-Yau manifold is dual to Heterotic theories on the 4-torus [139, 152–154], while Type I theory is dual to the Heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory [140, 155–160].

Non-perturbative S-dualities map the weak-coupling regime of one theory to the strong-coupling regime of another or the same theory. Proving the existence of an S-duality is notoriously difficult if not, at present, out of reach. Nonetheless, string perturbation theory can be used to provide indirect, yet powerful, tests of some of the implications of S-dualities that usually relate different loop orders. We shall now illustrate such tests and offer future targets.

One of the best studied non-perturbative dualities is the $SL(2,\mathbb{Z})$ self-duality of Type IIB string theory. The low-energy expansion of string theory, in powers of α', consists of supergravity plus local effective interactions produced by integrating out the tower of massive string modes. The coefficients of these effective interactions depend on the complexified axion-dilaton field Ω whose imaginary part sets the string coupling g_s. Self-duality requires the coefficients to be modular forms in Ω under $SL(2,\mathbb{Z})$, while supersymmetry imposes differential equations on these modular forms. Remarkably, $SL(2,\mathbb{Z})$ duality and supersymmetry combine to imply relations between different perturbative contributions to various effective interactions, and it is these predictions that can be tested against predictions from perturbative string theory.

The lowest order in α' corrections to Type II supergravity contain the $\mathcal{R}^4$ effective interaction evaluated at tree-level in [161, 162], where $\mathcal{R}^4$ symbolically represents a quartic scalar combination of the Riemann tensor whose tensor structure is fixed by supersymmetry. Combining $SL(2,\mathbb{Z})$ duality, D-instanton effects, and supersymmetry, the coefficient of $\mathcal{R}^4$ was found to be the non-holomorphic Eisenstein series $E_{\frac{3}{2}}(\Omega)$ [163–165]. The modular form $E_{\frac{3}{2}}(\Omega)$ produces perturbative contributions at tree and one-loop levels only, and predicts the vanishing of all higher-order perturbative contributions. This prediction was confirmed at two loop order in [20] and at three loop order in [54]. These non-renormalization effects may be attributed to the fact that $\mathcal{R}^4$ is BPS-protected.

Higher-order corrections in the α' expansion to Type II supergravity contain $D^{2k}\mathcal{R}^\ell$ effective interactions for $k \geq 0$ and $\ell \geq 4$, where D is the covariant derivative. The tree-level contributions to these effective interactions may be obtained by expanding the tree-level closed-string amplitudes in powers of α'. As will be detailed in the next section, these expansions can be imported at all multiplicities from the algorithmic expansion of open-string amplitudes [108, 166] in terms of multiple zeta values (MZVs) [167–169]. Going beyond perturbation theory, D-instanton effects have been further analyzed in [170–177].

Combining the tree-level contributions to BPS protected operators at higher order in α' with $SL(2,\mathbb{Z})$ duality, supersymmetry, and D-instanton effects, the coefficient of $D^4\mathcal{R}^4$ was found to be $E_{\frac{5}{2}}(\Omega)$ [178], while the coefficient of $D^6\mathcal{R}^4$ satisfies a modular differential equation [95, 179, 180]. The structure of these modular forms again predicts non-trivial relations between various perturbative corrections, or their absence. One-loop contributions to $D^{2k}\mathcal{R}^4$ and $D^{2k-2}\mathcal{R}^5$ for $k \leq 6$ were extracted from the four- and five-point amplitudes in [181–183] and [34, 184, 185], respectively, and successfully matched against the predictions of $SL(2,\mathbb{Z})$ duality. Two-loop perturbative contributions were evaluated and matched as follows: $D^4\mathcal{R}^4$ in [186, 187]; $D^6\mathcal{R}^4$ in [188, 189]; $D^2\mathcal{R}^5$ in [190]; and $D^4\mathcal{R}^5$ in [191].

The three-loop contribution to $D^6 \mathcal{R}^4$ was matched in [54, 95].

Future targets: An immediate target is to analyze the implications of $SL(2, \mathbb{Z})$ duality in Type IIB on the axion-dilaton field dependence of the coefficients of low-energy effective interactions that are less or not BPS-protected, such as $D^{2k} \mathcal{R}^4$ for $k \geq 4$ and terms with higher powers of the curvature. Tree-level results for $D^{2k} \mathcal{R}^\ell$ with $\ell \geq 5$ already predict the appearance of (single-valued) MZVs of arbitrary depth. In the low-energy asymptotics of the corresponding string amplitudes at loop level, these operators are invariably accompanied by non-local contributions, due to the exchange of massless states in loops, which complicates their precise definition. Although supersymmetry non-renormalization theorems are expected to apply with less predictive power, early studies suggest the presence of non-trivial mathematical structure [60, 183, 192–196]. Progress on such non-renormalization theorems would also help address questions regarding the ultraviolet properties of supergravity in four space-time dimensions.

A closely related issue is as follows. In spite of detailed control over the tensor structure of open- and closed-string tree amplitudes and their α'-expansions [31, 166, 169], the explicit form of the associated interactions, such as $D^{2k} R^\ell$, is unknown beyond the order of $(\alpha')^4$. An important goal is to develop a method by which the simplicity of string tree-level amplitudes can be used to construct the detailed tensor structure of the corresponding low-energy effective interactions (see for instance [197, 198] for progress in taming the complexity of traditional methods).

GAUGE/GRAVITY DUALITY

The AdS/CFT correspondence conjectures the quantum equivalence between Type IIB superstring theory on the $AdS_5 \times S^5$ space-time and $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) with gauge group $SU(N)$ and coupling g on four-dimensional flat space-time [199]. The conjecture provides an explicit relation between on-shell scattering amplitudes in Type IIB superstring theory and conformal correlators of gauge-invariant operators in $\mathcal{N} = 4$ SYM [200, 201]. The large N limit at fixed 't Hooft coupling $\lambda = g^2 N$ corresponds to classical superstring theory on $AdS_5 \times S^5$ while the further limit of large λ corresponds to the supergravity limit as the radii of both AdS_5 and S^5 tend to infinity. It is especially the correspondence involving the supergravity limit (including field theories with less or no supersymmetry and without conformal or Poincaré symmetry) that has spawned a record volume of research over the past 25 years (see [202–204] for reviews and [205, 206] for textbooks on the subject). In this section, we focus on progress towards the quantization of Type IIB strings on AdS backgrounds, and the interplay between conformal correlators in strongly coupled $\mathcal{N} = 4$ SYM and $SL(2, \mathbb{Z})$ duality. A separate White Paper is dedicated to progress [207] on establishing the map between conformal correlators and string perturbation theory.

One of the great challenges of perturbative string theory is to quantize strings in space-times other than flat ten-dimensional Minkowski space-time or toroidal compactifications thereof. This problem is unsolved, even at string tree-level on space-times with large isometry groups that preserve maximal supersymmetry such as $AdS_5 \times S^5$. A key obstacle in the RNS formulation is the difficulty of incorporating Ramond flux backgrounds [208]. Progress has been made for backgrounds with pure NS-NS background fields, such as string theory on $AdS_3 \times S^3$ with NS-NS flux through the sphere [209–212]. To describe the superstring in an $AdS_5 \times S^5$ background with finite radius, one needs to use either the Green-Schwarz [208] or pure-spinor formalism [213]. Although more computations have been performed using the Green-Schwarz formalism, the pure-spinor formalism has the advantage of allowing quantization in a manner that preserves the full $PSU(2, 2|4)$ symmetry [214].

To compute $AdS_5 \times S^5$ scattering amplitudes in which the external states have large R-charge, it is convenient to gauge-fix to a light-cone version of the Green-Schwarz superstring in which $PSU(1, 1|2) \times PSU(1, 1|2)$ symmetry is manifest. In the limit of infinite R-charge, the worldsheet action is quadratic and describes a plane-wave background [215], and numerous tests of the gauge-gravity duality have been performed by expanding around this background (see references in the reviews of [216–219]).

When the AdS_5 radius is small, the superstring amplitudes should coincide with perturbative SYM correlators and one can try to explicitly prove the AdS_5/CFT_4 correspondence. Using the hybrid formalism for the superstring in $AdS_3 \times S^3$ [220] at small radius, the AdS_3/CFT_2 correspondence was proven in [221] and it was shown that similar methods might be possible for the AdS_5/CFT_4 case [222]. Although less is known about the $AdS_5 \times S^5$ superstring at small radius, it has been conjectured that the $AdS_5 \times S^5$ superstring at zero radius is described by a topological string theory [223] and preliminary evidence for this conjecture was presented in [224].

The study of strongly coupled conformal field theories in various dimensions is of central importance for many areas of theoretical physics. The prime example is $\mathcal{N} = 4$ SYM, whose exact conformal invariance is guaranteed by supersymmetry, whose $SL(2, \mathbb{Z})$ duality is well-established, and whose correlators are related by holography to on-shell

amplitudes in Type IIB string theory on $AdS_5 \times S^5$. Here, we shall focus on recent results, obtained by supersymmetric localization techniques, that provide the realization of $SL(2,\mathbb{Z})$ duality on certain *integrated correlators* at arbitrary coupling.

The starting point for the study of integrated correlators is the fact that $\mathcal{N} = 4$ SYM theory can be expressed as a limit of the $\mathcal{N} = 2^*$ theory in which the hyper-multiplet mass m vanishes. Generalizing results of [225, 226], supersymmetric localization techniques were used in [227] to determine the partition function $Z_G(\tau, \bar{\tau}; m)$ for $\mathcal{N} = 2^*$ SYM on S^4 with any classical gauge group G and complex gauge coupling τ. In [228] the quantity $(\text{Im}\tau)^2 \partial_\tau \partial_{\bar{\tau}} \partial_m^2 Z_G(\tau, \bar{\tau}; m)|_{m=0}$ was shown to give the correlator of four superconformal primaries in the stress-tensor multiplet of $\mathcal{N} = 4$ SYM for gauge group $G = SU(N)$, integrated over their positions with a specific measure that preserves supersymmetry. Although detailed information about the space-time dependence is sacrificed by this integration the high degree of supersymmetry of such integrated correlators enables their exact evaluation. The large N limit for fixed 't Hooft coupling $\lambda = g^2 N$ is holographically dual to the perturbative expansion of the Type IIB superstring amplitude in $AdS_5 \times S^5$ [228, 229]. The analysis is facilitated by using Mellin transform techniques [230, 231] and was generalized to an arbitrary classical gauge group G in [232].

While Yang-Mills instantons are suppressed in the large N expansion at fixed 't Hooft coupling $\lambda = g^2 N$, they are unsuppressed in the large N expansion for fixed g^2 considered in [233]. The successive powers of $1/N$ now correspond to successive terms in the low-energy expansion of the Type IIB amplitude. The coefficient of each power is a sum of non-holomorphic $SL(2,\mathbb{Z})$ Eisenstein series and Montonen–Olive duality [234–236] is manifest. This is the gauge theory image of $SL(2,\mathbb{Z})$ S-duality of Type IIB superstring theory. Combined with information gathered from other integrated correlators [237, 238], these results reproduce the known coefficients of the $\mathcal{R}^4$ and $D^4\mathcal{R}^4$ effective interactions, but also contain sub-leading corrections that are suppressed by powers of the AdS_5 length scale.

Extracting more detailed information from the partition function $Z_{SU(N)}(\tau, \bar{\tau}; m)$ is complicated by the complexity of the multi-instanton sectors. However, in [239] an elegant reformulation of the integrated correlator was conjectured in terms of a two-dimensional lattice sum, that is well-defined for all values of N and τ, may be expressed formally as an infinite sum of non-holomorphic Eisenstein series, has a simple spectral representation [240], and generalizes to any classical gauge group [241]. For groups other than $SU(N)$ Montonen-Olive duality extends to include Goddard-Olive-Nuyts duality [242], which relates a theory with gauge group G to a theory with the Langlands dual gauge group, ^LG. Furthermore, for any classical gauge group G the integrated correlators satisfy Laplace difference equations that relate them to the integrated correlators for $SU(2)$. For classical groups other than $SU(N)$ the background geometry of the holographically dual string theory is an orientifold associated with $AdS_5 \times S^5/\mathbb{Z}_2$ [243].

