

Fault Isolation of a Class of Uncertain Nonlinear Parabolic PDE Systems

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Abstract: This paper proposes a novel fault isolation (FI) scheme for distributed parameter systems modeled by a class of parabolic partial differential equations (PDEs) with nonlinear uncertain dynamics. A key feature of the proposed FI scheme is its capability of dealing with the effects of system uncertainties for accurate FI. Specifically, an approximate ordinary differential equation (ODE) system is first derived to capture the dominant dynamics of the original PDE system. An adaptive dynamics identification approach using radial basis function neural network is then proposed based on this ODE system, to achieve locally-accurate identification of the uncertain system dynamics under faulty modes. A bank of FI estimators with associated adaptive thresholds are finally designed for real-time FI decision making. Rigorous analysis on the fault isolatability is provided. Simulation study on a representative transport-reaction process is conducted to demonstrate the effectiveness of the proposed approach.

Keywords: Partial differential equations, fault isolation, adaptive dynamics identification, neural network, distributed parameter systems.

1. INTRODUCTION

Many advanced engineering systems, e.g., biological process, convection-diffusion-reaction process and thermal process, are distributed parameter systems (DPSs) that are usually modeled by partial differential equations (PDEs). Due to the ever-increasing technical demands, fault diagnosis of DPSs has been an area of significantly growing interests. It is a critical step to realize fault tolerant operations for minimizing performance degradation and avoiding dangerous situations, so as to guarantee safety and reliability of DPSs. The past decades have witnessed tremendous progress in the research of fault diagnosis of DPSs, leading to a large variety of methods, see, e.g., Demetriou and Armaou (2012); Cai et al. (2016); Fischer and Deutscher (2020) and the references therein.

As opposed to the substantially growing body of literature on fault detection (FD) of DPSs (e.g., Demetriou and Armaou (2012); Cai et al. (2016); Dey et al. (2019); Zhang et al. (2020)), study on the fault isolation (FI) problem has gained quite limited success, especially for those DPSs with nonlinear unstructured uncertain dynamics. Some research efforts have been devoted to the development of FI methods for DPSs with precisely known models. For example, Baniamernian and Khorasani (2012) proposed an FI scheme using a finite-dimensional geometric approach. In El-Farra and Ghasemlou (2007), the FI problem for DPSs with various actuator faults has been investigated. For the

FI problem of DPSs with nonlinear uncertain dynamics, the research is still under-explored. One of the technical difficulties is that the dynamics of faults occurring in the system could be hidden within the system's general uncertain dynamics (e.g., unmodeled uncertainties), such that the fault feature could not be accurately identified for FI purpose. Some attempts have been made to overcome this difficulty. The FI scheme proposed in Cai and Jagannathan (2016) is able to distinguish the effects between occurring fault and system uncertainties. Ghasemlou and El-Farra (2009, 2007) developed a Lyapunov function-based FI scheme for DPSs, in which system uncertainties were handled by active control strategies. However, all these existing schemes have not appropriately dealt with the system uncertainties in the sense that occurring faults are typically required to have sufficiently large magnitudes (e.g., larger than those of the system uncertainties), limiting their wider applicability in practice.

In this paper, we aim to investigate an effective FI approach for general faults occurring in DPSs modeled by a class of parabolic PDEs with nonlinear unstructured uncertain dynamics. Specifically, we first utilize the Galerkin method (see, El-Farra and Ghasemlou (2007); Wu and Li (2008)) to derive an approximate ODE model for capturing the dominant dynamics of the PDE system. A radial basis function neural network (RBF NN) based adaptive dynamics identification approach is then developed based

on this ODE system, to accurately identify the uncertain system dynamics under various faulty modes. The identified knowledge can be obtained and stored in constant RBF NN models. Afterwards, a bank of FI estimators are designed with these constant models. Their generated residuals can be used to characterize the dynamics of the occurring fault and distinguish it from the system uncertain dynamics for accurate FI. Adaptive thresholds associated with such FI residuals are further designed to facilitate real-time FI decision making. We stress that our FI scheme does not require the faults to be of any special type (e.g., actuator faults as required in Ghantasala and El-Farra (2009, 2007); Baniamernian and Khorasani (2012) and/or sensor faults as required in Cai et al. (2016); Cai and Jagannathan (2016)), but is applicable to general system faults. Rigorous analysis on the fault isolatability is provided. Extensive simulations applied to a representative transport-reaction process are also conducted to demonstrate the effectiveness of our methodologies.

The contributions of this paper are summarized as follows: (i) The FI problem for uncertain nonlinear parabolic PDE systems is addressed using a novel Galerkin-RBFNN-based adaptive dynamics identification approach, which can achieve locally-accurate identification of the dominant uncertain dynamics of the PDE system under various faulty modes; (ii) New adaptive-threshold-based FI decision making schemes are proposed, which are capable of dealing with general faults occurring in parabolic PDE systems; (iii) Rigorous analysis on the fault isolatability is provided to demonstrate the effectiveness of the proposed approaches.

The remainder of this paper is organized as follows. Some preliminaries and the problem statement are provided in Section 2. The adaptive dynamics learning approach is presented in Section 3. The FI scheme is given in Section 4. Simulation studies are presented in Section 5. Section 6 concludes the paper.

