



The Q_k flow on complete non-compact graphs

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Abstract

We establish the long time existence of complete non-compact weakly convex and smooth hypersurfaces Σ_t evolving by the Q_k -flow. We show that the maximum existence time T depends on the dimension d_W of the vector space $W := \{w \in \mathbb{R}^{n+1} : \sup_{X \in \Sigma_0} |\langle X, w \rangle| = +\infty\}$ which contains each direction in which our initial data Σ_0 is infinite. If $d_W = \dim(W) \geq n - k + 1$, then the solution Σ_t exists for all time $t \in (0, +\infty)$; if $d_W = \dim(W) \leq n - k$, then the solution Σ_t exist up to some finite time $T < +\infty$. In the latter case, the *trace at infinity* Γ_t of the solution Σ_t is a closed convex viscosity solution of the $(n - d_W)$ -dimensional Q_k flow on $t \in (0, T)$.

1 Introduction

In this his work we study the long time existence of a family of complete non-compact strictly convex hypersurfaces Σ_t embedded in \mathbb{R}^{n+1} which evolve by the Q_k -flow. Given a complete and convex hypersurface Σ_0 embedded in \mathbb{R}^{n+1} , we assume that $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ is an immersion with $F_0(M^n) = \Sigma_0$. We say that the one-parameter family of immersions

$$F : M^n \times (0, T) \rightarrow \mathbb{R}^{n+1}$$

is a solution of the Q_k -flow ($1 \leq k \leq n$), if $F(M^n, t) = \Sigma_t$ are complete convex hypersurfaces for all $t \in (0, T)$ and $F(\cdot, t)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} F(p, t) &= Q_k(p, t) \vec{n}(p, t) \\ \lim_{t \rightarrow 0} F(p, t) &= F_0(p). \end{cases} \quad (*_k^n)$$

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where $\vec{n}(p, t)$ is the unit normal vector pointing inside the convex hull of Σ_t . The speed

$$Q_k(p, t) := \frac{S_k(p, t)}{S_{k-1}(p, t)}$$

is the quotient of the elementary successive polynomials of the principal curvatures $\{\lambda_1(p, t), \dots, \lambda_n(p, t)\}$ of Σ_t at $F(p, t)$, given by

$$S_0(p, t) = 1, \quad S_k(p, t) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1}(p, t) \cdots \lambda_{i_k}(p, t) \quad \text{for } 1 \leq k \leq n.$$

In [1], B. Andrews showed the existence of strictly convex closed solutions of a class of nonlinear flow which includes the Q_k -flow. S. Diater extended the results to closed convex solutions with the positive S_{k-1} curvature in [9]. Moreover, Caputo, Daskalopoulos, and Sesum showed the existence of compact convex $C^{1,1}$ viscosity solutions with flat sides in [3] and in [4]. Closed non-convex solutions of the Q_2 -flow in \mathbb{R}^3 , the *Harmonic mean curvature flow*, were considered by Daskalopoulos and Hamilton in [6] and by Daskalopoulos, Hamilton and Sesum in [8].

The equation $(*_k^n)$ is fully-nonlinear except from the case of $k = 1$ which is the flow by Mean curvature. The evolution of *entire graphs* by the Mean curvature flow was studied by Ecker and G. Huisken in [10, 11]. Sáez and Schnürer [12] showed the existence of complete solutions of the *Mean curvature flow* for an initial hypersurface which is a graph $\Sigma_0 = \{(x, u_0(x)) : x \in \Omega_0\}$ over a bounded domain Ω_0 , and $u_0(x) \rightarrow +\infty$ as $x \rightarrow \partial\Omega_0$.

The Ecker and Huisken result in [11] shows that in some sense the Mean curvature flow behaves better than the heat equation on \mathbb{R}^n , namely an entire graph solution exists for all time independently from the growth of the initial surface at infinity. The initial entire graph is assumed to be only locally Lipschitz. This result is based on a local gradient estimate which is then combined with the evolution of the norm of the second fundamental form $|A|^2$ to give a local bound on $|A|^2$, which is independent from the behavior of the solution at spatial infinity. The latter is achieved by adopting the well known technique of Caffarelli, Nirenberg and Spruck in [2] in this geometric setting.

An open question between the experts in the field is whether the techniques of Ecker and Huisken in [10, 11] can be extended to the fully-nonlinear setting. Recently in [5], the authors jointly with L. Kim and K.-A. Lee, established the all time existence for complete non-compact and convex solutions to the flow by positive powers of the Gauss curvature. In addition, L. Xiao [17] obtained the existence of admissible solutions to a certain class of fully nonlinear flows.

In this work we will show the existence of complete non-compact solutions of the Q_k -flow under the assumption of weak convexity. Let Σ_0 denote our initial surface. We will assume that Σ_0 is a *smooth weakly convex graph* $\Sigma_0 = \{(x, u_0(x)) : x \in \Omega\}$ with the *positive Q_k curvature* defined by a function $u_0 : \Omega \rightarrow \mathbb{R}$ on an open convex domain $\Omega \subset \mathbb{R}^n$ such that:

- (i) if $\Omega_0 \neq \mathbb{R}^n$, then for all $x_0 \in \partial\Omega_0$, $\lim_{x \rightarrow x_0} u_0(x) = +\infty$ holds;
- (ii) if Ω_0 is unbounded, then $\lim_{|x| \rightarrow +\infty} u_0(x) = +\infty$ holds.

Let W denote the vector space

$$W = \{w \in \mathbb{R}^{n+1} : \sup_{X \in \Sigma_0} |\langle X, w \rangle| = +\infty\}$$

which contains each direction in which Σ_0 is infinite. Then, $\sup_{X \in \Sigma_0} \sup_{w \in W^\perp} |\langle X, w \rangle|$ is bounded by some constant R . Namely, Σ_0 is contained in a cylinder $B_R^{n+1-dw} \times \mathbb{R}^{dw}$ where B_R^{n+1-dw}

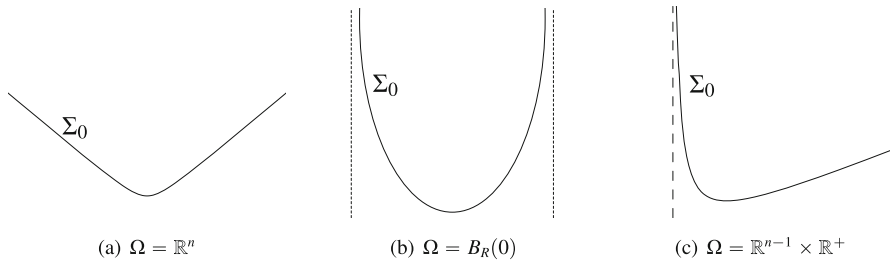


Fig. 1 Examples of the initial hypersurface Σ_0

is a $(n + 1 - d_W)$ -dimensional ball of radius R . Moreover, the convexity of Σ_0 implies that the trace at infinity Γ_0 of Σ_0 is a $(n - d_W)$ -dimensional closed convex hypersurface such that

$$\lim_{|F_0| \rightarrow +\infty} \Sigma_0 = \Gamma_0 \times \mathbb{R}^{d_W}. \quad (1)$$

For example, if $d_W = 1$ then Σ_0 is a graph over a bounded domain Ω and $\Gamma_0 = \partial\Omega$. In particular, if $n = d_W$ then Γ_0 consists of two points, namely Σ_0 is contained two parallel hyperplanes.

We will see in this work that the time of existence T for a solution Σ_t of $(*_k^n)$ depends on the dimension $d_W := \dim(W)$. If $d_W \geq n - k + 1$, then the solution Σ_t will exist for all time $t \in (0, +\infty)$. However, if $d_W = \dim(W) \leq n - k$, then the solution Σ_t will exist only up to some finite time T . In the latter case, we will show that the time of existence T also depends on the trace at infinity Γ_0 of Σ_0 . In fact, we will show that if Γ_t is the trace at infinity of the solution Σ_t , namely if

$$\lim_{|\Gamma_t| \rightarrow +\infty} \Sigma_t = \Gamma_t \times \mathbb{R}^{d_W} \quad (2)$$

then Γ_t is an $(n - d_W)$ -dimensional closed convex hypersurface which also evolves by the Q_k -flow $(*_k^n)$. Note that for this we need that $k \leq n - d_W$ which is equivalent to $d_W \leq n - k$.