GRAVITATIONAL WAVES AND BLACK HOLES

The detection of gravitational waves from binary systems composed of black holes and/or neutron stars by the LIGO/Virgo collaboration [244, 245] has initiated a new era for studying the cosmos. Gravitational phenomena are again at the core of fundamental physics, both in experiments and theory, and we are heading towards a bright high-precision frontier. Highly precise analytic calculations in general relativity are indispensable to build waveform models for the inspiral phase of binary mergers, and thus infer unique astrophysical and fundamental physics information hidden in the gravitational-wave signals. Remarkably, QFT and string theory provide advanced mathematical tools associated to scattering amplitudes that display stunning simplicity when applied to the non-linear problem of classical gravity [246–286]. A separate White Paper [287] is dedicated to the emerging subfield of applying modern amplitudes methods to the problem of gravitational-wave physics, see further references therein.

With higher sensitivity LIGO, Virgo and KAGRA observing runs in the next few years, and with upcoming detectors (Einstein Telescope, Cosmic Explorer, LISA) it is crucial to further improve the accuracy of theoretical predictions. Even with state-of-the-art analytical effective field theory (EFT) methods the needed post-Newtonian (PN) or post-Minkowskian (PM) calculations are highly challenging. The complexity can be traced back to the sheer number of terms appearing in the gravitational Feynman diagrams, as well as to the problem of analytic integration. For the integrals, the challenge is ameliorated by the increased understanding of integral bases, integration by parts identities, function spaces graded by transcendental weight, and differential equations – tools originally developed for QFT and string scattering amplitudes. The complexity of the integrands can be managed by exploiting the double-copy relationship between gravity and gauge theory – originally in the form of the KLT relations between open- and closed-string amplitudes. In combination with unitarity-based methods [288–290], this has launched a revolutionary path to access precision gravity via vastly simpler gauge theory calculations, recently reaching new heights at 3PM and 4PM orders [251, 272] for Schwarzschild black holes.

The PM calculations are necessary for describing scattering events of pairs of unbound black holes to all orders in the velocity, and they provide vital cross-checks and simplifications to the PN calculations needed for bound systems. Scattering of black holes benefits from the eikonal approach [273, 291], which in string theory was explored at high energy in the Amati-Ciafaloni-Veneziano (ACV) formalism [292, 293]. The latter approach recently led to the resolution of an open question with the conservative dynamics at 3PM [268]. Further, tangentially related QFT approaches based on Weinberg's soft theorem [294], gravitational memory [295] and celestial holography (see e.g. [296–299] and references therein), have lead to novel perspectives on gravitational wave emission [300–302].

Properly incorporating spin effects in general relativity is paramount for mergers involving astrophysical Kerr black holes, with accurate parameter estimation. In an EFT approach [303, 304], the spin multipole moments are included to match results from black hole perturbation theory and self-force calculations, but vast theoretical challenges remain. For a classical Kerr black hole, there is currently no satisfactory EFT action valid for higher-order calculations in both the gravitational coupling and spin. The worldline EFT approach to Kerr encounters challenges starting with unknown Wilson coefficients of the Riemann-squared operators at order S^4 [305], where S is the classical spin vector. Likewise the Compton scattering amplitude for a putative Kerr black hole develops spurious poles beyond quantum spin $s > 2$ [306–308].

Future targets: Can string theory provide us us with the correct Compton amplitude for a classical Kerr black hole? Recently, it has been shown that all known Kerr scattering amplitudes are compatible with established tree-level unitarity constraints from the higher-spin literature [309]. Higher-spin theories are close cousins of string theory and share many of the stringy features, which suggest synergies. The massive string modes are prime examples of how higher-spin states can be embedded in a unitary and mathematically consistent framework, and may be of use for modelling classical Kerr amplitudes.

Since string theory provides a microscopic description of extremal black holes [310] it should provide a natural framework for exploring scattering amplitudes of such objects, also in the macroscopic limit. Observations of near-extremal Kerr black holes are expected from astrophysics, and it would be interesting to describe their scattering amplitudes, and more generally their waveforms, using results such as the Kerr/CFT correspondence [311].

While quantum and stringy effects for astrophysical black holes are expected to be very small, it nevertheless is an interesting question in principle how to incorporate such effects in scattering amplitudes and waveform calculations. In particular, properly describing Hawking radiation [312] from an EFT [313] or amplitudes perspective could lead to significant progress in the understanding of quantum gravity.

MATHEMATICAL STRUCTURES

The interplay between duality, supersymmetry, and the low-energy expansion of superstring amplitudes has resulted in the discovery of unexpected and exciting algebraic and arithmetic structures. Numerous mathematical properties of string amplitudes are now understood in terms of recent developments in number theory, modular forms, and algebraic geometry. Conversely, the low-energy expansion of string amplitudes and their integrands has spawned new directions of mathematical research related to single-valued integration and generalizations of non-holomorphic modular forms.

Already at genus zero, the pattern of multiple zeta values (MZVs) in higher-point functions is elegantly understood in terms of Hopf-algebra structures of motivic MZVs and the Drinfeld associator [169, 314]. As a result, tree-level amplitudes in general string theories furnish clean showcases of the coaction conjecture [315, 316] on the stability of amplitudes under the motivic coaction. So far, this coaction conjecture has found manifestations in ϕ^4-theory [317], the magnetic moment of the electron [318], $\mathcal{N} = 4$ super-Yang–Mills [319], and various families of Feynman integrals [320–322]. Therefore, understanding the structure of string amplitudes may contribute to unravelling number-theoretic symmetries shared by quantum field theories and string theories.

Furthermore, the low-energy expansion of tree-level closed-string amplitudes can be obtained from the low-energy expansion of open-string amplitudes through a formal operation on motivic MZVs [323–327], referred to as the single-valued map [315, 328]. This process of inferring closed-string results from the open string goes considerably beyond the standard KLT formula of string theory [76] and combinatorially realizes a field theory double copy [169, 323] similar to that of open strings [106, 111]. The single-valued map between tree-level amplitudes connects different perturbative string theories beyond the scope of any known duality [324], e.g. the gauge sectors of Heterotic and Type-I superstrings at weak coupling.

An ambitious long-term goal of both conceptual and technical appeal is to reduce closed-string loop amplitudes to open-string building blocks. The appearance of single-valued MZVs in closed-string tree-level amplitudes found

several echoes at genus one [329–332] and led to a proposal for a single-valued map between configuration-space integrals in open- and closed-string one-loop amplitudes [333]. This proposal calls for extensions to the integration over modular parameters at genus one and to iterated integrals on higher-genus surfaces. The perturbative and possibly non-perturbative systematics of relating open- and closed-string interactions may have valuable lessons for old and new string dualities.

Another mathematical viewpoint on open-closed-string relations that may advance loop-level developments is furnished by twisted de Rham theory. The α'-dependent Koba-Nielsen factor aligns moduli-space integrals of string tree amplitudes into twisted cohomologies which have been studied in the mathematics literature of the 80's and 90's. The KLT relations at tree-level then become a consequence of linear algebra in these cohomologies [334] through the so-called twisted period relations [335]. The chiral variant of the KLT relations [336] is another corollary of the twisted period relations [337] that reproduces ambi-twistor-string amplitudes from a formal $\alpha' \to \infty$ limit [338].

Similarly, monodromy relations [79–82] among open-string amplitudes descend from properties of twisted homologies [84, 339]. The latter also shed light on mixed open- and closed-string amplitudes in the gravity sector of Type I as exemplified in the recent one-loop discussion [340], see [80, 341] for similar relations at tree-level. Methods of twisted de Rham theory also became increasingly important in particle physics, for instance for basis decompositions of Feynman integrals [342–345].

At one loop and beyond, string amplitudes produce a wealth of elliptic and modular generalizations of MZVs. Closely related elliptic polylogarithms [346] drive the algorithmic integration of open-string insertions on a genus one worldsheet [38, 347] and organize the α'-expansion of configuration-space integrals in terms of elliptic MZVs [348], see [349, 350] for recent all-order results. Closed-string applications of elliptic polylogarithms have been pioneered in [351, 352] and furnish rewarding targets for future research. At the same time, elliptic polylogarithms and the associated iterated integrals of modular forms substantially improved the computational reach for Feynman integrals beyond genus-zero polylogarithms [353, 354], see in particular the references in the White Paper [355].

Uniform transcendentality, familiar from dimensionally regularized Feynman integrals [356–358], enjoys a Type I and Type II superstring amplitude counterpart in their α'-expansions at tree-level [166, 359] and at one loop [183, 349, 360]. However, uniform transcendentality of Type II superstrings may get challenged at higher α'-orders of one-loop amplitudes [361] and may be violated in two-loop amplitudes [362]. Moreover, tree-level amplitudes of Heterotic and bosonic strings [110] violate uniform transcendentality. This resonates with the comparison of loop amplitudes in different gauge and supergravity theories whose transcendentality properties are sensitive to the amount of space-time supersymmetry [363–368].

The low-energy expansion of closed-string amplitudes at one-loop order and beyond has introduced fascinating families of non-holomorphic modular forms, dubbed modular graph functions [181, 182, 193, 330] and modular graph forms [369–371], see [372] for an overview and [373] for a MATHEMATICA package. Modular graph forms not only play a vital role in string-theory computations [374–377] but also developed their own life in shaping mathematical research directions [332, 378–381]. Their description via iterated integrals of holomorphic objects [331, 382] exposes the intriguing web of differential and algebraic relations among modular graph forms discovered in [182, 369, 370, 383], sheds light on the interplay between open and closed strings [333] and cross-fertilizes with studies of Poincaré series representations [371, 384, 385].

Similarly, building blocks of multi-loop closed-string amplitudes led to the definition of higher-genus modular graph functions [195] with applications to the low-energy expansion of two-loop amplitudes at four [196] and five points [191]. In fact, the study of algebraic [191] and differential relations [189, 386] among modular graph functions beyond genus one motivates their generalization to modular graph tensors [387]. Motivated in part by a construction of Kawazumi [388, 389], modular graph tensors are introduced as non-holomorphic functions on the Torelli space of compact Riemann surfaces of genus g which transform as tensors under the modular group $Sp(2g, \mathbb{Z})$.

Future targets: At higher genus, the study of modular graph functions [195] and tensors [387] has just begun. A natural first step is to explore the differential structure of modular graph tensors in order to reveal the systematics of their algebraic relations, for instance via iterated-integral representations. This analysis should clarify the relations of modular graph tensors to non-holomorphic Siegel modular forms [390] and improve the computational reach for multi-loop string amplitudes and their low-energy expansions.

The open-string analogues of modular graph tensors are uncharted terrain. By analogy with genus one, the meromorphic building blocks of multiloop open-string amplitudes are expected to inform the construction of higher-genus polylogarithms, in particular the relevant classes of iterated integrals and their integration kernels. The meromorphic function spaces obtained from open-string input are likely to find applications to Feynman integrals in quantum field

theory – not only to compute further classes beyond elliptic polylogarithms but also as a case study in preparation for Feynman integrals associated with K3 or Calabi-Yau geometries and beyond.

The rich mathematical structures of string amplitudes and their moduli-space integrals are not fully manifest from their standard worldsheet prescriptions. Instead, the newly discovered mathematical structures might guide us to find radical reformulations of string perturbation theory from number-theoretic and algebraic-geometric principles, possibly without any reference to worldsheets. A grand goal would be to find a unified picture for the coaction properties of open-string tree-level amplitudes [169], the derivation algebra governing modular graph forms at genus one [382] and similar structures to be found at higher genus.

CONCLUDING REMARKS

In this White Paper, we have attempted to illustrate the richness of string perturbation theory through a lightning overview of recent progress and a small selection of future targets. Current developments in string amplitudes are not only strengthening the ties between different directions within string theory but are also triggering a striking confluence of ideas between string theorists, particle physicists, relativists, cosmologists, condensed matter physicists, and mathematicians. No doubt, the future targets listed in this White Paper merely reflect a tip of the iceberg of important technical and conceptual avenues for future research.

ACKNOWLEDGEMENTS

[1] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory Vol. 1: 25th Anniversary Edition*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2012).

[2] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory Vol. 2: 25th Anniversary Edition*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2012).

[3] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2007).

[4] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2007).

[5] B. Zwiebach, *A first course in string theory* (Cambridge University Press, 2006).

[6] K. Becker, M. Becker, and J.H. Schwarz, *String Theory and M-theory* (Cambridge University Press, 2007).

[7] R. Blumenhagen, D. Lüst, and S. Theisen, *Basic concepts of string theory*, Theoretical and Mathematical Physics (Springer, Heidelberg, 2013).