Notation. $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}_+$ denote, respectively, the set of real numbers, the set of positive real numbers and the set of positive integers; $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrices; $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors; $|\cdot|$ is the absolute value of a real number; $\|\cdot\|$ is the 2-norm of a vector or a matrix, i.e. $\|x\| = (x^\top x)^{\frac{1}{2}}$.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Radial Basis Function Neural Networks

The RBF NNs can be described by $f_{nn}(Z) = \sum_{i=1}^{N_n} w_i s_i(Z) = W^\top S(Z)$ (see, Poewell (1992)), where $Z \in \Omega_Z \subset \mathbb{R}^q$ is the input vector, $W = [w_1, \dots, w_{N_n}]^\top \in \mathbb{R}^{N_n}$ is the weight vector, N_n is the NN node number, and $S(Z) = [s_1(\|Z - \varsigma_1\|), \dots, s_{N_n}(\|Z - \varsigma_{N_n}\|)]^\top$, with $s_i(\cdot)$ being a radial basis function (RBF), and ς_i ($i = 1, 2, \dots, N_n$) being distinct points in state space. The Gaussian function $s_i(\|Z - \varsigma_i\|) = \exp[-\frac{(Z - \varsigma_i)^\top (Z - \varsigma_i)}{\nu_i^2}]$ is one of the most commonly used RBFs, where $\varsigma_i = [\varsigma_{i1}, \varsigma_{i2}, \dots, \varsigma_{iq}]^\top$ is the center of the receptive field and ν_i is the width of the receptive field. The Gaussian function belongs to the class of localized RBFs in the sense that $s_i(\|Z - \varsigma_i\|) \rightarrow 0$ as $\|Z\| \rightarrow \infty$. It has been shown in Poewell (1992) that for

any continuous function $f(Z) : \Omega_Z \rightarrow \mathbb{R}$ where $\Omega_Z \subset \mathbb{R}^q$ is a compact set, and for the NN approximator, where the node number N_n is sufficiently large, there exists an ideal constant weight vector W^* , such that for any $\epsilon^* > 0$, $f(Z) = W^{*\top} S(Z) + \epsilon$, $\forall Z \in \Omega_Z$, where $|\epsilon| < \epsilon^*$ is the ideal approximation error. The ideal weight vector W^* is an “artificial” quantity used for analysis, which is defined as the value of W that minimizes $|\epsilon|$ for all $Z \in \Omega_Z \subset \mathbb{R}^q$, i.e., $W^* \triangleq \operatorname{argmin}_{W \in \mathbb{R}^{N_n}} \{\sup_{Z \in \Omega_Z} |f(Z) - W^\top S(Z)|\}$.

2.2 Problem Formulation

Consider a class of nonlinear parabolic PDE systems in one spatial dimension with a state-space description in the form of:

$$\frac{\partial x(z, t)}{\partial t} = a_1 \frac{\partial x(z, t)}{\partial z} + a_2 \frac{\partial^2 x(z, t)}{\partial z^2} + f(x, u) + \phi^k(x, u), \quad (1)$$

subject to the following initial condition $x(z, t_0) = x_0(z)$ and boundary conditions $m_i x(z_i, t) + n_i \frac{\partial x}{\partial z}(z_i, t) = d_i$ ($i = 1, 2$), where $x(z, t) \in \mathbb{R}$ is system state; $u \in \mathbb{R}^q$ is system input; $z \in [z_1, z_2]$ is the spatial coordinate; $t \in [t_0, \infty)$ is the time; $f(x, u) \in \mathbb{R}$ and $\phi^k(x, u) \in \mathbb{R}$ are unknown nonlinear functions satisfying locally Lipschitz continuous, which represent nonlinear uncertain system dynamics and deviations in system dynamics due to fault $k \in \{1, 2, \dots, N\}$ ($N \in \mathbb{N}_+$), respectively; $\frac{\partial x}{\partial z}$ and $\frac{\partial^2 x}{\partial z^2}$ are the first-order and second-order spatial derivatives of $x(z, t)$, respectively; $a_1, a_2, m_1, m_2, n_1, n_2, d_1, d_2$ are known constants.

Assumption 1. For the PDE system (1), the system input $u(t)$ and state $x(z_0, t)$ at any spatial point $z_0 \in [z_1, z_2]$ are bounded and recurrent for all $t \in [t_0, \infty)$.

Denote $\mathcal{H}$ as a Hilbert space of 1-D functions defined on $[z_1, z_2]$ that satisfies the boundary conditions of (1), with inner product and norm: $\langle \zeta_1, \zeta_2 \rangle = \int_{z_1}^{z_2} \zeta_1(z) \zeta_2(z) dz$, $\|\zeta_1\|_2 = \langle \zeta_1, \zeta_1 \rangle^{\frac{1}{2}}$, where $\zeta_1(z), \zeta_2(z)$ are two elements of $\mathcal{H}$. According to El-Farra and Ghantasala (2007); Wu and Li (2008), the PDE system (1) can be formulated as an infinite-dimensional system:

$$\dot{\chi} = \mathcal{A}\chi + f(\chi, u) + \phi^k(\chi, u), \quad \chi(t_0) = \chi_0, \quad (2)$$

where $\chi(t) = x(z, t)$ is the state function defined in $\mathcal{H}$, and $\mathcal{A}$ is a differential operator in $\mathcal{H}$ defined as $\mathcal{A}x = a_1 \frac{\partial x}{\partial z} + a_2 \frac{\partial^2 x}{\partial z^2}$, $x \in D(\mathcal{A}) := \{x \in \mathcal{H} | \mathcal{A}x \in \mathcal{H}, m_i x(z_i, t) + n_i \frac{\partial x}{\partial z}(z_i, t) = d_i, i = 1, 2\}$. The eigenvalue problem of $\mathcal{A}$ is defined as $\mathcal{A}\varphi_j = \lambda_j \varphi_j$ ($j = 1, 2, \dots, \infty$), where λ_j denotes an eigenvalue, and φ_j denotes an eigenfunction. The eigenspectrum of $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, is defined as the set of all eigenvalues of $\mathcal{A}$, i.e., $\sigma(\mathcal{A}) = \{\lambda_1, \lambda_2, \dots, \lambda_\infty\}$. The following assumption will be made for $\sigma(\mathcal{A})$.

