

Easy, Medium, and Hard: Structuring Space in 2D and 3D by Way of Linear Combinations

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Understanding linear combinations is at the core of linear algebra and impacts their understanding of basis and linear transformations. This research will focus on how students understand linear combinations after playing a video game created to help students link the algebraic and geometric representations of linear combinations. I found that having students reflect upon the game and create their own 3D version of the game illustrated which elements of 2D understanding could be translated into 3D. Also, students creation of easy, medium, and hard levels provided insight into how students progressively structure space.

Keywords: Linear Algebra, Linear Combinations, Game-based Learning

Understanding linear combinations in a variety of dimensions is core understanding linear algebra. In addition, it is important that students can reason through a variety of representations of multiple concepts including linear combinations (Hillel, 2000; Sierpinska, 2000; Larson & Zandieh, 2013). Well-designed games created with learning theories in mind can help students associate multiple representations and problem-solve (Ke & Clark, 2019). *Vector Unknown* (Zandieh et al., 2018; Mauntel et al, 2019; Mauntel et al., in press) is a game designed using the tenants of Realistic Mathematics Education (RME). The focus of the game is to help students connect geometric and algebraic representations of linear combinations with vector equations. This paper is part of a larger project where the goal is to analyze how students structure two-dimensional space after playing the game *Vector Unknown*, and how this two-dimensional space structuring informs their structuring of three-dimensional space.

Literature Review

Vector Unknown is based on the Magic Carpet Ride task (Wawro et al., 2013) which is the first task in a sequence that cover linear combinations, span, linear independence. The task uses travel as a metaphor for taking linear combinations of vectors. Stewart and Thomas (2010) found that linear combinations were central to students' understanding of basis, span, and linear independence. Dogan-Dunlap (2010) utilized Sierpinska's modes of reasoning (2000) and found that students who were able utilize both geometric modes of thinking were also able to utilize analytic modes of reasoning. This research indicates that bridging the gap between geometric and algebraic representations of vectors and linear combinations can be beneficial for students learning linear algebra. Coordinating the ideas of linear independence and span is a key area of difficulty related to organizing linear combinations and transitioning from two-dimensional to three-dimensional space. These areas could be addressed using game-based learning as they can be represented in multiple ways including a visual mode that may difficult to represent on paper.

Well-designed games follow good learning principles (Gee, 2003). Gee (2003) postulates that one of the reasons video games are so popular is because they keep students pleasantly frustrated with well-ordered problems that engage them in complex reasoning. Furthermore, intermittent periods of reflection and gameplay have been shown to be successful in helping students learn from video game environments (Foster & Shah, 2015). For this reason, this

research will involve having students play the game *Vector Unknown*, reflect upon the game, and then use this reflection to design their own three-dimensional space version of the game.

Study Context

The Game Vector Unknown

In this section I describe the game *Vector Unknown* (<https://tinyurl.com/linearbunny>) its creation, and describe some gameplay. The game *Vector Unknown* was developed as a coordination between math educators, experts in game-based learning, and programmed by a team of capstone students at Arizona State University to help students understand the connection between a vector equation and its geometric representation. Zandieh et al. (2018) designed the game *Vector Unknown* based upon a theoretical framework that intersects GBL with RME (Zandieh et al., 2018). The goal of the game is to have the player select two vectors, adjust the scalars, and press GO to guide the rabbit to the basket. In the first level (Figure 2), the game presents with a collection of four different vectors. Once players have chosen at least



Figure 2. First level of the game *Vector Unknown* from Mauntel et al. (2019)

one vector, they can adjust the scalars in front of the vectors. Feedback received by the players includes a red predicted path generated from the rabbit illustrating each component of the vector equation. The result of the vector equation adjusts as the player adjusts the scalars in the vector equation. Above the log, players can view their position and the position of the basket location. The map in the upper left-hand coordinates provides an alternate view of the map. After the player presses GO, the rabbit moves along the path towards the basket and the log provides all vector equations used.

Prior Research on Vector Unknown

Mauntel et al. (in press) previously categorized game-play strategies from two main lenses: geometric or numeric. The geometric lens is when students focus on utilizing the graphical components of the game relating to the graph in particular the Predicted Path. Numeric thinking relates to when the player focuses more on the Vector Equation. One of the main goals of the

game is to encourage the player to transition fluidly between the two different lenses. Mauntel et al. noticed several strategies that students employed while playing the game.

One of the strategies, quadrant-based reasoning, is when players utilize direction to anticipate to make their vector choices. The term quadrant is used because players start at the origin and often they would choose vectors that corresponded to the quadrant of the basket location. In the numeric lens this presented itself as a player choosing vectors that shared the same sign as the basket. In the geometric lens, students would often drag a vector and try it to see its direction to determine if it were in the same direction as the basket. Often players would transition between lenses for instance there are times when players would find the slope between two points by counting the change in y and change x on the Cartesian plane and then using this information to find the vector switching to a more numerical lens.

