Received: 10 April 2020 | Revised: 30 May 2021 | Accepted: 16 September 2021

DOI: 10.1112/jlms.12542

RESEARCH ARTICLE

Journal of the London Mathematical Society

Generalized cluster structures related to the Drinfeld double of GL_n

Misha Gekhtman[1] | **Michael Shapiro**[2,3] | **Alek Vainshtein**[4]

[1]Department of Mathematics, University of Notre Dame, Notre Dame, Indiana

[2]Department of Mathematics, Michigan State University, East Lansing, Michigan, USA

[3]National Research University Higher School of Economics, Moscow, Russia

[4]Department of Mathematics & Department of Computer Science, University of Haifa, Haifa, Mount Carmel, Israel

Correspondence
Michael Shapiro, Department of Mathematics, Michigan State University, East Lansing, MI 48823, USA.
Email: mshapiro@math.msu.edu

Funding information
NSF, Grant/Award Number: #1702054; ISF, Grant/Award Number: #1144/16; Research Institute for Mathematical Sciences; Sapienza University of Rome; Michigan State University; University of Notre Dame

Abstract

We prove that the regular generalized cluster structure on the Drinfeld double of GL_n constructed in Gekhtman, Shapiro, and Vainshtein (Int. Math. Res. Notes, 2022, to appear, arXiv:1912.00453) is complete and compatible with the standard Poisson–Lie structure on the double. Moreover, we show that for $n = 4$ this structure is distinct from a previously known regular generalized cluster structure on the Drinfeld double, even though they have the same compatible Poisson structure and the same collection of frozen variables. Further, we prove that the regular generalized cluster structure on band periodic matrices constructed in Gekhtman, Shapiro, and Vainshtein (Int. Math. Res. Notes, 2022, to appear, arXiv:1912.00453) possesses similar compatibility and completeness properties.

MSC (2020)
13F60, 53D17 (primary)

1 | INTRODUCTION

It is by now well known that many important algebraic varieties arising in Lie theory, representation theory and theory of integrable systems support a cluster structure. The first example of this kind is already present in the foundational paper [6] where it was shown that the homogeneous coordinate ring of the Grassmannian of 2-planes in $\mathbb{C}^{n+3}$ is naturally isomorphic to the cluster algebra of finite type A_n. Among the examples that followed were Grassmannians [9, 19], double Bruhat cells [1] and strata in flag varieties [17]. All of these examples share two key features:

(i) the variety in question is equipped with a Poisson brackets *compatible* with the cluster structure in a sense reviewed in Section 2.1, and (ii) cluster transformations that connect distinguished coordinate charts within a ring of regular functions are modeled on three-term relations such as short Plücker relations, Desnanot–Jacobi identities and their Lie-theoretic generalizations. The first feature led us to development of an approach for constructing a cluster structure in Poisson varieties possessing a particular nice coordinate chart (see, for example, [9]). However, there are situations when reliance on three-term relations (equivalently, usual cluster transformations) turns out to be too restrictive and when certain multinomial versions of cluster transformation are needed. These were first considered in [4] and termed *generalized cluster transformations*. The first geometric example of this sort was studied in [10, 12] where we used a more general form of transformations defined in [4] to construct an initial seed Σ_n for a complete generalized cluster structure $\mathcal{GC}_n^D$ in the standard Drinfeld double $D(GL_n)$ and proved that this structure is compatible with the standard Poisson–Lie structure on $D(GL_n)$.

In [13, Section 4], we presented a rich source of identities that can serve as generalized cluster transformation and, as one of the applications, constructed a different seed $\bar{\Sigma}_n$ for a regular generalized cluster structure $\overline{\mathcal{GC}}_n^D$ on $D(GL_n)$. In this paper, we prove that $\overline{\mathcal{GC}}_n^D$ shares all the properties of $\mathcal{GC}_n^D$: it is complete and compatible with the standard Poisson–Lie structure on $D(GL_n)$. Moreover, we prove that the seeds $\bar{\Sigma}_4(X,Y)$ and $\Sigma_4(Y^T,X^T)$ are not mutationally equivalent. This answers the question posed by Keel: 'Do there exist two different regular cluster structures on the same variety with the same compatible Poisson bracket and the same collection of frozen variables?' by providing an explicit example of two different regular complete generalized cluster structures on $D(GL_4)$ with the same compatible Poisson structure and the same collection of frozen variables. Further, from the above properties of $\overline{\mathcal{GC}}_n^D$ we derive that the generalized cluster structure in the ring of regular functions on band periodic matrices built in [13, Section 5] is complete and compatible with the restriction of the standard Poisson–Lie structure. Apart from possible applications to cluster integrable systems, the latter generalized cluster structure is closely related to the conjectural ones in cyclic symmetry loci in Grassmannians considered in the recent preprint [8] that appeared while this paper was under review, and in the Grothendieck rings of the quantum affine algebras $U_q\widehat{\mathfrak{sl}_k}$ at roots of unity [14]. These connections will be explored in a joint work of Gekhtman with Fraser and Trampel.

Section 2 below contains all necessary information about generalized cluster structures borrowed mainly from [13] to make this text self-contained. Section 3 is devoted to the study of $\overline{\mathcal{GC}}_n^D$. The initial seed is described in Section 3.1. The main result of this section is Theorem 3.1 which claims that $\overline{\mathcal{GC}}_n^D$ is compatible with the standard Poisson–Lie structure on $D(GL_n)$ and complete. The former statement is proved in Sections 3.2 and 3.3, and the latter in Section 3.4. In Section 4, we compare two generalized cluster structures on $D(GL_4)$: $\overline{\mathcal{GC}}_4^D$ and $\mathcal{GC}_4^D(Y^T,X^T)$ described in [12]. These two structures have the same set of frozen variables, and we prove that they are distinct, that is, the two seeds are not mutationally equivalent. Finally, Section 5 treats the case of periodic band matrices. The initial seed Σ_{kn} for the generalized cluster structure on the space $\mathcal{L}_{kn}$ of $(k+1)$-diagonal n-periodic band matrices is described in Section 5.1. The main result of this section is Theorem 5.1 which claims that $\mathcal{GC}(\Sigma_{kn})$ is compatible with the restriction of the standard Poisson–Lie structure on $D(\mathrm{Mat}_n)$ and complete. The former statement is proved in Section 5.2, and the latter in Section 5.3.

2 | PRELIMINARIES

Following [12], we remind the definition of a generalized cluster structure represented by a quiver with multiplicities. Let $(Q, d_1, \dots, d_N)$ be a quiver on N mutable and M frozen vertices with positive integer multiplicities d_i at mutable vertices. A vertex is called *special* if its multiplicity is greater than 1. A frozen vertex is called *isolated* if it is not connected to any other vertices. Let $\mathbb{F}$ be the field of rational functions in $N+M$ independent variables with rational coefficients. There are M distinguished variables corresponding to frozen vertices; they are denoted $x_{N+1}, \dots, x_{N+M}$ and called *stable*, or *frozen* variables. The *coefficient group* is a free multiplicative abelian group of Laurent monomials in stable variables, and its integer group ring is $\bar{\mathbb{A}} = \mathbb{Z}[x_{N+1}^{\pm 1}, \dots, x_{N+M}^{\pm 1}]$ (we write $x^{\pm 1}$ instead of x, x^{-1}).

An *extended seed* (of *geometric type*) in $\mathbb{F}$ is a triple $\Sigma = (\mathbf{x}, Q, \mathcal{P})$, where $\mathbf{x} = (x_1, \dots, x_N, x_{N+1}, \dots, x_{N+M})$ is a transcendence basis of $\mathbb{F}$ over the field of fractions of $\bar{\mathbb{A}}$ and $\mathcal{P}$ is a set of N *strings*. The ith string is a collection of monomials $p_{ir} \in \mathbb{A} = \mathbb{Z}[x_{N+1}, \dots, x_{N+M}]$, $0 \leqslant r \leqslant d_i$, such that $p_{i0} = p_{id_i} = 1$; it is called *trivial* if $d_i = 1$, and hence both elements of the string are equal to 1. The monomials p_{ir} are called *exchange coefficients*.

Given a seed as above, the *adjacent cluster* in direction k, $1 \leqslant k \leqslant N$, is defined by $\mathbf{x}' = (\mathbf{x} \setminus \{x_k\}) \cup \{x_k'\}$, where the new cluster variable x_k' is given by the *generalized exchange relation*

$$x_k x_k' = \sum_{r=0}^{d_k} p_{kr} u_{k;>}^r v_{k;>}^{[r]} u_{k;<}^{d_k - r} v_{k;<}^{[d_k - r]}; \tag{2.1}$$

here $u_{k;>}$ and $u_{k;<}$, $1 \leqslant k \leqslant N$, are defined by

$$u_{k;>} = \prod_{k \to i \in Q} x_i, \qquad u_{k;<} = \prod_{i \to k \in Q} x_i,$$

where the products are taken over all edges between k and mutable vertices, and *stable* τ-*monomials* $v_{k;>}^{[r]}$ and $v_{k;<}^{[r]}$, $1 \leqslant k \leqslant N$, $0 \leqslant r \leqslant d_k$, defined by

$$v_{k;>}^{[r]} = \prod_{N+1 \leqslant i \leqslant N+M} x_i^{\lfloor r b_{ki}/d_k \rfloor}, \qquad v_{k;<}^{[r]} = \prod_{N+1 \leqslant i \leqslant N+M} x_i^{\lfloor r b_{ik}/d_k \rfloor}, \tag{2.2}$$

where b_{ki} is the number of edges from k to i and b_{ik} is the number of edges from i to k; here, as usual, the product over the empty set is assumed to be equal to 1. In what follows, we write $v_{k;>}$ instead of $v_{k;>}^{[d_k]}$ and $v_{k;<}$ instead of $v_{k;<}^{[d_k]}$. The right-hand side of (2.1) is called a *generalized exchange polynomial*.

The standard definition of the *quiver mutation* in direction k is modified as follows: if both vertices i and j in a path $i \to k \to j$ are mutable, then this path contributes d_k edges $i \to j$ to the mutated quiver Q'; if one of the vertices i or j is frozen, then the path contributes d_j or d_i edges $i \to j$ to Q'. The multiplicities at the vertices do not change. Note that isolated vertices remain isolated in Q'.

The *exchange coefficient mutation* in direction k is given by

$$p_{ir}' = \begin{cases} p_{i,d_i - r}, & \text{if } i = k; \\ p_{ir}, & \text{otherwise.} \end{cases} \tag{2.3}$$

Given an extended seed $\Sigma = (\mathbf{x}, Q, \mathcal{P})$, we say that a seed $\Sigma' = (\mathbf{x}', Q', \mathcal{P}')$ is *adjacent* to Σ (in direction k) if $\mathbf{x}'$, Q' and $\mathcal{P}'$ are as above. Two such seeds are *mutation equivalent* if they can be connected by a sequence of pairwise adjacent seeds. The set of all seeds mutation equivalent to Σ is called the *generalized cluster structure* (of geometric type) in $\mathbb{F}$ associated with Σ and denoted by $\mathcal{GC}(\Sigma)$.

Fix a ground ring $\widehat{\mathbb{A}}$ such that $\mathbb{A} \subseteq \widehat{\mathbb{A}} \subseteq \bar{\mathbb{A}}$. The *generalized upper cluster algebra* $\overline{\mathcal{A}}(\mathcal{GC}) = \overline{\mathcal{A}}(\mathcal{GC}(\Sigma))$ is the intersection of the rings of Laurent polynomials over $\widehat{\mathbb{A}}$ in cluster variables taken over all seeds in $\mathcal{GC}(\Sigma)$. Let V be a quasi-affine variety over $\mathbb{C}$, $\mathbb{C}(V)$ be the field of rational functions on V, and $\mathcal{O}(V)$ be the ring of regular functions on V. A generalized cluster structure $\mathcal{GC}(\Sigma)$ in $\mathbb{C}(V)$ is an embedding of $\mathbf{x}$ into $\mathbb{C}(V)$ that can be extended to a field isomorphism between $\mathbb{F}_{\mathbb{C}} = \mathbb{F} \otimes \mathbb{C}$ and $\mathbb{C}(V)$. It is called *regular on* V if any cluster variable in any cluster belongs to $\mathcal{O}(V)$, and *complete* if $\overline{\mathcal{A}}(\mathcal{GC})$ tensored with $\mathbb{C}$ is isomorphic to $\mathcal{O}(V)$. The choice of the ground ring is discussed in [12, Section 2.1].

Let $\{\cdot,\cdot\}$ be a Poisson bracket on the ambient field $\mathbb{F}$, and $\mathcal{GC}$ be a generalized cluster structure in $\mathbb{F}$. We say that the bracket and the generalized cluster structure are *compatible* if any extended cluster $\widetilde{\mathbf{x}} = (x_1, \dots, x_{N+M})$ is *log-canonical* with respect to $\{\cdot,\cdot\}$, that is, $\{x_i, x_j\} = \omega_{ij} x_i x_j$, where $\omega_{ij} \in \mathbb{Z}$ are constants for all i, j, $1 \leqslant i, j \leqslant N+M$.

For any mutable vertex $k \in Q$, define the y-variable

$$y_k = \frac{u_{k;>}^{d_k} v_{k;>}}{u_{k;<}^{d_k} v_{k;<}}. \tag{2.4}$$

The following statement is an immediate corollary of [12, Proposition 2.5].

Proposition 2.1. *Assume that for any mutable vertex $j \in Q$*

$$\{\log x_i, \log y_j\} = \lambda d_j \delta_{ij} \quad \textit{for any } i \in Q,$$

where λ is a rational number not depending on j, δ_{ij} is the Kronecker symbol, and all Laurent monomials

$$\hat{p}_{kr} = \frac{\left(p_{kr} v_{k;>}^{[r]} v_{k;<}^{[d_k - r]}\right)^{d_k}}{v_{k;>}^{r} v_{k;<}^{d_k - r}}$$

are Casimirs of the bracket $\{\cdot,\cdot\}$. Then the bracket $\{\cdot,\cdot\}$ is compatible with $\mathcal{GC}(\Sigma)$.

The notion of compatibility extends to Poisson brackets on $\mathbb{F}_{\mathbb{C}}$ without any changes.

Fix an arbitrary extended cluster $\widetilde{\mathbf{x}} = (x_1, \dots, x_{N+M})$ and define a *local toric action* of rank s as a map $\mathcal{T}_{\mathbf{q}}^{W} : \mathbb{F}_{\mathbb{C}} \to \mathbb{F}_{\mathbb{C}}$ given on the generators of $\mathbb{F}_{\mathbb{C}} = \mathbb{C}(x_1, \dots, x_{N+M})$ by the formula

$$\mathcal{T}_{\mathbf{q}}^{W}(\widetilde{\mathbf{x}}) = \left(x_i \prod_{\alpha=1}^{s} q_\alpha^{w_{i\alpha}} \right)_{i=1}^{N+M}, \qquad \mathbf{q} = (q_1, \dots, q_s) \in (\mathbb{C}^*)^s, \tag{2.5}$$

where $W = (w_{i\alpha})$ is an integer $(N+M) \times s$ *weight matrix* of full rank, and extended naturally to the whole $\mathbb{F}_{\mathbb{C}}$.

Let $\widetilde{\mathbf{x}}'$ be another extended cluster in $\mathcal{GC}$, then the corresponding local toric action defined by the weight matrix W' is *compatible* with the local toric action (2.5) if it commutes with the sequence of (generalized) cluster transformations that takes $\widetilde{\mathbf{x}}$ to $\widetilde{\mathbf{x}}'$. If local toric actions at all clusters are compatible, they define a *global toric action* $\mathcal{T}_{\mathbf{q}}$ on $\mathbb{F}_{\mathbb{C}}$ called a $\mathcal{GC}$-extension of the local toric action (2.5). As shown in [9, Section 5.2], for a global toric action to be well defined, it suffices that local toric actions at all seeds adjacent to the initial one are compatible. The following statement is equivalent to [12, Proposition 2.6].

Proposition 2.2. *The local toric action (2.5) is uniquely $\mathcal{GC}$-extendable to a global action of $(\mathbb{C}^*)^s$ if all y-variables y_k and all Casimirs $\hat{p}_{ir}$ are invariant under (2.5).*

3 | THE STRUCTURE $\overline{\mathcal{GC}}_n^D$

In this section, we provide a description of the seed $\bar{\Sigma}_n$ and prove that the corresponding generalized cluster structure $\overline{\mathcal{GC}}_n^D$ is complete and compatible with the standard Poisson–Lie structure on $D(GL_n)$.