Assumption 2. (i) $\operatorname{Re}\{\lambda_1\} \geq \operatorname{Re}\{\lambda_2\} \geq \dots \geq \operatorname{Re}\{\lambda_j\} \geq \dots$, where $\operatorname{Re}\{\lambda_j\}$ denotes the real part of λ_j ; (ii) $\sigma(\mathcal{A})$ can be partitioned as $\sigma(\mathcal{A}) = \sigma_s(\mathcal{A}) + \sigma_f(\mathcal{A})$, where $\sigma_s(\mathcal{A})$ consists of the first m number of eigenvalues, that is, $\sigma_s(\mathcal{A}) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$, and $\left| \frac{\operatorname{Re}\{\lambda_1\}}{\operatorname{Re}\{\lambda_m\}} \right| = O(1)$; (iii) $\operatorname{Re}\{\lambda_{m+1}\} < 0$ and $\left| \frac{\operatorname{Re}\{\lambda_m\}}{\operatorname{Re}\{\lambda_{m+1}\}} \right| = O(\iota)$, where $\iota := \left| \frac{\operatorname{Re}\{\lambda_1\}}{\operatorname{Re}\{\lambda_{m+1}\}} \right| < 1$ is a small positive constant.

Remark 1. Assumption 2 can be satisfied by the vast majority of parabolic PDE systems, including both diffusion-

convection-reaction processes, e.g., the packed-bed reactor and the tubular reactor, and many classes of dissipative fluid dynamics systems.

Based on this assumption, consider the decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_f$, in which $\mathcal{H}_s = \text{span}\{\varphi_1, \dots, \varphi_m\}$ denotes the finite dimensional space spanned by the slow eigenfunctions corresponding to $\sigma_s(\mathcal{A})$, and $\mathcal{H}_f = \text{span}\{\varphi_{m+1}, \dots, \varphi_\infty\}$ denotes the infinite dimensional complement one spanned by the fast eigenfunctions corresponding to $\sigma_f(\mathcal{A})$. The PDE system (2) can be rewritten as:

$$\begin{aligned}\dot{x}_s &= A_s x_s + f_s(x_s, \chi_f, u) + \phi_s^k(x_s, \chi_f, u), \\ \dot{\chi}_f &= A_f \chi_f + f_f(x_s, \chi_f, u) + \phi_f^k(x_s, \chi_f, u),\end{aligned}\quad (3)$$

with $x_s(t_0) = x_{s_0}$, $\chi_f(t_0) = \chi_{f_0}$, where $x_s \in \mathbb{R}^m$, $\chi_f \in \mathbb{R}^\infty$, $A_s = \text{diag}\{\lambda_1, \dots, \lambda_m\}$, $A_f = \text{diag}\{\lambda_{m+1}, \dots, \lambda_\infty\}$, $f_s = \langle \varphi_s, f \rangle$, $f_f = \langle \varphi_f, f \rangle$, $\phi_s^k = \langle \varphi_s, \phi^k \rangle$, $\phi_f^k = \langle \varphi_f, \phi^k \rangle$, $x_{s_0} = \langle \varphi_s, \chi_0 \rangle$, $\chi_{f_0} = \langle \varphi_f, \chi_0 \rangle$ with $\varphi_s = [\varphi_1, \dots, \varphi_m]^\top$ and $\varphi_f = [\varphi_{m+1}, \dots, \varphi_\infty]^\top$. By neglecting the fast modes, we can obtain the following ODE model to characterize the dominant dynamics of the PDE system in (2):

$$\dot{x}_s = A_s x_s + f_s(x_s, u) + \phi_s^k(x_s, u), \quad x_s(t_0) = x_{s_0}. \quad (4)$$

3. IDENTIFICATION OF SYSTEM UNCERTAIN DYNAMICS

In this section, an RBF NN-based adaptive dynamics learning approach will be developed to achieve accurate identification of the uncertain dynamics $f_s(x_s, u)$ and $\phi_s^k(x_s, u)$ in system (4) under all faulty modes. Particularly, since the system uncertainty $f_s(x_s, u)$ and occurring fault $\phi_s^k(x_s, u)$ cannot be decoupled, we consider them together and define a general fault function $\eta^k(x_s, u) := f_s(x_s, u) + \phi_s^k(x_s, u)$, such that the system (4) can be rewritten as:

$$\dot{x}_{s_i} = \lambda_i x_{s_i} + \eta_i^k(x_s, u), \quad i = 1, \dots, m; \quad k = 1, \dots, N. \quad (5)$$

Considering the unknown function $\eta_i^k(x_s, u)$ in (5), according to the RBF NN approximation theory as presented in Section 2.1, we know that there exists an ideal constant NN weight vector $W_i^{k*} \in \mathbb{R}^{N_n}$ (with N_n denoting the number of NN nodes) such that

$$\eta_i^k(x_s, u) = W_i^{k* \top} S(x_s, u) + \varepsilon_{i_0}^k, \quad (6)$$

where $S(x_s, u) : \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^{N_n}$ is a smooth RBF vector and $\varepsilon_{i_0}^k$ is the estimation error satisfying $|\varepsilon_{i_0}^k| < \varepsilon_i^*$ with ε_i^* being a positive constant that can be made arbitrarily small given a sufficiently large number of neurons, according to Yuan and Wang (2011). Based on this, an adaptive dynamics identifier can be constructed:

$$\begin{aligned}\dot{\hat{x}}_i &= -a_i(\hat{x}_i - x_{s_i}) + \lambda_i x_{s_i} + \hat{W}_i^{k \top} S(x_s, u), \\ \dot{\hat{W}}_i^k &= -\sigma_i \Gamma_i \hat{W}_i^k - \Gamma_i(\hat{x}_i - x_{s_i}) S(x_s, u),\end{aligned}\quad (7)$$

where $\hat{x}_i$ is the identifier state, x_{s_i} is the state of system (5), $\hat{W}_i^k \in \mathbb{R}^{N_n}$ is the estimate of W_i^{k*} in (6), $a_i > 0$, $\Gamma_i = \Gamma_i^\top > 0$, $\sigma_i > 0$ are design constants with σ_i being a small number.