[8] E. Kiritsis, *String theory in a nutshell* (Princeton University Press, USA, 2019).

[9] M.B. Green and J.H. Schwarz, "Supersymmetrical Dual String Theory," Nucl. Phys. B **181**, 502–530 (1981).

[10] M.B. Green and J.H. Schwarz, "Supersymmetrical String Theories," Phys. Lett. B **109**, 444–448 (1982).

[11] D.J. Gross, J.A. Harvey, E.J. Martinec, and R. Rohm, "The Heterotic String," Phys. Rev. Lett. **54**, 502–505 (1985).

[12] D. Friedan, E.J. Martinec, and S.H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," Nucl. Phys. B **271**, 93–165 (1986).

[13] E. D'Hoker and D.H. Phong, "The Geometry of String Perturbation Theory," Rev. Mod. Phys. **60**, 917 (1988).

[14] E. D'Hoker, *String Theory*, edited by P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, and E. Witten, Quantum Fields and Strings: A Course for Mathematicians, Vol. II (American Mathematical Society, Institute for Advanced Study, 1999) pp. 807–1012.

[15] N. Berkovits and H. Gomez, "An Introduction to Pure Spinor Superstring Theory," in *9th Summer School on Geometric, Algebraic and Topological Methods for Quantum Field Theory*, Mathematical Physics Studies (2017) pp. 221–246, arXiv:1711.09966 [hep-th].

[16] E. D'Hoker, "Lecture Series on String Amplitudes," *Amplitudes 2018 Summer School*, UC Davis (2018).

[17] O. Schlotterer, "Lecture series on Superstring Amplitudes in the RNS and pure spinor formalism," *String theory from the worldsheet perspective*, GGI, Florence, Italy (2019).

[18] E. D'Hoker and D.H. Phong, "Two loop superstrings. 1. Main formulas," Phys. Lett. B **529**, 241–255 (2002), arXiv:hep-th/0110247.

[19] E. D'Hoker and D.H. Phong, "Lectures on two loop superstrings," Conf. Proc. C **0208124**, 85–123 (2002), arXiv:hep-th/0211111.

[20] E. D'Hoker and D.H. Phong, "Two-loop superstrings VI: Non-renormalization theorems and the 4-point function," Nucl. Phys. B **715**, 3–90 (2005), arXiv:hep-th/0501197.

[21] N. Berkovits, "Super Poincare covariant quantization of the superstring," JHEP **04**, 018 (2000), arXiv:hep-th/0001035.

[22] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **09**, 047 (2004), arXiv:hep-th/0406055.

[23] N. Berkovits, "Pure spinor formalism as an N=2 topological string," JHEP **10**, 089 (2005), arXiv:hep-th/0509120.

[24] E. Witten, "Notes On Super Riemann Surfaces And Their Moduli," Pure Appl. Math. Quart. **15**, 57–211 (2019), arXiv:1209.2459.

[25] E. Witten, "Superstring Perturbation Theory Revisited," (2012), arXiv:1209.5461.

[26] E. Witten, "More On Superstring Perturbation Theory: An Overview Of Superstring Perturbation Theory Via Super Riemann Surfaces," (2013), arXiv:1304.2832 [hep-th].

[27] N. Berkovits, "Origin of the Pure Spinor and Green-Schwarz Formalisms," JHEP **07**, 091 (2015), arXiv:1503.03080 [hep-th].

[28] N. Berkovits, "Untwisting the pure spinor formalism to the RNS and twistor string in a flat and $AdS_5 \times S^5$ background," JHEP **06**, 127 (2016), arXiv:1604.04617 [hep-th].

[29] N. Berkovits, "Manifest spacetime supersymmetry and the superstring," JHEP **10**, 162 (2021), arXiv:2106.04448 [hep-th].

[30] E. D'Hoker and D.H. Phong, "Conformal Scalar Fields and Chiral Splitting on Superriemann Surfaces," Commun. Math. Phys. **125**, 469 (1989).

[31] C.R. Mafra, O. Schlotterer, and S. Stieberger, "Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation," Nucl. Phys. **B873**, 419–460 (2013), arXiv:1106.2645.

[32] C.R. Mafra and O. Schlotterer, "Multiparticle SYM equations of motion and pure spinor BRST blocks," JHEP **07**, 153 (2014), arXiv:1404.4986 [hep-th].

[33] M.B. Green, J.H. Schwarz, and L. Brink, "N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories," Nucl. Phys. **B198**, 474–492 (1982).

[34] M.B. Green, C.R. Mafra, and O. Schlotterer, "Multiparticle one-loop amplitudes and S-duality in closed superstring theory," JHEP **10**, 188 (2013), arXiv:1307.3534 [hep-th].

[35] C.R. Mafra and O. Schlotterer, "One-loop superstring six-point amplitudes and anomalies in pure spinor superspace," JHEP **04**, 148 (2016), arXiv:1603.04790 [hep-th].

[36] C.R. Mafra and O. Schlotterer, "Towards the n-point one-loop superstring amplitude. Part III. One-loop correlators and their double-copy structure," JHEP **08**, 092 (2019), arXiv:1812.10971 [hep-th].

[37] A. Tsuchiya, "More on One Loop Massless Amplitudes of Superstring Theories," Phys. Rev. **D39**, 1626 (1989).

[38] J. Broedel, C.R. Mafra, N. Matthes, and O. Schlotterer, "Elliptic multiple zeta values and one-loop superstring amplitudes," JHEP **07**, 112 (2015), arXiv:1412.5535.

[39] E. D'Hoker and D.H. Phong, "Two loop superstrings. 2. The Chiral measure on moduli space," Nucl. Phys. B **636**, 3–60 (2002), arXiv:hep-th/0110283.

[40] E. D'Hoker and D.H. Phong, "Two loop superstrings. 3. Slice independence and absence of ambiguities," Nucl. Phys. B **636**, 61–79 (2002), arXiv:hep-th/0111016.

[41] E. D'Hoker and D.H. Phong, "Two loop superstrings 4: The Cosmological constant and modular forms," Nucl. Phys. B **639**, 129–181 (2002), arXiv:hep-th/0111040.

[42] E. Witten, "Notes On Holomorphic String And Superstring Theory Measures Of Low Genus," (2013), arXiv:1306.3621 [hep-th].

[43] E. D'Hoker and D.H. Phong, "Two-loop superstrings. V. Gauge slice independence of the N-point function," Nucl. Phys. B **715**, 91–119 (2005), arXiv:hep-th/0501196.

[44] N. Berkovits, "Super-Poincare covariant two-loop superstring amplitudes," JHEP **01**, 005 (2006), arXiv:hep-th/0503197.

[45] N. Berkovits and C.R. Mafra, "Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms," Phys. Rev. Lett. **96**, 011602 (2006), arXiv:hep-th/0509234.

[46] E. D'Hoker, C.R. Mafra, B. Pioline, and O. Schlotterer, "Two-loop superstring five-point amplitudes. Part I. Construction via chiral splitting and pure spinors," JHEP **08**, 135 (2020), arXiv:2006.05270.

[47] E. D'Hoker and O. Schlotterer, "Two-loop superstring five-point amplitudes. Part III. Construction via the RNS formulation: even spin structures," JHEP **12**, 063 (2021), arXiv:2108.01104 [hep-th].

[48] E. D'Hoker and D.H. Phong, "Two-loop vacuum energy for Calabi-Yau orbifold models," Nucl. Phys. B **877**, 343–400 (2013), arXiv:1307.1749.

[49] N. Berkovits and E. Witten, "Supersymmetry Breaking Effects using the Pure Spinor Formalism of the Superstring," JHEP **06**, 127 (2014), arXiv:1404.5346.

[50] S.L. Cacciatori, F. Dalla Piazza, and B. van Geemen, "Modular Forms and Three Loop Superstring Amplitudes," Nucl. Phys. B **800**, 565–590 (2008), arXiv:0801.2543 [hep-th].

[51] E. Witten, "The Super Period Matrix With Ramond Punctures," J. Geom. Phys. **92**, 210–239 (2015), arXiv:1501.02499 [hep-th].

[52] S. Grushevsky and R. Salvati Manni, "The Vanishing of two-point functions for three-loop superstring scattering amplitudes," Commun. Math. Phys. **294**, 343–352 (2010), arXiv:0806.0354 [hep-th].

[53] M. Matone and R. Volpato, "Superstring measure and non-renormalization of the three-point amplitude," Nucl. Phys. B **806**, 735–747 (2009), arXiv:0806.4370 [hep-th].

[54] H. Gomez and C.R. Mafra, "The closed-string 3-loop amplitude and S-duality," JHEP **10**, 217 (2013), arXiv:1308.6567.

[55] Y. Geyer, R. Monteiro, and R. Stark-Muchão, "Superstring Loop Amplitudes from the Field Theory Limit," Phys. Rev. Lett. **127**, 211603 (2021), arXiv:2106.03968 [hep-th].

[56] L. Mason and D. Skinner, "Ambitwistor strings and the scattering equations," JHEP **07**, 048 (2014), arXiv:1311.2564 [hep-th].

[57] N. Berkovits, "Infinite Tension Limit of the Pure Spinor Superstring," JHEP **03**, 017 (2014), arXiv:1311.4156 [hep-th].

[58] T. Adamo, E. Casali, and D. Skinner, "Ambitwistor strings and the scattering equations at one loop," JHEP **04**, 104 (2014), arXiv:1312.3828 [hep-th].

[59] N. Berkovits, "New higher-derivative R^4 theorems," Phys. Rev. Lett. **98**, 211601 (2007), arXiv:hep-th/0609006.

[60] N. Berkovits, M.B. Green, J.G. Russo, and P. Vanhove, "Non-renormalization conditions for four-gluon scattering in supersymmetric string and field theory," JHEP **11**, 063 (2009), arXiv:0908.1923 [hep-th].

[61] E. D'Hoker and D.H. Phong, "The Super Period Matrix with Ramond Punctures in the supergravity formulation," Nucl. Phys. B **899**, 772–809 (2015), arXiv:1501.02675 [hep-th].

[62] E. D'Hoker and D.H. Phong, "Momentum analyticity and finiteness of the one loop superstring amplitude," Phys. Rev. Lett. **70**, 3692–3695 (1993), arXiv:hep-th/9302003.

[63] E. D'Hoker and D.H. Phong, "Dispersion relations in string theory," Theor. Math. Phys. **98**, 306–316 (1994), arXiv:hep-th/9404128.

[64] E. D'Hoker and D.H. Phong, "The Box graph in superstring theory," Nucl. Phys. B **440**, 24–94 (1995), arXiv:hep-th/9410152.

[65] R. Donagi and E. Witten, "Supermoduli Space Is Not Projected," Proc. Symp. Pure Math. **90**, 19–72 (2015), arXiv:1304.7798.

[66] A. Sen and E. Witten, "Filling the gaps with PCO's," JHEP **09**, 004 (2015), arXiv:1504.00609.

[67] J. Kaidi, J. Parra-Martinez, and Y. Tachikawa, "Classification of String Theories via Topological Phases," Phys. Rev. Lett. **124**, 121601 (2020), arXiv:1908.04805 [hep-th].

[68] J. Kaidi, J. Parra-Martinez, Y. Tachikawa, and with a mathematical appendix by A. Debray, "Topological Superconductors on Superstring Worldsheets," SciPost Phys. **9**, 10 (2020), arXiv:1911.11780 [hep-th].

[69] Y. Aisaka and N. Berkovits, "Pure Spinor Vertex Operators in Siegel Gauge and Loop Amplitude Regularization," JHEP **07**, 062 (2009), arXiv:0903.3443 [hep-th].

[70] P.A. Grassi and P. Vanhove, "Higher-loop amplitudes in the non-minimal pure spinor formalism," JHEP **05**, 089 (2009), arXiv:0903.3903 [hep-th].

[71] T. Adamo, J.J.M. Carrasco, M. Carrillo-González, M. Chiodaroli, H. Elvang, H. Johansson, D. O'Connell, R. Roiban, and O. Schlotterer, "Snowmass White Paper: the Double Copy and its Applications," (2022) arXiv:2204.06547 [hep-th].

[72] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov, and J. Trnka, *Grassmannian Geometry of Scattering Amplitudes* (Cambridge University Press, 2016) arXiv:1212.5605 [hep-th].

[73] H. Elvang and Y-t. Huang, *Scattering Amplitudes in Gauge Theory and Gravity* (Cambridge University Press, 2015).