First, we list some terms and notations that will be used in what follows. A notation A_I^J is reserved for a submatrix of a matrix A with a row set I and a column set J. If I (respectively, J) is not specified, it is assumed that all rows (respectively, columns) are selected. An interval notation $[i, j]$ is used for the index set $[i, i+1, \dots, j]$. We call a submatrix or minor of A *dense* if both its row and column sets are intervals. A dense minor of A is called *trailing* if it contains the lower right entry of A.

3.1 | The initial seed

Let $(X, Y) \in D(GL_n) = GL_n \times GL_n$. Following [13], define an $N \times N$ matrix

$$\Phi = \Phi(X,Y) = \begin{pmatrix} Y_{[2,n]} & & & & \\ X_{[2,n]} & Y_{[2,n]} & & & \\ & \ddots & \ddots & & \\ & & & X_{[2,n]} & Y_{[2,n]} \\ & & & & X_{[2,n]} \end{pmatrix} \tag{3.1}$$

with $N = (n-1)n$ and put $\varphi_i = \det \Phi_{[i,N]}^{[i,N]}$ for $1 \leqslant i \leqslant N$. Further, put $\det(\lambda Y + \mu X) = \sum_{i=0}^{n} c_i(X,Y)\mu^i \lambda^{n-i}$.

Next, we define $g_{ij} = \det X_{[i,n]}^{[j,j+n-i]}$ for $1 \leqslant j \leqslant i \leqslant n$, and $h_{ij} = \det Y_{[i,i+n-j]}^{[j,n]}$ for $1 \leqslant i \leqslant j \leqslant n$; note that $\varphi_i = g_{i-N+n-1,i-N+n-1}$ for $i > N-n+1$. The family $\bar{\mathcal{F}}_n$ of $2n^2$ functions in the ring of regular functions on $D(GL_n)$ is defined as

$$\bar{\mathcal{F}}_n = \left\{ \{\varphi_i\}_{i=1}^{N-n+1}; \{g_{ij}\}_{1\leqslant j\leqslant i\leqslant n}; \{h_{ij}\}_{1\leqslant i\leqslant j\leqslant n}; \{\tilde{c}_i\}_{i=1}^{n-1} \right\}$$

with $\tilde{c}_i(X,Y) = (-1)^{i(n-1)} c_i(X,Y)$ for $1 \leqslant i \leqslant n-1$.

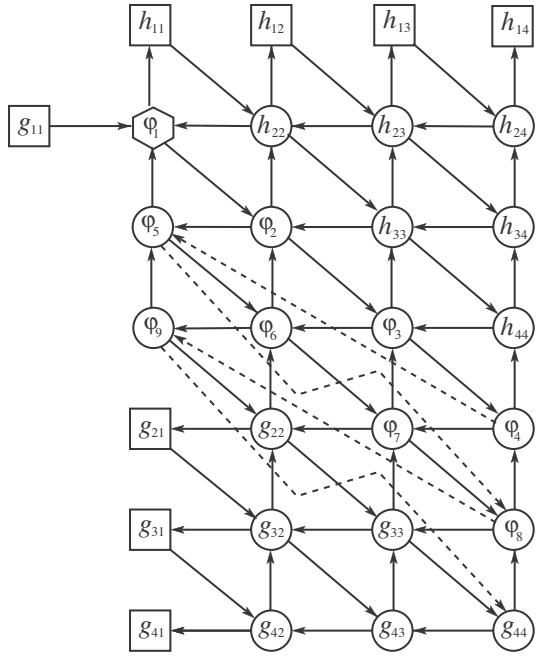

FIGURE 1 Quiver $\bar{Q}_4$

The corresponding quiver $\bar{Q}_n$ is defined below and illustrated, for the $n = 4$ case, in Figure 1. It has $2n^2$ vertices corresponding to the functions in $\bar{\mathcal{F}}_n$. The $n-1$ vertices corresponding to $\tilde{c}_i(X, Y)$, $1 \leqslant i \leqslant n-1$, are isolated; they are not shown. There are $2n$ frozen vertices corresponding to g_{i1}, $1 \leqslant i \leqslant n$, and h_{1j}, $1 \leqslant j \leqslant n$; they are shown as squares in the figure below. All vertices except for one are arranged into a $(2n-1) \times n$ grid; we will refer to vertices of the grid using their position in the grid numbered top to bottom and left to right. The edges of $\bar{Q}_n$ are $(i, j) \to (i+1, j+1)$ for $i = 1, \dots, 2n-2$, $j = 1, \dots, n-1$, $(i, j) \to (i, j-1)$ and $(i, j) \to (i-1, j)$ for $i = 2, \dots, 2n-1$, $j = 2, \dots, n$, and $(i, 1) \to (i-1, 1)$ for $i = 2, \dots, n$. Additionally, there is an oriented path

$$(n+1, n) \to (3, 1) \to (n+2, n) \to (4, 1) \to \cdots \to (n, 1) \to (2n-1, n).$$

The edges in this path are depicted as dashed in Figure 1 (the dashed style does not indicate any special features of these edges, it is for visualization purposes only). The vertex (2,1) is special; it is shown as a hexagon in the figure. The last remaining vertex of $\bar{Q}_n$ is placed to the left of the special vertex and there is an edge pointing from the former one to the latter.

Functions h_{ij} are attached to the vertices (i, j), $1 \leqslant i \leqslant j \leqslant n$, and all vertices in the upper row of $\bar{Q}_n$ are frozen. Functions g_{ij} are attached to the vertices $(n+i-1, j)$, $1 \leqslant j \leqslant i \leqslant n$, $(i, j) \neq (1, 1)$, and all such vertices in the first column are frozen. The function g_{11} is attached to the vertex to the left of the special one, and this vertex is frozen. Functions φ_{kn+i} are attached to the vertices $(i+k+1, i)$ for $1 \leqslant i \leqslant n$, $0 \leqslant k \leqslant n-3$; the function φ_{N-n+1} is attached to the vertex $(n, 1)$. All these vertices are mutable. One can identify in $\bar{Q}_n$ three regions associated with three families $\{g_{ij}\}, \{h_{ij}\}, \{\varphi_k\}$. We will call vertices in these regions g-, h- and φ-vertices, respectively. The set of strings $\bar{\mathcal{P}}_n$ contains a unique nontrivial string $(1, \tilde{c}_1(X, Y), \dots, \tilde{c}_{n-1}(X, Y), 1)$ corresponding to the unique special vertex.

Theorem 3.1. *The seed $\bar{\Sigma}_n = (\bar{F}_n, \bar{Q}_n, \bar{P}_n)$ defines a complete generalized cluster structure $\overline{\mathcal{GC}}_n^D$ in the ring of regular functions on the Drinfeld double $D(GL_n)$ compatible with the standard Poisson–Lie structure on $D(GL_n)$.*

Proof. Regularity of $\overline{\mathcal{GC}}_n^D$ is proved in [13, Theorem 4.1]. To prove compatibility, it suffices to check that the family $\bar{F}_n$ is log-canonical with respect to the bracket $\{\cdot,\cdot\}_D$, which is done in Section 3.2, and to check the compatibility conditions of Proposition 2.1, which is done in Section 3.3. To prove the completeness, we establish a connection between cluster dynamics for standard cluster structures on rectangular matrices and that for $\overline{\mathcal{GC}}_n^D$ with certain vertices frozen, see Section 3.4. As a consequence, we prove in Proposition 3.7 that all matrix entries in Y, and all matrix entries in X except for the first row are cluster variables in $\overline{\mathcal{GC}}_n^D$. The entries in the first row of X are treated separately in Lemma 3.8. □

3.2 | Log-canonicity

Denote by $\mathfrak{b}_\pm$ Borel subalgebras of upper/lower triangular matrices in $\mathfrak{gl}_n$ and by $\mathfrak{n}_\pm$ the corresponding nilpotent ideals. Let $\pi_{>0}$, $\pi_{<0}$ be the projections of an element of $\mathfrak{gl}_n$ onto $\mathfrak{n}_+$, $\mathfrak{n}_-$, π_0 be the projection onto the diagonal subalgebra, $R_+ = \frac{1}{2}\pi_0 + \pi_{>0}$. As explained in [12, Section 2.2], the *standard Poisson–Lie bracket* $\{\cdot,\cdot\} = \{\cdot,\cdot\}_D$ on $D(GL_n)$ can be written as

$$\begin{aligned}\{f_1, f_2\} &= \big\langle R_+(E_L f_1), E_L f_2\big\rangle - \big\langle R_+(E_R f_1), E_R f_2\big\rangle \\ &\quad + \langle X\nabla_X f_1, Y\nabla_Y f_2\rangle - \langle \nabla_X f_1 \cdot X, \nabla_Y f_2 \cdot Y\rangle \\ &= \big\langle R_+(E_L f_1), E_L f_2\big\rangle - \big\langle R_+(E_R f_1), E_R f_2\big\rangle \\ &\quad + \langle E_R f_1, Y\nabla_Y f_2\rangle - \langle E_L f_1, \nabla_Y f_2 \cdot Y\rangle,\end{aligned} \tag{3.2}$$

where $\nabla_X f = (\frac{\partial f}{\partial x_{ji}})_{i,j=1}^n$ and $\nabla_Y f = (\frac{\partial f}{\partial y_{ji}})_{i,j=1}^n$ are the gradients of f with respect to X and Y, respectively, the operators E_R and E_L are defined via

$$E_R f = X\nabla_X f + Y\nabla_Y f, \qquad E_L f = \nabla_X f \cdot X + \nabla_Y f \cdot Y,$$

and $\langle A, B\rangle = \operatorname{Tr} AB$ is the trace form; in what follows we will omit the comma and write just $\langle AB\rangle$.

Note that the functions $\tilde{c}_i$ are Casimirs for $\{\cdot,\cdot\}$. One way to see this is by observing that any function f on $D(GL_n)$ that has a property, shared by all $\tilde{c}_i$, that $f(AXB, AYB) = \det A \det B \cdot f(X,Y)$ for any $X, Y, A, B \in GL_n$ is a Casimir. For such f, $E_R f = E_L f$ is a scalar multiple of the identity matrix and so the claim follows from the second formula in (3.2) and the identity $\operatorname{Tr} AB = \operatorname{Tr} BA$. Thus, we only need to treat the functions in the three other subfamilies in $\bar{F}_n$.

Lemma 3.2.

(i) *For any $1 \leqslant j \leqslant i \leqslant n$ and $1 \leqslant k \leqslant N$,*

$$\begin{aligned}&g_{ij}(X) = g_{ij}(N_+X), \qquad h_{ji}(Y) = h_{ji}(YN_-),\\ &\varphi_k(X,Y) = \varphi_k(N_+XN_-, N_+YN_-),\end{aligned} \tag{3.3}$$

where N_+ is an arbitrary unipotent upper triangular matrix and N_- is an arbitrary unipotent lower triangular matrix. In addition, g_{ij} and h_{ij} are homogeneous with respect to right and left multiplication of their arguments by arbitrary diagonal matrices, and φ_k are homogeneous with respect to right and left multiplication of X, Y by the same pair of diagonal matrices.

(ii) *Let g, h and φ be three functions possessing invariance properties (3.3), respectively. Then*

$$\begin{aligned}\{\varphi, g\} &= \frac{1}{2}\langle E_L\varphi, \nabla_X g\cdot X\rangle_0 - \frac{1}{2}\langle E_R\varphi, X\nabla_X g\rangle_0,\\ \{\varphi, h\} &= \frac{1}{2}\langle E_R\varphi, Y\nabla_Y h\rangle_0 - \frac{1}{2}\langle E_L\varphi, \nabla_Y h\cdot Y\rangle_0,\\ \{g, h\} &= \frac{1}{2}\langle X\nabla_X g, Y\nabla_Y h\rangle_0 - \frac{1}{2}\langle \nabla_X g\cdot X, \nabla_Y h\cdot Y\rangle_0,\end{aligned} \tag{3.4}$$

where $\langle A, B\rangle_0 = \langle AB\rangle_0 = \langle \pi_0(A)\pi_0(B)\rangle$.

Proof.

(i) Invariance properties of functions g_{ij}, h_{ij}, φ_k follow easily from their definition.

(ii) Infinitesimally, equations (3.3) imply that $X\nabla_X g, E_R\varphi \in \mathfrak{b}_+$ and $\nabla_Y h\cdot Y, E_L\varphi \in \mathfrak{b}_-$. Taking into account that $R_+(\xi) = \frac{1}{2}\pi_0(\xi)$ for any $\xi \in \mathfrak{b}_\pm$ and that $\mathfrak{b}_\pm \perp \mathfrak{n}_\pm$ with respect to $\langle\cdot,\cdot\rangle$, the result follows from (3.2). □

The fact that any two of the three functions g_{ij}, h_{pq}, φ_k are log-canonical follows immediately from the above lemma. Indeed, the infinitesimal version of homogeneity properties described in Lemma 3.2(i) states that for functions $g = \log g_{ij}$, $h = \log h_{pq}$, $\varphi = \log\varphi_k$, projections of $\nabla_X g\cdot X$, $X\nabla_X g$, $Y\nabla_Y h$, $\nabla_Y h\cdot Y$, $E_L\varphi$, $E_R\varphi$ onto the Cartan subalgebra are constant diagonal matrices and the claim then follows from (3.4). Log-canonicity of the families $\{g_{ij}\}$ and $\{h_{ij}\}$ is established in [12, Lemma 5.4].

It remains to show that $\{\log\varphi_k, \log\varphi_l\}$ is constant for any k, l. In fact, it suffices to consider the case $1 \leqslant k \leqslant l \leqslant N-n+1$, since for $k > N-n+1$ the function φ_k belongs to the family $\{g_{ij}\}$.

Denote $\nabla_X\log\varphi_k, \nabla_Y\log\varphi_k, E_R\log\varphi_k, E_L\log\varphi_k$ by $\nabla_X^k, \nabla_Y^k, E_R^k, E_L^k$, respectively. As mentioned in the proof of Lemma 3.2 (ii), $E_R^k \in \mathfrak{b}_+$, $E_L^k \in \mathfrak{b}_-$. Consequently, (3.2) gives

$$\begin{aligned}\{\log\varphi_k, \log\varphi_l\} = \frac{1}{2}\left\langle E_L^k E_L^l\right\rangle_0 - \frac{1}{2}\left\langle E_R^k E_R^l\right\rangle_0\\ + \left\langle X\nabla_X^k Y\nabla_Y^l\right\rangle - \left\langle \nabla_X^k X\nabla_Y^l Y\right\rangle.\end{aligned} \tag{3.5}$$

It follows from the homogeneity of functions φ_k (Lemma 3.2(i)) that the first two terms above are constant. Thus, we need to evaluate

$$\left\langle X\nabla_X^k Y\nabla_Y^l\right\rangle - \left\langle \nabla_X^k X\nabla_Y^l Y\right\rangle. \tag{3.6}$$

For a smooth function F on Mat_N, we write its gradient $\nabla_\Phi F$ in a block form as $\nabla_\Phi F = (\nabla_{pq})_{p=1,q=1}^{n-1,n}$, where the blocks ∇_{pq} are of size $n\times(n-1)$. For functions φ_k viewed as functions on Mat_N, we denote $\nabla_\Phi\log\varphi_k$ by $\nabla_\Phi^k = (\nabla_{pq}^k)_{p=1,q=1}^{n-1,n}$.