Theorem 1. Consider the adaptive learning system consisting of the plant (5) and the identifier (7). Under Assumption 1, with initial condition $\hat{W}_i^k(0) = 0$, for all $i = 1, \dots, m$ and $k = 1, \dots, N$, we have: (i) all signals in the system remain bounded; (ii) the estimation error

$|\hat{x}_i - x_{s_i}|$ converges to a small neighborhood around the origin; and (iii) a locally-accurate approximation of the unknown function $\eta_i^k(x_s, u)$ is achieved by $\hat{W}_i^{k \top} S(x_s, u)$ as well as $\bar{W}_i^{k \top} S(x_s, u)$ along the recurrent system trajectory (x_s, u) , where $\bar{W}_i^k := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \hat{W}_i^k(\tau) d\tau$ with $[t_1, t_2]$ representing a time segment after the transient process.

Detailed proof can be completed by following a similar line of the proof of (Wang and Hill, 2009, Th. 3.1), thus is omitted here.

Through the above learning process, the knowledge of unknown function $\eta_i^k(x_s, u)$ of (5) can finally be obtained and stored in the constant RBF NN model $\bar{W}_i^{k \top} S(x_s, u)$, i.e.,

$$\eta_i^k(x_s, u) = \bar{W}_i^{k \top} S(x_s, u) + \varepsilon_i^k, \quad (8)$$

for all $i = 1, 2, \dots, m$ and $k = 1, 2, \dots, N$, where the approximation error ε_i^k satisfies $|\varepsilon_i^k| = O(\varepsilon_i^*) < \xi_i^*$, with ξ_i^* being a positive constant that can be made arbitrarily small by constructing a sufficiently large number of neurons, according to Yuan and Wang (2011).

4. FAULT ISOLATION SCHEME

With the obtained identification results, an FI scheme will be proposed in this section, and the associated analysis of FI performance will also be presented. Before proceeding, we assume without loss of generality that an unknown fault l' that is similar to the trained fault l ($l \in \{1, \dots, N\}$) is occurring in system (4), i.e.,

$$\dot{x}_s = A_s x_s + f_s(x_s, u) + \phi_{s_i}^{l'}(x_s, u). \quad (9)$$

4.1 FI Estimator Design and Decision Making

With the constant models $\bar{W}_i^{k \top} S(x_s, u)$ of (8), we propose to construct a bank of FI estimators as:

$$\dot{\hat{x}}_i^k = -b_i(\hat{x}_i^k - x_{s_i}) + \lambda_i x_{s_i} + \bar{W}_i^{k \top} S(x_s, u), \quad (10)$$

where $i = 1, \dots, m$, $k = 1, \dots, N$, $\hat{x}_i^k$ is the estimator state with initial condition $\hat{x}_i^k(t_0) = x_{s_i}(t_0)$, x_{s_i} is the i -th state of system (9), λ_i is the i -th diagonal element of matrix A_s in (9), b_i is a positive design constant, and $\bar{W}_i^{k \top} S(x_s, u)$ approximates the function $\eta_i^k(x_s, u) = f_{s_i}(x_s, u) + \phi_{s_i}^k(x_s, u)$ of system (5).

Comparing the FI estimators (10) with the monitored system (9), and based on (8), the residual systems (with residual $\hat{x}_i^k := \hat{x}_i^k - x_{s_i}$) can be derived as follows:

$$\dot{\hat{x}}_i^k = -b_i \hat{x}_i^k - \varepsilon_i^k + \phi_{s_i}^k(x_s, u) - \phi_{s_i}^{l'}(x_s, u), \quad (11)$$

where ε_i^k is the approximation error of model $\bar{W}_i^{k \top} S(x_s, u)$ for function $\eta_i^k(x_s, u)$ as defined in (8), $\phi_{s_i}^k(x_s, u)$ is the k -th faulty dynamics that has been learned/trained in Section 3, and $\phi_{s_i}^{l'}(x_s, u)$ is the faulty dynamics occurring in system (9). For the purpose of analysis, we introduce a so-called fault mismatch function $\rho_i^{k, l'}(x_s, u) := \phi_{s_i}^k(x_s, u) - \phi_{s_i}^{l'}(x_s, u)$ to represent the dynamics difference between the trained fault k and occurring fault l' . Then, the residual system (11) can be rewritten as:

$$\dot{\hat{x}}_i^k = -b_i \hat{x}_i^k - \varepsilon_i^k + \rho_i^{k, l'}(x_s, u). \quad (12)$$

The $\mathcal{L}_1$ norm of residual signal $\tilde{x}_i^k$ in (12), i.e., $\|\tilde{x}_i^k(t)\|_1 = \frac{1}{T} \int_{t-T}^t |\tilde{x}_i^k(\tau)| d\tau$ with T being a design parameter, will be utilized for real-time FI decision making.

For FI decision making, an adaptive threshold, denoted as $\bar{e}_i^k(t)$, will be further designed to upper bound the residual signal $\|\tilde{x}_i^k(t)\|_1$ of (12) when the occurring fault l' in (9) is similar to the trained fault l . To this end, the following assumption on the dynamics difference between the occurring fault l' and similar fault l is made.