[74] Z. Bern, J.J.M. Carrasco, M. Chiodaroli, H. Johansson, and R. Roiban, "The Duality Between Color and Kinematics and its Applications," (2019), arXiv:1909.01358.

[75] G. Travaglini *et al.*, "The SAGEX Review on Scattering Amplitudes," (2022), arXiv:2203.13011 [hep-th].

[76] H. Kawai, D.C. Lewellen, and S.H.H. Tye, "A Relation Between Tree Amplitudes of Closed and Open Strings," Nucl. Phys. **B269**, 1–23 (1986).

[77] Z. Bern, J.J.M. Carrasco, and H. Johansson, "Perturbative Quantum Gravity as a Double Copy of Gauge Theory," Phys. Rev. Lett. **105**, 061602 (2010), arXiv:1004.0476.

[78] Z. Bern, J.J.M. Carrasco, and H. Johansson, "New Relations for Gauge-Theory Amplitudes," Phys. Rev. **D78**, 085011 (2008), arXiv:0805.3993.

[79] N.E.J. Bjerrum-Bohr, P.H. Damgaard, and P. Vanhove, "Minimal Basis for Gauge Theory Amplitudes," Phys. Rev. Lett. **103**, 161602 (2009), arXiv:0907.1425 [hep-th].

[80] S. Stieberger, "Open & Closed vs. Pure Open String Disk Amplitudes," (2009), arXiv:0907.2211 [hep-th].

[81] P. Tourkine and P. Vanhove, "Higher-loop amplitude monodromy relations in string and gauge theory," Phys. Rev. Lett. **117**, 211601 (2016), arXiv:1608.01665 [hep-th].

[82] S. Hohenegger and S. Stieberger, "Monodromy Relations in Higher-Loop String Amplitudes," Nucl. Phys. **B925**, 63–134 (2017), arXiv:1702.04963 [hep-th].

[83] P. Tourkine, "Integrands and loop momentum in string and field theory," Phys. Rev. D **102**, 026006 (2020), arXiv:1901.02432 [hep-th].

[84] E. Casali, S. Mizera, and P. Tourkine, "Monodromy relations from twisted homology," JHEP **12**, 087 (2019), arXiv:1910.08514.

[85] E. Casali, S. Mizera, and P. Tourkine, "Loop amplitudes monodromy relations and color-kinematics duality," JHEP **03**, 048 (2021), arXiv:2005.05329 [hep-th].

[86] C.R. Mafra, O. Schlotterer, and S. Stieberger, "Explicit BCJ Numerators from Pure Spinors," JHEP **07**, 092 (2011), arXiv:1104.5224.

[87] C.R. Mafra and O. Schlotterer, "Towards one-loop SYM amplitudes from the pure spinor BRST cohomology," Fortsch. Phys. **63**, 105–131 (2015), arXiv:1410.0668.

[88] S. He, R. Monteiro, and O. Schlotterer, "String-inspired BCJ numerators for one-loop MHV amplitudes," JHEP **01**, 171 (2016), arXiv:1507.06288 [hep-th].

[89] P. Tourkine, "Tropical Amplitudes," Annales Henri Poincare **18**, 2199–2249 (2017), arXiv:1309.3551 [hep-th].

[90] S. Mizera, "Kinematic Jacobi Identity is a Residue Theorem: Geometry of Color-Kinematics Duality for Gauge and Gravity Amplitudes," Phys. Rev. Lett. **124**, 141601 (2020), arXiv:1912.03397 [hep-th].

[91] M. Ben-Shahar and M. Guillen, "10D super-Yang-Mills scattering amplitudes from its pure spinor action," JHEP **12**, 014 (2021), arXiv:2108.11708 [hep-th].

[92] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson, and R. Roiban, "Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes," Phys. Rev. **D85**, 105014 (2012), arXiv:1201.5366 [hep-th].

[93] Z. Bern, S. Davies, T. Dennen, A.V. Smirnov, and V.A. Smirnov, "Ultraviolet Properties of N=4 Supergravity at Four Loops," Phys. Rev. Lett. **111**, 231302 (2013), arXiv:1309.2498 [hep-th].

[94] Z. Bern, S. Davies, and T. Dennen, "Enhanced ultraviolet cancellations in $\mathcal{N} = 5$ supergravity at four loops," Phys. Rev. **D90**, 105011 (2014), arXiv:1409.3089 [hep-th].

[95] M.B. Green and P. Vanhove, "Duality and higher derivative terms in M theory," JHEP **01**, 093 (2006), arXiv:hep-th/0510027 [hep-th].

[96] M.B. Green, J.G. Russo, and P. Vanhove, "String theory dualities and supergravity divergences," JHEP **06**, 075 (2010), arXiv:1002.3805 [hep-th].

[97] P. Tourkine and P. Vanhove, "An R^4 non-renormalisation theorem in N=4 supergravity," Class. Quant. Grav. **29**, 115006 (2012), arXiv:1202.3692 [hep-th].

[98] B. Pioline, "String theory integrands and supergravity divergences," JHEP **02**, 148 (2019), arXiv:1810.11343 [hep-th].

[99] Z. Bern, J.J.M. Carrasco, W.M. Chen, A. Edison, H. Johansson, J. Parra-Martinez, R. Roiban, and M. Zeng, "Ultraviolet Properties of $\mathcal{N} = 8$ Supergravity at Five Loops," Phys. Rev. D **98**, 086021 (2018), arXiv:1804.09311.

[100] J. Bjornsson and M.B. Green, "5 loops in 24/5 dimensions," JHEP **08**, 132 (2010), arXiv:1004.2692.

[101] F. Cachazo, S. He, and E.Y. Yuan, "Scattering equations and Kawai-Lewellen-Tye orthogonality," Phys. Rev. **D90**, 065001 (2014), arXiv:1306.6575 [hep-th].

[102] F. Cachazo, S. He, and E.Y. Yuan, "Scattering of Massless Particles in Arbitrary Dimensions," Phys. Rev. Lett. **113**, 171601 (2014), arXiv:1307.2199 [hep-th].

[103] H. Gomez and E.Y. Yuan, "N-point tree-level scattering amplitude in the new Berkovits' string," JHEP **04**, 046 (2014), arXiv:1312.5485 [hep-th].

[104] Y. Geyer and R. Monteiro, "Two-Loop Scattering Amplitudes from Ambitwistor Strings: from Genus Two to the Nodal Riemann Sphere," JHEP **11**, 008 (2018), arXiv:1805.05344 [hep-th].

[105] S. He, F. Teng, and Y. Zhang, "String amplitudes from field-theory amplitudes and vice versa," Phys. Rev. Lett. **122**, 211603 (2019), arXiv:1812.03369 [hep-th].

[106] J. Broedel, O. Schlotterer, and S. Stieberger, "Polylogarithms, Multiple Zeta Values and Superstring Amplitudes," Fortsch. Phys. **61**, 812–870 (2013), arXiv:1304.7267.

[107] J.J.M. Carrasco, C.R. Mafra, and O. Schlotterer, "Abelian Z-theory: NLSM amplitudes and α'-corrections from the open string," JHEP **06**, 093 (2017), arXiv:1608.02569 [hep-th].

[108] C.R. Mafra and O. Schlotterer, "Non-abelian Z-theory: Berends-Giele recursion for the α'-expansion of disk integrals," JHEP **01**, 031 (2017), arXiv:1609.07078 [hep-th].

[109] J.J.M. Carrasco, C.R. Mafra, and O. Schlotterer, "Semi-abelian Z-theory: NLSM+ϕ^3 from the open string," JHEP **08**, 135 (2017), arXiv:1612.06446 [hep-th].

[110] Y-t. Huang, O. Schlotterer, and C. Wen, "Universality in string interactions," JHEP **09**, 155 (2016), arXiv:1602.01674 [hep-th].

[111] T. Azevedo, M. Chiodaroli, H. Johansson, and O. Schlotterer, "Heterotic and bosonic string amplitudes via field theory," JHEP **10**, 012 (2018), arXiv:1803.05452.

[112] H. Johansson and J. Nohle, "Conformal Gravity from Gauge Theory," (2017), arXiv:1707.02965.

[113] Z. Bern, J.J.M. Carrasco, W.M. Chen, H. Johansson, R. Roiban, and M. Zeng, "Five-loop four-point integrand of $N = 8$ supergravity as a generalized double copy," Phys. Rev. **D96**, 126012 (2017), arXiv:1708.06807 [hep-th].

[114] E. Bridges and C.R. Mafra, "Local BCJ numerators for ten-dimensional SYM at one loop," JHEP **07**, 031 (2021), arXiv:2102.12943 [hep-th].

[115] Z. Bern, J.J.M. Carrasco, W.M. Chen, H. Johansson, and R. Roiban, "Gravity Amplitudes as Generalized Double Copies of Gauge-Theory Amplitudes," Phys. Rev. Lett. **118**, 181602 (2017), arXiv:1701.02519 [hep-th].

[116] S. He and O. Schlotterer, "New Relations for Gauge-Theory and Gravity Amplitudes at Loop Level," Phys. Rev. Lett. **118**, 161601 (2017), arXiv:1612.00417 [hep-th].

[117] S. He, O. Schlotterer, and Y. Zhang, "New BCJ representations for one-loop amplitudes in gauge theories and gravity," Nucl. Phys. **B930**, 328–383 (2018), arXiv:1706.00640 [hep-th].

[118] C.R. Mafra and O. Schlotterer, "Double-Copy Structure of One-Loop Open-String Amplitudes," Phys. Rev. Lett. **121**, 011601 (2018), arXiv:1711.09104.

[119] T. Erler, "Four Lectures on Closed String Field Theory," Phys. Rept. **851**, 1–36 (2020), arXiv:1905.06785 [hep-th].

[120] T. Erler, "Four Lectures on Analytic Solutions in Open String Field Theory," (2019), arXiv:1912.00521 [hep-th].

[121] A. Sen, "Tachyon condensation on the brane anti-brane system," JHEP **08**, 012 (1998), arXiv:hep-th/9805170.

[122] A. Sen and B. Zwiebach, "Tachyon condensation in string field theory," JHEP **03**, 002 (2000), arXiv:hep-th/9912249.

[123] N. Berkovits, A. Sen, and B. Zwiebach, "Tachyon condensation in superstring field theory," Nucl. Phys. B **587**, 147–178 (2000), arXiv:hep-th/0002211.

[124] R. Pius, A. Rudra, and A. Sen, "Mass Renormalization in String Theory: Special States," JHEP **07**, 058 (2014), arXiv:1311.1257 [hep-th].

[125] R. Pius, A. Rudra, and A. Sen, "Mass Renormalization in String Theory: General States," JHEP **07**, 062 (2014), arXiv:1401.7014 [hep-th].

[126] A. Sen, "One Loop Mass Renormalization of Unstable Particles in Superstring Theory," JHEP **11**, 050 (2016), arXiv:1607.06500 [hep-th].

[127] A. Sen, "Gauge Invariant 1PI Effective Superstring Field Theory: Inclusion of the Ramond Sector," JHEP **08**, 025 (2015), arXiv:1501.00988 [hep-th].

[128] C. de Lacroix, H. Erbin, S.P. Kashyap, A. Sen, and M. Verma, "Closed Superstring Field Theory and its Applications," Int. J. Mod. Phys. A **32**, 1730021 (2017), arXiv:1703.06410 [hep-th].

[129] M.B. Green and J.H. Schwarz, "Superstring Field Theory," Nucl. Phys. B **243**, 475–536 (1984).

[130] N. Berkovits, "SuperPoincare invariant superstring field theory," Nucl. Phys. B **450**, 90–102 (1995), [Erratum: Nucl.Phys.B 459, 439–451 (1996)], arXiv:hep-th/9503099.

[131] T. Erler, "Relating Berkovits and A_∞ superstring field theories; large Hilbert space perspective," JHEP **02**, 121 (2016), arXiv:1510.00364 [hep-th].

[132] T. Erler, "Superstring Field Theory and the Wess-Zumino-Witten Action," JHEP **10**, 057 (2017), arXiv:1706.02629 [hep-th].

[133] T. Erler and H. Matsunaga, "Mapping between Witten and lightcone string field theories," JHEP **11**, 208 (2021), arXiv:2012.09521 [hep-th].

[134] E. Witten, "Noncommutative Geometry and String Field Theory," Nucl. Phys. B **268**, 253–294 (1986).