Our main result in this work states as follows:

Theorem 1.1 Assume that $\Sigma_0 = \{(x, u_0(x)) : x \in \Omega\}$ is a smooth weakly convex graph with the positive Q_k curvature defined by a function $u_0 : \Omega \rightarrow \mathbb{R}$ such that the conditions (i)-(ii) above hold. Let W denote the vector space $W = \{w \in \mathbb{R}^{n+1} : \sup_{X \in \Sigma_0} |\langle X, w \rangle| = +\infty\}$ which contains each direction in which Σ_0 is infinite. Then, given a smooth immersion F_0 of $F_0(M^n) = \Sigma_0$, the following holds:

- if $d_W = \dim(W) \geq n - k + 1$, then there is a complete convex solution Σ_t of $(*_k^n)$ with initial data Σ_0 existing for all time $t \in (0, +\infty)$;
- if $d_W = \dim(W) \leq n - k$, then there exists a complete convex smooth solution Σ_t of $(*_k^n)$ with initial data Σ_0 which is defined on $t \in (0, T)$, for some finite $T < \infty$. Moreover, the trace at infinity Γ_t of the solution Σ_t , defined by (2), is a closed convex viscosity solution of the $(n - d_W)$ -dimensional Q_k flow $(*_k^n)$ on $t \in (0, T)$.

Remark 1.2 (Different cases in Theorem 1.1) In the case $d_W \leq n - k$, the trace at infinity Γ_t is a continuous solution of the $(n - d_W)$ -dimensional Q_k flow $(*_k^n)$ which is defined in the viscosity sense (c.f. [4]). Since Γ_t is a closed hypersurface, it develops singularity at some finite time \hat{T} . However, Σ_t may possibly develop a singularity or become flat at time $T < \hat{T}$. In this paper, we only consider the maximum existence time rather than its limit profile.

In the case $k = 1$, the result in [12] shows that if a convex complete solution Σ_t to the mean curvature flow is a family of graphs on an evolving bounded domain Ω_t , then the boundary $\partial\Omega_t$ is also a solution to the mean curvature flow. In this case, in our Theorem 1.1, $\Gamma_t = \partial\Omega_t$.

In the case $k = n$, Theorem 1.1 shows that strictly convex non-compact complete solutions exist for all time, which is the same result to the Gauss curvature flow in [5]. Although the σ_n curvature is different from the Gauss curvature, they go to zero at the infinity, which yields the all time existence.

Remark 1.3 (General initial data) A complete and strictly convex hypersurface in \mathbb{R}^{n+1} can be expressed as the graph of a function such that conditions (i)-(ii), see in [16]. Thus, Theorem 1.1 shows the existence of a complete convex solution Σ_t of $(*_k^n)$ for any complete smooth strictly convex hypersurface Σ_0 .

Discussion of the proof of Theorem 1.1: The proof of Theorem 1.1 mainly relies on three a priori local estimates: the local gradient bound shown in Theorem 2.4, the local speed estimate given in Theorem 3.1 and a local bound from above on the second fundamental form $|A|^2$ given in Theorem 4.2. The gradient and the speed estimates use the well known technique by Caffarelli, Nirenberg and Spruck in [2] also used by Esker and Huisken in the context of the Mean curvature flow in [11]. Then, by using the concavity of the $Q_k(\lambda)$ function, we derive a local bound on $|A|^2$ by modifying the elliptic estimate by W. Sheng, J. Urbas and X.-J. Wang in [13] to the parabolic setting. The long time existence is shown by approximation with compact hypersurfaces and applying the local a priori estimates.

Notation 1.4 We summarize the following notation, which will be frequently used in this paper.

- (i) We recall the second fundamental form $h_{ij} := \langle \nabla_i \nabla_j F, \vec{n} \rangle$ and the metric $g_{ij} := \langle F_i, F_j \rangle$, where $F_i := \nabla_i F$.
- (ii) We denote by $\bar{u} : M^n \rightarrow \mathbb{R}$ the height function $\bar{u}(p, t) := \langle F(p, t), \vec{e}_{n+1} \rangle$. Also, given a constant $M \in \mathbb{R}$, we define a cut-off function ψ by

$$\psi(p, t) := (M - \bar{u}(p, t))_+ = \max(M - \bar{u}, 0).$$

- (iii) $v := (\vec{n}, \vec{e}_{n+1})^{-1}$ denote the gradient function (as in [11]).
- (iv) We denote by \mathcal{L} the linearized operator,

$$\mathcal{L} := \frac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j.$$

In addition, $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ denotes the inner product $\langle \nabla f, \nabla g \rangle_{\mathcal{L}} = \frac{\partial Q_k}{\partial h_{ij}} \nabla_i f \nabla_j g$, where f, g are differentiable functions on M^n , and $\| \cdot \|_{\mathcal{L}}$ denotes the \mathcal{L} -norm given by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$.

- (v) For the principal curvatures $\{\lambda_1, \dots, \lambda_n\}$, we denote by λ_{\max} the largest principal curvature $\lambda_{\max} := \max\{\lambda_1, \dots, \lambda_n\}$. Also, denote the following functions of the principal curvatures

$$S_{k;i}(\lambda) := \frac{\partial S_{k+1}(\lambda)}{\partial \lambda_i}, \quad |A|_k^2(\lambda) := \sum_{i=1}^n \frac{\partial Q_k(\lambda)}{\partial \lambda_i} \lambda_i^2, \quad D_i Q_k = \frac{\partial Q_k(\lambda)}{\partial \lambda_i}, \quad D_{ij} Q_k = \frac{\partial^2 Q_k(\lambda)}{\partial \lambda_i \partial \lambda_j}.$$

2 Preliminaries

In this section, we will review some properties of the symmetric function $Q_k(\lambda)$ of λ , and we will derive some basic evolution equations under the Q_k -flow. We will also establish a local gradient estimate and a local lower bound on the speed, as straightforward consequence of the evolution equations.

Proposition 2.1 *Assume that Σ is a convex smooth hypersurface in \mathbb{R}^{n+1} with the positive Q_k curvature and $F : M^n \rightarrow \mathbb{R}^{n+1}$ is a smooth immersion satisfying $F(M^n) = \Sigma$. Let us choose an orthonormal frame at some point $F(p)$ satisfying $g_{ij}(p) = \delta_{ij}$, $h_{ij}(p) = \delta_{ij}\lambda_i(p)$. Then, the following holds at $F(p)$ for each $m \in \{1, \dots, n\}$*

$$\sum_{i,j,p,q} \frac{\partial^2 Q_k}{\partial h_{ij} \partial h_{pq}} \nabla^m h_{ij} \nabla_m h_{pq} \leq 2 \sum_{i < j} \frac{D_i Q_k - D_j Q_k}{\lambda_i - \lambda_j} \nabla^m h_{ij} \nabla_m h_{ij}.$$

Proof We recall the following identity which holds on homogeneous of degree one functions of matrices, given in [1] (see also in [4]):

$$\sum_{i,j,p,q} \frac{\partial^2 Q_k}{\partial h_{ij} \partial h_{pq}} \nabla^m h_{ij} \nabla_m h_{pq} = \sum_{i,j} D_{ij} Q_k \nabla^m h_{ii} \nabla_m h_{jj} + \sum_{i \neq j} \frac{D_i Q_k - D_j Q_k}{\lambda_i - \lambda_j} \nabla^m h_{ij} \nabla_m h_{ji}.$$

By the concavity of $Q_k(\lambda)$ we have $\sum_{i,j} D_{ij} Q_k \nabla^m h_{ii} \nabla_m h_{jj} \leq 0$, hence the desired inequality follows. \square