Assumption 3. The dynamics difference between any pair of the occurring fault l' and its similar fault l ($l \in \{1, \dots, N\}$), denoted by $\rho_i^{l,l'}(x_s, u)$, is bounded by a known function $\bar{\rho}_i^l(x_s, u)$, i.e., $|\rho_i^{l,l'}(x_s, u)| \leq \bar{\rho}_i^l(x_s, u)$ for all $i = 1, \dots, m$.

Based on the above setup, to design the FI adaptive threshold, we consider the l -th residual system in (12), its time-domain solution can be derived as:

$$\tilde{x}_i^l(t) = \tilde{x}_i^l(t_0)e^{-b_i(t-t_0)} + \int_{t_0}^t e^{-b_i(t-\tau)}(\rho_i^{l,l'}(x_s, u) - \varepsilon_i^l)d\tau. \quad (13)$$

Note that $\tilde{x}_i^l(t_0) = 0$ and $|\varepsilon_i^l| < \xi_i^*$ from (8), under Assumption 3, we have:

$$\begin{aligned} |\tilde{x}_i^l(t)| &\leq \int_{t_0}^t e^{-b_i(t-\tau)}(|\varepsilon_i^l| + |\rho_i^{l,l'}(x_s, u)|)d\tau \\ &< \frac{\xi_i^*}{b_i} + \int_{t_0}^t e^{-b_i(t-\tau)}\bar{\rho}_i^l(x_s, u)d\tau. \end{aligned} \quad (14)$$

It guarantees that the FI residual signal $\|\tilde{x}_i^l(t)\|_1$ satisfies:

$$\|\tilde{x}_i^l(t)\|_1 < \frac{\xi_i^*}{b_i} + \left\| \int_{t_0}^t e^{-b_i(t-\tau)}\bar{\rho}_i^l(x_s, u)d\tau \right\|_1. \quad (15)$$

Thus, the FI adaptive threshold $\bar{e}_i^l(t)$ can be designed as:

$$\bar{e}_i^l(t) := \frac{\xi_i^*}{b_i} + \left\| \int_{t_0}^t e^{-b_i(t-\tau)}\bar{\rho}_i^l(x_s, u)d\tau \right\|_1, \quad (16)$$

for all $i = 1, \dots, m$, where ξ_i^* is a small constant given in (8), b_i is a design constant from (10), and $\bar{\rho}_i^l(x_s, u)$ is a known function defined in Assumption 3.

Consequently, for the monitored system (9), the proposed FI scheme consists of the FI estimators (10) and the adaptive threshold (16). The FI decision making scheme can be devised as follows.

Fault isolation decision making: Compare the FI residual signals $\|\tilde{x}_i^k(t)\|_1$ with the FI adaptive thresholds $\bar{e}_i^k(t)$ for time $t \geq t_0$ and all $i = 1, \dots, m$, $k = 1, \dots, N$. If there exists a unique $l \in \{1, \dots, N\}$ such that: (i) $\forall i = 1, \dots, m$, $\|\tilde{x}_i^l(t)\|_1 \leq \bar{e}_i^l(t)$ holds for all time $t \geq t_0$; and (ii) $\forall k \in \{1, \dots, N\}/\{l\}$, $\exists i \in \{1, \dots, m\}$, $\|\tilde{x}_i^k(t^k)\|_1 > \bar{e}_i^k(t^k)$ holds at some time instant $t^k > t_0$. Then, the occurring fault l' can be identified similar to the fault l , and the isolation time can be obtained as: $t_{iso} = \max\{t^k, k \in \{1, \dots, N\}/\{l\}\}$.

4.2 Isolatability Condition

To analyze the performance of the proposed FI scheme, the fault isolatability condition will be studied, i.e., under

what conditions the occurring fault l' in system (9) can be identified similar to a unique trained fault l .

Theorem 2. Consider the monitored system (9) and the fault isolation system consisting of estimators (10) and adaptive thresholds (16). For each $k \in \{1, \dots, N\}/\{l\}$ and some $i \in \{1, \dots, m\}$, if there exists a time interval $I^k = [t_a^k, t_b^k] \subseteq [t_b^k - T, t_b^k]$ with $t_a^k \geq t_0$, such that

$$|\rho_i^{k,l'}(x_s, u)| > \bar{\rho}_i^k(x_s, u) + 2\xi_i^*, \quad \forall t \in I^k, \quad (17)$$

and

$$\begin{aligned} l^k := t_a^k - t_b^k &\geq \frac{2\xi_i^* + 2\bar{\rho}_{i_{\max}}^k}{\mu_i + 2\bar{\rho}_{i_{\max}}^k} \left(T + \frac{1}{b_i} \ln \frac{\mu_i + 2\bar{\rho}_{i_{\max}}^k + \xi_i^*}{\mu_i - 2\xi_i^*} \right) \\ &\quad + \frac{\mu_i - 2\xi_i^*}{\mu_i + 2\bar{\rho}_{i_{\max}}^k} \left(\frac{1}{b_i} \ln \frac{3\mu_i + 4\bar{\rho}_{i_{\max}}^k}{\mu_i - 2\xi_i^*} \right), \end{aligned} \quad (18)$$

where $\mu_i := \min\{|\rho_i^{k,l'}| - \bar{\rho}_i^k, \forall t \in I^k\}$, $\bar{\rho}_{i_{\max}}^k := \max\{\bar{\rho}_i^k, \forall t \geq t_0\}$, then, $\|\tilde{x}_i^k(t_b^k)\|_1 > \bar{e}_i^k(t_b^k)$ holds, the occurring fault l' will be identified similar to fault l , and the isolation time is: $t_{iso} = \max\{t_b^k, \forall k \in \{1, \dots, N\}/\{l\}\}$.