[135] R. Gopakumar and C. Vafa, "On the gauge theory / geometry correspondence," Adv. Theor. Math. Phys. **3**, 1415–1443 (1999), arXiv:hep-th/9811131.

[136] D. Gaiotto and L. Rastelli, "A Paradigm of open / closed duality: Liouville D-branes and the Kontsevich model," JHEP **07**, 053 (2005), arXiv:hep-th/0312196.

[137] N. Berkovits, "Perturbative Super-Yang-Mills from the Topological $AdS_5 \times S^5$ Sigma Model," JHEP **09**, 088 (2008), arXiv:0806.1960 [hep-th].

[138] Y. Okawa, "Nonperturbative definition of closed string theory via open string field theory," (2020), arXiv:2006.16449 [hep-th].

[139] C.M. Hull and P.K. Townsend, "Unity of superstring dualities," Nucl. Phys. B **438**, 109–137 (1995), arXiv:hep-th/9410167.

[140] E. Witten, "String theory dynamics in various dimensions," Nucl. Phys. **B443**, 85–126 (1995), arXiv:hep-th/9503124.

[141] D.Z. Freedman and A. Van Proeyen, *Supergravity* (Cambridge Univ. Press, Cambridge, UK, 2012).

[142] M.J. Duff, ed., *The World in Eleven Dimensions : Supergravity, supermembranes and M-theory* (Taylor & Francis, 1999).

[143] N.A. Obers and B. Pioline, "U duality and M theory," Phys. Rept. **318**, 113–225 (1999), arXiv:hep-th/9809039.

[144] N.A. Obers and B. Pioline, "Eisenstein series and string thresholds," Commun. Math. Phys. **209**, 275–324, arXiv:hep-th/9903113.

[145] M.B. Green, S.D. Miller, J.G. Russo, and P. Vanhove, "Eisenstein series for higher-rank groups and string theory amplitudes," Commun. Num. Theor. Phys. **4**, 551–596 (2010), arXiv:1004.0163 [hep-th].

[146] P. Fleig, H.P.A. Gustafsson, A. Kleinschmidt, and D. Persson, *Eisenstein series and automorphic representations*, Cambridge Studies in Advanced Mathematics, Vol. 176 (Cambridge University Press, Cambridge, 2018) arXiv:1511.04265.

[147] T. Hubsch, *Calabi-Yau manifolds: A Bestiary for physicists* (World Scientific, Singapore, 1994).

[148] R. Blumenhagen, B. Körs, D. Lüst, and S. Stieberger, "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes," Phys. Rept. **445**, 1–193 (2007), arXiv:hep-th/0610327 [hep-th].

[149] S.-T. Yau, ed., *Essays on Mirror Manifolds* (International Press, Hong-Kong, 1992).

[150] B. Greene and S.-T. Yau, eds., *Mirror symmetry II.* (Am. Math. Soc, Providence, 1997).

[151] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, Clay mathematics monographs, Vol. 1 (AMS, Providence, USA, 2003).

[152] C. Vafa and E. Witten, "A One loop test of string duality," Nucl. Phys. B **447**, 261–270 (1995), arXiv:hep-th/9505053.

[153] I. Antoniadis, S. Ferrara, R. Minasian, and K.S. Narain, "R^4 couplings in M and type II theories on Calabi-Yau spaces," Nucl. Phys. B **507**, 571–588 (1997), arXiv:hep-th/9707013.

[154] E. Kiritsis, N.A. Obers, and B. Pioline, "Heterotic / type II triality and instantons on K(3)," JHEP **01**, 029 (2000), arXiv:hep-th/0001083.

[155] J. Polchinski and E. Witten, "Evidence for heterotic - type I string duality," Nucl. Phys. **B460**, 525–540 (1996), arXiv:hep-th/9510169 [hep-th].

[156] A.A. Tseytlin, "On SO(32) heterotic type I superstring duality in ten-dimensions," Phys. Lett. B **367**, 84–90 (1996), arXiv:hep-th/9510173.

[157] A.A. Tseytlin, "Heterotic type I superstring duality and low-energy effective actions," Nucl. Phys. B **467**, 383–398 (1996), arXiv:hep-th/9512081.

[158] C. Bachas, C. Fabre, E. Kiritsis, N.A. Obers, and P. Vanhove, "Heterotic / type I duality and D-brane instantons," Nucl. Phys. B **509**, 33–52 (1998), arXiv:hep-th/9707126.

[159] S. Stieberger and T.R. Taylor, "NonAbelian Born-Infeld action and type I. Heterotic duality (1): Heterotic F^6 terms at two loops," Nucl. Phys. B **647**, 49–68 (2002), arXiv:hep-th/0207026.

[160] S. Stieberger and T.R. Taylor, "NonAbelian Born-Infeld action and type 1. - heterotic duality 2: Nonrenormalization theorems," Nucl. Phys. **B648**, 3–34 (2003), arXiv:hep-th/0209064 [hep-th].

[161] D.J. Gross and E. Witten, "Superstring Modifications of Einstein's Equations," Nucl. Phys. B **277**, 1 (1986).

[162] M.T. Grisaru, A.E.M. van de Ven, and D. Zanon, "Four Loop beta Function for the N=1 and N=2 Supersymmetric Nonlinear Sigma Model in Two-Dimensions," Phys. Lett. B **173**, 423–428 (1986).

[163] M.B. Green and M. Gutperle, "Effects of D instantons," Nucl. Phys. **B498**, 195–227 (1997), arXiv:hep-th/9701093.

[164] M.B. Green, M. Gutperle, and P. Vanhove, "One loop in eleven-dimensions," Phys. Lett. **B409**, 177–184 (1997), arXiv:hep-th/9706175.

[165] M.B. Green and S. Sethi, "Supersymmetry constraints on type IIB supergravity," Phys. Rev. D **59**, 046006 (1999), arXiv:hep-th/9808061.

[166] J. Broedel, O. Schlotterer, S. Stieberger, and T. Terasoma, "All order α'-expansion of superstring trees from the Drinfeld associator," Phys. Rev. **D89**, 066014 (2014), arXiv:1304.7304.

[167] T. Terasoma, "Selberg integrals and multiple zeta values," Compositio Math. **133**, 1–24 (2002).

[168] F. Brown, "Multiple zeta values and periods of moduli spaces $\mathcal{M}_{0,n}(\mathbb{R})$," Annales Sci. Ecole Norm. Sup. **42**, 371 (2009), arXiv:math/0606419 [math.AG].

[169] O. Schlotterer and S. Stieberger, "Motivic Multiple Zeta Values and Superstring Amplitudes," J. Phys. **A46**, 475401 (2013), arXiv:1205.1516.

[170] B. Balthazar, V.A. Rodriguez, and X. Yin, "ZZ Instantons and the Non-Perturbative Dual of c = 1 String Theory," (2019), arXiv:1907.07688 [hep-th].

[171] A. Sen, "D-instanton Perturbation Theory," JHEP **08**, 075 (2020), arXiv:2002.04043 [hep-th].

[172] A. Sen, "Cutkosky rules and unitarity (violation) in D-instanton amplitudes," JHEP **07**, 205 (2021), arXiv:2012.00041 [hep-th].

[173] A. Sen, "Normalization of D-instanton amplitudes," JHEP **11**, 077 (2021), arXiv:2101.08566 [hep-th].

[174] A. Sen, "Normalization of type IIB D-instanton amplitudes," JHEP **12**, 146 (2021), arXiv:2104.11109 [hep-th].

[175] A. Sen, "Muti-instanton amplitudes in type IIB string theory," JHEP **12**, 065 (2021), arXiv:2104.15110 [hep-th].

[176] S. Alexandrov, A. Sen, and B. Stefański, "D-instantons in Type IIA string theory on Calabi-Yau threefolds," JHEP **11**, 018 (2021), arXiv:2108.04265 [hep-th].

[177] S. Alexandrov, A. Sen, and B. Stefański, "Euclidean D-branes in type IIB string theory on Calabi-Yau threefolds," JHEP **12**, 044 (2021), arXiv:2110.06949 [hep-th].

[178] M.B. Green, H-h. Kwon, and P. Vanhove, "Two loops in eleven-dimensions," Phys. Rev. **D61**, 104010 (2000), arXiv:hep-th/9910055 [hep-th].

[179] M.B. Green, S.D. Miller, and P. Vanhove, "$SL(2, \mathbb{Z})$-invariance and D-instanton contributions to the $D^6 R^4$ interaction," Commun. Num. Theor. Phys. **09**, 307–344 (2015), arXiv:1404.2192 [hep-th].

[180] G. Bossard, A. Kleinschmidt, and B. Pioline, "1/8-BPS Couplings and Exceptional Automorphic Functions," SciPost Phys. **8**, 054 (2020), arXiv:2001.05562 [hep-th].

[181] M.B. Green and P. Vanhove, "The Low-energy expansion of the one loop type II superstring amplitude," Phys.Rev. **D61**, 104011 (2000), arXiv:hep-th/9910056.

[182] E. D'Hoker, M.B. Green, and P. Vanhove, "On the modular structure of the genus-one Type II superstring low energy expansion," JHEP **08**, 041 (2015), arXiv:1502.06698.

[183] E. D'Hoker and M.B. Green, "Exploring transcendentality in superstring amplitudes," JHEP **07**, 149 (2019), arXiv:1906.01652.

[184] D.M. Richards, "The One-Loop Five-Graviton Amplitude and the Effective Action," JHEP **10**, 042 (2008), arXiv:0807.2421 [hep-th].

[185] A. Basu, "Simplifying the one loop five graviton amplitude in type IIB string theory," Int. J. Mod. Phys. **A32**, 1750074 (2017), arXiv:1608.02056 [hep-th].

[186] E. D'Hoker, M. Gutperle, and D.H. Phong, "Two-loop superstrings and S-duality," Nucl. Phys. B **722**, 81–118 (2005), arXiv:hep-th/0503180.

[187] H. Gomez and C.R. Mafra, "The Overall Coefficient of the Two-loop Superstring Amplitude Using Pure Spinors," JHEP **05**, 017 (2010), arXiv:1003.0678 [hep-th].

[188] E. D'Hoker and M.B. Green, "Zhang-Kawazumi Invariants and Superstring Amplitudes," J. Number Theor. **144**, 111 (2014), arXiv:1308.4597 [hep-th].

[189] E. D'Hoker, M.B. Green, B. Pioline, and R. Russo, "Matching the $D^6 R^4$ interaction at two-loops," JHEP **01**, 031 (2015), arXiv:1405.6226.

[190] H. Gomez, C.R. Mafra, and O. Schlotterer, "Two-loop superstring five-point amplitude and S-duality," Phys. Rev. D **93**, 045030 (2016), arXiv:1504.02759 [hep-th].

[191] E. D'Hoker, C.R. Mafra, B. Pioline, and O. Schlotterer, "Two-loop superstring five-point amplitudes. Part II. Low energy expansion and S-duality," JHEP **02**, 139 (2021), arXiv:2008.08687.

[192] M.B. Green, J.G. Russo, and P. Vanhove, "Non-renormalisation conditions in type II string theory and maximal supergravity," JHEP **02**, 099 (2007), arXiv:hep-th/0610299 [hep-th].

[193] M.B. Green, J.G. Russo, and P. Vanhove, "Low energy expansion of the four-particle genus-one amplitude in type II superstring theory," JHEP **02**, 020 (2008), arXiv:0801.0322.

[194] M.B. Green, J.G. Russo, and P. Vanhove, "Modular properties of two-loop maximal supergravity and connections with string theory," JHEP **07**, 126 (2008), arXiv:0807.0389 [hep-th].

[195] E. D'Hoker, M.B. Green, and B. Pioline, "Higher genus modular graph functions, string invariants, and their exact asymptotics," Commun. Math. Phys. **366**, 927–979 (2019), arXiv:1712.06135.

[196] E. D'Hoker, M.B. Green, and B. Pioline, "Asymptotics of the $D^8\mathcal{R}^4$ genus-two string invariant," Commun. Num. Theor. Phys. **13**, 351–462 (2019), arXiv:1806.02691 [hep-th].

[197] P. Koerber and A. Sevrin, "The NonAbelian D-brane effective action through order α'^4," JHEP **10**, 046 (2002), arXiv:hep-th/0208044.