Observe that if one views $l \times l$ trailing principal minor of a square matrix A as a function $f(A)$ of A, then

$$A\nabla f(A) = \begin{pmatrix} 0 & \star \\ 0 & \mathbf{1}_l \end{pmatrix}, \qquad \nabla f(A)A = \begin{pmatrix} 0 & 0 \\ \star & \mathbf{1}_l \end{pmatrix}.$$

Thus

$$\Phi\nabla_\Phi^k = \begin{pmatrix} 0 & \star \\ 0 & \mathbf{1}_{N-k+1} \end{pmatrix}, \qquad \nabla_\Phi^k\Phi = \begin{pmatrix} 0 & 0 \\ \star & \mathbf{1}_{N-k+1} \end{pmatrix}. \tag{3.7}$$

Denote $\mathcal{X} = X_{[2,n]}$, $\mathcal{Y} = Y_{[2,n]}$. Then (3.7) translates into

$$\begin{aligned} &\mathcal{X}\nabla_{pq}^k + \mathcal{Y}\nabla_{p+1,q}^k = 0, \quad p \geqslant q, \\ &\mathcal{X}\nabla_{n-1,q}^k = 0, \quad q < n, \\ &\nabla_{pq}^k\mathcal{Y} + \nabla_{p,q+1}^k\mathcal{X} = 0, \quad p < q. \end{aligned} \tag{3.8}$$

Clearly,

$$\left\langle X\nabla_X^k Y\nabla_Y^l \right\rangle = \left\langle \mathcal{X}\sum_{p=1}^{n-1}\nabla_{p,p+1}^k\mathcal{Y}\sum_{q=1}^{n-1}\nabla_{qq}^l \right\rangle,$$

$$\left\langle \nabla_X^k X\nabla_Y^l Y \right\rangle = \left\langle \sum_{p=1}^{n-1}\nabla_{p,p+1}^k\mathcal{X}\sum_{q=1}^{n-1}\nabla_{qq}^l\mathcal{Y} \right\rangle.$$

Denote $A_{pq} = \langle \mathcal{X}\nabla_{p,p+1}^k\mathcal{Y}\nabla_{qq}^l \rangle$, $B_{pq} = \langle \nabla_{p,p+1}^k\mathcal{X}\nabla_{qq}^l\mathcal{Y} \rangle$. Using (3.8), we obtain

$$\begin{aligned} A_{pq} &= -\left\langle \mathcal{X}\nabla_{p,p+2}^k\mathcal{X}\nabla_{qq}^l \right\rangle = \left\langle \mathcal{X}\nabla_{p,p+2}^k\mathcal{Y}\nabla_{q+1,q}^l \right\rangle = \cdots \\ &= -\left\langle \mathcal{X}\nabla_{p,p+s+1}^k\mathcal{X}\nabla_{q+s-1,q}^l \right\rangle = \left\langle \mathcal{X}\nabla_{p,p+s+1}^k\mathcal{Y}\nabla_{q+s,q}^l \right\rangle = \cdots \\ &= \begin{cases} -\left\langle \mathcal{X}\nabla_{p,p-q+n+1}^k\mathcal{X}\nabla_{n-1,q}^l \right\rangle = 0, & p < q, \\ \left\langle \mathcal{X}\nabla_{pn}^k\mathcal{Y}\nabla_{q-p+n-1,q}^l \right\rangle, & p \geqslant q. \end{cases} \end{aligned}$$

Similarly,

$$B_{pq} = \begin{cases} 0, & p \leqslant q, \\ -\left\langle \nabla_{pn}^k\mathcal{Y}\nabla_{q-p+n,q}^l\mathcal{Y} \right\rangle, & p > q. \end{cases}$$

Therefore,

$$\begin{aligned} &\left\langle X\nabla_X^k Y\nabla_Y^l \right\rangle - \left\langle \nabla_X^k X\nabla_Y^l Y \right\rangle \\ &\quad = \sum_{1\leqslant q\leqslant p\leqslant n-1} \left\langle \mathcal{X}\nabla_{pn}^k\mathcal{Y}\nabla_{q-p+n-1,q}^l \right\rangle + \sum_{1\leqslant q<p\leqslant n-1} \left\langle \mathcal{Y}\nabla_{pn}^k\mathcal{Y}\nabla_{q-p+n,q}^l \right\rangle \end{aligned}$$

$$= \sum_{1\leqslant q\leqslant p\leqslant n-1} \left\langle \mathcal{X}\nabla^k_{pn}\mathcal{Y}\nabla^l_{q-p+n-1,q}\right\rangle + \sum_{1\leqslant q\leqslant p\leqslant n-2} \left\langle \mathcal{Y}\nabla^k_{p+1,n}\mathcal{Y}\nabla^l_{q-p+n-1,q}\right\rangle$$

$$= \sum_{1\leqslant q\leqslant n-1} \left\langle (\Phi\nabla^k_\Phi)_{nn}\mathcal{Y}\nabla^l_{qq}\right\rangle + \sum_{1\leqslant q\leqslant p\leqslant n-2} \left\langle (\Phi\nabla^k_\Phi)_{p+1,n}\mathcal{Y}\nabla^l_{q-p+n-1,q}\right\rangle. \tag{3.9}$$

Since $(\Phi\nabla^k_\Phi)_{nn} = \mathbf{1}_{n-1}$ for $k \leqslant N-n+1$, the first sum above is equal to $\sum_{1\leqslant q\leqslant n-1}\langle \mathcal{Y}\nabla^l_{qq}\rangle$. Furthermore, $\nabla^l_{q-p+n-1,q} = 0$ for $l > q(n-1)+1$, while $(\Phi\nabla^k_\Phi)_{p+1,n} = 0$ for $k \leqslant p(n-1)+1$. Thus, the summand in the second sum in (3.9) is zero unless $p(n-1)+1 < k \leqslant l \leqslant q(n-1)+1$, which is impossible since $q \leqslant p$. We conclude that

$$\left\langle X\nabla^k_X Y\nabla^l_Y\right\rangle - \left\langle \nabla^k_X X\nabla^l_Y Y\right\rangle = \sum_{1\leqslant q\leqslant n-1}\left\langle \mathcal{Y}\nabla^l_{qq}\right\rangle$$

for $1 \leqslant k \leqslant l \leqslant N-n+1$.

Let us show that the right-hand side above is constant. To this end, first observe that

$$\sum_{1\leqslant q\leqslant n-1}\left\langle \mathcal{Y}\nabla^l_{qq}\right\rangle = \left.\frac{d}{dt}\right|_{t=0} \log\varphi_l\left(X, e^tY\right).$$

Furthermore,

$$\Phi\left(X, e^tY\right) = \operatorname{diag}\left(e^{nt}\mathbf{1}_{n-1},\dots,e^t\mathbf{1}_{n-1}\right)\Phi(X,Y)\operatorname{diag}\left(e^{-(n-1)t}\mathbf{1}_n,\dots,e^{-t}\mathbf{1}_n\right).$$

Then $\varphi_l(X,e^tY) = \det\Phi(X,e^tY)^{[l,N]}_{[l,N]} = e^{\varkappa_l t}\varphi_l(X,Y)$, where constant coefficients $\varkappa_l$ can be arranged into a matrix

$$\left(\varkappa_{\mu(n-1)+\sigma}\right)^{n-1\ n-1}_{\mu=0\ \sigma=1} = \begin{pmatrix} \binom{n}{2} & \binom{n}{2}-1 & \binom{n}{2}-2 & \cdots & \binom{n-1}{2}+1 \\ \binom{n-1}{2} & \binom{n-1}{2} & \binom{n-1}{2}-1 & \cdots & \binom{n-2}{2}+1 \\ \binom{n-2}{2} & \binom{n-2}{2} & \binom{n-2}{2} & \cdots & \binom{n-3}{2}+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.10}$$

Therefore, $\sum_{1\leqslant q\leqslant n-1}\langle \mathcal{Y}\nabla^l_{qq}\rangle = \varkappa_l$ and we are done.

3.3 | Compatibility

To prove the compatibility statement, we start with the following lemma, which is a direct analog of [12, Theorem 6.1].

Lemma 3.3. *The action*

$$(X,Y) \mapsto (T_1XT_2, T_1YT_2) \tag{3.11}$$

of right and left multiplication by diagonal matrices is $\overline{\mathcal{GC}}^D_n$-extendable to a global toric action on $\mathcal{F}_{\mathbb{C}}$.

Proof. As explained in [12], Proposition 2.2 implies that it suffices to check that y-variables (2.4) are homogeneous functions of degree zero with respect to the action (3.11) (note that the functions $\hat{p}_{1r}$ for $\overline{\mathcal{GC}}_n^D$ are the same as for $\mathcal{GC}_n^D$). To this effect, we define weights $\xi_L(f) = \pi_0(E_L \log f)$ and $\xi_R(f) = \pi_0(E_R \log f)$. For $1 \leqslant i \leqslant j \leqslant n$, let $\Delta(i,j)$ denote a diagonal matrix with ones in the entries $(i,i), \dots, (j,j)$ and zeros everywhere else. Direct computation shows that

$$\begin{aligned}
&\xi_L(g_{ij}) = \Delta(j, n+j-i), \quad \xi_R(g_{ij}) = \Delta(i,n),\\
&\xi_L(h_{ij}) = \Delta(j,n), \quad \xi_R(h_{ij}) = \Delta(i, n+i-j),\\
&\xi_L(\varphi_k) = (n-2-\lambda_k)\Delta(1,n) + \Delta(\rho_k, n),\\
&\xi_R(\varphi_k) = (n-1-\mu_k)\Delta(2,n) + \Delta(\sigma_k+1, n),
\end{aligned} \tag{3.12}$$

where λ_k, ρ_k, μ_k and σ_k are defined via $k = \lambda_k n + \rho_k$, $1 \leqslant \rho_k \leqslant n$, and $k = \mu_k(n-1) + \sigma_k$, $1 \leqslant \sigma_k \leqslant n-1$. Now the verification of the claim above becomes straightforward. It is based on the description of $\bar{Q}_n$ in Section 3.1 and the fact that for a Laurent monomial in homogeneous functions $M = \psi_1^{\alpha_1}\psi_2^{\alpha_2}\cdots$ the right and left weights are $\xi_{R,L}(M) = \alpha_1\xi_{R,L}(\psi_1) + \alpha_2\xi_{R,L}(\psi_2) + \cdots$.

For example, let v be the vertex associated with the function φ_k, $n+1 \leqslant k \leqslant N-n$. To treat the left weight of y_v we have to consider the following three cases.

Case 1: $\lambda_{k-1} = \lambda_k = \lambda_{k+1} = \lambda$. Consequently, $\lambda_{k-n} = \lambda_{k-n+1} = \lambda - 1$, $\lambda_{k+n-1} = \lambda_{k+n} = \lambda + 1$ and $\rho_{k-1} = \rho_{k+n-1} = \rho$, $\rho_{k-n} = \rho_{k+n} = \rho + 1$, $\rho_{k+1} = \rho_{k-n+1} = \rho + 2$. Therefore, (3.12) yields

$$\begin{aligned}
&\xi_L(y_v)\\
&\quad = \xi_L(\varphi_{k-1}) + \xi_L(\varphi_{k+n}) + \xi_L(\varphi_{k-n+1}) - \xi_L(\varphi_{k+1}) - \xi_L(\varphi_{k-n}) - \xi_L(\varphi_{k+n-1})\\
&\quad = (n-2-\lambda)\Delta(1,n) + \Delta(\rho,n) + (n-2-\lambda-1)\Delta(1,n) + \Delta(\rho+1,n)\\
&\quad + (n-2-\lambda+1)\Delta(1,n) + \Delta(\rho+2,n) - (n-2-\lambda)\Delta(1,n) - \Delta(\rho+2,n)\\
&\quad - (n-2-\lambda+1)\Delta(1,n) - \Delta(\rho+1,n) - (n-2-\lambda-1)\Delta(1,n) - \Delta(\rho,n) = 0.
\end{aligned}$$

Case 2: $\lambda_{k-1} = \lambda_k = \lambda$, $\lambda_{k+1} = \lambda + 1$. Consequently, $\lambda_{k-n} = \lambda - 1$, $\lambda_{k-n+1} = \lambda$, $\lambda_{k+n-1} = \lambda_{k+n} = \lambda + 1$ and $\rho_{k-1} = \rho_{k+n-1} = n-1$, $\rho_{k-n} = \rho_{k+n} = n$, $\rho_{k+1} = \rho_{k-n+1} = 1$. Therefore, (3.12) yields

$$\begin{aligned}
&\xi_L(y_v)\\
&\quad = \xi_L(\varphi_{k-1}) + \xi_L(\varphi_{k+n}) + \xi_L(\varphi_{k-n+1}) - \xi_L(\varphi_{k+1}) - \xi_L(\varphi_{k-n}) - \xi_L(\varphi_{k+n-1})\\
&\quad = (n-2-\lambda)\Delta(1,n) + \Delta(n-1,n) + (n-2-\lambda-1)\Delta(1,n) + \Delta(n,n)\\
&\quad + (n-2-\lambda)\Delta(1,n) + \Delta(1,n) - (n-2-\lambda-1)\Delta(1,n) - \Delta(1,n)\\
&\quad - (n-2-\lambda+1)\Delta(1,n) - \Delta(n,n) - (n-2-\lambda-1)\Delta(1,n) - \Delta(n-1,n) = 0.
\end{aligned}$$

Case 3: $\lambda_{k-1} = \lambda$, $\lambda_k = \lambda_{k+1} = \lambda + 1$. Consequently, $\lambda_{k-n} = \lambda_{k-n+1} = \lambda$, $\lambda_{k+n-1} = \lambda + 1$, $\lambda_{k+n} = \lambda + 2$ and $\rho_{k-1} = \rho_{k+n-1} = n$, $\rho_{k-n} = \rho_{k+n} = 1$, $\rho_{k+1} = \rho_{k-n+1} = 2$. Therefore, (3.12)

yields

$$\begin{aligned}
&\xi_L(y_v)\\
&\quad = \xi_L(\varphi_{k-1}) + \xi_L(\varphi_{k+n}) + \xi_L(\varphi_{k-n+1}) - \xi_L(\varphi_{k+1}) - \xi_L(\varphi_{k-n}) - \xi_L(\varphi_{k+n-1})\\
&\quad = (n-2-\lambda)\Delta(1,n) + \Delta(n,n) + (n-2-\lambda-2)\Delta(1,n) + \Delta(1,n)\\
&\quad + (n-2-\lambda)\Delta(1,n) + \Delta(2,n) - (n-2-\lambda-1)\Delta(1,n) - \Delta(2,n)\\
&\quad - (n-2-\lambda)\Delta(1,n) - \Delta(1,n) - (n-2-\lambda-1)\Delta(1,n) - \Delta(n,n) = 0.
\end{aligned}$$

The right weight of y_v is treated in a similar way. □

The verification of compatibility conditions in Proposition 2.1 is based on relations (3.4), (3.5) and (3.12). The case when the vertices i and j belong to different regions in $\bar{Q}_n$, or are simultaneously non-diagonal g- or h-vertices, is treated in the same way as in [12]. For the case when both i and j are φ-vertices, the check is based on the following claim in which we assume $\varkappa_{N+1} = 0$.

Proposition 3.4.

(i) *For any k, $1 \leqslant k \leqslant N$, either $\lambda_k = \mu_k$ and $\varkappa_k = \varkappa_{k+1} + 1$ or $\lambda_k = \mu_k - 1$ and $\varkappa_k = \varkappa_{k+1}$.*
(ii) *For any k, $1 \leqslant k \leqslant N - n$,*

$$\varkappa_k - \varkappa_{k+1} = \begin{cases} \varkappa_{k+n} - \varkappa_{k+n+1}, & \sigma_k \neq n-1,\\ \varkappa_{k+n} - \varkappa_{k+n+1} + 1, & \sigma_k = n-1. \end{cases} \tag{3.13}$$

Proof. Follows immediately from (3.10) and the definitions of λ_k, ρ_k, μ_k and σ_k. □

Let us check the compatibility condition for the case when v is the vertex associated with the function φ_i and u is the vertex associated with the function φ_j, $n+1 \leqslant j \leqslant N-n$. There are five possible cases: (i) $i \leqslant j-n$; (ii) $j-n+1 \leqslant i \leqslant j-1$; (iii) $i = j$; (iv) $j+1 \leqslant i \leqslant j+n-1$; (v) $j+n \leqslant i$.

In the first case, the left-hand side of the condition in Proposition 2.1 is computed directly via (3.5). The first term equals

$$\begin{aligned}
\frac{1}{2}\langle \xi_L(\varphi_i)\xi_L(y_u)\rangle &= \frac{1}{2}\langle \xi_L(\varphi_i)\xi_L\big(\varphi_{j-1}/\varphi_j\big)\rangle - \frac{1}{2}\langle \xi_L(\varphi_i)\xi_L\big(\varphi_{j+n-1}/\varphi_{j+n}\big)\rangle\\
&\quad + \frac{1}{2}\langle \xi_L(\varphi_i)\xi_L\big(\varphi_j/\varphi_{j+1}\big)\rangle - \frac{1}{2}\langle \xi_L(\varphi_i)\xi_L\big(\varphi_{j-n}/\varphi_{j-n+1}\big)\rangle
\end{aligned}$$

and vanishes by Lemma 3.3. Similarly, the second term equals

$$\begin{aligned}
-\frac{1}{2}\langle \xi_R(\varphi_i)\xi_R(y_u)\rangle &= -\frac{1}{2}\langle \xi_R(\varphi_i)\xi_R\big(\varphi_{j-1}/\varphi_j\big)\rangle + \frac{1}{2}\langle \xi_R(\varphi_i)\xi_R\big(\varphi_{j+n-1}/\varphi_{j+n}\big)\rangle\\
&\quad - \frac{1}{2}\langle \xi_R(\varphi_i)\xi_R\big(\varphi_j/\varphi_{j+1}\big)\rangle + \frac{1}{2}\langle \xi_R(\varphi_i)\xi_R\big(\varphi_{j-n}/\varphi_{j-n+1}\big)\rangle
\end{aligned}$$

and vanishes by Lemma 3.3. The third term equals

$$(\varkappa_{j-1} - \varkappa_j) - (\varkappa_{j+n-1} - \varkappa_{j+n}) + (\varkappa_j - \varkappa_{j+1}) - (\varkappa_{j-n} - \varkappa_{j-n+1})$$

and vanishes by (3.13) since $\sigma_{j-1} = \sigma_{j-n}$.