Proposition 2.2 *If $Q_k(\lambda_1, \dots, \lambda_n) > 0$ and $\lambda_i \geq 0$ for all $i \in \{1, \dots, n\}$, then the following hold*

$$n^{-2} Q_k^2 \lambda_{\max}^{-2} \leq D_i Q_k \leq 1 \quad (2.1)$$

$$D_i Q_k \leq \lambda_i^{-2} Q_k^2 \quad (2.2)$$

$$\frac{k}{n-k+1} Q_k^2 \leq |A|_k^2 \leq n Q_k^2 \quad (2.3)$$

Proof The case $k = 1$ is obvious. We assume that $k \geq 2$ and $\lambda_1 = \lambda_{\max}$. we begin by recalling the following from the proof of Lemma 3.6 in [9]

$$\frac{n}{k(n-k+1)} \frac{S_{k-1;i}^2}{S_{k-1}^2} \leq \frac{\partial Q_k}{\partial \lambda_i} = \frac{S_{k-1;i}^2 - S_{k;i} S_{k-2;i}}{S_{k-1}^2} \leq \frac{S_{k-1;i}^2}{S_{k-1}^2}. \quad (2.4)$$

Hence, we have the right hand side inequality of (2.1). For the left hand side inequality, we observe that $\lambda_i \leq \lambda_1$ implies $\lambda_i S_{k-1;i} \leq \lambda_1 S_{k-1;1}$. Therefore, we obtain

$$n \lambda_1 S_{k-1;1} \geq \sum_{i=1}^n \lambda_i S_{k-1;i} = \frac{n \binom{n-1}{k-1}}{\binom{n}{k}} S_k = k S_k.$$

Also, $S_{k-1;i} \geq S_{k-1;1}$ and combining the above yields

$$\frac{\partial Q_k}{\partial \lambda_i} \geq \frac{n}{k(n-k+1)} \frac{S_{k-1;1}^2}{S_{k-1}^2} \geq \frac{1}{k \lambda_1^2} \frac{\lambda_1^2 S_{k-1;1}^2}{S_{k-1}^2} \geq \frac{k}{n^2 \lambda_1^2} \frac{S_k^2}{S_{k-1}^2} \geq \frac{1}{n^2} \frac{Q_k^2}{\lambda_1^2}.$$

Next, to show (2.2), we employ (2.4) again to obtain

$$\frac{\partial Q_k}{\partial \lambda_i} \leq \frac{S_{k-1;i}^2}{S_{k-1}^2} = \frac{1}{\lambda_i^2} \frac{\lambda_i^2 S_{k-1;i}^2}{S_{k-1}^2} \leq \frac{1}{\lambda_i^2} \frac{S_k^2}{S_{k-1}^2} = \frac{Q_k^2}{\lambda_i^2}.$$

The left hand side of (2.3) is proven in Lemma 3.7 in [9]. The right hand side inequality readily follows by (2.2), since $|A|_k^2 := \sum_{i=1}^n \lambda_i^2 D_i Q_k \leq n Q_k^2$. \square

Proposition 2.3 Assume Ω_0 and Σ_0 satisfy the assumptions in Theorem 1.1. Let Σ_t be a convex complete smooth graph solution of $(*_k^n)$ with the positive Q_k curvature. Then, the following hold

$$\partial_t \psi = \mathcal{L} \psi \quad (2.5)$$

$$\partial_t g_{ij} = -2Q_k h_{ij} \quad (2.6)$$

$$\partial_t g^{ij} = 2Q_k h^{ij} \quad (2.7)$$

$$\partial_t \vec{n} = -(\nabla_j Q_k) F^j \quad (2.8)$$

$$\partial_t h_{ij} = \mathcal{L} h_{ij} + \frac{\partial^2 Q_k}{\partial h_{pq} \partial h_{rs}} \nabla_i h_{pq} \nabla_j h_{rs} - 2Q_k h_{il} h_j^l + |A|_k^2 h_{ij} \quad (2.9)$$

$$\partial_t Q_k = \mathcal{L} Q_k + |A|_k^2 Q_k \quad (2.10)$$

$$\partial_t v^2 = \mathcal{L} v^2 - 6\|\nabla v\|_{\mathcal{L}}^2 - 2|A|_k^2 v^2 \quad (2.11)$$

Proof (2.6) – (2.10) are given in [1] (see also in [9]). Equation (2.5) readily follows from the definition $\psi := (M - \bar{u})_+$, where $\bar{u} := \langle F, \vec{e}_{n+1} \rangle$ and

$$\begin{aligned} \mathcal{L} \bar{u} &= \mathcal{L} \langle F, \vec{e}_{n+1} \rangle = \left\langle \frac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j F, \vec{e}_{n+1} \right\rangle = \left\langle \frac{\partial Q_k}{\partial h_{ij}} h_{ij} \vec{n}, \vec{e}_{n+1} \right\rangle = \langle Q_k \vec{n}, \vec{e}_{n+1} \rangle \\ &= \langle \partial_t F, \vec{e}_{n+1} \rangle = \partial_t \bar{u}. \end{aligned}$$

To show (2.11), we derive from $v := \langle \vec{n}, \vec{e}_{n+1} \rangle^{-1}$ that $\nabla_i v = -\langle \nabla_i \vec{n}, \vec{e}_{n+1} \rangle v^2 = \langle h_{ij} F^j, \vec{e}_{n+1} \rangle v^2$. Hence,

$$\begin{aligned} \mathcal{L} v &= \frac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j v = \frac{\partial Q_k}{\partial h_{ij}} \nabla_i (\langle h_{jm} F^m, \vec{e}_{n+1} \rangle v^2) \\ &= \left\langle \left(\frac{\partial Q_k}{\partial h_{ij}} \nabla_i h_{jm} \right) F^m, \vec{e}_{n+1} \right\rangle v^2 + \left\langle \frac{\partial Q_k}{\partial h_{ij}} h_{ij}^m h_{jm} \vec{n}, \vec{e}_{n+1} \right\rangle v^2 \\ &\quad + 2 \frac{\partial Q_k}{\partial h_{ij}} \langle h_{jm} F^m, \vec{e}_{n+1} \rangle \langle h_{il} F^l, \vec{e}_{n+1} \rangle v^3 \\ &= \langle (\nabla_m Q_k) F^m, \vec{e}_{n+1} \rangle v^2 + |A|_k^2 v + 2v^{-1} \|\nabla v\|_{\mathcal{L}}^2. \end{aligned}$$

On the other hand, (2.8) gives $\partial_t v = \langle (\nabla_j Q_k) F^j, \vec{e}_{n+1} \rangle v^2$. Therefore,

$$\begin{aligned} \partial_t v^2 &= 2v \partial_t v = 2v \langle (\nabla_m Q_k) F^m, \vec{e}_{n+1} \rangle v^2 \\ &= 2v \mathcal{L} v - 4\|\nabla v\|_{\mathcal{L}}^2 - 2|A|_k^2 v^2 = \mathcal{L} v^2 - 6\|\nabla v\|_{\mathcal{L}}^2 - 2|A|_k^2 v^2. \end{aligned}$$

\square

If $\psi := (M - \bar{u})_+$, for a given $M > 0$, then we have the following two estimates.

Theorem 2.4 (Gradient estimate) Assume Ω_0 and Σ_0 satisfy the assumptions in Theorem 1.1. Let Σ_t be a convex complete smooth graph solution of $(*_k^n)$ with the positive Q_k curvature defined on $M^n \times [0, T]$, for some $T > 0$. Then

$$\psi(p, t)v(p, t) \leq \sup_{p \in M^n} \psi(p, 0)v(p, 0).$$

Proof By combining (2.5) and (2.11), we have

$$\partial_t(v^2\psi^2) = \mathcal{L}(v^2\psi^2) - \langle 6\psi\nabla v + 2v\nabla\psi, \nabla(v\psi) \rangle_{\mathcal{L}} - 2|A|_k^2 v^2\psi^2.$$

Since the conditions (ii), (iii) in Theorem 1.1 mean that ψ is compactly supported, it follows by the maximum principle that $\sup_{p \in M^n} v(p, t)\psi(p, t) \leq \sup_{p \in M^n} v(p, 0)\psi(p, 0)$, which yields the desired result. \square

Theorem 2.5 (Lower bound of speed) Assume Ω_0 and Σ_0 satisfy the assumptions in Theorem 1.1. Let Σ_t be a convex complete smooth graph solution of $(*_k^n)$ with the positive Q_k curvature defined on $M^n \times [0, T]$, for some $T > 0$. Then,

$$\psi(p, t)^{-1}Q_k(p, t) \geq \inf_{p \in M^n} \psi(p, 0)^{-1}Q_k(p, 0).$$

Proof From (2.5), (2.10), we derive

$$\partial_t(\psi Q_k^{-1}) = \mathcal{L}(\psi Q_k^{-1}) - 2Q_k \langle \nabla(\psi Q_k^{-1}), \nabla Q_k^{-1} \rangle_{\mathcal{L}} - |A|_k^2 Q_k^{-1}\psi.$$

Thus, the maximum principle gives the desired result. \square

3 Speed estimate

In this section we will obtain a local upper bound on the speed Q_k . We will use the gradient function v to localize our estimate in the spirit of the well known Caffarelli, Nirenberg, and Spruck estimate in [2]. A similar technique was used by Ecker and Huisken [11] in the context of the Mean curvature flow to obtain a local bound on $|A|^2$. Our proof is similar to that in [11].