For the strategy focus on one vector, players would choose a single vector and then scale that vector as close to the basket as possible and then choose another vector to complete the trip to the basket. This theme presented itself in the numeric lens as scaling a vector until it was as close to the basket location in the vector equation as possible. In the geometric lens, players would scale the vector until the predicted path was close to the basket location, and then choose another vector to complete the trip. The fourth theme, focus on one coordinate, involved the players trying to use a vector or combination of vectors to reach a value or destination that matched the x - or y -coordinate of the basket location. This theme presented itself frequently when standard basis vectors were involved.

Theoretical Framing

My research will focus on analyzing student's mathematical activity in the sense of Freudenthal (1971, as cited by Gravemeijer & Terwell, 2000) as it relates to the taking linear combinations of vectors in two dimensions and the emerging activity that it generates. From this mathematical activity, I want to characterize a new type of emergent activity which I call structuring space use the term emerging activity as opposed to emerging models to indicate that tasks are not designed to guide the students to a fixed emergent model or formalize a particular concept, rather this research is focused on what models are possible to inform future development of an RME sequence. I call the students' emerging activity their structuring of space. After the students structure two-dimensional space, they will be asked to structure three-dimensional space. This transition is meant to induce a model of/model for transition as their structuring of two-dimensional space is utilized for structuring three-dimensional space. This research will utilize a realistic starting point of the video game Vector Unknown. Also, I will be looking at a how the student's structuring of two-dimensional space impacts their gameplay to gain more insight into how their ideas of structuring space relate to the game context. My research questions are:

1. How did students structure two-dimensional space with respect to linear combinations in relation to the game Vector Unknown?
2. How did students adapt their structure of two-dimensional space to a three-dimensional setting when designing a three-dimensional game based upon Vector Unknown?

This paper is part of a larger dissertation project and in this paper I will present a portion of my findings for research questions 1 and 2 for one group of students.

Interview Protocol

The interview took place over the course of four sessions. During the first session students played the game Vector Unknown individually. In the second section students were

paired with another student based upon their gaming experience and were given a set of four vectors taken from the Vector Selection in the game and asked to list all possible locations to place the basket. Then, they were asked to create an easy, medium, and hard difficult set of vectors that could reach the basket location at $(-3, 5)$. Finally, during the third session, participants were asked to design a three-dimensional space game based upon the two-dimensional space game by either creating an Easy, Medium, and Hard difficulty game with vectors and potential basket locations or design three sets of vectors (Easy, Medium, and Hard) that could be used to reach a basket placed at $(3, -4, 5)$.

Participants, Data Sources, and Methods of Analysis

For this paper I will be looking at two students from a large Southern University. Neither student had taken linear algebra at the time of the interview. Students were asked to self-identify their race and gender. Both students identified as women, with one identifying her race as white (given the pseudonym Gabby) and the other identifying her race as other (given the pseudonym Delores). Also, both students played video games for less than 5 hours per week and thus were classified as non-gamers.

Interviews were conducted over Zoom, recorded to a secure server, and auto-transcribed using Zoom's automatic transcription service. The videos and associated transcripts for the second and third sessions were then reviewed for instances of structuring space in two and three dimensions respectively. These instances were then detailed using captured images and quotations from the transcripts edited for accuracy when necessary. The two-dimensional and three-dimensional structuring spaces were then reviewed for connections and cross-cutting themes consistent with grounded theory (Strauss & Corbin, 1990).

Findings

In this section I present several examples of structuring space induced by the game *Vector Unknown* and the post-play reflections. Delores and Gabby had several ways of thinking about linear combinations of vectors in two- and three-dimensional space. In two-dimensional space, Delores and Gabby focused on creating increasing more complex linear combinations of the vectors that would reach the basket of $(-3,5)$. When designing the three-dimensional space game, they created easy, medium, and hard version of the game and chose to illustrate all the possible locations that could be reached by the vectors. With the basket location not fixed their method of creating different difficulties of the game was based upon creating basket locations that were more difficult to reach rather than creating different sets of vectors.

Easy, Medium, and Hard in two-dimensional space: Numerical Structuring leads to Vector Equations

Gabby and Delores's work in creating Easy, Medium, and Hard vectors led them think about vector equations and how they could be solved. Gabby and Delores designed their Easy vectors by thinking about the numerical relationships of the vectors in relation to the basket at $\langle -3, 5 \rangle$ (Figure 2). They wanted the vectors to sum to the basket location without having to scale. This activity resulted in breaking apart the vector $\langle -3, 5 \rangle$ in terms of x and y coordinates (for $\langle 0, 5 \rangle$ and $\langle -3, 0 \rangle$) or finding numbers that sum to -3 and 5 , for example $\langle -1, 3 \rangle$ and $\langle -2, 2 \rangle$. Gabby described their strategy for finding Medium vectors by comparing it to their strategy for finding easy vectors:

So that one would be more medium, I guess, because, instead of just having the numbers right there you actually have to multiply the numbers to get the ones you want. For medium, we can do like negative one zero and then zero negative one, so that way it's less addition and more multiplication.