Theorem 2 can be proved by investigating the relation between the FI residuals $\|\tilde{x}_i^k\|_1$ of (12) and the fault mismatch function $\rho_i^{k,l'}(x_s, u)$, which can be conducted based on the locally Lipschitz continuity of function $\rho_i^{k,l'}(x_s, u)$. Detailed proof can be completed by following a similar line of the proof of (Zhang et al., 2022, Th. 3), thus is omitted here.

5. SIMULATION STUDIES

Consider a typical transport-reaction process in chemical industry, i.e., a long, thin catalytic rod in a reactor, which is borrowed from El-Farra and Ghantasala (2007); Wu and Li (2008). The system is described by the parabolic PDE of (1), where $a_1 = 0$, $a_2 = 1$, $f(x, u) = \beta_T(e^{-\frac{\gamma}{1+x}} - e^{-\gamma}) + \beta_u(b(z)u(t) - x)$, with $\beta_T = 50$, $\gamma = 4$, $\beta_u = 2$, $b(z) = 1.5 \sin(z) + 1.8 \sin(2z) + 2 \sin(3z)$, and the boundary and initial conditions: $x(0, t) = 0$, $x(\pi, t) = 0$, $x(z, 0) = 15 \sin(z)$. Three types of faults are considered. (i) Fault 1: an actuator fault with a faulty actuator distribution function: $b'(z) = 1.8 \sin(z) + 1.8 \sin(2z) + 2 \sin(3z)$, leading to the associated fault function $\phi^1(x, u) = \tilde{b}(z)\beta_u u$ with $\tilde{b}(z) = b(z) - b'(z) = -0.3 \sin(z)$. (ii) Fault 2: a state fault with fault function $\phi^2(x, u) = \tilde{h}(z)x$, where $\tilde{h}(z) = h(z - 1) - h(z - 1.3)$ and $h(\cdot)$ is a heaviside function. (iii) Fault 3: a component fault with a faulty system parameter: $\beta'_T = 48$, and the associated fault function is thus $\phi^3(x, u) = \tilde{\beta}_T(e^{-\frac{\gamma}{1+x}} - e^{-\gamma})$ with $\tilde{\beta}_T = \beta_T - \beta'_T = 2$.

For the above-mentioned PDE system, the eigenvalue problem of the spatial differential operator in the form of (2) has the solution: $\lambda_i = -i^2$, $\varphi_i(z) = \sqrt{\frac{2}{\pi}} \sin(iz)$ with $i = 1, \dots, \infty$. Choosing the first $m = 3$ number of eigenvalues as dominant ones, we can obtain the ODE system in the form of (4) to describe the dominant dynamics of the PDE system (1).

Identification process for system (4) is implemented according to (7). The RBF network $\hat{W}_i^{k\top} S(x_s, u)$ is constructed in a regular lattice, with nodes $N_n = 14 \times 9 \times 8 \times$

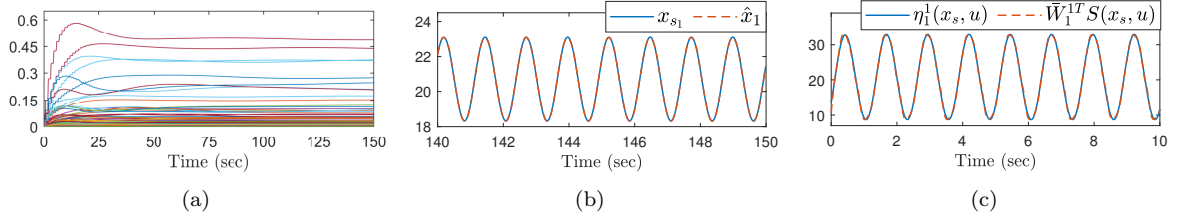


Fig. 1. Identification of function $\eta_1^1(x_s, u)$ with identifier (7). (a) Convergence of NN weight $\hat{W}_1^1$; (b) estimation performance of x_{s1} by $\hat{x}_1$; and (c) function approximation performance of $\eta_1^1(x_s, u)$ by $\hat{W}_1^{1T} S(x_s, u)$.