[198] L.A. Barreiro and R. Medina, "Revisiting the S-matrix approach to the open superstring low energy effective lagrangian," JHEP **10**, 108 (2012), arXiv:1208.6066 [hep-th].

[199] J.M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113–1133 (1999), [Adv. Theor. Math. Phys.2,231(1998)], arXiv:hep-th/9711200 [hep-th].

[200] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253–291 (1998), arXiv:hep-th/9802150.

[201] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105–114 (1998), arXiv:hep-th/9802109.

[202] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183–386 (2000), arXiv:hep-th/9905111 [hep-th].

[203] E. D'Hoker and D.Z. Freedman, "Supersymmetric gauge theories and the AdS / CFT correspondence," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and EXTRA Dimensions* (2002) pp. 3–158, arXiv:hep-th/0201253.

[204] N. Beisert *et al.*, "Review of AdS/CFT Integrability: An Overview," Lett. Math. Phys. **99**, 3–32 (2012), arXiv:1012.3982 [hep-th].

[205] M. Ammon and J. Erdmenger, *Gauge/gravity duality: Foundations and applications* (Cambridge University Press, Cambridge, 2015).

[206] H. Nastase, *Introduction to the ADS/CFT Correspondence* (Cambridge University Press, 2015).

[207] R. Gopakumar, E. Perlmutter, S.S. Pufu, and X. Yin, "Snowmass White Paper: Bootstrapping String Theory," (2022), arXiv:2202.07163 [hep-th].

[208] R.R. Metsaev and A.A. Tseytlin, "Type IIB superstring action in $AdS_5 \times S^5$ background," Nucl. Phys. B **533**, 109–126 (1998), arXiv:9805028.

[209] J.M. Maldacena and H. Ooguri, "Strings in AdS(3) and the SL(2,R) WZW model. Part 3. Correlation functions," Phys. Rev. D **65**, 106006 (2002), arXiv:hep-th/0111180.

[210] J.M. Maldacena, H. Ooguri, and J. Son, "Strings in AdS(3) and the SL(2,R) WZW model. Part 2. Euclidean black hole," J. Math. Phys. **42**, 2961–2977 (2001), arXiv:hep-th/0005183.

[211] J.M. Maldacena and H. Ooguri, "Strings in AdS(3) and SL(2,R) WZW model 1.: The Spectrum," J. Math. Phys. **42**, 2929–2960 (2001), arXiv:hep-th/0001053.

[212] M.R. Gaberdiel and S. Gerigk, "The massless string spectrum on $AdS_3 \times S^3$ from the supergroup," JHEP **10**, 045 (2011), arXiv:1107.2660 [hep-th].

[213] L. Mazzucato, "Superstrings in AdS," Phys. Rept. **521**, 1–68 (2012), arXiv:1104.2604 [hep-th].

[214] N. Berkovits, "Quantum consistency of the superstring in $AdS_5 \times S^5$ background," JHEP **03**, 041 (2005), arXiv:hep-th/0411170.

[215] R.R. Metsaev, "Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background," Nucl. Phys. B **625**, 70–96 (2002), arXiv:hep-th/0112044.

[216] A.A. Tseytlin, "Review of AdS/CFT Integrability, Chapter II.1: Classical $AdS_5 \times S^5$ string solutions," Lett. Math. Phys. **99**, 103–125 (2012), arXiv:1012.3986 [hep-th].

[217] T. McLoughlin, "Review of AdS/CFT Integrability, Chapter II.2: Quantum Strings in $AdS_5 \times S^5$," Lett. Math. Phys. **99**, 127–148 (2012), arXiv:1012.3987 [hep-th].

[218] M. Magro, "Review of AdS/CFT Integrability, Chapter II.3: Sigma Model, Gauge Fixing," Lett. Math. Phys. **99**, 149–167 (2012), arXiv:1012.3988 [hep-th].

[219] S. Schafer-Nameki, "Review of AdS/CFT Integrability, Chapter II.4: The Spectral Curve," Lett. Math. Phys. **99**, 169–190 (2012), arXiv:1012.3989 [hep-th].

[220] N. Berkovits, C. Vafa, and E. Witten, "Conformal field theory of AdS background with Ramond-Ramond flux," JHEP **03**, 018 (1999), arXiv:hep-th/9902098.

[221] L. Eberhardt, M.R. Gaberdiel, and R. Gopakumar, "Deriving the AdS$_3$/CFT$_2$ correspondence," JHEP **02**, 136 (2020), arXiv:1911.00378 [hep-th].

[222] M.R. Gaberdiel and R. Gopakumar, "The worldsheet dual of free super Yang-Mills in 4D," JHEP **11**, 129 (2021), arXiv:2105.10496 [hep-th].

[223] N. Berkovits, "Simplifying and Extending the $AdS_5 \times S^5$ Pure Spinor Formalism," JHEP **09**, 051 (2009), arXiv:0812.5074 [hep-th].

[224] N. Berkovits, "Sketching a Proof of the Maldacena Conjecture at Small Radius," JHEP **06**, 111 (2019), arXiv:1903.08264 [hep-th].

[225] N.A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," Adv. Theor. Math. Phys. **7**, 831–864 (2003), arXiv:hep-th/0206161.

[226] N. Nekrasov and A. Okounkov, "Seiberg-Witten theory and random partitions," Prog. Math. **244**, 525–596 (2006), arXiv:hep-th/0306238.

[227] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," Commun. Math. Phys. **313**, 71–129 (2012), arXiv:0712.2824 [hep-th].

[228] D.J. Binder, S.M. Chester, S.S. Pufu, and Y. Wang, "$\mathcal{N} = 4$ Super-Yang-Mills correlators at strong coupling from string theory and localization," JHEP **12**, 119 (2019), arXiv:1902.06263 [hep-th].

[229] S.M. Chester, "Genus-2 holographic correlator on AdS$_5 \times$ S^5 from localization," JHEP **04**, 193 (2020), arXiv:1908.05247 [hep-th].

[230] G. Mack, "D-independent representation of Conformal Field Theories in D dimensions via transformation to auxiliary Dual Resonance Models. Scalar amplitudes," (2009), arXiv:0907.2407 [hep-th].

[231] J. Penedones, "Writing CFT correlation functions as AdS scattering amplitudes," JHEP **03**, 025 (2011), arXiv:1011.1485 [hep-th].

[232] L.F. Alday, S.M. Chester, and T. Hansen, "Modular invariant holographic correlators for $\mathcal{N} = 4$ SYM with general gauge group," JHEP **12**, 159 (2021), arXiv:2110.13106 [hep-th].

[233] S.M. Chester, M.B. Green, S.S. Pufu, Y. Wang, and C. Wen, "Modular invariance in superstring theory from $\mathcal{N} = 4$ super-Yang-Mills," JHEP **11**, 016 (2020), arXiv:1912.13365.

[234] C. Montonen and D.I. Olive, "Magnetic Monopoles as Gauge Particles?" Phys. Lett. B **72**, 117–120 (1977).

[235] E. Witten and D.I. Olive, "Supersymmetry Algebras That Include Topological Charges," Phys. Lett. B **78**, 97–101 (1978).

[236] H. Osborn, "Topological Charges for N=4 Supersymmetric Gauge Theories and Monopoles of Spin 1," Phys. Lett. B **83**, 321–326 (1979).

[237] S.M. Chester and S.S. Pufu, "Far beyond the planar limit in strongly-coupled $\mathcal{N} = 4$ SYM," JHEP **01**, 103 (2021), arXiv:2003.08412 [hep-th].

[238] S.M. Chester, M.B. Green, S.S. Pufu, Y. Wang, and C. Wen, "New modular invariants in $\mathcal{N} = 4$ Super-Yang-Mills theory," JHEP **04**, 212 (2021), arXiv:2008.02713 [hep-th].

[239] D. Dorigoni, M.B. Green, and C. Wen, "Exact properties of an integrated correlator in $\mathcal{N} = 4$ SU(N) SYM," JHEP **05**, 089 (2021), arXiv:2102.09537 [hep-th].

[240] S. Collier and E. Perlmutter, "Harnessing S-Duality in $\mathcal{N} = 4$ SYM & Supergravity as $SL(2, \mathbb{Z})$-Averaged Strings," (2022), arXiv:2201.05093 [hep-th].

[241] D. Dorigoni, M.B. Green, and C. Wen, "Exact results for duality-covariant integrated correlators in $\mathcal{N} = 4$ SYM with general classical gauge groups," (2022), arXiv:2202.05784 [hep-th].

[242] P. Goddard, J. Nuyts, and D.I. Olive, "Gauge Theories and Magnetic Charge," Nucl. Phys. B **125**, 1–28 (1977).

[243] E. Witten, "Baryons and branes in anti-de Sitter space," JHEP **07**, 006 (1998), arXiv:hep-th/9805112.

[244] B.P. Abbott *et al.* (LIGO Scientific, Virgo), "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116**, 061102 (2016), arXiv:1602.03837 [gr-qc].

[245] B.P. Abbott *et al.* (LIGO Scientific, Virgo), "GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral," Phys. Rev. Lett. **119**, 161101 (2017), arXiv:1710.05832 [gr-qc].

[246] A. Luna, R. Monteiro, I. Nicholson, D. O'Connell, and C.D. White, "The double copy: Bremsstrahlung and accelerating black holes," JHEP **06**, 023 (2016), arXiv:1603.05737 [hep-th].

[247] N.E.J. Bjerrum-Bohr, P.H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, "General Relativity from Scattering Amplitudes," Phys. Rev. Lett. **121**, 171601 (2018), arXiv:1806.04920 [hep-th].

[248] C-H. Shen, "Gravitational Radiation from Color-Kinematics Duality," JHEP **11**, 162 (2018), arXiv:1806.07388 [hep-th].

[249] C. Cheung, I.Z. Rothstein, and M.P. Solon, "From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion," Phys. Rev. Lett. **121**, 251101 (2018), arXiv:1808.02489 [hep-th].

[250] D.A. Kosower, B. Maybee, and D. O'Connell, "Amplitudes, Observables, and Classical Scattering," JHEP **02**, 137 (2019), arXiv:1811.10950 [hep-th].

[251] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M.P. Solon, and M. Zeng, "Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order," Phys. Rev. Lett. **122**, 201603 (2019), arXiv:1901.04424 [hep-th].

[252] J. Blümlein, A. Maier, and P. Marquard, "Five-Loop Static Contribution to the Gravitational Interaction Potential of Two Point Masses," Phys. Lett. B **800**, 135100 (2020), arXiv:1902.11180 [gr-qc].

[253] B. Maybee, D. O'Connell, and J. Vines, "Observables and amplitudes for spinning particles and black holes," JHEP **12**, 156 (2019), arXiv:1906.09260 [hep-th].

[254] H. Johansson and A. Ochirov, "Double copy for massive quantum particles with spin," JHEP **09**, 040 (2019), arXiv:1906.12292 [hep-th].

[255] P.H. Damgaard, K. Haddad, and A. Helset, "Heavy Black Hole Effective Theory," JHEP **11**, 070 (2019), arXiv:1908.10308 [hep-ph].

[256] Z. Bern, C. Cheung, R. Roiban, C-H. Shen, M.P. Solon, and M. Zeng, "Black Hole Binary Dynamics from the Double Copy and Effective Theory," JHEP **10**, 206 (2019), arXiv:1908.01493 [hep-th].

[257] G. Kälin and R.A. Porto, "From Boundary Data to Bound States," JHEP **01**, 072 (2020), arXiv:1910.03008 [hep-th].

[258] G. Kälin and R.A. Porto, "From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist)," JHEP **02**, 120 (2020), arXiv:1911.09130 [hep-th].

[259] M. Levi, A.J. Mcleod, and M. Von Hippel, "N^{3}LO gravitational spin-orbit coupling at order G^4," JHEP **07**, 115 (2021), arXiv:2003.02827 [hep-th].

[260] M-Z. Chung, Y-t. Huang, J-W. Kim, and S. Lee, "Complete Hamiltonian for spinning binary systems at first post-Minkowskian order," JHEP **05**, 105 (2020), arXiv:2003.06600 [hep-th].

[261] C. Cheung and M.P. Solon, "Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams," JHEP **06**, 144 (2020), arXiv:2003.08351 [hep-th].

[262] C. Cheung and M.P. Solon, "Tidal Effects in the Post-Minkowskian Expansion," Phys. Rev. Lett. **125**, 191601 (2020), arXiv:2006.06665 [hep-th].