Medium difficulty is the first instance that Gabby and Delores transition to using linear combinations of vectors that included addition and multiplication. Gabby's idea for the first two vectors led to Delores's idea for finding the remaining vectors by writing down a vector equation.

Easy	Medium	Hard
$\langle -1, 3 \rangle, \langle -2, 2 \rangle, \langle 0, 5 \rangle, \langle -3, 0 \rangle$	$\langle -1, 0 \rangle, \langle 0, -1 \rangle, \langle -1, 2 \rangle, \langle -1, 1 \rangle$	$\langle 6, 7 \rangle, \langle -9, -2 \rangle, \langle 4, 7 \rangle, \langle 9, 26 \rangle$

Figure 2. Easy, Medium, and Hard Vectors Selections

Here Delores first started with trying to find two vectors that would add up to $\langle -3, 5 \rangle$ knowing that she wanted to use -2 as a weight (Figure 3). She eventually dropped the negative and through a process of guess and check was able to find vectors that worked. For the Hard set of vectors both Gabby and Delores agreed that larger numbers were appropriate. Delores created the pair of vectors $\langle 6, 7 \rangle$ and $\langle -9, -2 \rangle$ to sum to the basket location of $\langle -3, 5 \rangle$. Gabby used a similar technique to Delores to design the Vectors $\langle 4, 7 \rangle$ and $\langle 9, 26 \rangle$ by choosing a scalar of -3, selecting the vector $\langle 4, 7 \rangle$ and multiplying it by -3, and finally finding solving for each coordinate to find the vector $\langle 9, 26 \rangle$. Gabby's work did not indicate clearly if she had guessed and checked both vectors or if she chose -3 and $\langle -4, 7 \rangle$ and found then solved for $\langle 9, 26 \rangle$.

$$-2 \begin{bmatrix} \square \\ \square \end{bmatrix} + \begin{bmatrix} \square \\ \square \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \quad \rightarrow \quad 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Figure 3. Vector Equations as drawn by Delores for Medium Difficulty

After designing Easy, Medium, and Hard vectors, the interviewer asked the participants how they would explain to a programmer a technique for creating vectors. They focused on describing their process for Medium and Hard difficulty vectors. Delores explained that you could create the vectors by “just take four random numbers and subtract them until I get my, until I get either the -3 or 5” (Figure 3). Here Delores is describing her process of trying random numbers in the vectors until you find a combination that works for the x and y coordinate. This indicates that Delores is thinking about using subtraction to identify individual coordinates instead of whole vectors. This corresponds strongly with the focus on one coordinate strategy in the Vector Unknown game.

$$n \begin{bmatrix} n \\ n \end{bmatrix} + \begin{bmatrix} n \\ n \end{bmatrix}$$

Figure 3. Random Vector Equation

Gabby built on this idea and stated that “you would already have the point specified that you want like (-3,5) and then picking random numbers for Vector 1 ... then the number you multiply it by and then solving for Vector 2.” This suggests that Gabby has started thinking about Vector Equations as equations in themselves, subtracting whole vectors as opposed to just thinking in terms of individual coordinates.

Easy, Medium, and Hard in three-dimensional space: Working with Vector Equations

When asked to design a three-dimensional space game, Gabby and Delores chose to create Easy, Medium, and Hard games and illustrate all the possible locations where the basket could be located. This contrasts to the exercise in the previous session where they were designing easy, medium, and hard vectors to reach a specific basket location. The interviewer introduced them to GeoGebra three-dimensional space since neither student had worked in three-dimensional space previously. They began by designing by choosing random numbers for easy vectors and taking all combinations of the easy vectors for possible locations of the basket indicated by points A-F which were sums of the vectors $\langle 0,2,0 \rangle$, $\langle -1,2,3 \rangle$, $\langle 2,0,1 \rangle$, and $\langle -4,3,-1 \rangle$ (Figure 4).

