

ROOTS OF CHARACTERISTIC EQUATION FOR SYMPLECTIC GROUPOID

L. O. Chekhov, M. Shapiro, and H. Shibo

Morphisms on the set $\mathcal{A}_n \subseteq gl_n$ of unipotent upper-triangular $n \times n$ matrices are transformations such that $\mathbb{A} \mapsto B\mathbb{A}B^T \in \mathcal{A}_n$. For $B = e^{\varepsilon g}$, linearized transformations $\delta_\varepsilon \mathbb{A} = g\mathbb{A} + \mathbb{A}g^T$ are analogous to Krichever–Novikov transformations [4]. The quantization of Bondal’s Poisson structure on $\mathcal{A}_n$ [1] obtained using the symplectic groupoid construction is the *reflection equation* with the trigonometric R -matrix

$$\mathcal{R}_n(q) \mathbb{A} \mathcal{R}_n^{t_1}(q) \mathbb{A}^2 = \mathbb{A} \mathcal{R}_n^{t_1}(q) \mathbb{A} \mathcal{R}_n(q) \quad (1)$$

Here $\mathbb{A}$ is an upper-triangular matrix with diagonal entries $q^{-1/2}$ and with self-adjoint operator entries $a_{i,j}$ for $i < j$. The combination $\mathbb{A}\mathbb{A}^{-\dagger} := \mathbb{A}[\mathbb{A}^\dagger]^{-1}$ then undergoes an adjoint transformation $\mathbb{A}\mathbb{A}^{-\dagger} \rightarrow B\mathbb{A}\mathbb{A}^{-\dagger}B^{-1}$, and we are to find its eigenvalues $\lambda_i \in \mathbb{C}$ determining $[n/2]$ independent Casimir elements [1].

We express entries of $\mathbb{A}$ using quantum *Fock–Goncharov variables* $Z_\alpha = Z_{(i,j,k)}$ parameterized in terms of barycentric coordinates ($i+j+k=n$) of vertices of the b_n -quiver [3] (left part of Fig. 1): A solid arrow from α to β means that $Z_\beta Z_\alpha = q^{-2} Z_\alpha Z_\beta$ and a dashed arrow gives $Z_\beta Z_\alpha = q^{-1} Z_\alpha Z_\beta$. A *directed network* N dual to the b_n -quiver is a directed graph of double arrows in Fig. 1.

With any oriented path $P : j \rightsquigarrow i$ in any planar directed network $\mathcal{N}$ we associate the *quantum weight* $w(P)$, which is Weyl ordered (denoted by the symbol $\bullet \cdots \bullet$; see [2]) product of variables Z_α of all faces of $\mathcal{N}$ lying to the right from the path.

In the case of the b_n -quiver, we define three $n \times n$ *quantum transport matrices*

$$(\mathcal{M}_1)_{i,j} = \sum_{\substack{\text{directed path } P: j \rightsquigarrow i' \\ \text{from right to left}}} w(P), \quad (\mathcal{M}_2)_{i,j} = \sum_{\substack{\text{directed path } P: j \rightsquigarrow i'' \\ \text{from right to bottom}}} w(P), \quad (\mathcal{M}_3)_{i,j} = \sum_{\substack{\text{directed path } P: j' \rightsquigarrow i'' \\ \text{from left to bottom}}} w(P),$$

where paths contributing to $\mathcal{M}_3$ are obtained by inverting all horizontal double arrows in N . Note that $\mathcal{M}_1$ and $\mathcal{M}_3$ are lower-triangular matrices and $\mathcal{M}_2$ is an upper-triangular matrix. We define a diagonal matrix $Q := \sum_{i=1}^n (q)^{-i+1/2} e_{i,i}$, an antidiagonal matrix $S = \sum_{i=1}^n (-1)^{i+1} e_{i,n+1-i}$, and let $T_i := \bullet \prod_{j=0}^{n-i} Z_{(j,n-j-i,i)} \bullet$ denote products of Fock–Goncharov variables along SE-diagonals of the b_n -quiver.

We have the *groupoid condition* [2] $\mathcal{M}_3 Q S \mathcal{M}_1 = \mathcal{M}_2$, and *Theorem 4.1* of [2] states that $\mathbb{A} := \mathcal{M}_1^T \mathcal{M}_3 Q S \mathcal{M}_1$ satisfies Eq. (1). Amalgamating variables $Z_{(i,0,n-i)}$ and $Z_{(0,n-i,i)}$ pairwise we obtain new Casimirs $\bullet \prod_{i=1}^l Z_{(i-1,n-l,i-1)} Z_{(l,n-l,0)}^2 \prod_{j=1}^{n-l} Z_{(l,j-1,n-l-j+1)} \bullet$ used to eliminate variables $Z_{(l,n-l,0)}$, and we obtain the $\mathcal{A}_n$ -quiver (right part of Fig. 1) whose Casimirs are $[n/2]$ elements $C_i = \bullet T_i T_{n-i} \bullet$ for $1 \leq i < n/2$ and $C_{n/2} = T_{n/2}$ for integer $n/2$.