In the second case, in order to use (3.5), one has to swap the arguments in the brackets $\{\log\varphi_i, \log\varphi_{j-n}\}_D$ and $\{\log\varphi_i, \log\varphi_{j-n+1}\}_D$. Consequently, the first term contributes

$$\langle \xi_L(\varphi_i)\xi_L(\varphi_{j-n}/\varphi_{j-n+1})\rangle,$$

the second term contributes

$$-\langle \xi_R(\varphi_i)\xi_R(\varphi_{j-n}/\varphi_{j-n+1})\rangle,$$

and the third term contributes $\varkappa_{j-n} - \varkappa_{j-n+1}$. By Proposition 3.4(i), either $\varkappa_{j-n} - \varkappa_{j-n+1} = 0$ and $\lambda_{j-n} = \mu_{j-n} - 1$ which implies

$$\langle \xi_L(\varphi_i)\xi_L(\varphi_{j-n}/\varphi_{j-n+1})\rangle = \langle \xi_R(\varphi_i)\xi_R(\varphi_{j-n}/\varphi_{j-n+1})\rangle = n - \lambda_{j-n} - 2,$$

or $\varkappa_{j-n} - \varkappa_{j-n+1} = 1$ and $\lambda_{j-n} = \mu_{j-n}$ which implies

$$\langle \xi_L(\varphi_i)\xi_L(\varphi_{j-n}/\varphi_{j-n+1})\rangle = n - \lambda_{j-n} - 2,$$
$$\langle \xi_R(\varphi_i)\xi_R(\varphi_{j-n}/\varphi_{j-n+1})\rangle = n - \lambda_{j-n} - 1.$$

In both cases, the total additional contribution vanishes.

Cases (iii)–(v) are treated in a similar way.

3.4 | Completeness

We start with the following proposition.

Proposition 3.5. *There exists an $(n-1)\times(n-1)$ unipotent upper triangular matrix $G = G(X, Y)$ such that:*

(i) *entries of G are rational functions in X, Y whose denominators are monomials in cluster variables φ_{kn+1}, $k = 1, \dots, n-2$; and*

(ii) *the $(2n-1)\times n$ matrix $S = \begin{bmatrix} Y \\ GX_{[2,n]} \end{bmatrix}$ satisfies*

$$\det S^{[n-i+1,n]}_{[n-i+1+k,n+k]} = \frac{\varphi_{kn-i+1}}{\varphi_{kn+1}}, \qquad k = 1, \dots, n-2, \quad i = 1, \dots, n. \tag{3.14}$$

Proof. We will establish the claim by applying to the matrix Φ a sequence of $n-2$ transformations that do not affect its trailing principal minors. After the kth transformation, every X-block of Φ will be multiplied on the left by the same unipotent upper triangular matrix G_k, while the Y-block

in the jth block row of Φ will be replaced by $(G_k)^{[j,n-1]}_{[j,n-1]} Y_{[j+1,n]}$ preceded with $j-1$ zero rows for $j = 2, \dots, k$ and by $Y_{[k+1,n]}$ preceded with $k-1$ zero rows for $j = k+1, \dots, n-1$.

On the first step, we use the submatrix $\Psi_1 = \Phi^{[n+1,N]}_{[n+1,N]}$ to eliminate all the entries in the row immediately above Ψ_1 in the submatrix $\Phi^{[n+1,N]}$. To this effect, we multiply the block rows from 2 to n of Φ on the left by a block Töplitz upper triangular matrix

$$\begin{bmatrix} G_{11} & G_{12} & \dots & G_{1,n-1} \\ 0 & G_{11} & \dots & G_{1,n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G_{11} \end{bmatrix},$$

where G_{1j} for $j > 1$ are $(n-1) \times (n-1)$ matrices with the only non-zero entries lying in the first row and G_{11} is a unipotent upper triangular matrix with the only off-diagonal non-zero entries lying in the first row. Note that the denominator of the non-trivial entries in G_{1j} is equal to $\det \Psi_1 = \varphi_{n+1}$. As a result, all X-blocks of Φ are multiplied by $G_1 = G_{11}$, and Y-blocks in the block rows $2, \dots, n-1$ are replaced by $Y_{[3,n]}$ preceded with the zero row. All zero blocks are not changed. The obtained matrix is block lower triangular, and hence (3.14) is valid for $k = 1$, $i = 1, \dots, n$, for the matrix $S_1 = \begin{bmatrix} Y \\ G_1 X_{[2,n]} \end{bmatrix}$.

On the second step, we use the submatrix $\Psi_2 = \Phi^{[2n+1,N]}_{[2n+1,N]}$ to eliminate all the entries in the row immediately above Ψ_2 in the submatrix formed by columns $[2n+1, N]$ of the matrix obtained on the previous step. To this effect, we multiply the block rows from 2 to n of this matrix on the left by a block upper triangular matrix

$$\begin{bmatrix} G_{21} & 0 & 0 & \dots & 0 \\ 0 & G_{21} & G_{22} & \dots & G_{2,n-2} \\ 0 & 0 & G_{21} & \dots & G_{2,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & G_{21} \end{bmatrix},$$

where G_{2j} for $j > 1$ are $(n-1) \times (n-1)$ matrices with the only non-zero entries lying in the second row and G_{21} is a unipotent upper triangular with the same property. Note that the denominator of the non-trivial entries in G_{2j} is equal to $\det \Psi_2 = \varphi_{2n+1}$. As a result, all X-blocks are multiplied by $G_2 = G_{21} G_{11}$, the Y-block in the second block row is replaced by $(G_2)^{[2,n-1]}_{[2,n-1]} Y_{[3,n]}$ preceded by the zero row and Y-blocks in the block rows $3, \dots, n-1$ are replaced by $Y_{[4,n]}$ preceded by two zero rows. The submatrix of the obtained matrix lying in rows and columns $n+1, \dots, N$ is block lower triangular, and hence (3.14) is valid for $k = 2$, $i = 1, \dots, n$, for the matrix $S_2 = \begin{bmatrix} Y \\ G_2 X_{[2,n]} \end{bmatrix}$. It is also valid for $k = 1$, $i = 1, \dots, n$, since the corresponding minors coincide with those for the matrix S_1.

Continuing in the same fashion, we define $G_{31}, \dots, G_{n-2,1}$ such that the product $G = G_{n-2,1} \cdots G_{11}$ satisfies properties (i) and (ii) of the claim. Note that the kth row of G coincides with the kth row of G_{k1}. □

The following proposition can be proved in a similar way.

Proposition 3.6. *There exists an $n \times n$ unipotent lower triangular matrix $H = H(X, Y)$ such that:*

(i) *entries of H are rational functions in X, Y whose denominators are monomials in cluster variables $\varphi_{k(n-1)+1}$, $k = 1, \dots, n-1$, and*
(ii) *the $(n-1) \times 2n$ matrix $T = [X_{[2,n]}\ Y_{[2,n]}H]$ satisfies*

$$\det T^{[n+k-i+1,n+k]}_{[n-i,n-1]} = \frac{\varphi_{(n-k)(n-1)-i+1}}{\varphi_{(n-k)(n-1)+1}}, \qquad k = 1, \dots, n-1, \quad i = 1, \dots, n-1. \tag{3.15}$$

We will use Propositions 3.5 and 3.6 to establish the following

Proposition 3.7. *All matrix entries of $X_{[2,n]}$ and Y are cluster variables in $\overline{\mathcal{GC}}^D_n$.*

Proof. We will use the comparison with the standard cluster structure $\widetilde{\mathcal{C}}_{l\times m}$ on $\mathrm{Mat}_{l\times m}$ which is isomorphic to the cluster structure on the big cell in the Grassmannian $\mathrm{Gr}(l, l+m)$ described in [9, Section 4.2.2] via the identification of a matrix $A \in \mathrm{Mat}_{l\times m}$ with a representative $[\mathbf{1}_l\ (AW_m)]$ of an element in $\mathrm{Gr}(l, l+m)$, where W_m is the $m \times m$ antidiagonal matrix. First, we establish the claim for matrix entries of $Y = (y_{ij})_{i,j=1}^n$. Let us temporarily treat the vertices in $\bar{Q}_n$ that correspond to variables φ_{kn+1}, $k = 0, \dots, n-2$, as frozen. Then the vertex corresponding to g_{11} becomes isolated. Denote the quiver formed by the rest of the non-isolated vertices by $\widehat{Q}_n$, and let $\widehat{\mathcal{F}}_n \subset \bar{\mathcal{F}}_n$ be the corresponding subset of cluster variables.

Define a new collection of variables $\tilde{\mathcal{F}}_n = \{\tilde{f} \colon f \in \widehat{\mathcal{F}}_n\}$ via $\tilde{\varphi}_{kn-i+1} = \frac{\varphi_{kn-i+1}}{\varphi_{kn+1}}$, $k = 1, \dots, n-2$, $i = 1, \dots n$, and $\tilde{f} = f$ for all other $f \in \widehat{\mathcal{F}}_n$. Denote by $\tilde{Q}_n$ the quiver obtained from $\widehat{Q}_n$ via deletion of all edges that are dashed in Figure 1.

Let us now define a $(2n-1) \times n$ matrix S as in Proposition 3.5. Then (3.14) ensures that the collection $\tilde{\mathcal{F}}_n$ consists of all dense minors of S containing entries of the last row or column of S, where the minor that has an (i, j)-entry of S in a top left corner is attached to the (i, j) vertex in the grid that describes $\tilde{Q}_n$. Viewed this way, $\widetilde{\Sigma} = (\tilde{\mathcal{F}}_n, \tilde{Q}_n)$ becomes the initial seed for the standard cluster structure $\widetilde{\mathcal{C}}_{(2n-1)\times n}$ on $\mathrm{Mat}_{(2n-1)\times n}$. Note that every exchange relation in this seed is obtained via dividing the corresponding exchange relation of $\overline{\mathcal{GC}}^D_n$ by an appropriate monomial in variables φ_{kn+1}, $k = 1, \dots, n-2$, which are frozen in $\widehat{\Sigma} = (\widehat{\mathcal{F}}_n, \hat{Q}_n)$. Applying [11, Lemma 8.4] repeatedly, we conclude that if φ is a cluster variable in $\overline{\mathcal{GC}}^D_n$ obtained via an arbitrary sequence of mutations not involving mutations at φ_{kn+1}, $k = 0, \dots, n-2$, then the result of the same sequence of mutations in $\widetilde{\mathcal{C}}_{(2n-1)\times n}$ is $\tilde{\varphi} = \frac{\varphi}{M}$, where M is a Laurent monomial in φ_{kn+1}, $k = 1, \dots, n-2$.

Since all matrix entries of S are cluster variables in $\widetilde{\mathcal{C}}_{(2n-1)\times n}$, the latter observation means that, for any $i, j \in [1, n]$, there is a cluster variable $\varphi \in \overline{\mathcal{GC}}^D_n$ obtained via a sequence of mutations not involving mutations at φ_{kn+1}, $k = 0, \dots, n-2$, and such that $\varphi = y_{ij}M$, where M is a Laurent monomial in φ_{kn+1}, $k = 1, \dots, n-2$. We will now show that $M = 1$. Indeed, since φ is a regular function in X, Y, and all functions in $\mathcal{F}_n$ are irreducible via [13, Lemma 4.2], all factors in M have non-negative degrees. On the other hand, in terms of elements of $\tilde{\mathcal{F}}_n$, φ is a polynomial in φ_{kn+1}, $k = 1, \dots, n-2$. Furthermore, as a polynomial in each φ_{kn+1} it has a non-zero constant term. This claim is obvious for every single mutation away from the initial cluster, and then is verified inductively. This implies that $M = 1$. Thus all matrix entries of Y are cluster variables in $\overline{\mathcal{GC}}^D_n$.

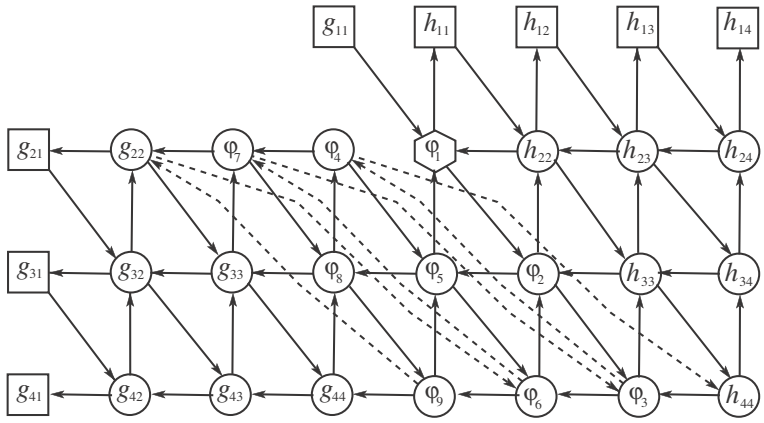

FIGURE 2 Rearranged quiver $\bar{Q}_4$

To treat matrix entries of $X_{[2,n]}$, we start with rearranging the vertices of $\bar{Q}_n$. All vertices except for those corresponding to g_{11} and h_{1i}, $i = 1, \dots, n$, are arranged into an $(n-1) \times 2n$ grid. It is obtained by moving the lower $(n-1) \times n$ part and placing it on the left from the upper $(n-1) \times n$ part; the remaining $n+1$ vertices are placed above as an additional row aligned on the right. All former dashed edges become regular, and the path

$$(n-1, n+1) \to (1,2) \to (n-1, n+2) \to (1,3) \to \cdots \to (1,n) \to (n-1, 2n)$$

becomes dashed; see Figure 2 for the rearranged version of $\bar{Q}_4$. To proceed further, we temporarily freeze vertices that correspond to $\varphi_{k(n-1)+1}$, $k = 0, \dots, n-2$, and h_{2j}, $j = 2, \dots, n$, and compare the result with an initial seed for the standard cluster structure on $\mathrm{Mat}_{(n-1)\times 2n}$ defined by the matrix T given in Proposition 3.6. □

Recall that the generalized exchange relation for φ_1 is given by

$$\varphi_1 \varphi_1' = \sum_{i=0}^{n} c_i(X,Y)\big((-1)^{n-1} h_{22} \varphi_{n+1}\big)^i \varphi_2^{n-i}, \tag{3.16}$$

where φ_1' is a polynomial in the entries of X and Y, see [13, Section 4]. Denote $\bar{\mathcal{F}}_n' = \bar{\mathcal{F}}_n \setminus \{\varphi_1\} \cup \{\varphi_1'\}$.

The following lemma implies that entries of the first row of X belong to the generalized upper cluster algebra $\overline{\mathcal{A}}(\overline{\mathcal{GC}}_n^D)$.

Lemma 3.8. *Every matrix entry x_{1i} can be expressed as $\frac{P(X,Y)}{\varphi_1}$, where P is a polynomial in matrix entries of Y, $X_{[2,n]}$ and functions $c_j(X,Y)$, $j = 1, \dots, n$. Alternatively, x_{1i} can be expressed as Laurent polynomials in terms of cluster variables in $\bar{\mathcal{F}}_n'$.*

Proof. Denote by $\bar{X}$ the matrix obtained from X by setting all entries of the first row to 0. For $1 \leqslant j \leqslant n$, $c_j(X,Y)$ is a linear function in matrix entries x_{1i}, and so we can write

$$c_j(X,Y) = c_j(\bar{X},Y) + \sum_{i=1}^{n} x_{1i} z_{ij}(\bar{X},Y), \tag{3.17}$$

where $z_{ij}(\bar{X},Y)$ are polynomials in the entries of $X_{[2,n]}$ and Y. Thus we obtain a linear system for the entries x_{1i} and $Z = (z_{ij}(\bar{X},Y))_{i,j=1}^{n}$ is the matrix of this system. Clearly, solutions of the system are polynomials in the entries of $X_{[2,n]}$ and Y divided by $\det Z$.