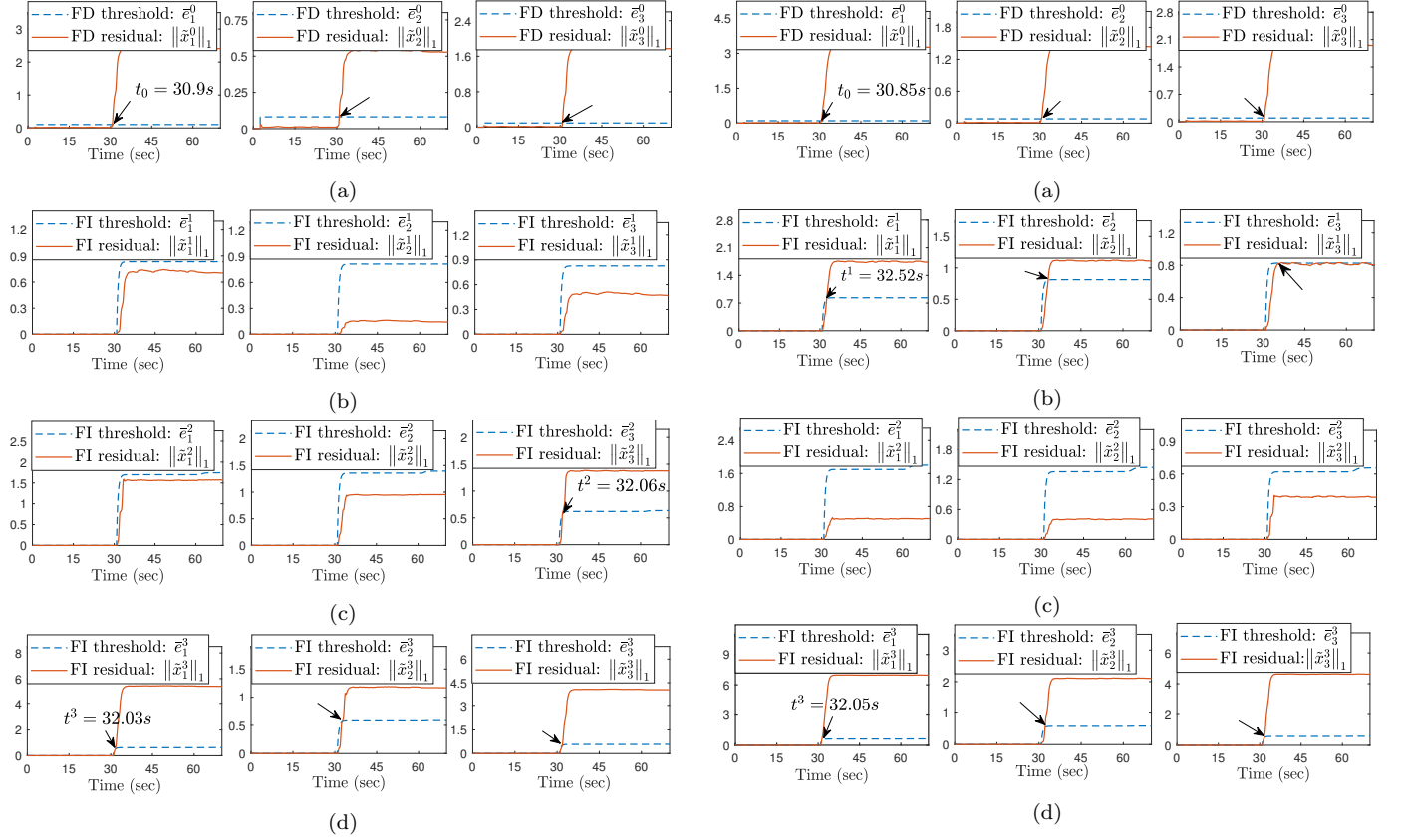


Fig. 2. FDI performance when fault 1' occurs at time $t = 30s$: (a) FD residuals and thresholds; (b) 1-st FI residuals and thresholds; (c) 2-nd FI residuals and thresholds; and (d) 3-rd FI residuals and thresholds.

13, the center evenly spaced on $[17.5, 24] \times [-1, 3] \times [0, 3.5] \times [-2, 4]$ and the widths $\nu_i = 0.5$. The design parameters in (7) are $a_i = 4$, $\Gamma_i = 0.35$, $\sigma_i = 0.001$, $\hat{W}_i^k(0) = 0$ ($\forall i = 1, 2, 3$) and $\hat{x}(0) = x_s(0)$. The identification performance of system (4) under faulty mode $k = 1$ is plotted in Fig. 1, showing that accurate identification for dynamics $\eta_1^1(x_s, u) = f_{s1}(x_s, u) + \phi_{s1}^1(x_s, u)$ can be achieved with the identifier (7). Simulation results for the cases of faulty modes $k = 2, 3$ are similar to those in Fig. 1 and thus omitted here. Consequently, based on the identification result, we can obtain constant NN models $\bar{W}_i^k S(x_s, u)$ by $\bar{W}_i^k = \frac{1}{10} \int_{140}^{150} \hat{W}_i^k(\tau) d\tau$ ($i = 1, 2, 3$, $k = 1, 2, 3$); and the parameters $\xi_1^* = 0.0860$, $\xi_2^* = 0.0430$, and $\xi_3^* = 0.0703$.

FI scheme is implemented according to (10) and (16). Associated parameters are $b_i = 2$, $T = 2.5s$, and $\bar{\rho}_i^k(x_s, u) = \int_0^\pi \bar{\phi}^k(x, u) |\varphi_i(z)| dz$, where $\bar{\phi}^1(x, u) = |\Delta_{\bar{b}} \beta_u u|$ with

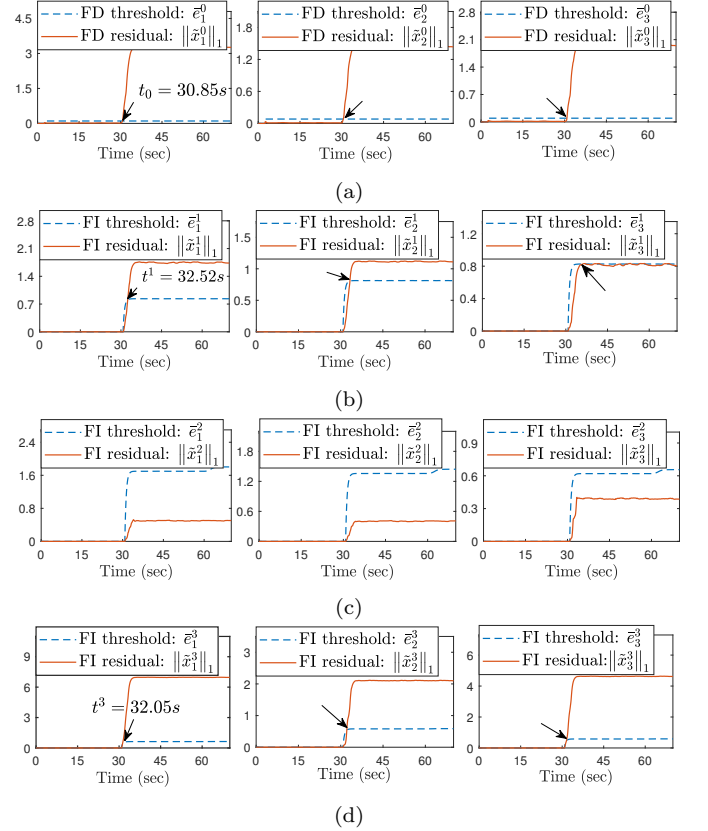


Fig. 3. FDI performance when fault 2' occurs at time $t = 30s$: (a) FD residuals and thresholds; (b) 1-st FI residuals and thresholds; (c) 2-nd FI residuals and thresholds; and (d) 3-rd FI residuals and thresholds.