Theorem 3.1 (Speed estimate) Assume Ω_0 and Σ_0 satisfy the assumptions in Theorem 1.1. Let Σ_t be a convex complete smooth graph solution of $(*_k^n)$ with the positive Q_k curvature defined on $M^n \times [0, T]$. Given a constant M , we have

$$(\psi Q_k)^2(p, t) \leq \max \left\{ 10n^2 \sup_{Q_M} v^4(\cdot, t), 2 \sup_{Q_M} v^2(\cdot, t) \sup_{p \in M^n} (Q_k \psi)^2(\cdot, 0) \right\}$$

where $Q_M = \{(p, s) \in M^n \times [0, t] : \bar{u}(p, s) \leq M\}$.

Proof Given a time $T_0 \in [0, T]$, we define the set $Q_M = \{(p, s) \in M^n \times [0, T_0] : \bar{u}(p, s) \leq M\}$ and we will prove that

$$(\psi Q_k)^2(p, T_0) \leq \max \left\{ 10n^2 \sup_{Q_M} v^4(\cdot, t), 2 \sup_{Q_M} v^2(\cdot, t) \sup_{p \in M^n} (Q_k \psi)^2(\cdot, 0) \right\}.$$

Let $\mathcal{K} := \sup_{Q_M} v^2$ and define the function φ depending on v^2 by

$$\varphi(v^2) = \frac{v^2}{2\mathcal{K} - v^2}.$$

The evolution equation of v^2 in (2.11) yields

$$\frac{\partial}{\partial t} \varphi(v^2) = \varphi'(\mathcal{L}v^2 - 6\|\nabla v\|_{\mathcal{L}}^2 - 2|A|_k^2 v^2) = \mathcal{L}\varphi - \varphi''\|\nabla v^2\|_{\mathcal{L}}^2 - \varphi'(6\|\nabla v\|_{\mathcal{L}}^2 + 2|A|_k^2 v^2)$$

which combined with (2.10) yields

$$\begin{aligned} \frac{\partial}{\partial t} (Q_k^2 \varphi) &= \mathcal{L}(Q_k^2 \varphi) - 2\langle \nabla \varphi, \nabla Q_k^2 \rangle_{\mathcal{L}} - 2\varphi \|\nabla Q_k\|_{\mathcal{L}}^2 \\ &\quad - (4\varphi''v^2 + 6\varphi')Q_k^2 \|\nabla v\|_{\mathcal{L}}^2 + 2|A|_k^2 Q_k^2 (\varphi - \varphi'v^2). \end{aligned}$$

Observe the following

$$\begin{aligned} -2\langle \nabla \varphi, \nabla Q_k^2 \rangle_{\mathcal{L}} &= -2Q_k \langle \nabla \varphi, \nabla Q_k \rangle_{\mathcal{L}} + \varphi^{-1} Q_k^2 \|\nabla \varphi\|_{\mathcal{L}}^2 - \varphi^{-1} \langle \nabla \varphi, \nabla (\varphi Q_k^2) \rangle_{\mathcal{L}} \\ &\leq 2\varphi \|\nabla Q_k\|_{\mathcal{L}}^2 + \frac{3}{2} \varphi^{-1} Q_k^2 \|\nabla \varphi\|_{\mathcal{L}}^2 - \varphi^{-1} \langle \nabla \varphi, \nabla (\varphi Q_k^2) \rangle_{\mathcal{L}}. \end{aligned}$$

Hence, the following inequality holds

$$\begin{aligned} \frac{\partial}{\partial t} (Q_k^2 \varphi) &\leq \mathcal{L}(Q_k^2 \varphi) - \varphi^{-1} \langle \nabla \varphi, \nabla (Q_k^2 \varphi) \rangle_{\mathcal{L}} \\ &\quad - (4\varphi''v^2 + 6\varphi' - 6\varphi^{-1} \varphi'^2 v^2) Q_k^2 \|\nabla v\|_{\mathcal{L}}^2 + 2|A|_k^2 Q_k^2 (\varphi - \varphi'v^2). \quad (3.1) \end{aligned}$$

On the other hand, a direct computation gives the following identities

$$\varphi - \varphi'v^2 = -\varphi^2, \quad \varphi^{-1} \nabla \varphi = 4\mathcal{K} \varphi v^{-3} \nabla v, \quad 4\varphi''v^2 + 6\varphi' - 6\varphi^{-1} \varphi'^2 v^2 = \frac{4\mathcal{K}}{(2\mathcal{K} - v^2)^2} \varphi.$$

Setting $f := Q_k^2 \varphi(v^2)$ in (3.1) and applying the identities above and also $Q_k^2 \leq n|A|_k^2$ (see in (2.3)) gives

$$\frac{\partial}{\partial t} f \leq \mathcal{L}f - 4\mathcal{K} \varphi v^{-3} \langle \nabla v, \nabla f \rangle_{\mathcal{L}} - \frac{4\mathcal{K}}{(2\mathcal{K} - v^2)^2} \|\nabla v\|_{\mathcal{L}}^2 f - 2|A|_k^2 \varphi f.$$

We will next consider the evolution of $f\psi^2$ for our given cut off function ϕ . We have seen in (2.5) that $\partial_t \psi^2 = \mathcal{L}\psi^2 - 2\|\nabla \psi\|_{\mathcal{L}}^2$, on the support of ψ . Combining this with the evolution of f yields

$$\begin{aligned} \frac{\partial}{\partial t} (f\psi^2) &\leq \mathcal{L}(f\psi^2) - 2\langle \nabla \psi^2, \nabla f \rangle_{\mathcal{L}} - 2f \|\nabla \psi\|_{\mathcal{L}}^2 \\ &\quad - 4\mathcal{K} \varphi v^{-3} \psi^2 \langle \nabla v, \nabla f \rangle_{\mathcal{L}} - \frac{4\mathcal{K}}{(2\mathcal{K} - v^2)^2} f\psi^2 \|\nabla v\|_{\mathcal{L}}^2 - \frac{2}{n} f^2 \psi^2 \end{aligned}$$

We compute the following

$$\begin{aligned} -4\mathcal{K} \varphi v^{-3} \psi^2 \langle \nabla v, \nabla f \rangle_{\mathcal{L}} &= -4\mathcal{K} \varphi v^{-3} \langle \nabla v, \nabla (f\psi^2) \rangle_{\mathcal{L}} + 8\mathcal{K} \varphi v^{-3} f\psi \langle \nabla v, \nabla \psi \rangle_{\mathcal{L}} \\ &\leq -4\mathcal{K} \varphi v^{-3} \langle \nabla v, \nabla (f\psi^2) \rangle_{\mathcal{L}} + \frac{4\mathcal{K} f\psi^2 \|\nabla v\|_{\mathcal{L}}^2}{(1 - \mathcal{K}v^2)^2} + 4\mathcal{K} \varphi^2 \frac{(1 - \mathcal{K}v^2)^2}{v^6} f \|\nabla \psi\|_{\mathcal{L}}^2 \\ &= -4\mathcal{K} \varphi v^{-3} \langle \nabla v, \nabla (f\psi^2) \rangle_{\mathcal{L}} + \frac{4\mathcal{K}}{(1 - \mathcal{K}v^2)^2} f\psi^2 \|\nabla v\|_{\mathcal{L}}^2 + 4\mathcal{K} v^{-2} f \|\nabla \psi\|_{\mathcal{L}}^2. \end{aligned}$$