[263] A. Cristofoli, P.H. Damgaard, P. Di Vecchia, and C. Heissenberg, "Second-order Post-Minkowskian scattering in arbitrary dimensions," JHEP **07**, 122 (2020), arXiv:2003.10274 [hep-th].

[264] Z. Bern, A. Luna, R. Roiban, C-H. Shen, and M. Zeng, "Spinning black hole binary dynamics, scattering amplitudes, and effective field theory," Phys. Rev. D **104**, 065014 (2021), arXiv:2005.03071 [hep-th].

[265] G. Kälin and R.A. Porto, "Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics," JHEP **11**, 106 (2020), arXiv:2006.01184 [hep-th].

[266] G. Kälin, Z. Liu, and R.A. Porto, "Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach," Phys. Rev. Lett. **125**, 261103 (2020), arXiv:2007.04977 [hep-th].

[267] M. Levi and F. Teng, "NLO gravitational quartic-in-spin interaction," JHEP **01**, 066 (2021), arXiv:2008.12280 [hep-th].

[268] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, "Universality of ultra-relativistic gravitational scattering," Phys. Lett. B **811**, 135924 (2020), arXiv:2008.12743 [hep-th].

[269] Z. Bern, J. Parra-Martinez, R. Roiban, E. Sawyer, and C-H. Shen, "Leading Nonlinear Tidal Effects and Scattering Amplitudes," JHEP **05**, 188 (2021), arXiv:2010.08559 [hep-th].

[270] G. Mogull, J. Plefka, and J. Steinhoff, "Classical black hole scattering from a worldline quantum field theory," JHEP **02**, 048 (2021), arXiv:2010.02865 [hep-th].

[271] M.A. Huber, A. Brandhuber, S. De Angelis, and G. Travaglini, "From amplitudes to gravitational radiation with cubic interactions and tidal effects," Phys. Rev. D **103**, 045015 (2021), arXiv:2012.06548 [hep-th].

[272] Z. Bern, J. Parra-Martinez, R. Roiban, M.S. Ruf, C.-H. Shen, M.P. Solon, and M. Zeng, "Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$," Phys. Rev. Lett. **126**, 171601 (2021), arXiv:2101.07254 [hep-th].

[273] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, "The eikonal approach to gravitational scattering and radiation at $\mathcal{O}(\mathrm{G}^3)$," JHEP **07**, 169 (2021), arXiv:2104.03256 [hep-th].

[274] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, "A new gauge-invariant double copy for heavy-mass effective theory," JHEP **07**, 047 (2021), arXiv:2104.11206 [hep-th].

[275] C. Dlapa, G. Kälin, Z. Liu, and R.A. Porto, "Dynamics of Binary Systems to Fourth Post-Minkowskian Order from the Effective Field Theory Approach," (2021), arXiv:2106.08276 [hep-th].

[276] P.H. Damgaard, L. Plante, and P. Vanhove, "On an exponential representation of the gravitational S-matrix," JHEP **11**, 213 (2021), arXiv:2107.12891 [hep-th].

[277] A. Cristofoli, R. Gonzo, D.A. Kosower, and D. O'Connell, "Waveforms from Amplitudes," (2021), arXiv:2107.10193 [hep-th].

[278] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, "Classical gravitational scattering from a gauge-invariant double copy," JHEP **10**, 118 (2021), arXiv:2108.04216 [hep-th].

[279] W-M. Chen, M-Z. Chung, Y-t. Huang, and J-W. Kim, "The 2PM Hamiltonian for binary Kerr to quartic in spin," (2021), arXiv:2111.13639 [hep-th].

[280] J-W. Kim, M. Levi, and Z. Yin, "Quadratic-in-spin interactions at fifth post-Newtonian order probe new physics," (2021), arXiv:2112.01509 [hep-th].

[281] G. Cho, G. Kälin, and R.A. Porto, "From Boundary Data to Bound States III: Radiative Effects," (2021), arXiv:2112.03976 [hep-th].

[282] Z. Bern, J. Parra-Martinez, R. Roiban, M.S. Ruf, C-H. Shen, M.P. Solon, and M. Zeng, "Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at $O(G^4)$," (2021), arXiv:2112.10750 [hep-th].

[283] G.U. Jakobsen and G. Mogull, "Conservative and radiative dynamics of spinning bodies at third post-Minkowskian order using worldline quantum field theory," (2022), arXiv:2201.07778 [hep-th].

[284] G. Veneziano and G.A. Vilkovisky, "Angular momentum loss in gravitational scattering, radiation reaction, and the Bondi gauge ambiguity," (2022), arXiv:2201.11607 [gr-qc].

[285] A. Edison and M. Levi, "A tale of tails through generalized unitarity," (2022), arXiv:2202.04674 [hep-th].

[286] A.V. Manohar, A.K. Ridgway, and C-H. Shen, "Radiated Angular Momentum and Dissipative Effects in Classical Scattering," (2022), arXiv:2203.04283 [hep-th].

[287] A. Buonanno, M. Khalil, D. O'Connell, R. Roiban, M.P. Solon, and M. Zeng, "Snowmass White Paper: Gravitational Waves and Scattering Amplitudes," (2022) arXiv:2204.05194 [hep-th].

[288] Z. Bern, Lance J. Dixon, D. C. Dunbar, and D. A. Kosower, "One loop n point gauge theory amplitudes, unitarity and collinear limits," Nucl. Phys. **B425**, 217–260 (1994), arXiv:hep-ph/9403226 [hep-ph].

[289] Z. Bern, Lance J. Dixon, D. C. Dunbar, and D. A. Kosower, "Fusing gauge theory tree amplitudes into loop amplitudes," Nucl. Phys. **B435**, 59–101 (1995), arXiv:hep-ph/9409265 [hep-ph].

[290] R. Britto, F. Cachazo, and B. Feng, "Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills," Nucl. Phys. B **725**, 275–305 (2005), arXiv:hep-th/0412103.

[291] M. Ciafaloni, D. Colferai, and G. Veneziano, "Infrared features of gravitational scattering and radiation in the eikonal approach," Phys. Rev. D **99**, 066008 (2019), arXiv:1812.08137 [hep-th].

[292] D. Amati, M. Ciafaloni, and G. Veneziano, "Superstring Collisions at Planckian Energies," Phys. Lett. B **197**, 81 (1987).

[293] D. Amati, M. Ciafaloni, and G. Veneziano, "Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions," Int. J. Mod. Phys. A **3**, 1615–1661 (1988).

[294] S. Weinberg, "Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass," Phys. Rev. **135**, B1049–B1056 (1964).

[295] Y.B. Zel'dovich and A.G. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars," Sov. Astron. **18**, 17 (1974).

[296] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge Theory," (2017), arXiv:1703.05448 [hep-th].

[297] A.-M. Raclariu, "Lectures on Celestial Holography," (2021), arXiv:2107.02075 [hep-th].

[298] S. Pasterski, "Lectures on celestial amplitudes," Eur. Phys. J. C **81**, 1062 (2021), arXiv:2108.04801 [hep-th].

[299] S. Pasterski, M. Pate, and A-M. Raclariu, "Celestial Holography," in *2022 Snowmass Summer Study* (2021) arXiv:2111.11392 [hep-th].

[300] A. Laddha and A. Sen, "Gravity Waves from Soft Theorem in General Dimensions," JHEP **09**, 105 (2018), arXiv:1801.07719 [hep-th].

[301] A. Addazi, M. Bianchi, and G. Veneziano, "Soft gravitational radiation from ultra-relativistic collisions at sub- and sub-sub-leading order," JHEP **05**, 050 (2019), arXiv:1901.10986 [hep-th].

[302] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, "Radiation Reaction from Soft Theorems," Phys. Lett. B **818**, 136379 (2021), arXiv:2101.05772 [hep-th].

[303] W.D. Goldberger and I.Z. Rothstein, "An Effective field theory of gravity for extended objects," Phys. Rev. D **73**, 104029 (2006), arXiv:hep-th/0409156.

[304] R.A. Porto, "Post-Newtonian corrections to the motion of spinning bodies in NRGR," Phys. Rev. D **73**, 104031 (2006), arXiv:gr-qc/0511061.

[305] N. Siemonsen and J. Vines, "Test black holes, scattering amplitudes and perturbations of Kerr spacetime," Phys. Rev. D **101**, 064066 (2020), arXiv:1909.07361 [gr-qc].

[306] N. Arkani-Hamed, T.-C. Huang, and Y. Huang, "Scattering amplitudes for all masses and spins," JHEP **11**, 070 (2021), arXiv:1709.04891 [hep-th].

[307] A. Guevara, A. Ochirov, and J. Vines, "Scattering of Spinning Black Holes from Exponentiated Soft Factors," JHEP **09**, 056 (2019), arXiv:1812.06895 [hep-th].

[308] N. Arkani-Hamed, Y. Huang, and D. O'Connell, "Kerr black holes as elementary particles," JHEP **01**, 046 (2020), arXiv:1906.10100 [hep-th].

[309] M. Chiodaroli, H. Johansson, and P. Pichini, "Compton black-hole scattering for s $\leq$ 5/2," JHEP **02**, 156 (2022), arXiv:2107.14779 [hep-th].

[310] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," Phys. Lett. B **379**, 99–104 (1996), arXiv:hep-th/9601029.

[311] M. Guica, T. Hartman, W. Song, and A. Strominger, "The Kerr/CFT Correspondence," Phys. Rev. D **80**, 124008 (2009), arXiv:0809.4266 [hep-th].

[312] S.W. Hawking, "Particle Creation by Black Holes," Commun. Math. Phys. **43**, 199–220 (1975), [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[313] W.D. Goldberger and I.Z. Rothstein, "Virtual Hawking Radiation," Phys. Rev. Lett. **125**, 211301 (2020), arXiv:2007.00726 [hep-th].

[314] J.M. Drummond and E. Ragoucy, "Superstring amplitudes and the associator," JHEP **08**, 135 (2013), arXiv:1301.0794.

[315] O. Schnetz, "Graphical functions and single-valued multiple polylogarithms," Commun. Num. Theor. Phys. **08**, 589–675 (2014), arXiv:1302.6445.

[316] F. Brown, "Notes on motivic periods," Communications in Number Theory and Physics **11**, 557–655 (2015), arXiv:1512.06410 [math.NT].

[317] E. Panzer and O. Schnetz, "The Galois coaction on ϕ^4 periods," Commun. Num. Theor. Phys. **11**, 657–705 (2017), arXiv:1603.04289 [hep-th].

[318] O. Schnetz, "The Galois coaction on the electron anomalous magnetic moment," Commun. Num. Theor. Phys. **12**, 335–354 (2018), arXiv:1711.05118 [math-ph].

[319] S. Caron-Huot, L.J. Dixon, F. Dulat, M. Von Hippel, A.J. McLeod, and G. Papathanasiou, "The Cosmic Galois Group and Extended Steinmann Relations for Planar $\mathcal{N} = 4$ SYM Amplitudes," JHEP **09**, 061 (2019), arXiv:1906.07116 [hep-th].

[320] S. Abreu, R. Britto, C. Duhr, and E. Gardi, "Diagrammatic Hopf algebra of cut Feynman integrals: the one-loop case," JHEP **12**, 090 (2017), arXiv:1704.07931 [hep-th].

[321] F. Brown and C. Dupont, "Lauricella hypergeometric functions, unipotent fundamental groups of the punctured Riemann sphere, and their motivic coactions," (2019), arXiv:1907.06603 [math.AG].

[322] S. Abreu, R. Britto, C. Duhr, E. Gardi, and J. Matthew, "The diagrammatic coaction beyond one loop," JHEP **10**, 131 (2021), arXiv:2106.01280 [hep-th].

[323] S. Stieberger, "Closed superstring amplitudes, single-valued multiple zeta values and the Deligne associator," J. Phys. **A47**, 155401 (2014), arXiv:1310.3259 [hep-th].

[324] S. Stieberger and T.R. Taylor, "Closed String Amplitudes as Single-Valued Open String Amplitudes," Nucl. Phys. **B881**, 269–287 (2014), arXiv:1401.1218.

[325] O. Schlotterer and O. Schnetz, "Closed strings as single-valued open strings: A genus-zero derivation," J. Phys. **A52**, 045401 (2019), arXiv:1808.00713.

[326] P. Vanhove and F. Zerbini, "Single-valued hyperlogarithms, correlation functions and closed string amplitudes," (2018), arXiv:1812.03018 [hep-th].