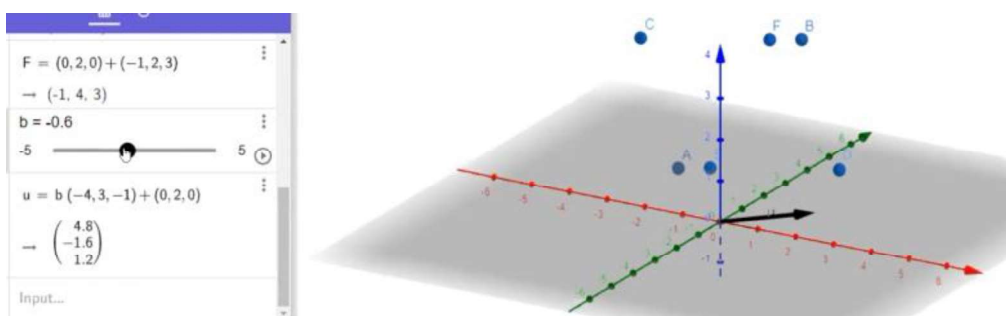


Figure 4. Creating the basket locations for the Easy and beginning goals for Medium

When they transitioned to Medium rather than change the vectors, they decided to change the possible basket locations by placing scalars in front of one vector in each sum that represented a basket in the Easy mode of the game. Figure 5 illustrates one basket location from easy and how it was adapted to medium and hard.

Basket Locations for Easy	Basket Locations for Medium	Basket Locations for Hard
$(2,0,1) + (-1,2,3)$	$b(2,0,1) + (-1,2,3)$ $(2,0,1) + b(-1,2,3)$	$b(2,0,1) + c(-1, 2, 3)$

Figure 5. Easy, Medium, and Hard basket location comparison

Both students focused on how they could adapt expression that they had developed to find the basket location. Here Gabby and Delores used reasoning about the Easy, Medium, and Hard vectors where an increase in difficulty resulted in more scalar multiplication in the context of their linear combinations.

After creating Easy, Medium, and Hard difficulty, Gabby and Delores were asked to describe geometrically some of their basket locations:

Interviewer: What does $G = (-4, 3, -1) + b(0,2,0)$ look like geometrically?

Delores: It would go all around the board because it has negatives and positives...[Delores adjusts her slider] It stays on one side of the board.

Gabby: I can't tell if the vector stays on one side because the graph is three-dimensional space. We cannot use the quadrants anymore.

Here Delores and Gabby are trying to build on different knowledge they used previously in the interview. Delores is trying to first use a mainly numerical argument using signs until she adjusts the sliders and discovers that the basket location seems to be on one side of the board. Gabby could not decide if she was convinced Delores's statement is correct. This indicates that Gabby is trying to adapt the notion of quadrants to three-dimensional space and what it means to be on "one side." This is important because one of the keyways that student's reason about the game in two-dimensional space was quadrant-based reasoning. If students were using quadrant-based reasoning to reason about three-dimensional space, they need a new way to reconceptualize quadrants. One possible way of dealing with this involves thinking about the controls of the video game and their relations to the scalars as suggested in the following conversation:

Interviewer: If you want to look at all the places you put a goal [basket] with the equation $b \langle 2, 0, 1 \rangle + c \langle -1, 2, 3 \rangle$, can you describe it?

Gabby: b makes it move the vector move up and down and c makes it move side to side

Interviewer: So if I were to plot all the goals [baskets], what would it look like?

Delores: It would look like an L-shape and on the left side.

This indicates that thinking about the way the basket moves with the slides similar to the controls of the game might be a way to think about the location movement of the basket. Delores's comments highlight the point that while she has conceptualized some points in the linear combination $b \langle 2, 0, 1 \rangle + c \langle -1, 2, 3 \rangle$, the basket locations that require moving both b and c eluded her description. This could be a result of the limited time of the session or because she thought there might be constraints on what scalars were used at what points. Either way it illustrates that points that involve changing the scalars on each vector simultaneously

Discussion

While students were reflecting upon the two-dimensional game and designing their own three-dimensional game, they provided valuable insight into how they organized linear combinations, solved vector equations, and conceptualized span (all possible basket locations). While this work only analyzed one group of students and thus lacks generalizability, it illustrates that certain aspects of two-dimensional space like quadrants, lines dividing space, and balancing multiple coordinates do not necessarily translate easily for students to three-dimensions. In addition, the linear combination $av_1 + bv_2$ for vectors v_1, v_2 and scalars a, b can prove difficult to understand and may need additional geometric representational support in order to be fully understood. Additionally, game-based scaffolds such as easy, medium, and hard type levels can provide insight into how students scaffold their conception of structuring space and taking linear combinations. This could be valuable as well-designed video games contain several scaffolds to create pleasantly frustrating experiences for the player (Gee, 2003). I can see having students analyze these scaffolds and design games with these scaffolds could be a rich area of research that allows students to structure their ideas and provide valuable insights into their thinking.

Future research includes interviewing more students and seeing how they structure space and transition from two-dimensional structuring to three-dimensional structuring. Also, additional analysis will compare the student's gameplay strategies from their playthrough of the game with their structuring of space in two and three dimensions.

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