In addition, we know

$$-2\langle \nabla \psi^2, \nabla f \rangle_{\mathcal{L}} = -4\psi^{-1} \langle \nabla \psi, \nabla (f\psi^2) \rangle_{\mathcal{L}} + 8f \|\nabla \psi\|_{\mathcal{L}}^2.$$

Therefore, by using the equations above, we can reduce the evolution equation of $f\psi^2$ to

$$\begin{aligned} \frac{\partial}{\partial t}(f\psi^2) &\leq \mathcal{L}(f\psi^2) - \langle 4\psi^{-1}\nabla\psi + 4\mathcal{K}\varphi v^{-3}\nabla v, \nabla(f\psi^2) \rangle_{\mathcal{L}} + (6 + 4\mathcal{K}v^{-2})f\|\nabla\psi\|_{\mathcal{L}}^2 \\ &\quad - \frac{2}{n}f^2\psi^2. \end{aligned}$$

Applying the inequality $D_i Q_k \leq 1$ shown in (2.1) yields

$$\begin{aligned} \|\nabla\psi\|_{\mathcal{L}}^2 &= \|\langle \nabla F, \vec{e}_{n+1} \rangle\|_{\mathcal{L}}^2 \leq \sum_{m=1}^{n+1} \|\nabla \langle F, \vec{e}_m \rangle\|_{\mathcal{L}}^2 = \sum_{m=1}^{n+1} \frac{\partial Q_k}{\partial h_{ij}} \nabla_i \langle F, \vec{e}_m \rangle \nabla_j \langle F, \vec{e}_m \rangle \\ &= \frac{\partial Q_k}{\partial h_{ij}} \sum_{m=1}^{n+1} \langle F_i, \vec{e}_m \rangle \langle F_j, \vec{e}_m \rangle = \frac{\partial Q_k}{\partial h_{ij}} \langle F_i, F_j \rangle = \frac{\partial Q_k}{\partial h_{ij}} g_{ij} = \sum_{i=1}^n D_i Q_k \leq n. \end{aligned}$$

Hence, by the definition of \mathcal{K} and $v \geq 1$, we have

$$(6 + 4\mathcal{K}v^{-2})f\|\nabla\psi\|_{\mathcal{L}}^2 \leq 10n\mathcal{K}v^{-2}f \leq 10n\mathcal{K}f$$

Combining the above inequalities we finally obtain

$$\frac{\partial}{\partial t}(f\psi^2) \leq \mathcal{L}(f\psi^2) - \langle 4\psi^{-1}\nabla\psi + 4\mathcal{K}\varphi v^{-3}\nabla v, \nabla(f\psi^2) \rangle_{\mathcal{L}} + 10n\mathcal{K}f - \frac{2}{n}f^2\psi^2.$$

Since ψ is compactly supported by the conditions (ii), (iii) in Theorem 1.1, $f\psi^2$ attains its maximum \mathcal{M} on $M^n \times [0, T_0]$ at some (p_0, t_0) . If $t_0 > 0$, then at (p_0, t_0) , we obtain

$$\frac{2}{n}\mathcal{M}f = \frac{2}{n}f^2\psi^2 \leq 10n\mathcal{K}f.$$

Since $f\psi^2 = \varphi(v^2)Q_k^2\psi^2 \leq (Q_k\psi)^2$, the following holds

$$\mathcal{M} \leq \max \left\{ 5n^2\mathcal{K}, \sup_{p \in M^n} f\psi^2(\cdot, 0) \right\} \leq \max \left\{ 5n^2\mathcal{K}, \sup_{p \in M} (Q_k\psi)^2(\cdot, 0) \right\}.$$

Finally, $(Q_k\psi)^2 \leq v^2 Q_k^2\psi^2 \leq 2\mathcal{K}\varphi(v^2)Q_k^2\psi^2 = 2\mathcal{K}f\psi^2$ and $f\psi^2(p, T_0) \leq \mathcal{M}$ imply

$$(Q_k\psi)^2(p, T_0) \leq \max \left\{ 10n^2 \sup_{Q_M} v^4(\cdot, t), 2 \sup_{Q_M} v^2(\cdot, t) \sup_{p \in M^n} (Q_k\psi)^2(\cdot, 0) \right\}.$$

□

4 Curvature estimate

In this section we will derive a local upper bound on the largest principal curvature λ_{\max} of M_t . We will employ a Pogorelov type computation with respect to h_{ii} using a technique that was introduced by Sheng, Urbas, and Wang in [13] for the elliptic setting. The following known formula and will be used in the proof.

Proposition 4.1 (The Euler's formula) *Let Σ be a smooth hypersurface, and $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion with $F(M^n) = \Sigma$. Then, for all $p \in M^n$ and $i \in \{1, \dots, n\}$, the following holds*

$$\frac{h_{ii}(p)}{g_{ii}(p)} \leq \lambda_{\max}(p).$$

Proof Assume $\{E_1(p), \dots, E_n(p)\}$ is an orthonormal basis of $T\Sigma_{F(p)}$ satisfying $L(E_j(p)) = \lambda_j(p)E_j(p)$, where L is the Weingarten map. Let $\nabla_i F := F_i = a_{ij}E_j$. Then, $g_{ii} = \sum_{j=1}^n (a_{ij})^2$. Thus,

$$h_{ii} = \langle L(F_i), F_i \rangle = \sum_{j=1}^n (a_{ij})^2 \lambda_j \leq \sum_{j=1}^n (a_{ij})^2 \lambda_{\max} = g_{ii} \lambda_{\max}.$$

□

Theorem 4.2 (Curvature estimate) Assume Ω_0 and Σ_0 satisfy the assumptions in Theorem 1.1. Let Σ_t be a convex complete smooth graph solution of $(*_k^n)$ with the positive Q_k curvature defined on $M^n \times [0, T)$. Then, for any given a constant M , we have

$$(\psi^2 \lambda_{\max})(p, t) \leq \exp(2nt \sup_{Q_M} Q_k^2) \max \left\{ 5M, \sup_{p \in M^n} (\psi^2 \lambda_{\max})(p, 0) \right\}$$

where $Q_M = \{(p, s) \in M^n \times [0, t] : \bar{u}(p, s) \leq M\}$.

Proof Given $T_0 \in [0, T)$, we define $Q_M = \{(p, s) \in M^n \times [0, T_0] : \bar{u}(p, s) \leq M\}$ and we will prove that

$$(\psi^2 \lambda_{\max})(p, T_0) \leq \exp(2nT_0 \sup_{Q_M} Q_k^2) \max \left\{ 5M, \sup_{p \in M^n} (\psi^2 \lambda_{\max})(p, 0) \right\}.$$

We set $\mathcal{A} = \sup_{Q_M} Q_k^2$. By the conditions (ii), (iii) in Theorem $(*_k^n)$, $\exp(-2nt\mathcal{A})\psi^2 \lambda_{\max}$ attains its maximum in $M^n \times [0, T_0]$ at some point (p_0, t_0) . If $t_0 = 0$, we obtain the desired result. So, we may assume $t_0 > 0$. First we choose a chart (U, φ) with $p_0 \in \varphi(U) \subset M^n$ such that the covariant derivatives $\{\nabla_i F(p_0, t_0) : i = 1, \dots, n\}$ form an orthonormal basis of $(T\Sigma_{t_0})$ satisfying

$$g_{ij}(p_0, t_0) = \delta_{ij}, \quad h_{ij}(p_0, t_0) = \delta_{ij} \lambda_i(p_0, t_0), \quad \lambda_1(p_0, t_0) = \lambda_{\max}(p_0, t_0).$$