Note that $z_{ij}(\bar{X},Y)$ is a polynomial of degree $n-j-1$ in the entries of X and of degree j in the entries of Y and so, both $\det Z$ and $\varphi_1(X,Y)$ are polynomials of total degree $\frac{n(n-1)}{2}$ in terms of both X and Y. We will now show that, up to a scalar multiple, $\det Z$ coincides with $\varphi_1(X,Y)$. To this end, we will demonstrate that $\det Z = 0$ implies $\varphi_1 = 0$. Since φ_1 is irreducible ([13, Lemma 4.2]), this means that $\det Z$ and $\varphi_1(X,Y)$ differ by a constant multiple.

To prove the implication $\det Z = 0 \Rightarrow \varphi_1 = 0$, suppose that $\det Z$ vanishes. Then the system (3.17) is still solvable, but its solution is not unique. Consequently, there exists a non-zero row vector v^T such that $\det(e_1 v^T + X + \lambda Y) = \det(X + \lambda Y)$. The determinant on the left is evaluated via the Schur complement:

$$\det \begin{bmatrix} X+\lambda Y & e_1 \\ v^T & -1 \end{bmatrix} = \det(X+\lambda Y)\big(1 + v^T (X+\lambda Y)^{-1} e_1\big), \tag{3.18}$$

which means that $v^T(X+\lambda Y)^{-1}e_1 = 0$ for any λ. Equivalently, $v^T Y^{-1} U^j e_1 = 0$ for $j = 0,1,2,\ldots$, where $U = XY^{-1}$, and hence $\det[U^{n-1}e_1\ U^{n-2}e_1 \ldots Ue_1 e_1] = 0$. However, by [13, Lemma 3.3],

$$\varphi_1(X,Y) = \pm(\det Y)^{n-1} \det\left[U^{n-1}e_1 U^{n-2}e_1 \cdots Ue_1\ e_1\right]$$

and so, whenever $\det Z$ vanishes so does φ_1. This proves the first claim of the Lemma.

To establish the second claim, let us consider the dependence of the coefficient matrix Z and functions $\bar{c}_j = c_j(\bar{X},Y)$ on the initial cluster variables. In view of the proof of Proposition 3.7, all z_{ij} and $\bar{c}_j$ are Laurent polynomials in variables from $\widehat{\mathcal{F}}_n$ and, moreover, are polynomials in φ_1. Write $Z = Z^0 + \varphi_1 Z^1$, where Z^0 does not depend on φ_1. Since $\det Z$ is a scalar multiple of φ_1, the entries of Z^{-1} are Laurent polynomials in cluster variables from $\widehat{\mathcal{F}}_n$. Furthermore, it is easy to see that $\det(Z^0 + \varphi_1 Z^1)$ is proportional to $\varphi_1^{n-\operatorname{rank}(Z^0)}$, and hence the rank of Z^0 is equal to $n-1$. Further, $Z^{-1} = \varphi_1^{-1} W^0 + W^1$, where W^1 is polynomial in φ_1, and W^0 does not depend on φ_1 and satisfies relations $Z^0 W^0 = W^0 Z^0 = 0$. It follows that W^0 is of rank 1, that is, $W^0 = w_1 w_2^T$, where w_1, w_2 are non-zero column vectors such that w_1 spans the kernel of Z^0. Therefore, the first row of X can be expressed as

$$X_{[1]} = \varphi_1^{-1}\left(\sum_{j=1}^{n}(c_j - \bar{c}_j)w_{1j}\right)w_2^T + \mathcal{L},$$

where $\mathcal{L}$ is a vector of Laurent polynomials in terms of cluster variables in $\bar{\mathcal{F}}'_n$.

Let us show that the vector

$$u = \left(\left((-1)^{n-1}h_{22}\varphi_{n+1}\right)^j \varphi_2^{n-j}\right)_{j=1}^{n}$$

spans the kernel of Z^0. Note that (3.16) implies

$$\sum_{j=1}^{n} c_j u_j + \det Y \varphi_2^n = \varphi_1 \varphi_1',$$

$$\sum_{j=1}^{n} \bar{c}_j u_j + \det Y \varphi_2^n = \varphi_1 \bar{\varphi}_1',$$

where $\bar{\varphi}_1'$ is a polynomial in the entries of $X_{[2,n]}$ and $Y_{[2,n]}$, and thus can be written as a Laurent polynomial in variables from $\widehat{\mathcal{F}}_n$ which is polynomial in φ_1, and hence as a Laurent polynomial in variables from $\bar{\mathcal{F}}_n'$. Consequently, $\sum_{j=1}^{n}(c_j - \bar{c}_j)u_j = \varphi_1(\varphi_1' - \bar{\varphi}_1')$, so that

$$\varphi_1(\varphi_1' - \bar{\varphi}_1') = X_{[1]} Z u = X_{[1]} Z^0 u + \varphi_1 X_{[1]} Z^1 u,$$

and hence $\varphi_1 = 0$ implies $X_{[1]} Z^0 u = 0$. Note that Z^0 and u does not depend on φ_1 and on $X_{[1]}$, which means that u spans the kernel of Z^0, as claimed. Therefore, we can choose $w_1 = u$, and since the entries of u are monomials in terms of variables from $\bar{\mathcal{F}}_n \setminus \{\varphi_1\}$, entries of w_2 are Laurent monomials in terms of the same variables. We conclude that

$$X_{[1]} = \varphi_1^{-1}\left(\sum_{j=1}^{n}(c_j - \bar{c}_j)u_j\right) w_2^T + \mathcal{L} = \varphi_1' w_2^T + (\mathcal{L} - \bar{\varphi}_1' w_2^T),$$

which proves the claim. □

It follows immediately from Lemma 3.8 that each x_{1i} can be written as a Laurent polynomial in terms of the cluster variables in the initial cluster and in any of its neighbors, since by Proposition 3.7, all entries of Y and $X_{[2,n]}$ are cluster variables, and hence Laurent polynomials in any cluster.

4 | TWO GENERALIZED CLUSTER STRUCTURES ON $D(GL_4)$

In [12], we described a generalized cluster structure $\mathcal{GC}_n^D$ on $D(GL_n)$. It is easy to see that the generalized cluster structures $\overline{\mathcal{GC}}_n^D$ described in this paper and $\mathcal{GC}_n^D(Y^T, X^T)$ have the same set of frozen variables. Moreover, for $n = 2$ the initial seeds $\bar{\Sigma}_2(X, Y)$ and $\Sigma_2(Y^T, X^T)$ coincide. For $n = 3$, a straightforward computation shows that the sequence of mutations at vertices (4,3), (3,2), (2,1) takes $\bar{\Sigma}_3(X, Y)$ to $\Sigma_3(Y^T, X^T)$. In this section, we prove that for $n = 4$ no such sequence exists, and hence generalized cluster structures $\overline{\mathcal{GC}}_4^D$ and $\mathcal{GC}_4^D(Y^T, X^T)$ are distinct. We conjecture that this holds true for any $n > 4$ as well, see Remark 4.4.

Proposition 4.1. *The generalized cluster structures $\overline{\mathcal{GC}}_4^D = \overline{\mathcal{GC}}_4^D(X, Y)$ and $\mathcal{GC}_4^D = \mathcal{GC}_4^D(Y^T, X^T)$ are distinct.*

Proof. We start with the seed $\bar{\Sigma}_4$ for $\overline{\mathcal{GC}}_4^D$ and perform a sequence of mutations at vertices (5,4), (4,3), (3,2), (6,4), (5,3), (4,2), (5,4) and (4,3). In this way, we get functions φ_4', φ_3', φ_2', φ_8', φ_7', φ_6', φ_4'',

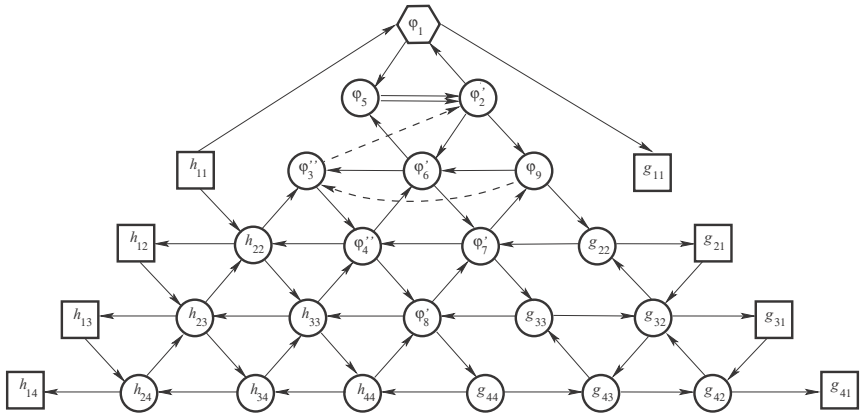

FIGURE 3 Modified $\bar{Q}_4$

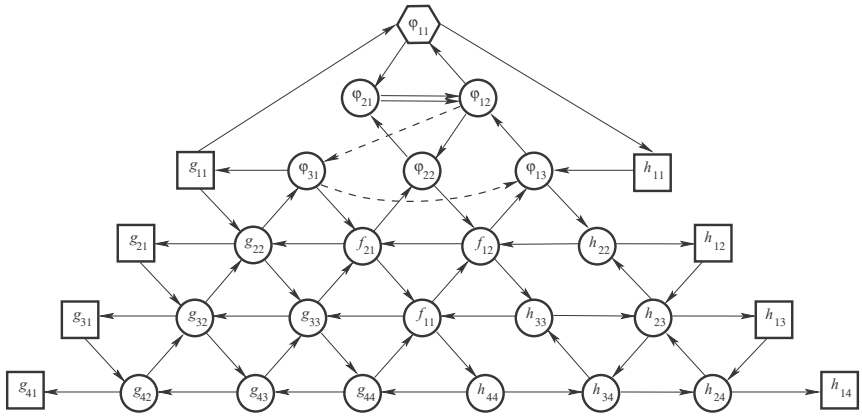

FIGURE 4 Quiver Q_4

φ''_3, respectively. A straightforward computation shows that

$$\varphi'_8 = \det\begin{bmatrix} y_{43} & y_{44} \\ x_{43} & x_{44} \end{bmatrix}, \quad \varphi'_7 = \det\begin{bmatrix} y_{42} & y_{43} & y_{44} \\ x_{32} & x_{33} & x_{34} \\ x_{42} & x_{43} & x_{44} \end{bmatrix}, \quad \varphi''_4 = \det\begin{bmatrix} y_{32} & y_{33} & y_{34} \\ y_{42} & y_{43} & y_{44} \\ x_{42} & x_{43} & x_{44} \end{bmatrix},$$

$$\varphi'_6 = \det\begin{bmatrix} y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}, \quad \varphi''_3 = \det\begin{bmatrix} y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}.$$

The corresponding quiver is shown in Figure 3. The quiver for the initial seed for $\mathcal{GC}_4^D$ constructed in [12] is shown in Figure 4. Recall that we are interested in the seed $\Sigma_4(Y^T, X^T)$, and hence in this

FIGURE 5 Subquivers of Q_4 and $\bar{Q}_4$

case $g_{ij}(Y^T, X^T) = \det(Y^T)^{[j,j+n-i]}_{[i,n]} = h_{ji}(X, Y)$ and $h_{ij}(Y^T, X^T) = g_{ji}(X, Y)$. Further, functions f_{ij} are defined via

$$f_{ij}(Y^T, X^T) = \det\left[(Y^T)^{[n-i+1,n]}\,(X^T)^{[n-j+1,n]}\right]_{[n-i-j+1,n]},$$

and hence $f_{11}(Y^T, X^T) = \varphi'_8(X, Y)$, $f_{21}(Y^T, X^T) = \varphi''_4(X, Y)$, and $f_{12}(Y^T, X^T) = \varphi'_7(X, Y)$. Finally, as explained in [12, Remark 3.1], functions φ_{ij} with $i + j = 4$ are defined via the same expression as f_{ij}, and hence $\varphi_{31}(Y^T, X^T) = \varphi''_3(X, Y)$, $\varphi_{22}(Y^T, X^T) = \varphi'_6(X, Y)$, and $\varphi_{13}(Y^T, X^T) = \varphi_9(X, Y)$. We thus see that the restrictions of both quivers to the three lower rows coincide, as well as the functions attached to the corresponding vertices. Moreover, the functions attached to the fourth row from below in both quivers coincide as well, as well as the arrows between the third and the fourth row. We will prove that the corresponding two seeds are not mutationally equivalent.

By [3, Theorem 3.6], if two seeds are mutationally equivalent and share a set of common cluster variables, there exists a sequence of mutations that connects these seeds and does not involve the common cluster variables.

Remark 4.2. In fact, the definition of a generalized cluster structure in [3] and in the preceding paper [2] is more restrictive, since it imposes a reciprocity condition on exchange coefficients, following [18]. However, this condition is only used in the proof of Lemma 4.20 in [2], which in turn is based on Proposition 3.3 in [18]. This proposition claims that every cluster variable can be written as a Laurent polynomial in cluster variables of the initial cluster and an ordinary polynomial in frozen variables. It is an analog of the corresponding statement for ordinary cluster structures and its proof extends to the case of generalized cluster structures as defined in Section 2 without any changes.

Consequently, if the above two seeds are equivalent, there should exist a sequence of mutations that involves only three vertices comprising the uppermost triangle. We will concentrate on two four-vertex subquivers that are formed by the uppermost triangle and the vertex corresponding to $\varphi_{13}(Y^T, X^T) = \varphi_9(X, Y)$. These two subquivers are shown in Figure 5. We claim that there is no sequence of mutations at the vertices 1, 2 and 3 that takes one subquiver to the other one. Note that although the mutations at vertex 4 are not allowed, it is not frozen.

To prove our claim, we consider the evolution of a more general quiver $Q(\alpha, \beta, \gamma)$ shown in Figure 6 under mutations at the vertices 1, 2 and 3. Here multiplicities α, β and γ can take any integer values except for $\alpha = \beta = \gamma = 0$. A negative value means that the direction of the corresponding arrow is reversed. Clearly, two quivers shown in Figure 5 are $Q(0, 1, 0)$ and $Q(0, -1, 0)$. To keep track of the mutations, it will be convenient to renumber the vertices so that the

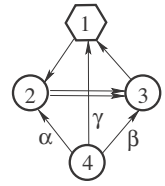

FIGURE 6 Quiver $Q(\alpha, \beta, \gamma)$

generalized vertex is always vertex 1, and the direction of arrows in the triangle is $1 \to 2 \to 3 \to 1$. Note that any mutation of $Q(\alpha, \beta, \gamma)$ transforms it into $Q(\alpha', \beta', \gamma')$ for certain values of α', β' and γ'.

Define the *charge* of $Q = Q(\alpha, \beta, \gamma)$ as $C(Q) = |\alpha| + |\beta| + |\gamma|$. The nodes of the 3-regular tree $\mathbb{T}$ that describes all possible mutations of $Q(\alpha, \beta, \gamma)$ can be classified into 10 possible types with respect to the charge. We encode these types by a triple $[i, j, k]$, where i stands for the number of mutations that increase the charge, j stands for the number of mutations that preserve the charge, and k stands for the number of mutations that decrease the charge, so that $i + j + k = 3$. Note that both quivers $Q(0, 1, 0)$ and $Q(0, -1, 0)$ have charge 1. Consequently, if they are mutation equivalent, then either all quivers along the simple path (the one that never returns to the same vertex) in $\mathbb{T}$ that connects $Q(0, 1, 0)$ and $Q(0, -1, 0)$ have charge 1, or this path contains two quivers Q_1, Q_2 that differ by one mutation, such that $C(Q_1) < C(Q_2)$ and $C(Q_2) \geqslant C(Q)$ for any other quiver Q along the path.

Consider the first possibility. A straightforward computation shows that mutations at vertices 1 and 2 preserve the charge and take $Q(0, 1, 0)$ to $Q(1, 0, 0)$, while mutation at vertex 3 increases the charge. Further, mutations of $Q(1, 0, 0)$ at vertices 1 and 3 preserve the charge and take $Q(1, 0, 0)$ to $Q(0, 1, 0)$, while mutation at vertex 2 increases the charge. Therefore, $Q(0, -1, 0)$ cannot be reached from $Q(0, 1, 0)$ along a path with the constant charge 1.