Theorem 1 *Eigenvalues $\lambda_i \in \mathbb{C}$, $1 \leq i \leq n$, of the operator $\mathbb{A}\mathbb{A}^{-\dagger}$ are*

$$\lambda_i = (-1)^{n-1} q^{-n} \times \begin{cases} \prod_{k=i}^{[n/2]} C_k & \text{for } 1 \leq i \leq [n/2]; \\ 1 & \text{for } i = (n+1)/2 \text{ for odd } n; \\ \prod_{k=n+1-i}^{[n/2]} C_k^{-1} & \text{for } n - [n/2] + 1 \leq i \leq n \end{cases}$$

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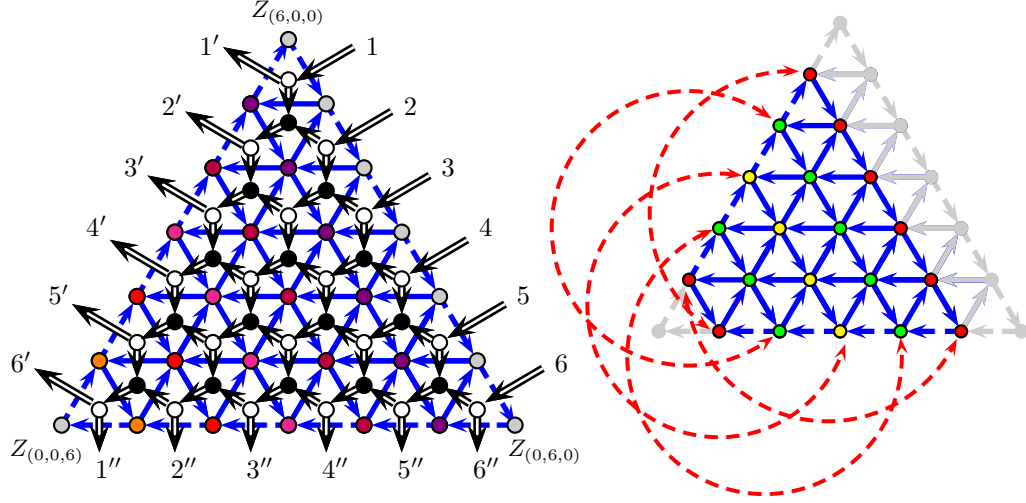


Рис. 1: On the left is the Fock-Goncharov b_6 -quiver and the directed network N dual to it; $\mathcal{M}_1$ transports from side $\{1 - 6\}$ to side $\{1' - 6'\}$ and $\mathcal{M}_2$ from side $\{1 - 6\}$ to side $\{1'' - 6''\}$. On the right is the $\mathcal{A}_n$ -quiver with amalgamations indicated by dashed arrows.

Proof. For $\mathbb{A} = \mathcal{M}_1^T \mathcal{M}_3 Q S \mathcal{M}_1$, since entries of all matrices $\mathcal{M}_i$ are self-adjoint, $\mathcal{M}_i^\dagger = \mathcal{M}_i^T$, $Q^\dagger = Q^{-1}$ and $S^\dagger = (-1)^{n+1} S$, we have $\mathbb{A}^\dagger = \mathcal{M}_1^T S^T Q^{-1} \mathcal{M}_3^T \mathcal{M}_1$, and only the block $S^T Q^{-1} \mathcal{M}_3^T$ sandwiched between $\mathcal{M}_1^T$ and $\mathcal{M}_1$ is changed, $\mathbb{A} - \lambda \mathbb{A}^\dagger = \mathcal{M}_1^T (\mathcal{M}_3 Q S - \lambda S^T Q^{-1} \mathcal{M}_3^T) \mathcal{M}_1$, so the singularity equation becomes $(\mathcal{M}_3 Q S - \lambda S^T Q^{-1} \mathcal{M}_3^T) \psi = 0$. The matrix $\mathcal{M}_3$ is lower-triangular; its diagonal entries are $m_1 = Z_{(n,0,0)}$, $m_i = \bullet Z_{(n,0,0)} \prod_{j=1}^{i-1} T_j$, $2 \leq i \leq n$, and the *crucial observation* is that the *both* matrices $\mathcal{M}_3 Q S$ and $S^T Q^{-1} \mathcal{M}_3^T$ are *upper-anti-diagonal*. The anti-diagonal components of the corresponding matrices are $\sum_{i=1}^n (-1)^{i+1} q^{i-1/2} m_i e_{n+1-i,i}$ and $\sum_{i=1}^n (-1)^{n-i} q^{-n+i-1/2} m_{n+1-i} e_{n+1-i,i}$, and the singularity equation has a nontrivial solution if their combination contains zero element, so the admissible values are $\lambda_i = (-1)^{n-1} q^{-n} m_{n+1-i} / m_i$; these quotients are just expressions in cases in Theorem 1. $\square$

For all λ_i distinct, the Jordan form of $\mathbb{A} \mathbb{A}^{-\dagger}$ is diagonal; an interesting case is when $\lambda_i^m = \lambda_j^m$ for all i, j for some natural $m \geq n$ since the whole form then trivializes, $[\mathbb{A} \mathbb{A}^{-\dagger}]^m \sim \mathbf{I}$.

Список литературы

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(L. O. Chekhov)

Steklov Mathematical Institute, Moscow, Russia,
National Research University Higher School of Economics, Moscow, Russia,
Michigan State University, East Lansing, USA.

(M. Shapiro)

Michigan State University, East Lansing, USA
National Research University Higher School of Economics, Moscow, Russia.

(H. Shibo)

Xi'an Jiaotong University, Xi'an, Shaanxi, P.R. China

E-mail: chekhov@mi-ras.ru, mshapiro@msu.edu