Then, $h_{11}(p_0, t_0) = \lambda_{\max}(p_0, t_0)$, $g_{11}(p_0, t_0) = 1$ hold. Next, we define the function $w : U \times [0, T_0] \rightarrow \mathbb{R}$ by

$$w := \exp(-2nt\mathcal{A})\psi^2 \frac{h_{11}}{g_{11}}.$$

Notice that if $t \neq t_0$, the covariant derivatives $\{\nabla_i F(p_0, t)\}_{i=1, \dots, n}$ may fail to form an orthonormal basis of $(T\Sigma)_{F(p_0, t)}$. However, Proposition 4.1 applies for every chart and immersion. So, for all points $(p, t) \in \varphi(U) \times [0, T_0]$, we have

$$w(p, t) \leq \exp(-2nt\mathcal{A})\psi^2 \lambda_{\max}(p, t) \leq \exp(-2nt_0\mathcal{A})\psi^2 \lambda_{\max}(p_0, t_0) = w(p_0, t_0)$$

implying that w attains its maximum at (p_0, t_0) . Since $\nabla g_{11} = 0$, the following holds on the support of ψ

$$\frac{\nabla_i w}{w} = 2 \frac{\nabla_i \psi}{\psi} + \frac{\nabla_i h_{11}}{h_{11}}. \quad (4.1)$$

Differentiating the equation above we obtain

$$\frac{\nabla_i \nabla_j w}{w} - \frac{\nabla_i w \nabla_j w}{w^2} = 2 \frac{\nabla_i \nabla_j \psi}{\psi} - 2 \frac{\nabla_i \psi \nabla_j \psi}{\psi^2} + \frac{\nabla_i \nabla_j h_{11}}{h_{11}} - \frac{\nabla_i h_{11} \nabla_j h_{11}}{(h_{11})^2}.$$

Multiplying by $\frac{\partial Q_k}{\partial h_{ij}}$ and summing over all i, j yields

$$\frac{\mathcal{L}w}{w} - \frac{\|\nabla w\|_{\mathcal{L}}^2}{w^2} = 2\frac{\mathcal{L}\psi}{\psi} - 2\frac{\|\nabla\psi\|_{\mathcal{L}}^2}{\psi^2} + \frac{\mathcal{L}h_{11}}{h_{11}} - \frac{\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2}.$$

On the other hand, on the support of ψ , the following holds

$$\frac{\partial_t w}{w} = -2n\mathcal{A} + 2\frac{\partial_t \psi}{\psi} + \frac{\partial_t h_{11}}{h_{11}} - \frac{\partial_t g_{11}}{g_{11}}.$$

Recall that $\partial_t \psi = \mathcal{L}\psi$ by (2.5), $\partial_t g_{11} = -2Q_k h_{11}$ by (2.6), and also that

$$\partial_t h_{11} = \mathcal{L}h_{11} + \frac{\partial^2 Q_k}{\partial h_{ij} \partial h_{ml}} \nabla_1 h_{ij} \nabla_1 h_{ml} - 2Q_k h_{1i} h_1^i + |A|_k^2 h_{11}$$

by in (2.9). Combining the equations above yields

$$\begin{aligned} \frac{\mathcal{L}w}{w} - \frac{\|\nabla w\|_{\mathcal{L}}^2}{w^2} - \frac{\partial_t w}{w} = & -2\frac{\|\nabla\psi\|_{\mathcal{L}}^2}{\psi^2} - \frac{\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2} - \frac{1}{h_{11}} \frac{\partial^2 Q_k}{\partial h_{ij} \partial h_{ml}} \nabla_1 h_{ij} \nabla_1 h_{ml} \\ & + 2n\mathcal{A} + \frac{2Q_k h_{1i} h_1^i}{h_{11}} - |A|_k^2 - 2Q_k \frac{h_{11}}{g_{11}}. \end{aligned} \quad (4.2)$$

Observe next that $|A|_k^2 \leq nQ_k^2 \leq n\mathcal{A}$ by (2.3). Also, at (p_0, t_0) , $\frac{2Q_k h_{1i} h_1^i}{h_{11}} - 2Q_k \frac{h_{11}}{g_{11}} = 0$ holds. Moreover, Proposition 2.1 implies that

$$-\frac{1}{h_{11}} \frac{\partial^2 Q_k}{\partial h_{ij} \partial h_{ml}} \nabla_1 h_{ij} \nabla_1 h_{ml} \geq 2 \sum_{i=2}^n -\frac{D_1 Q_k - D_i Q_k}{\lambda_1(\lambda_1 - \lambda_i)} |\nabla_1 h_{1i}|^2$$

holds at the point (p_0, t_0) . Furthermore, by the definition of the operator \mathcal{L} , at the point (p_0, t_0) we have

$$\frac{\|\nabla w\|_{\mathcal{L}}^2}{w^2} \geq 0, \quad \frac{\|\nabla\psi\|_{\mathcal{L}}^2}{\psi^2} = \sum_{i=1}^n \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2}, \quad \frac{\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2} = \sum_{i=1}^n \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i h_{11}|^2}{\lambda_1^2}.$$

We conclude from (4.2) that at the maximum point (p_0, t_0) of w , the following holds

$$n\mathcal{A} \leq 2 \sum_{i=1}^n \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2} + \sum_{i=1}^n \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i h_{11}|^2}{\lambda_1^2} + \sum_{i=2}^n \frac{2(D_1 Q_k - D_i Q_k)}{\lambda_1(\lambda_1 - \lambda_i)} |\nabla_i h_{11}|^2 \quad (4.3)$$

Next, we define the following sets

$$I = \{i \in (1, \dots, n) : \frac{\partial Q_k}{\partial \lambda_i} < 4\frac{\partial Q_k}{\partial \lambda_1}\} \quad \text{and} \quad J = \{j \in (1, \dots, n) : \frac{\partial Q_k}{\partial \lambda_j} \geq 4\frac{\partial Q_k}{\partial \lambda_1}\}.$$

Since w attains its maximum at (p_0, t_0) , $\nabla w(p_0, t_0) = 0$ holds. Thus, by (4.1)

$$\begin{aligned} 2 \sum_{i=1}^n \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2} &= 2 \sum_{i \in I} \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2} + 2 \sum_{j \in J} \frac{\partial Q_k}{\partial \lambda_j} \frac{|\nabla_j \psi|^2}{\psi^2} = 2 \sum_{i \in I} \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2} \\ &+ \frac{1}{2} \sum_{j \in J} \frac{\partial Q_k}{\partial \lambda_j} \frac{|\nabla_j h_{11}|^2}{h_{11}^2} \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^n \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i h_{11}|^2}{(h_{11})^2} &= \sum_{i \in I} \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i h_{11}|^2}{(h_{11})^2} + \sum_{j \in J} \frac{\partial Q_k}{\partial \lambda_j} \frac{|\nabla_j h_{11}|^2}{(h_{11})^2} = 4 \sum_{i \in I} \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2} \\ &+ \sum_{j \in J} \frac{\partial Q_k}{\partial \lambda_j} \frac{|\nabla_j h_{11}|^2}{h_{11}^2} \end{aligned}$$

and by adding the two equations above we obtain

$$2 \sum_{i=1}^n \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2} + \sum_{i=1}^n \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i h_{11}|^2}{\lambda_1^2} = 6 \sum_{i \in I} \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2} + \frac{3}{2} \sum_{j \in J} \frac{\partial Q_k}{\partial \lambda_j} \frac{|\nabla_j h_{11}|^2}{\lambda_1^2}.$$

However, we know $1 \notin J$ and $\lambda_1 \neq \lambda_j$. Hence, for $j \in J$, the definition of J leads to

$$\frac{\partial Q_k}{\partial \lambda_1} - \frac{\partial Q_k}{\partial \lambda_j} \leq -\frac{3}{4} \frac{\partial Q_k}{\partial \lambda_j}.$$

On the other hand $\lambda_1(\lambda_1 - \lambda_j) \leq (\lambda_1)^2$ holds. Hence, for $j \in J$, we obtain

$$\frac{2(D_1 Q_k - D_j Q_k)}{\lambda_1(\lambda_1 - \lambda_j)} |\nabla_j h_{11}|^2 \leq -\frac{3}{2} \frac{\partial Q_k}{\partial \lambda_j} \frac{|\nabla_j h_{11}|^2}{\lambda_1^2}.$$