Consider the second possibility. Let $[i, j, k]$ be the type of Q_2, then $k \geqslant 1$ and $j + k \geqslant 2$, so we remain with the following possibilities: [1,1,1], [1,0,2], [0,2,1], [0,1,2], [0,0,3]. We are interested in finding conditions on α, β and γ that would guarantee that the type of Q_2 is indeed one of the types listed above. One can distinguish eight possible cases according to the signs of α, β and γ. Let us consider in detail one of the nontrivial cases.

Let $\alpha < 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$. Mutation at vertex 1 takes $Q = Q(\alpha, \beta, \gamma)$ to $Q' = Q(\beta, \alpha + 4\gamma, -\gamma)$, so that $C(Q') = C(Q) + \alpha + |\alpha + 4\gamma|$, and hence $C(Q') < C(Q)$ if $\alpha + 2\gamma < 0$ and $\gamma \neq 0$. Mutation at vertex 2 takes Q to $Q' = Q(\beta, -\alpha, \gamma + \alpha)$, so that $C(Q') = C(Q) - \gamma + |\alpha + \gamma|$, and hence $C(Q') < C(Q)$ if $\alpha + 2\gamma > 0$. Mutation at vertex 3 takes Q to $Q' = Q(-\beta, \alpha, \gamma + \beta)$, so that $C(Q') = C(Q) + \beta \geqslant C(Q)$. Consequently, the type of $Q(\alpha, \beta, \gamma)$ is [1,1,1] if $\alpha + 2\gamma > 0$ and $\beta = 0$, or if $\alpha + 2\gamma < 0$, $\gamma \neq 0$ and $\beta = 0$. For the other values of parameters, the type of $Q(\alpha, \beta, \gamma)$ is either $[i, j, 0]$ with $i + j = 3$ or [2,0,1].

Results of similar considerations in all the remaining cases are summarized in Table 1 that contains, for each case, the values of the charge after the three possible mutations and possible types of Q depending on the values of α, β, and γ.

It follows from the results presented in the table that the only possible candidates for the quiver Q_2 are quivers of type [1,1,1] in Cases 2–4. In all three cases, the next node in the path should have the same charge. If Q_2 is as in Case 2, then the next node is obtained by mutation at vertex 3, and the resulting quiver is $Q' = Q(0, \alpha, \gamma)$ with $\alpha < 0$ and $\gamma > 0$. This situation is covered by Case 3, and

TABLE 1 Types of Q

	Case 1: $\alpha \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0$	*Case 2*: $\alpha < 0, \beta \geqslant 0, \gamma \geqslant 0$
$C(\mu_1(Q))$	$C(Q) + 4\gamma \geqslant C(Q)$	$C(Q) + \alpha + \lvert\alpha + 4\gamma\rvert$
$C(\mu_2(Q))$	$C(Q) + 2\alpha \geqslant C(Q)$	$C(Q) - \gamma + \lvert\alpha + \gamma\rvert$
$C(\mu_3(Q))$	$C(Q) + \beta \geqslant C(Q)$	$C(Q) + \beta$
type of Q	$[i, j, 0]$	$\begin{cases} [1,1,1], & \text{if } \alpha + 2\gamma > 0, \beta = 0 \\ & \text{or if } \alpha + 2\gamma < 0, \\ & \gamma \neq 0, \beta = 0 \\ [i,j,0] \text{ or} & \\ [2,0,1] & \text{otherwise.} \end{cases}$
	Case 3: $\alpha \geqslant 0, \beta < 0, \gamma \geqslant 0$	*Case 4*: $\alpha \geqslant 0, \beta \geqslant 0, \gamma < 0$
$C(\mu_1(Q))$	$C(Q) + 4\gamma \geqslant C(Q)$	$C(Q) - \beta + \lvert\beta + 4\gamma\rvert$
$C(\mu_2(Q))$	$C(Q) + \beta + \lvert\beta + 2\gamma\rvert$	$C(Q) + 2\alpha \geqslant C(Q)$
$C(\mu_3(Q))$	$C(Q) - \alpha + \lvert\alpha + 2\beta\rvert$	$C(Q) + \gamma + \lvert\gamma + \beta\rvert$
type of Q	$\begin{cases} [1,1,1], & \text{if } \alpha + \beta > 0, \gamma = 0 \\ & \text{or if } \alpha + \beta < 0, \\ & \alpha \neq 0, \gamma = 0 \\ [i,j,0] \text{ or} & \\ [2,0,1] & \text{otherwise.} \end{cases}$	$\begin{cases} [1,1,1], & \text{if } \beta + 2\gamma > 0, \alpha = 0 \\ & \text{or if } \beta + 2\gamma < 0, \\ & \gamma \neq 0, \alpha = 0 \\ [i,j,0] \text{ or} & \\ [2,0,1] & \text{otherwise.} \end{cases}$
	Case 5: $\alpha < 0, \beta < 0, \gamma \geqslant 0$	*Case 6*: $\alpha < 0, \beta \geqslant 0, \gamma < 0$
$C(\mu_1(Q))$	$C(Q) + \alpha + \lvert\alpha + 4\gamma\rvert$	$C(Q) - \beta + \lvert\beta + 4\gamma\rvert$
$C(\mu_2(Q))$	$C(Q) - \gamma + \lvert\alpha + \gamma\rvert$	$C(Q) - \alpha > C(Q)$
$C(\mu_3(Q))$	$C(Q) - 2\beta > C(Q)$	$C(Q) + \gamma + \lvert\gamma + \beta\rvert$
type of Q	$[i, j, 0]$ or $[2, 0, 1]$	$[i, j, 0]$ or $[2, 0, 1]$
	Case 7: $\alpha \geqslant 0, \beta < 0, \gamma < 0$	*Case 8*: $\alpha < 0, \beta < 0, \gamma < 0$
$C(\mu_1(Q))$	$C(Q) - 4\gamma > C(Q)$	$C(Q) - 4\gamma > C(Q)$
$C(\mu_2(Q))$	$C(Q) + \beta + \lvert\beta + 2\alpha\rvert$	$C(Q) - \alpha > C(Q)$
$C(\mu_3(Q))$	$C(Q) - \alpha + \lvert\alpha + 2\beta\rvert$	$C(Q) - 2\beta > C(Q)$
type of Q	$[i, j, 0]$ or $[2, 0, 1]$	[3,0,0]

the other two mutations of Q' yield $C(Q') + 4\gamma > C(Q')$ and $C(Q') + |2\alpha| > C(Q')$. Consequently, the maximality condition for the charge of Q_2 fails.

The remaining two cases are analyzed in a similar way, with Case 3 leading to Case 2 and Case 4 leading to Case 4. Therefore, $Q(0, -1, 0)$ cannot be reached from $Q(0, 1, 0)$ along a path with a varying charge, which completes the proof. □

Remark 4.3. In a recent preprint [21], a technique of scattering diagrams is used to construct a log Calabi–Yau variety with two non-equivalent cluster structures both associated with the Markov quiver. This variety is obtained by a certain augmentation of a cluster A-variety with principal coefficients in the sense of [15]. In contrast to this, the example we presented above gives two explicitly defined non-equivalent generalized cluster structures with the same set of frozen variables in an affine variety obtained by deleting a divisor with a normal crossing from an affine space.

Remark 4.4. For $n > 4$, one can build a sequence of mutations that takes the seed $\bar{\Sigma}_n$ to a seed $\Sigma = (\mathbf{x}, Q, \mathcal{P})$ with the following properties. Let $\Sigma_n(Y^T, X^T) = (\mathbf{x}_n, Q_n, \mathcal{P}_n)$, and assume that Q_n and Q are arranged in $2n-2$ rows, as in Figures 3 and 4, then the restrictions of Q and Q_n to $n-1$ lower rows coincide, as well as the functions attached to the corresponding vertices. So do the functions attached to the nth row from below in both quivers and the arrows between the nth and the $(n-1)$th row. The restrictions of Q and Q_n to the remaining $n-2$ upper rows also coincide, however, the functions attached to the corresponding vertices differ. Finally, $\varphi_{n-2} \to \varphi_{N-n+1}$ is an edge in Q, $\varphi_{1,n-1} \to \varphi_{1,n-2}$ is an edge in Q_n, $\varphi_{N-n+1}(X,Y) = \varphi_{1,n-1}(Y^T, X^T)$, vertices φ_{n-2} and $\varphi_{1,n-2}$ correspond each other in the upper parts of Q and Q_n, respectively, and there are no edges between φ_{N-n+1} and the upper part of Q ($\varphi_{1,n-1}$ and the upper part of Q_n, respectively). We believe that similarly to the case $n = 4$, one can prove that it is impossible to invert the arrow in question via a sequence of mutations at the vertices of the upper part, which would imply that the generalized cluster structures $\overline{\mathcal{GC}}_n^D(X,Y)$ and $\mathcal{GC}_n^D(Y^T, X^T)$ are distinct.

The example presented above describes two different generalized cluster structures $\mathcal{GC}_1 = \overline{\mathcal{GC}}_4^D$, $\mathcal{GC}_2 = \mathcal{GC}_4^D$ such that:

- the corresponding upper cluster algebras coincide with the ring $\mathbb{C}[D(GL_4)]$ of regular functions on the Drinfeld double of GL_4;
- both generalized cluster algebras are compatible with the standard Poisson–Lie bracket on $D(GL_4)$ and have the same collection of frozen cluster variables.

We believe that both generalized cluster structures $\mathcal{GC}_1$ and $\mathcal{GC}_2$ can be related to the same ordinary cluster structure $\mathcal{C}$ using a conjectural construction outlined below for the case of general n.

The cluster structure $\mathcal{C}$ is associated with the moduli space $\mathcal{A}_{G,S}$ introduced by Fock and Goncharov in the study of G-local systems on a marked Riemann surface S, see [5]. In our example $G = GL_n$, S is the punctured disk with four marked points on the boundary. The variety $\mathcal{A}_{GL_n,S}$ is homeomorphic to the configuration space of triples $(\mathbf{F}, M, \Phi)$ modulo the GL_n-action where $\mathbf{F}$ denotes a quadruple of decorated flags at the marked points, M is the GL_n-monodromy about the puncture amd Φ is a flag at the puncture which is invariant under the monodromy. Note that the invariant flag at the puncture is not uniquely defined by the monodromy: there are $n!$ choices corresponding to different orderings of monodromy eigenvalues. The Weyl group $W = S_n$ acts on $\mathcal{A}_{GL_n,S}$ by reordering eigenvalues, which results in a different choice of the invariant flag.

The parametrization of $\mathcal{A}_{GL_n,S}$ introduced in [5] endows it with a cluster structure which, in turn, leads to a compatible Poisson bracket. This Poisson structure has corank $2n$ with n Casimirs given by the coefficients of the characteristic polynomial of the monodromy and n additional Casimirs. Fixing values of n additional Casimirs to 1, we obtain a codimension n subvariety V. The action of W restricts to V. Further, V is a cluster variety whose coordinate ring is equipped with the cluster structure $\mathcal{C}$. In particular, $\mathcal{C}$ inherits the W-action.

There is a natural projection $\pi : V \to D(GL_n)$ with a fiber $W \times H$ where H is the Cartan subgroup. We conjecture that the projection π provides a natural connection between generalized cluster structures $\mathcal{GC}_i$ in $\mathbb{C}[D(GL_n)]$ for $i = 1, 2$ and $\mathcal{C}$. More precisely, the pullbacks of all cluster variables in $\mathcal{GC}_i$ are W-invariant cluster variables in $\mathcal{C}$. Furthermore, each seed Σ of the generalized cluster structure $\mathcal{GC}_i$ contains one cluster variable $g(\Sigma)$ attached to the special vertex of the quiver and satisfying a generalized mutation rule, while the remaining cluster variables obey the usual

mutation rules. The rank of $\mathcal{C}$ is $n-1$ more than the rank of $\mathcal{GC}_i$. Any seed Σ of $\mathcal{GC}_i$ corresponds to a seed $\tilde{\Sigma}$ of $\mathcal{C}$ in which $g(\Sigma)$ corresponds to an n-tuple of cluster variables $\tilde{g}_j(\tilde{\Sigma})$, $j \in [1,n]$, such that $\pi^* g(\Sigma) = \prod_{j=1}^n \tilde{g}_j(\tilde{\Sigma})$. The remaining cluster variables of the seed $\tilde{\Sigma}$ and frozen variables of $\mathcal{C}$ are obtained as pullbacks π^* of the corresponding cluster variables of Σ and frozen variables in $\mathcal{GC}_i$. The generalized mutation of $g(\Sigma)$ corresponds to the composition $s(\tilde{\Sigma})$ of n mutations at all $\tilde{g}_j(\tilde{\Sigma})$ taken in any order (mutations of $\tilde{g}_j(\tilde{\Sigma})$ commute). Namely, let g' denote the function obtained by the generalized mutation of $g(\Sigma)$ and $\tilde{g}'_j = s(\tilde{\Sigma})(\tilde{g}_j(\tilde{\Sigma}))$, then $\pi^* g' = \prod_{j=1}^n \tilde{g}'_j$. Further, the set of seeds of $\mathcal{C}$ corresponding to $\mathcal{GC}_1$ is disjoint from the set of seeds corresponding to $\mathcal{GC}_2$.

A detailed proof will be presented elsewhere.

Remark 4.5. Let G be a semisimple complex Lie group with the Lie algebra $\mathfrak{g}$. The group G is equipped with the standard Poisson–Lie structure. In [20], the semiclassical limit of $\mathcal{U}_q(\mathfrak{g})$ is realized as a quotient by an ideal generated by Poisson central elements of the W-invariant subring of the coordinate ring of the second moduli space $\mathcal{X}_{G,S_{0,1,2}}$ of Fock–Goncharov cluster ensemble, where $S_{0,1,2}$ is a once punctured disk with two marked points on the boundary. This construction seems to be closely related to the projection π above. However, no cluster structure on the W-invariant subring was considered in [20].

5 | REDUCTION TO A GENERALIZED CLUSTER STRUCTURE ON BAND PERIODIC MATRICES

In [13], we presented a framework for constructing generalized cluster structures. It is based on certain identities associated with periodic staircase shaped matrices. One of the examples considered in [13] was the generalized cluster structure $\overline{\mathcal{GC}}_n^D$ that we treated in previous sections. Another example was a generalized cluster structure on the space of $(k+1)$ diagonal n-periodic band matrices with $k \leqslant n$. In this section, we will show how the latter structure, denoted here by $\mathcal{GC}(\mathcal{L}_{kn})$ can be obtained as a restriction of the former. In particular, this will allow us to obtain an analogue of Theorem 3.1 for $\mathcal{GC}(\mathcal{L}_{kn})$.

5.1 | Initial cluster

In the case of the Drinfeld double $D(GL_n)$, the periodic staircase matrix mentioned above is an infinite block bidiagonal matrix

$$L = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & & \\ & 0 & X & Y & 0 & & \\ & & 0 & X & Y & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{5.1}$$

that corresponds to $(X,Y) \in D(GL_n)$. Now we drop the invertibility requirement for X and choose Y to be a lower triangular band matrix with $\min(k+1,n)$ non-zero diagonals (including the main diagonal) and X to be a matrix with zeros everywhere outside of the $k \times k$ upper triangular block

in the upper right corner:

$$X = \begin{bmatrix} 0 & \cdots & 0 & a_{11} & \cdots & a_{k1} \\ 0 & \cdots & 0 & 0 & a_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & \cdots & 0 & a_{1k} \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$Y = \begin{bmatrix} a_{k+1,1} & 0 & \cdots & \cdots & \cdots & 0 \\ a_{k2} & a_{k+1,2} & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{1,k+1} & a_{2,k+1} & \cdots & a_{k+1,k+1} & 0 & \cdots \\ 0 & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{1n} & a_{2n} & \cdots & a_{k+1,n} \end{bmatrix} \quad \text{for } k < n, \tag{5.2}$$

$$Y = \left(a_{n+1-i+j,i}\right)_{i,j=1}^{n} \quad \text{for } k = n,$$

where we assume that $a_{ti} = 0$ when $t > k+1$. Then L in (5.1) is a $(k+1)$-diagonal n-periodic band matrix. We denote by $\mathcal{L}_{kn}$ the space of such matrices with an additional condition that all entries of the lowest and the highest diagonals are non-zero. Let $\bar{\mathcal{L}}_{kn}$ be the closure of $\mathcal{L}_{kn}$ in the space $D(\mathrm{Mat}_n) = \mathrm{Mat}_n \times \mathrm{Mat}_n$. Every element of $\bar{\mathcal{L}}_{kn}$ is identified with a pair of matrices of the form (5.2). In particular, vanishing of the lowest diagonal yields an inclusion $\mathcal{L}_{k-1,n} \subset \bar{\mathcal{L}}_{kn}$.