$\Delta_{\bar{b}} = 0.25$, $\bar{\phi}^2(x, u) = |\Delta_{\bar{h}} x|$ with $\Delta_{\bar{h}} = h(z - 1) - h(z - 1.3)$, and $\bar{\phi}^3(x, u) = |\Delta_{\bar{\beta}_T}(e^{-\frac{\gamma}{1+x}} - e^{-\gamma})|$ with $\Delta_{\bar{\beta}_T} = 1$. For testing purpose, we assume three occurring faults to be detected and isolated, including fault 1': $\phi^{1'}(x, u) = \tilde{b}'(z)\beta_u u$ with $\tilde{b}'(z) = -0.5\sin(z)$; fault 2': $\phi^{2'}(x, u) = \tilde{h}'(z)x$ with $\tilde{h}'(z) = h(z - 1) - h(z - 1.2)$; and fault 3': $\phi^{3'}(x, u) = \tilde{\beta}_T'(e^{-\frac{\gamma}{1+x}} - e^{-\gamma})$ with $\tilde{\beta}_T' = 49$.

For testing purpose, we consider the faults $l' = 1', 2', 3'$ occur in system (1) at time $t = 30s$, respectively. For better study, an FD scheme proposed in our previous work Zhang et al. (2020) will be used to detect the occurrence of fault l' , and after the fault being detected (at time t_0), the FI scheme proposed in the current paper can be activated to identify the type of fault l' . Due to the limited space, the implementation of FD scheme is omitted here. We first consider the case of fault 1', and associated fault

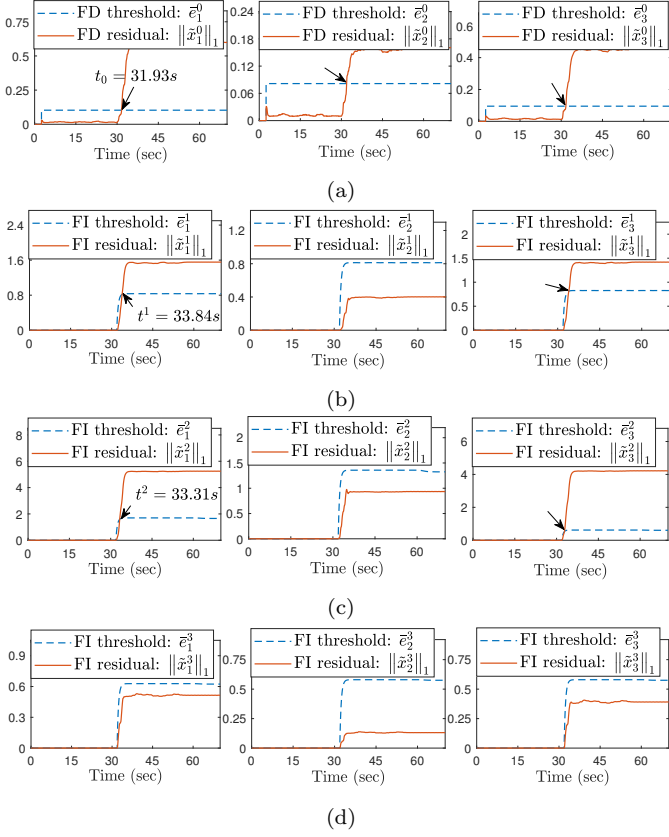


Fig. 4. FDI performance when fault 3' occurs at time $t = 30s$: (a) FD residuals and thresholds; (b) 1-st FI residuals and thresholds; (c) 2-nd FI residuals and thresholds; and (d) 3-rd FI residuals and thresholds.

detection and isolation (FDI) performances are displayed in Fig. 2. Specifically, in Fig. 2a, FD residual signals become larger than the FD threshold at time $t_0 = 30.9s$, indicating that the fault is detected and the FI scheme can be activated at time $t_0 = 30.9s$; in Figs. 2b–2d, according to FI decision making principle of Sec. 4.1, the fault 1' can be identified to be similar to the fault 1, and the isolation time is obtained at: $t_{iso} = \max\{t^2, t^3\} = 32.06s$. Further considering the cases of faults 2' in Fig. 3 and fault 3' in Fig. 4, the occurring fault 2' is detected at time $t_0 = 30.85s$ and isolated at time $t_{iso} = 32.52s$; while the occurring fault 3' is detected at time $t_0 = 31.93s$ and isolated at time $t_{iso} = 33.84s$. These simulation results demonstrate feasibility and effectiveness of our proposed FI scheme.

6. CONCLUSIONS

We have addressed the FI problem of parabolic PDE systems with uncertain nonlinear dynamics in this paper. The developed FI scheme possesses an important capability of dealing with the effects of system uncertainties for accurate FI. Specifically, we have first utilized the Galerkin method to generate a reduced-order ODE model to approximate the parabolic PDE system. Based on this model, the system dominant uncertain nonlinear dynamics has been locally-accurately identified with an RBF NN-based adaptive learning scheme. A bank of FI estimators have been further proposed with these learned results. Adaptive thresholds associated with these estimators were designed for real-time decision making. Rigorous analysis

on the fault isolatability has been provided. Simulation studies have also been conducted to verify the effectiveness of the proposed methodologies.

ACKNOWLEDGEMENTS

This work was supported in part by the National Science Foundation under Grant CMMI-1929729.

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