Thus, (4.3) can be reduced to

$$n\mathcal{A} \leq 6 \sum_{i \in I} \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2}.$$

Applying $|\nabla_i \psi|^2 = |\langle F_i, \vec{e}_{n+1} \rangle|^2 \leq |F_i|^2 = g_{ii} = 1$ and the definition of I , we obtain

$$n\mathcal{A} \leq 6 \sum_{i \in I} \frac{\partial Q_k}{\partial \lambda_i} \frac{|\nabla_i \psi|^2}{\psi^2} \leq 24 \sum_{i \in I} \frac{\partial Q_k}{\partial \lambda_1} \frac{1}{\psi^2} \leq 24n \frac{\partial Q_k}{\partial \lambda_1} \psi^{-2}.$$

Using that $D_1 \lambda \leq Q_k^2 \lambda_1^{-2} = Q_k^2 \lambda_{\max}^{-2}$ by (2.2), $\psi \leq M$, and $Q_k^2 \leq \mathcal{A}$, in the inequality above yields

$$\mathcal{A} \psi^4 \lambda_{\max}^2 \leq 24 Q_k^2 \psi^2 \leq 24 \mathcal{A} M^2$$

implying that $\psi^2 \lambda_{\max}(p_0, t_0) \leq 5M$ holds. In conclusion,

$$w(p, t) \leq w(p_0, t_0) := \exp(-2nt_0 \mathcal{A}) \psi^2 \lambda_{\max}(p_0, t_0) \leq \psi^2 \lambda_{\max}(p_0, t_0) \leq 5M$$

which finishes the proof of our estimate. \square

5 Long time existence

In this final section, we will establish the long time existence of the Q_k -flow $(*_k^n)$, as stated in our main Theorem 1.1. Our proof will be based on the a priori estimates in Sect. 2–4. Before we present the proof of Theorem 1.1, we will introduce some extra notation and preliminary results.

Notation 5.1 We have:

- (i) Given a set $A \in \mathbb{R}^{n+1}$, we denote by $\text{Conv}(A)$ its convex hull $\{tx + (1-t)y : x, y \in A, t \in [0, 1]\}$.
- (ii) Let Σ be a convex complete (or closed) hypersurface. If a set V is a subset of $\text{Conv}(\Sigma)$, we say V is enclosed by Σ and use the notation

$$V \preceq \Sigma.$$

In particular, if $V \cap \Sigma = \emptyset$ and $V \preceq \Sigma$, we use $V \prec \Sigma$.

- (iii) For a constant $r > 0$ and a point $(x_0, t_0) \in \mathbb{R}^n \times (0, +\infty)$, $Q_r((x_0, t_0)) =: B_r(x_0) \times (t_0 - r^2, t_0]$ denotes the parabolic cube centered at (x_0, t_0) . Also, for a constant $\alpha \in (0, 1)$, $C_{x,t}^{\alpha, \alpha/2}(Q_r)$ denotes the standard Hölder space with respect to the parabolic distance.
- (iv) $B_R^{n+1}(Y) := \{X \in \mathbb{R}^{n+1} : |X - Y| < R\}$ denotes the $(n+1)$ -ball of radius R centered at $Y \in \mathbb{R}^{n+1}$.
- (v) For a convex hypersurface Σ and $\eta > 0$, we denote by Σ^η the η -envelope of Σ .

$$\Sigma^\eta = \{Y \in \mathbb{R}^{n+1} : d(Y, \Sigma) = \eta, Y \notin \text{Conv}(\Sigma)\}$$

where d is the distance function.

- (vi) For a convex closed hypersurface Σ , we define the support function $S : S^n \rightarrow \mathbb{R}$ by

$$S(v) = \max_{Y \in \Sigma} \langle v, Y \rangle.$$

- (vii) For a convex C^2 hypersurface Σ and a point $X \in \Sigma$, we denote by $\lambda_{\min}(\Sigma)(X)$ the smallest principal curvature. Also, for any convex hypersurface Σ and a point $X \in \Sigma$, we define

$$\lambda_{\min}(\Sigma)(X) = \sup \{ \lambda_{\min}(\Phi)(X) : \Phi \text{ complete (or closed) } C^2 \text{ hypersurface, } \Sigma \preceq \Phi, X \in \Phi \}$$

and also

$$\lambda_{\min}^{\text{loc}}(\Sigma)(X) = \liminf_{r \rightarrow 0} \left\{ \lambda_{\min}(\Sigma)(Y) : Y \in \Sigma \cap B_r^{n+1}(X) \right\}.$$

Proposition 5.2 ($C^{2,\alpha}$ estimate) Let $u : Q_r \rightarrow \mathbb{R}$ be a convex function whose graph is a solution of $(*_k^n)$ with the positive Q_k curvature and $Q_r =: Q_r((x_0, t_0))$. Then, there are some constant $\alpha \in (0, 1)$ such that for all $\theta \in (0, 1)$ the following holds

$$\|D^2u\|_{C_{x,t}^{\alpha, \alpha/2}(Q_{\theta r})} \leq C(r, \theta, \alpha, n, \sup_{Q_r} v, \sup_{Q_r} \lambda_{\max}, \inf_{Q_r} Q_k)$$

Proof The function $u(x, t)$ satisfies the following parabolic equation

$$\partial_t u = (1 + |Du|^2)^{\frac{1}{2}} Q_k(D^2u, Du),$$

where $Q_k(D^2u, Du)$ denotes the Q_k curvature of the graph of u . We consider the immersion $F(x, t) = (x, u(x, t))$ with the corresponding image $\Sigma_t = F(B_r(0), t)$. Then, we have

$$g_{ij} = \delta_{ij} + u_i u_j, \quad g^{ij} = \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2}, \quad \vec{n} = \frac{(-Du, 1)}{(1 + |Du|^2)^{\frac{1}{2}}}, \quad h_{ij} = \frac{u_{ij}}{(1 + |Du|^2)^{\frac{1}{2}}}.$$

The concavity of $Q_k(\lambda)$ in $(\lambda_1, \dots, \lambda_n)$ implies the concavity of $Q_k(D^2u, Du)$ in D^2u as follows

$$\frac{\partial^2 Q_k}{\partial u_{ij} \partial u_{kl}} N_{ij} N_{kl} = \frac{\partial^2 Q_k}{\partial h_{ij} \partial h_{kl}} N_{ij} N_{kl} (1 + |Du|^2)^{-1} \leq 0,$$

where $N_{ij} \in \mathbb{R}^{n \times n}$ is a matrix.

Now, we let $\{E_1, \dots, E_n\}$ be an orthonormal basis of $(T\Sigma_t)_{F(x,t)}$ with the corresponding diagonalized second fundamental form $h_{ij}^o = \text{diag}[\lambda_1, \dots, \lambda_n]$. Also, we let c_{ij} be the matrix satisfying $F^i = c_{ij} E_j$. Then, $\frac{\partial Q_k}{\partial h_{ij}} F_i \otimes F_j = \frac{\partial Q_k}{\partial h_{ab}^o} E_a \otimes E_b$ gives

$$(1 + |Du|^2)^{\frac{1}{2}} \frac{\partial Q_k}{\partial u_{ij}} = \frac{\partial Q_k}{\partial h_{ij}} = \frac{\partial Q_k}{\partial h_{ab}^o} \langle E_a, F^i \rangle \langle E_b, F^j \rangle = \sum_{a=1}^n \frac{\partial Q_k(\lambda)}{\partial \lambda_a} c_{ai} c_{aj}.$$

Recall (2.1), $n^{-2} \lambda_{\max}^{-2} Q_k^2 \leq \partial_a Q_k \leq 1$. Then, $g^{ij} = \langle F^i, F^j \rangle = c_{ia} c_{aj}$ leads to

$$Q_k^2 n^{-2} \lambda_{\max}^{-2} g^{ij} = Q_k^2 n^{-2} \lambda_{\max}^{-2} c_{ai} c_{aj} \leq \sum_{a=1}^n \partial_a Q_k c_{ai} c_{aj} \leq c_{ai} c_{aj} = g^{ij}.$$

Hence, the elliptic coefficients of $(1 + |Du|^2)^{\frac{1}{2}} Q_k(D^2u, Du)$ are bounded by n , $\sup_{Q_r} v$, $\sup_{Q_r} \lambda_{\max}$, $\inf_{Q_r} Q_k$. In conclusion, we can employ the $C^{2,\alpha}$ estimate in [14] which yields the desired result. \square

Since we will approximate the initial hypersurface Σ_0 by its envelopes $(\Sigma_0)^\eta$, which are of class $C^{1,1}$, in order to regularized them, we introduce the convolution on the sphere.