[327] F. Brown and C. Dupont, "Single-valued integration and superstring amplitudes in genus zero," Commun. Math. Phys. **382**, 815–874 (2021), arXiv:1910.01107 [math.NT].

[328] F. Brown, "Single-valued Motivic Periods and Multiple Zeta Values," SIGMA **2**, e25 (2014), arXiv:1309.5309 [math.NT].

[329] F. Zerbini, "Single-valued multiple zeta values in genus 1 superstring amplitudes," Commun. Num. Theor. Phys. **10**, 703–737 (2016), arXiv:1512.05689.

[330] E. D'Hoker, M.B. Green, Ö. Gürdogan, and P. Vanhove, "Modular graph functions," Commun. Num. Theor. Phys. **11**, 165–218 (2017), arXiv:1512.06779.

[331] J. Broedel, O. Schlotterer, and F. Zerbini, "From elliptic multiple zeta values to modular graph functions: open and closed strings at one loop," JHEP **01**, 155 (2019), arXiv:1803.00527.

[332] D. Zagier and F. Zerbini, "Genus-zero and genus-one string amplitudes and special multiple zeta values," Commun. Num. Theor. Phys. **14**, 413–452 (2020), arXiv:1906.12339.

[333] J.E. Gerken, A. Kleinschmidt, C.R. Mafra, O. Schlotterer, and B. Verbeek, "Towards closed strings as single-valued open strings at genus one," (2020), arXiv:2010.10558.

[334] S. Mizera, "Combinatorics and Topology of Kawai-Lewellen-Tye Relations," JHEP **08**, 097 (2017), arXiv:1706.08527 [hep-th].

[335] K. Cho and K. Matsumoto, "Intersection theory for twisted cohomologies and twisted Riemann's period relations I," Nagoya Math. J. **139**, 67–86 (1995).

[336] Y-t. Huang, W. Siegel, and E.Y. Yuan, "Factorization of Chiral String Amplitudes," JHEP **09**, 101 (2016), arXiv:1603.02588 [hep-th].

[337] S. Mizera, "Scattering Amplitudes from Intersection Theory," Phys. Rev. Lett. **120**, 141602 (2018), arXiv:1711.00469 [hep-th].

[338] E. Casali and P. Tourkine, "On the null origin of the ambitwistor string," JHEP **11**, 036 (2016), arXiv:1606.05636 [hep-th].

[339] S. Mizera, *Aspects of Scattering Amplitudes and Moduli Space Localization*, Ph.D. thesis, Perimeter Inst. Theor. Phys. (2019), arXiv:1906.02099 [hep-th].

[340] S. Stieberger, "Open & Closed vs. Pure Open String One-Loop Amplitudes," (2021), arXiv:2105.06888 [hep-th].

[341] S. Stieberger and T.R. Taylor, "Disk Scattering of Open and Closed Strings (I)," Nucl. Phys. B **903**, 104–117 (2016), arXiv:1510.01774 [hep-th].

[342] P. Mastrolia and S. Mizera, "Feynman Integrals and Intersection Theory," JHEP **02**, 139 (2019), arXiv:1810.03818 [hep-th].

[343] S. Abreu, R. Britto, C. Duhr, E. Gardi, and J. Matthew, "From positive geometries to a coaction on hypergeometric functions," JHEP **02**, 122 (2020), arXiv:1910.08358 [hep-th].

[344] S. Mizera, "Status of Intersection Theory and Feynman Integrals," PoS **MA2019**, 016 (2019), arXiv:2002.10476 [hep-th].

[345] H. Frellesvig, F. Gasparotto, S. Laporta, M.K. Mandal, P. Mastrolia, L. Mattiazzi, and S. Mizera, "Decomposition of Feynman Integrals by Multivariate Intersection Numbers," JHEP **03**, 027 (2021), arXiv:2008.04823 [hep-th].

[346] F. Brown and A. Levin, "Multiple elliptic polylogarithms," (2011), arXiv:1110.6917 [math.NT].

[347] J. Broedel, N. Matthes, G. Richter, and O. Schlotterer, "Twisted elliptic multiple zeta values and non-planar one-loop open-string amplitudes," J. Phys. **A51**, 285401 (2018), arXiv:1704.03449 [hep-th].

[348] B. Enriquez, "Analogues elliptiques des nombres multizétas," Bull. Soc. Math. France **144**, 395–427 (2016), 1301.3042.

[349] C.R. Mafra and O. Schlotterer, "One-loop open-string integrals from differential equations: all-order α'-expansions at n points," JHEP **03**, 007 (2020), arXiv:1908.10830.

[350] J. Broedel and A. Kaderli, "Amplitude recursions with an extra marked point," Commun. Num. Theor. Phys. **16**, 75–158 (2022), arXiv:1912.09927 [hep-th].

[351] E. Panzer, "Talk on modular graph functions as iterated Eisenstein integrals," *Elliptic Integrals in Mathematics and Physics*, Ascona, Switzerland (2018).

[352] J. Broedel and A. Kaderli, "Functional relations for elliptic polylogarithms," J. Phys. A **53**, 245201 (2020), arXiv:1906.11857 [hep-th].

[353] L. Adams and S. Weinzierl, "Feynman integrals and iterated integrals of modular forms," Commun. Num. Theor. Phys. **12**, 193–251 (2018), arXiv:1704.08895.

[354] J. Broedel, C. Duhr, F. Dulat, and L. Tancredi, "Elliptic polylogarithms and iterated integrals on elliptic curves. Part I: general formalism," JHEP **05**, 093 (2018), arXiv:1712.07089.

[355] J.L. Bourjaily *et al.*, "Functions Beyond Multiple Polylogarithms for Precision Collider Physics," (2022) arXiv:2203.07088 [hep-ph].

[356] A.V. Kotikov, "Differential equations method: New technique for massive Feynman diagrams calculation," Phys. Lett. **B254**, 158–164 (1991).

[357] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, and J. Trnka, "Local Integrals for Planar Scattering Amplitudes," JHEP **06**, 125 (2012), arXiv:1012.6032 [hep-th].

[358] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, "Elliptic Feynman integrals and pure functions," JHEP **01**, 023 (2019), arXiv:1809.10698 [hep-th].

[359] J.M. Henn, "Multiloop integrals in dimensional regularization made simple," Phys. Rev. Lett. **110**, 251601 (2013), arXiv:1304.1806.

[360] L. Adams and S. Weinzierl, "The ε-form of the differential equations for Feynman integrals in the elliptic case," Phys. Lett. **B781**, 270–278 (2018), arXiv:1802.05020.

[361] E. D'Hoker and N. Geiser, "Integrating three-loop modular graph functions and transcendentality of string amplitudes," JHEP **02**, 019 (2022), arXiv:2110.06237 [hep-th].

[362] A. Basu, "Transcendentality violation in type IIB string amplitudes," JHEP **02**, 034 (2020), arXiv:1912.12661 [hep-th].

[363] D.C. Dunbar and P.S. Norridge, "Calculation of graviton scattering amplitudes using string based methods," Nucl. Phys. **B433**, 181–208 (1995), arXiv:hep-th/9408014 [hep-th].

[364] C. Boucher-Veronneau and L.J. Dixon, "$\mathcal{N} \geq 4$ Supergravity Amplitudes from Gauge Theory at Two Loops," JHEP **12**, 046 (2011), arXiv:1110.1132 [hep-th].

[365] H. Johansson and A. Ochirov, "Pure Gravities via Color-Kinematics Duality for Fundamental Matter," JHEP **11**, 046 (2015), arXiv:1407.4772 [hep-th].

[366] M. Leoni, A. Mauri, and A. Santambrogio, "On the amplitude/Wilson loop duality in N=2 SCQCD," Phys. Lett. B **747**, 325–330 (2015), arXiv:1502.07614 [hep-th].

[367] C. Duhr, H. Johansson, G. Kälin, G. Mogull, and B. Verbeek, "Full-Color Two-Loop Four-Gluon Amplitude in $\mathcal{N}{=}2$ Supersymmetric QCD," Phys. Rev. Lett. **123**, 241601 (2019), arXiv:1904.05299 [hep-th].

[368] G. Kälin, G. Mogull, A. Ochirov, and B. Verbeek, "Infrared and transcendental structure of two-loop supersymmetric QCD amplitudes," JHEP **01**, 068 (2020), arXiv:1911.10218 [hep-th].

[369] E. D'Hoker and M.B. Green, "Identities between modular graph forms," J. Number Theory **189**, 25–80 (2018), arXiv:1603.00839.

[370] E. D'Hoker and J. Kaidi, "Hierarchy of Modular Graph Identities," JHEP **11**, 051 (2016), arXiv:1608.04393.

[371] E. D'Hoker and J. Kaidi, "Modular graph functions and odd cuspidal functions. Fourier and Poincaré series," JHEP **04**, 136 (2019), arXiv:1902.04180.

[372] J.E. Gerken, "Modular Graph Forms and Scattering Amplitudes in String Theory," (2020), arXiv:2011.08647.

[373] J.E. Gerken, "Basis Decompositions and a Mathematica Package for Modular Graph Forms," J. Phys. A **54**, 195401 (2021), arXiv:2007.05476 [hep-th].

[374] A. Basu, "Proving relations between modular graph functions," Class. Quant. Grav. **33**, 235011 (2016), arXiv:1606.07084.

[375] A. Basu, "Low momentum expansion of one loop amplitudes in heterotic string theory," JHEP **11**, 139 (2017), arXiv:1708.08409 [hep-th].

[376] J.E. Gerken, A. Kleinschmidt, and O. Schlotterer, "Heterotic-string amplitudes at one loop: modular graph forms and relations to open strings," JHEP **01**, 052 (2019), arXiv:1811.02548 [hep-th].

[377] A. Basu, "Eigenvalue equation for the modular graph $C_{a,b,c,d}$," JHEP **07**, 126 (2019), arXiv:1906.02674.

[378] E. D'Hoker and W. Duke, "Fourier series of modular graph functions," J. Number Theory **192**, 1–36 (2018), arXiv:1708.07998 [math.NT].

[379] F. Brown, "A class of non-holomorphic modular forms I," Res. Math. Sci. **5**, 5:7 (2018), arXiv:1707.01230.

[380] F. Brown, "A class of non-holomorphic modular forms II: equivariant iterated Eisenstein integrals," Forum of Mathematics, Sigma **8**, 1 (2020), arXiv:1708.03354.

[381] M. Berg, K. Bringmann, and T. Gannon, "Massive deformations of Maass forms and Jacobi forms," Commun. Num. Theor. Phys. **15**, 575–603 (2021), arXiv:1910.02745 [math.NT].

[382] J.E. Gerken, A. Kleinschmidt, and O. Schlotterer, "Generating series of all modular graph forms from iterated Eisenstein integrals," JHEP **07**, 190 (2020), arXiv:2004.05156 [hep-th].

[383] J.E. Gerken and J. Kaidi, "Holomorphic subgraph reduction of higher-point modular graph forms," JHEP **01**, 131 (2019), arXiv:1809.05122 [hep-th].

[384] D. Dorigoni and A. Kleinschmidt, "Modular graph functions and asymptotic expansions of Poincaré series," Commun. Num. Theor. Phys. **13**, 569–617 (2019), arXiv:1903.09250.

[385] D. Dorigoni, A. Kleinschmidt, and O. Schlotterer, "Poincaré series for modular graph forms at depth two. Part I. Seeds and Laplace systems," JHEP **01**, 133 (2022), arXiv:2109.05017 [hep-th].

[386] A. Basu, "Poisson equation for genus two string invariants: a conjecture," JHEP **04**, 050 (2021), arXiv:2101.04597 [hep-th].

[387] E. D'Hoker and O. Schlotterer, "Identities among higher genus modular graph tensors," Commun. Num. Theor. Phys. **16**, 35–74 (2022), arXiv:2010.00924 [hep-th].

[388] N. Kawazumi, "Some tensor field on the Teichmüller space," *Lecture at* MCM2016, OIST (2016).

[389] N. Kawazumi, "Differential forms and functions on the moduli space of Riemann surfaces," *Séminaire Algèbre et topologie*, Université de Strasbourg (2017).

[390] B. Pioline, "A Theta lift representation for the Kawazumi-Zhang and Faltings invariants of genus-two Riemann surfaces," J. Number Theory **163**, 520 (2016), arXiv:1504.04182.