Note that when such matrices are substituted into (3.1), Φ becomes reducible with a leading irreducible block $\Phi^{(k)}$ of size $(k-1)(n-1) \times (k-1)(n-1)$. For $i = 1, \dots, (k-1)(n-1)$, we define

$$\tilde{\varphi}_i = \tilde{\varphi}_i^{(k)} = \det \Phi_{[i,(k-1)(n-1)]}^{[i,(k-1)(n-1)]}. \tag{5.3}$$

By [13], a generalized cluster structure in the space of regular functions on $\mathcal{L}_{kn}$ is defined by the following data.

Define the family $\mathcal{F}_{kn}$ of $(k+1)n$ regular functions on $\mathcal{L}_{kn}$ via

$$\mathcal{F}_{kn} = \left\{ \{\tilde{\varphi}_i^{(k)}\}_{i=1}^{(k-1)(n-1)};\ \tilde{a}_{11};\ \{a_{1i}\}_{i=2}^{n};\ \{a_{k+1,i}\}_{i=1}^{n};\ \{\tilde{c}_i(X,Y)\}_{i=1}^{k-1} \right\}, \tag{5.4}$$

where $\tilde{a}_{11} = (-1)^{k(n-1)} a_{11}$ and $\tilde{c}_i(X,Y) = (-1)^{i(n-1)} c_i(X,Y)$ for $1 \leqslant i \leqslant k-1$ with $c_i(X,Y)$ satisfying the identity $\det(\lambda Y + \mu X) = \lambda^{n-k} \sum_{i=0}^{k} c_i(X,Y) \mu^i \lambda^{k-i}$.

Let Q_{kn} be the quiver with $(k+1)n$ vertices, of which $k-1$ vertices are isolated and are not shown in the figure below, $(k+1)(n-1)$ are arranged in an $(n-1) \times (k+1)$ grid and denoted (i,j), $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant k+1$, and the remaining two are placed on top of the leftmost and the rightmost columns in the grid and denoted (0,1) and $(0, k+1)$, respectively. All vertices in the leftmost and in the rightmost columns are frozen. The vertex $(1,k)$ is special, and its multiplicity equals k. All other vertices are regular mutable vertices.

The edge set of Q_{kn} consists of the edges $(i,j) \to (i+1,j)$ for $i = 1, \dots, n-2$, $j = 2, \dots, k$; $(i,j) \to (i,j-1)$ for $i = 1, \dots, n-1$, $j = 2, \dots, k$, $(i,j) \neq (1,k)$; $(i+1,j) \to (i,j+1)$ for $i = 1, \dots, n-2$, $j = 2, \dots, k$, shown by solid lines. In addition, there are edges $(n-1,3) \to (1,2)$,

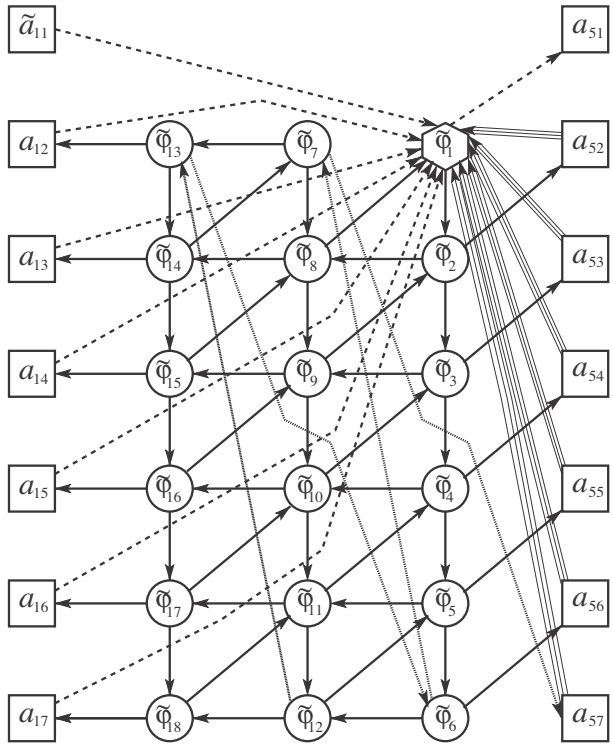

FIGURE 7 Quiver Q_{47}

$(1,2) \to (n-1,4)$, $(n-1,4) \to (1,3), \dots, (1,k-1) \to (n-1,k+1)$ that form a directed path (shown by dotted lines). Save for this path, and the missing edge $(1,k) \to (1,k-1)$, mutable vertices of Q_{kn} form a mesh of consistently oriented triangles

Finally, there are edges between the special vertex $(1,k)$ and frozen vertices $(i,1)$, $(i,k+1)$ for $i = 0, \dots n-1$. There are $k-1$ parallel edges between $(1,k)$ and $(i,k+1)$ for $i = 1, \dots, n-1$, and one edge between $(1,k)$ and all other frozen vertices (including $(0,k+1)$). The edge to $(0,k+1)$ is directed from $(1,k)$; if $k > 2$, all other edges are directed toward $(1,k)$, and if $k = 2$, the direction of the edge between (1,1) and $(1,k)$ is reversed.

Quiver Q_{47} is shown in Figure 7.

We attach functions $\tilde{a}_{11}, a_{12}, \dots, a_{1n}$, in a top to bottom order, to the vertices of the leftmost column in Q_{kn}, and functions $a_{k+1,1}, \dots, a_{k+1,n}$, in the same order, to the vertices of the rightmost column in Q_{kn}. Functions $\tilde{\varphi}_i$ are attached, in a top to bottom, right to left order, to the remaining vertices of Q_{kn}, starting with $\tilde{\varphi}_1$ attached to the special vertex $(1,k)$. The set of strings $\mathcal{P}_{kn}$ contains a unique non-trivial string $(1, \tilde{c}_1(X,Y), \dots, \tilde{c}_{k-1}(X,Y), 1)$ corresponding to the unique special vertex.

Theorem 5.1. *The seed $\Sigma_{kn} = (\mathcal{F}_{kn}, Q_{kn}, \mathcal{P}_{kn})$ defines a complete generalized cluster structure $\mathcal{GC}(\Sigma_{kn})$ in the ring of regular functions on $\mathcal{L}_{kn}$ compatible with the restriction of the standard Poisson–Lie structure on $D(\mathrm{Mat}_n)$.*

Proof. The proof follows closely that for Theorem 3.1. Regularity of $\mathcal{GC}(\Sigma_{kn})$ is borrowed from [13, Theorem 5.1]. The proof of log-canonicity and of compatibility is based on the downward

induction on k, see Section 5.2. The proof of completeness is a modification of a similar statement for the Drinfeld double and relies on the same ideas, see Section 5.3. □

5.2 | Compatible Poisson bracket

Let us re-write the Poisson bracket (3.2) in terms of matrix entries of a pair of matrices (X, Y):

$$\begin{aligned}
\{x_{ij}, x_{pq}\} &= \frac{1}{2}(\operatorname{sign}(p-i) + \operatorname{sign}(q-j))x_{iq}x_{pj},\\
\{y_{ij}, y_{pq}\} &= \frac{1}{2}(\operatorname{sign}(p-i) + \operatorname{sign}(q-j))y_{iq}y_{pj},\\
\{y_{ij}, x_{pq}\} &= \frac{1}{2}\big(1 + \operatorname{sign}(q-j))y_{iq}x_{pj} - (1 + \operatorname{sign}(i-p))x_{iq}y_{pj}\big).
\end{aligned} \tag{5.5}$$

This Poisson bracket can be extended to $D(\mathrm{Mat}_n)$. It follows from (5.5) that every inclusion in the chain

$$D(\mathrm{Mat}_n) \supset \bar{\mathcal{L}}_{nn} \supset \bar{\mathcal{L}}_{n-1,n} \supset \cdots \supset \bar{\mathcal{L}}_{kn} \supset \cdots \supset \bar{\mathcal{L}}_{2n}$$

is a Poisson map. The same is true about inclusions $\mathcal{L}_{kn} \subset \bar{\mathcal{L}}_{kn}$.

Proposition 5.2. *The family $\mathcal{F}_{kn}$ defined in (5.4) is log-canonical with respect to the restriction of the Poisson bracket (5.5) to $\mathcal{L}_{kn}$.*

Proof. For $k = n$, substitute $(X, Y) \in \mathcal{L}_{nn}$ into (3.1). As was mentioned above, Φ becomes reducible with an irreducible $(n-1)^2 \times (n-1)^2$ upper left block $\Phi^{(n)}$ and, for $i \leqslant (n-1)^2$, functions φ_i restricted to $\mathcal{L}_{nn}$ factor as

$$\varphi_i = \tilde{\varphi}_i^{(n)} \varphi_{(n-1)^2+1}. \tag{5.6}$$

By Theorem 3.1, Poisson brackets $\{\log \varphi_i, \log \varphi_j\} = \omega_{ij}$ are constant on $D(GL_n)$ and, by extension, on $D(\mathrm{Mat}_n)$. Since $\mathcal{L}_{nn}$ is a Poisson submanifold in $D(\mathrm{Mat}_n)$, we obtain $\{\log \tilde{\varphi}_i^{(n)}, \log \varphi_{(n-1)^2+1}\} = \omega_{i,(n-1)^2+1}$ for $i \leqslant (n-1)^2$, and thus

$$\{\log \tilde{\varphi}_i^{(n)}, \log \tilde{\varphi}_j^{(n)}\} = \omega_{ij} - \omega_{i,(n-1)^2+1} + \omega_{j,(n-1)^2+1}$$

is constant on $\mathcal{L}_{nn}$ for any $i, j = 1, \dots, (n-1)^2$; we denote this constant by $\omega_{ij}^{(n)}$. Furthermore, on $\mathcal{L}_{nn}$ we have $h_{ii}(Y) = a_{n+1,i} \cdots a_{n+1,n}$, $g_{ii}(X) = a_{1i} \cdots a_{1n}$, and so the log-canonicity of the entire family $\mathcal{F}_{nn}$ follows from the log-canonicity of $\bar{\mathcal{F}}_n$. By extension, we get the log-canonicity of the family $\mathcal{F}_{nn}$ on $\bar{\mathcal{L}}_{nn}$.

Using induction, assume that $\{\log \tilde{\varphi}_i^{(k)}, \log \tilde{\varphi}_j^{(k)}\} = \omega_{ij}^{(k)}$ is constant on $\bar{\mathcal{L}}_{kn}$ for any $i, j = 1, \dots, (k-1)(n-1)$. Substituting $(X, Y) \in \mathcal{L}_{k-1,n}$ into the $(k-1)(n-1) \times (k-1)(n-1)$ matrix $\Phi^{(k)}$ makes it reducible with an irreducible $(k-2)(n-1) \times (k-2)(n-1)$ upper left block $\Phi^{(k-1)}$ and functions $\tilde{\varphi}_i^{(k)}$ restricted to $\mathcal{L}_{k-1,n}$ factor as $\tilde{\varphi}_i^{(k)} = \tilde{\varphi}_i^{(k-1)} \tilde{\varphi}_{(k-2)(n-1)+1}^{(k)}$ for $i \leqslant (k-2)(n-1)$.

In addition, $\tilde{\varphi}^{(k)}_{(k-2)(n-1)+i} = a_{1,i+1}\cdots a_{1n}$ for $i = 2,\dots,n$. Arguing precisely as above, we conclude that

$$\{\log\tilde{\varphi}_i^{(k-1)}, \log\tilde{\varphi}_j^{(k-1)}\} = \omega_{ij}^{(k)} - \omega^{(k)}_{i,(k-2)(n-1)+1} + \omega^{(k)}_{j,(k-2)(n-1)+1} \tag{5.7}$$

is a constant on $\mathcal{L}_{k-1,n}$ for any $i, j = 1,\dots,(k-2)(n-1)$ and denote it by $\omega_{ij}^{(k-1)}$. Therefore, the log-canonicity of the entire family $\mathcal{F}_{k-1,n}$ on $\mathcal{L}_{k-1,n}$, and hence on $\bar{\mathcal{L}}_{k-1,n}$, follows from the log-canonicity of $\mathcal{F}_{kn}$ on $\bar{\mathcal{L}}_{kn}$. □

Proposition 5.3. *The Poisson bracket (5.5) is compatible with the generalized cluster structure $\mathcal{GC}(\Sigma_{kn})$ on $\mathcal{L}_{kn}$.*

Proof. We will use induction again. Within this proof, it will be convenient to refer to the vertex in $\bar{Q}_n$ to which the variable φ_i is attached and to the vertex in Q_{nn} to which the variable $\tilde{\varphi}_i$ is attached as the vertex i in the corresponding quiver. Assume first that $k = n$, and let y_i be the y-variable corresponding to the vertex i in $\bar{Q}_n$ and $y_i^{(n)}$ be the y-variable corresponding to the vertex i in Q_{nn}. We claim that on $\mathcal{L}_{nn}$, $y_i^{(n)} = y_i$ for all mutable vertices in Q_{nn}.

Indeed, for $n < i \leqslant (n-2)(n-1)$, the neighborhood of the vertex labeled i in Q_{nn} is identical to the neighborhood of the vertex labeled i in $\bar{Q}_n$. We claim that on $\mathcal{L}_{nn}$, $y_i^{(n)} = y_i$.

For $(n-2)(n-1) < i \leqslant (n-1)^2$, let $i' = i - (n-2)(n-1)$. Then

$$y_i^{(n)} = \frac{\tilde{\varphi}^{(n)}_{i+1}\tilde{\varphi}^{(n)}_{i-n}a_{1,i'+1}}{\tilde{\varphi}^{(n)}_{i-1}\tilde{\varphi}^{(n)}_{i-n+1}} = \frac{\varphi_{i+1}\varphi_{i-n}g_{i'+1,i'+1}}{\varphi_{i-1}\varphi_{i-n+1}g_{i'+2,i'+2}} = y_i,$$

by (5.6) and since $g_{jj} = a_{1j}\cdots a_{1n}$ on $\mathcal{L}_{nn}$.

For $1 < i \leqslant n$,

$$y_i^{(n)} = \frac{\tilde{\varphi}^{(n)}_{i+1}\tilde{\varphi}^{(n)}_{i+n-1}a_{n+1,i}}{\tilde{\varphi}^{(n)}_{i-1}\tilde{\varphi}^{(n)}_{i+n}} = \frac{\varphi_{i+1}\varphi_{i+n-1}h_{ii}}{\varphi_{i-1}\varphi_{i+n}h_{i+1,i+1}} = y_i,$$

by (5.6) and since $h_{jj} = a_{n+1,j}\cdots a_{n+1,n}$ on $\mathcal{L}_{nn}$.

Finally,

$$y_1^{(n)} = \left(\frac{\tilde{\varphi}^{(n)}_2}{\tilde{\varphi}^{(n)}_{n+1}}\right)^n \frac{a_{n+1,1}}{\prod_{j=1}^n a_{1j}\left(\prod_{j=2}^n a_{n+1,j}\right)^{n-1}} = \left(\frac{\varphi_2}{\varphi_{n+1}h_{22}}\right)^n \frac{h_{11}}{g_{11}} = y_1.$$

Therefore $\{\log y_i^{(n)}, \log\tilde{\varphi}_j^{(n)}\} = \{\log y_i, \log\varphi_j - \log\varphi_{(n-1)^2+1}\} = d_i\delta_{ij}$, where $d_1 = n$ and $d_i = 1$ otherwise. The induction step is performed in precisely the same fashion by showing that on $\mathcal{L}_{k-1,n}$, for all mutable vertices in $Q_{k-1,n}$, $y_i^{(k-1)} = y_i^{(k)}$. □

Remark 5.4. When restricted to $\mathcal{L}_{kn}$, the Poisson structure 5.5 coincides with the one considered in a recent paper [16] on the space of *properly bounded n-periodic difference operators*. A modification of that Poisson bracket for spaces of *sparse* pseudo-difference operators was also considered in [16]

in order to derive complete integrability of a family of pentagram-like maps. It would be interesting to see if such a modification has a cluster-algebraic meaning as well.

5.3 | Completeness

The next two propositions are analogous to Propositions 3.5, 3.6 and can be proved in exactly the same way.

Proposition 5.5. *There exists a $(k-1)\times(k-1)$ unipotent upper triangular matrix $G = G(X,Y)$ such that:*

(i) *entries of G are rational functions in X, Y whose denominators are monomials in cluster variables $\tilde{\varphi}_{jn+1}$, $j = 1,\dots,k-2$; and*

(ii) *the $(2k-2)\times(k-1)$ matrix $S = \begin{bmatrix} Y^{[n-k+2,n]}_{[n-k+2,n]} \\ GX^{[n-k+2,n]}_{[1,k-1]} \end{bmatrix}$ satisfies*

$$\det G^{[k-i,k-1]}_{[k+j-i,k+j-1]} = \frac{\tilde{\varphi}_{jn-i+1}}{\tilde{\varphi}_{jn+1}}, \qquad j = 1,\dots,k-2, \quad i = 1,\dots,k-1. \tag{5.8}$$

Proposition 5.6. *There exists a $(k-1)\times(k-1)$ unipotent lower triangular matrix $H = H(X,Y)$ such that:*

(i) *entries of H are rational functions in X, Y whose denominators are monomials in cluster variables $\tilde{\varphi}_{j(n-1)+1}$, $j = 1,\dots,k-2$; and*

(ii) *the $(n-1)\times(n+k-1)$ band matrix*

$$T = \begin{bmatrix} X^{[n-k+2,n]}_{[2,n]} & Y^{[1,n-k+1]}_{[2,n]} & Y^{[n-k+2,n]}_{[2,n]} H \end{bmatrix}$$

satisfies

$$\det T^{[n+j-i+1,n+j]}_{[n-i,n-1]} = \frac{\tilde{\varphi}_{(n-j)(n-1)-i+1}}{\tilde{\varphi}_{(k-j)(n-1)+1}}, \qquad j = 2,\dots,k-1, \quad i = 1,\dots,n-1. \tag{5.9}$$

The completeness statement for $\mathcal{GC}(\mathcal{L}_{kn})$ is based on the following result.

Proposition 5.7. *All matrix entries a_{ij}, $i = 2,\dots,k$, $j = 2,\dots n$, are cluster variables in $\mathcal{GC}(\mathcal{L}_{kn})$.*

Proof. We use an argument similar to that in the proof of Proposition 3.7. Namely, we will temporarily freeze certain subsets of vertices in Q_{kn} and compare the result with initial seeds of appropriate previously studied cluster structures. First, freeze the vertices in the top row of Q_{kn}, that is, those that correspond to $\tilde{\varphi}_{j(n-1)+1}$, $j = 1,\dots,k-2$. Then the vertices that correspond to a_{11}, a_{12}, $a_{k+1,1}$, $a_{k+1,n}$ become isolated. The subquiver $\tilde{Q}_{kn}$ of Q_{kn} formed by the remaining vertices is closely related to an initial quiver $Q'_{k,n-1}$ for the regular cluster structure $\mathcal{C}(\mathcal{L}'_{k,n-1})$ on the space $\mathcal{L}'_{k,n-1}$ of $(n-1)\times(n+k-1)$ band matrices with $k+1$ diagonals that was constructed in [7, Section 10] via a quasi-isomorphism from the regular cluster structure on the affine cone $\widehat{\mathrm{Gr}}(n-1,n+k-1)$ over the Grassmannian $\mathrm{Gr}(n-1,n+k-1)$. The difference is that in $Q'_{k,n-1}$

there are no edges between the vertices in the top row and vertices in the bottom row. Let T be an element in $\mathcal{L}'_{k,n-1}$:

$$T = \begin{bmatrix} t_{12} & t_{22} & \cdots & t_{k+1,2} & 0 & \cdots & 0 \\ 0 & t_{13} & t_{23} & \cdots & t_{k+1,3} & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots \\ 0 & \cdots & 0 & t_{1n} & t_{2n} & \cdots & t_{k+1,n} \end{bmatrix};$$

note that $t_{ij} = T_{j-1,i+j-2}$. Initial cluster variables that correspond to $Q'_{k,n-1}$ in $\mathcal{C}(\mathcal{L}'_{k,n-1})$ are functions $\psi_{ij}(T)$, $i = 2, \dots, k$, $j = 2, \dots, n$, where $\psi_{ij}(T)$ is the maximal dense minor of T with t_{ij} in the upper left corner, and matrix entries t_{1j}, $j = 3, \dots, n$, $t_{k+1,j}$, $j = 2, \dots, n-1$. The latter are frozen and attached to the same vertices in Q'_{kn} that a_{1j}, $j = 3, \dots, n$, and $a_{k+1,j}$, $j = 2, \dots, n-1$, are attached in $\tilde{Q}_{kn}$. The function $\psi_{ij}(T)$ is attached to the vertex $(i-1, j)$ of Q'_{kn}. In addition to the frozen variables mentioned above, the variables attached to the vertices of the first row of Q'_{kn} are also frozen. *All* the irreducible row-dense minors of T are cluster variables in $\mathcal{C}(\mathcal{L}'_{k,n-1})$.

Note that in [7] an initial seed for $\mathcal{C}(\mathcal{L}'_{k,n-1})$ is not described explicitly. To justify our explicit description of the seed above, we rely on two observations. First, the functions $\psi_{ij}(T)$ are images under the quasi-isomorphism of [7] of cluster variables of the initial seed for $\widehat{\mathrm{Gr}}(n-1, n+k-1)$ as described in [9, Chapter 4]. This means that subquivers formed by non-frozen vertices in the initial quivers for these two structures coincide, the only difference is in the arrows that connect frozen and non-frozen variables. Second, the edges between frozen and non-frozen variables in the initial quiver for $\mathcal{C}(\mathcal{L}'_{k,n-1})$ are uniquely determined by the regularity of this cluster structures. To see this, one needs to analyze, in a bottom to top order, the exchange relations for left-most and right-most mutable vertices in $Q'_{k,n-1}$ and apply the standard Desnanot–Jacobi identities.

Now, assume that $T \in \mathcal{L}'_{k,n-1}$ is the matrix defined in Proposition 5.6. Then it follows from (5.9) that $t_{1j} = a_{1j}$ for $j = 3, \dots, n$, $t_{k+1,j} = a_{k+1,j}$ for $j = 2, \dots, n-1$ and $\psi_{ij}(T) = \frac{\tilde{\varphi}_{(k-i)(n-1)+j-1}}{\tilde{\varphi}_{(k-i+1)(n-1)+1}}$ for $i = 2, \dots, k$, $j = 2, \dots, n$. Then, just like in the proof of Proposition 3.7, we conclude that sequences of mutations in $\mathcal{GC}(\mathcal{L}_{kn})$ that do not involve functions $\tilde{\varphi}_{j(n-1)+1}$, $j = 1, \dots, k-1$, result in corresponding sequences of mutations in $\mathcal{C}(\mathcal{L}'_{k,n-1})$ with the initial seed defined by Q'_{kn} and functions t_{1j}, $t_{k+1,j}$, $\psi_{ij}(T)$. Since every t_{ij} is a cluster variable in $\mathcal{C}(\mathcal{L}'_{k,n-1})$ and $t_{ij} = a_{ij}$ unless $(i, j) \in R = \{(l, m),\ l = 3, \dots, k,\ m = n-l+3, \dots, n\}$, we use the argument in the proof of Proposition 3.7 to conclude that, for $(i, j) \notin R$, a_{ij} is a cluster variable in $\mathcal{GC}(\mathcal{L}_{kn})$.

The case of $(i, j) \in R$ is treated in a similar way. Namely, consider the vertices corresponding to $\tilde{\varphi}_{jn+1}$, $j = 0, \dots, k-2$, and to $\tilde{\varphi}_{(k-2)(n-1)-jn}$, $j = 0, \dots, k-3$, in Q_{kn}. The vertices in the first set form an anti-diagonal that starts in the upper right corner of the grid formed by non-frozen vertices of Q_{kn}, and the vertices in the second set lie immediately below the anti-diagonal that starts in the lower left corner of this grid. Let us temporarily freeze the vertices in both sets as well as all the vertices lying between them. The quiver $\hat{Q}_{kn}$ obtained by deleting all isolated vertices is, once again, similar to the quiver of the initial seed for the cluster structure $\mathcal{C}(\mathcal{L}'_{2k-2,k-1})$ on a set of finite band matrices, this time $(2k-2) \times (k-1)$ matrices with k diagonals. To see this, one just needs to move the vertices of $\hat{Q}_{kn}$ around in a way illustrated in Figure 8 for the case $n = 7, k = 5$.

The latter quiver is isomorphic to $Q'_{k-1,2k-2}$, but we denote it by $Q^T_{k-1,2k-2}$ to reflect the fact that the initial seed for $\mathcal{C}(\mathcal{L}'_{2k-2,k-1})$ can be easily obtained from the one for $\mathcal{C}(\mathcal{L}'_{k-1,2k-2})$ via transposition. To obtain $Q^T_{k-1,2k-2}$ from $\hat{Q}_{kn}$ one simply needs to erase vertices in the bottom row that

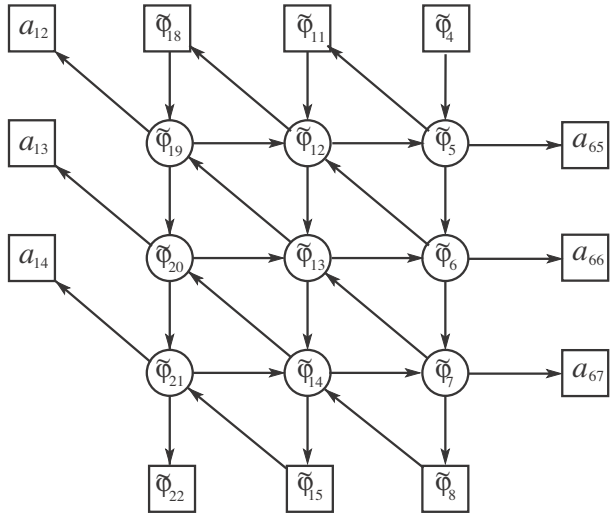

FIGURE 8 Quiver $\hat{Q}_{57}$

correspond to functions $\tilde{\varphi}_{jn+1}$, $j = 1, \dots, k-2$. Then the functions

$$\frac{\tilde{\varphi}_{jn-i+1}}{\tilde{\varphi}_{jn+1}}, \qquad j = 1, \dots, k-2, \quad i = 1, \dots, k-1$$

are subject to exchange relations in $\mathcal{C}(\mathcal{L}'_{2k-2,k-1})$. At the same time, by Proposition 5.5, these functions represent the minors of the band matrix A that, together with the frozen variables a_{1i}, $i = 2, \dots, k-1$, and $a_{k+1,i}$, $i = n-k+3, \dots, n$, form an initial seed for $\mathcal{C}(\mathcal{L}'_{2k-2,k-1})$. Then the argument concludes exactly as above. □

To establish completeness of $\mathcal{GC}(\mathcal{L}_{\underline{kn}})$, it now remains to show that a_{i1}, $i = 2, \dots, k$, belong to the generalized upper cluster algebra $\mathcal{A}(\mathcal{GC}(\mathcal{L}_{kn}))$. Since these are the entries of the first row of X as defined in (5.2), one applies a modification of Lemma 3.8 and its proof. To this end, we replace the system (3.17) with

$$\tilde{c}_j(X, Y) = \tilde{c}_j(\bar{X}, Y) + \sum_{i=1}^{k} a_{i1} z_{ij}(\bar{X}, Y), \tag{5.10}$$

where $\bar{X}$ is defined as in the proof of Lemma 3.8. The implication $\det Z = \det(z_{ij}(\bar{X}, Y))_{i,j=1}^{k} = 0 \Rightarrow \tilde{\varphi}_1 = 0$ is established via the same reasoning as before, except now $XY^{-1} = \begin{bmatrix} U & \star \\ 0 & 0 \end{bmatrix}$, where the block U is $k \times k$ and (3.18) implies that $\det[U^{k-1}e_1 U^{k-2}e_1 \cdots Ue_1\ e_1] = 0$. As before, [13, Lemma 3.3] states that the determinant in the last equation is a non-zero multiple of $\tilde{\varphi}_1$ and the desired implication is confirmed. The rest of the arguments in the proof of Lemma 3.8 transfer to the current situation in a straightforward way.

ACKNOWLEDGEMENTS

Our research was supported in part by the NSF research grant DMS #1702054 (M.G.), NSF research grant DMS #1702115 and International Laboratory of Cluster Geometry NRU HSE, RF Government grant, ag. # 075-15-2021-608 from 08.06.2021 (M.S.), and ISF grant #1144/16 (A.V.). While

working on this project, we benefited from support of the following institutions and programs: Research Institute for Mathematical Sciences, Kyoto (M.G., M.S., A.V., Spring 2019), Research in Pairs Program at the Mathematisches Forschungsinstitut Oberwolfach (M.S. and A.V., Summer 2019), Istituto Nazionale di Alta Matematica Francesco Severi and the Sapienza University of Rome (A.V., Fall 2019), University of Haifa (M.G., Fall 2019), Mathematical Science Research Institute, Berkeley (M.S., Fall 2019), Michigan State University (A.V., Spring 2020), University of Notre Dame (A.V., Spring 2020). We are grateful to all these institutions for their hospitality and outstanding working conditions they provided. Special thanks are due to Peigen Cao and Fang Li who in response to our request provided a generalization [3] of their previous results, to Linhui Shen for pointing out to us the preprint [21], to Alexander Shapiro and Gus Schrader for many fruitful discussions, and to the reviewers for constructive suggestions.

JOURNAL INFORMATION

REFERENCES

1. A. Berenstein, S. Fomin, and A. Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52.
2. P. Cao and F. Li, *Some conjectures on generalized cluster algebras via the cluster formula and D-matrix pattern*, J. Algebra **493** (2018), 57–78.
3. P. Cao and F. Li, *On some combinatorial properties of generalized cluster algebras*, J. Pure Appl. Algebra **225** (2021), 106650.
4. L. Chekhov and M. Shapiro, *Teichmüller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables*, Int. Math. Res. Notes (2014), no. 10, 2746–2772.
5. V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 1–211.
6. S. Fomin and A. Zelevinsky, *Cluster algebras I: foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.
7. C. Fraser, *Quasi-homomorphisms of cluster algebras*, Adv. Appl. Math. **81** (2016), 40–77.
8. C. Fraser, *Cyclic symmetry loci in Grassmannians*, arXiv:2010.05972v1, 2020.
9. M. Gekhtman, M. Shapiro, and A. Vainshtein, *Cluster algebras and Poisson geometry*, Mathematical Surveys and Monographs, vol. 167, Amer. Math. Soc., Providence, R.I., 2010.
10. M. Gekhtman, M. Shapiro, and A. Vainshtein, *Generalized cluster structure on the Drinfeld double of GL_n*, C. R. Math. Acad. Sci. Paris **354** (2016), 345–349.
11. M. Gekhtman, M. Shapiro, and A. Vainshtein, *Exotic cluster structures on SL_n: the Cremmer–Gervais case*, Mem. Amer. Math. Soc. **246** (2017), no. 1165.
12. M. Gekhtman, M. Shapiro, and A. Vainshtein, *Drinfeld double of GL_n and generalized cluster structures*, Proc. Lond. Math. Soc. **116** (2018), 429–484.
13. M. Gekhtman, M. Shapiro, and A. Vainshtein, *Periodic staircase matrices and generalized cluster structures*, Int. Math. Res. Notes, 2022, to appear, arXiv:1912.00453.
14. A.- S. Gleitz, *Representations of $U_q(L\mathfrak{sl}_2)$ at roots of unity and generalised cluster algebras*, European J. Combin. **57** (2016), 94–108.
15. M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), 497–608.
16. A. Izosimov, *Pentagram maps and refactorization in Poisson-Lie groups*, arXiv:1803.00726, 2018.
17. B. Leclerc, *Cluster structures on strata of flag varieties*, Adv. Math. **300** (2016), 190–228.

18. T. Nakanishi, *Structure of seeds in generalized cluster algebras*, Pacific J. Math. **277** (2015), 201–218.
19. J. S. Scott, *Grassmannians and cluster algebras*, Proc. Lond. Math. Soc. **92** (2006), 345–380.
20. L. Shen, *Duals of semisimple Poisson-Lie groups and cluster theory of moduli spaces of G-local systems*, Int. Math. Res. Notes, 2021; https://doi.org/10.1093/imrn/rnab094
21. Y. Zhou, *Cluster structures and subfans in scattering diagrams*, SIGMA **16** (2020), 013.