Proposition 5.3 (Convolution on S^n) For $\epsilon \in (0, 1)$, let $\varphi_\epsilon : S^n \times S^n \rightarrow \mathbb{R}$ be a smooth function satisfying

- (i) $\varphi_\epsilon(v, w) = \eta_\epsilon(\langle v, w \rangle)$ for a non-negative function $\eta_\epsilon : [-1, 1] \rightarrow [0, +\infty)$,
- (ii) $\eta_\epsilon(r) = 0$ for all $r \in [-1, 1 - \epsilon]$
- (iii) $\int_{S^n} \varphi_\epsilon(v) ds = 1$, where ds denotes the surface measure on S^n

and define the convolution $f * \varphi_\epsilon$ with a function $f : S^n \rightarrow \mathbb{R}$ by

$$f * \varphi_\epsilon(v) = \int_{S^n} f(w) \varphi_\epsilon(v, w) ds_w.$$

Assume that f is of class $C^m(U)$ on an open subset $U \subset S^n$, then $f * \varphi_\epsilon$ uniformly converge to f in $C^m(K)$ on any compact subset K of U .

Proof The proof is standard but we include it here for completeness. Since $\epsilon \in (0, 1)$, for each $v \in S^n$, the support of $\varphi_\epsilon(v, \cdot)$ is compactly embedded in the hemisphere centered at v . We choose a rotation matrix $Q \in O(n)$ satisfying $Q(-\vec{e}_{n+1}) = v$. We define a chart $\xi : \mathbb{R}^n \rightarrow S^n$ and a differential operator $\tilde{\nabla} : C^1(\mathbb{R}^n) \rightarrow (C^0(\mathbb{R}^n))^n$ by

$$\xi(x) = Q((x, -1)(1 + |x|^2)^{-\frac{1}{2}}) \quad \text{and} \quad \tilde{\nabla}_i h(y) = \partial_i h(y) + y_i \sum_{j=1}^n y_j \partial_j h(y)$$

where $h \in C^1(\mathbb{R}^n)$. Then, by direct computation, we have

$$\tilde{\nabla}((f * \varphi_\epsilon) \circ \xi)(x) = \int_{\mathbb{R}^n} \tilde{\nabla}(f \circ \xi)(y) \varphi_\epsilon(\xi(x), \xi(y)) (1 + |y|^2)^{-\frac{n+1}{2}} dy \quad (5.1)$$

which gives the desired result. \square

Let φ_ϵ be as in Proposition 5.3 and let Σ be a strictly convex closed hypersurface. We will next show how to regularize Σ by convolving the support function of its η -envelope Σ_η (which is a $C^{1,1}$ hypersurface) with the function φ_ϵ . This is a standard argument which we include here for the reader's convenience.

Proposition 5.4 *Let Σ^η denote the η -envelope of a convex closed hypersurface Σ with a uniform lower bound for $\lambda_{\min}(\Sigma)(X)$. and let S denote the support function of Σ^η . Assume Σ encloses the origin. Then, there is a small constant $\alpha(\eta, \Sigma) > 0$ such that for each $\epsilon \in (0, \alpha)$, $S * \varphi_\epsilon$ is the support function of a strictly convex smooth closed hypersurface Σ_ϵ^η . In addition, $S * \varphi_\epsilon \rightarrow S$, as $\epsilon \rightarrow 0$, uniformly on $C^1(S^n)$, and the following holds*

$$\liminf_{\epsilon \rightarrow 0} \lambda_{\min}(\Sigma_\epsilon^\eta)(X_\epsilon) \geq \lambda_{\min}^{\text{loc}}(\Sigma^\eta)(X)$$

where $\{X_\epsilon\}$ is a set of points $X_\epsilon \in \Sigma_\epsilon^\eta$ converging to $X \in \Sigma^\eta$ as $\epsilon \rightarrow 0$.

Proof Let \bar{g}_{ij} denote the standard metric on S^n and let $\bar{\nabla}$ be the connection on S^n defined by \bar{g}_{ij} . Notice that for a function $f : S^n \rightarrow \mathbb{R}^+$, if $\bar{\nabla}_i \bar{\nabla}_j f + f \bar{g}_{ij}$ is a positive definite matrix with respect to the metric \bar{g}_{ij} , then f is the support function of a strictly convex hypersurface, and the eigenvalues of $\bar{\nabla}_i \bar{\nabla}_j f + f \bar{g}_{ij}$ are the principal radii of curvature of the hypersurface (c.f. [15]).

Since Σ^η is a uniformly convex hypersurface of class $C^{1,1}$, its support function S is of class $C^{1,1}(S^n)$, namely $\bar{\nabla} \bar{\nabla} S$ exists almost everywhere. In addition, since the principal radii of curvature of the η -envelope are bounded from below by η , we have

$$\eta \bar{g}_{ij} \leq \bar{\nabla}_i \bar{\nabla}_j S + S \bar{g}_{ij} \leq \sup_{X \in \Sigma^\eta} \lambda_{\min}(\Sigma^\eta)^{-1}(X) \bar{g}_{ij} = \left(\eta + \sup_{X \in \Sigma} \lambda_{\min}(\Sigma)^{-1}(X) \right) \bar{g}_{ij} \quad (5.2)$$

at points where $\bar{\nabla} \bar{\nabla} S$ exists.

Recall the chart ξ and the differential operator $\tilde{\nabla}$ in Proposition 5.3. Then, direct computations yield

$$\begin{aligned} \partial_i &= \tilde{\nabla}_i - \frac{x_i x_j \tilde{\nabla}_j}{1 + |x|^2}, & \partial_i \partial_j &= \tilde{\nabla}_i \tilde{\nabla}_j - \frac{x_i^2}{1 + |x|^2} \tilde{\nabla}_i \tilde{\nabla}_j - \frac{x_j^2}{1 + |x|^2} \tilde{\nabla}_i \tilde{\nabla}_j + \frac{x_i x_j x_k x_l}{(1 + |x|^2)^2} \tilde{\nabla}_k \tilde{\nabla}_l \\ & & & - \frac{x_i x_j x_k}{(1 + |x|^2)^2} \tilde{\nabla}_k - \frac{(\delta_{ij} - x_i x_j)}{1 + |x|^2} \tilde{\nabla}_k - \frac{x_j}{1 + |x|^2} \tilde{\nabla}_i. \end{aligned} \quad (5.3)$$

Since $(\bar{\nabla}_i \bar{\nabla}_j f) \circ \xi = \partial_i \partial_j (f \circ \xi) - \Gamma_{ij}^k \partial_k (f \circ \xi)$ holds for $f \in C^2(S^n)$, we have a linear map L_x satisfying

$$(\bar{\nabla}_i \bar{\nabla}_j f) \circ \xi(x) + \bar{g}_{ij} f \circ \xi(x) = L_x(\tilde{\nabla} \tilde{\nabla} (f \circ \xi)(x), \tilde{\nabla} (f \circ \xi)(x), f \circ \xi(x)).$$

For convenience, we denote $f \circ \xi$ and $L_x(\tilde{\nabla} \tilde{\nabla} f, \tilde{\nabla} f, f)$ by f and $L_x(f)$, respectively. Then, since S is a.e. second order differentiable, (5.1) gives

$$\begin{aligned} (\bar{\nabla}_i \bar{\nabla}_j S * \varphi_\epsilon + \bar{g}_{ij} S * \varphi_\epsilon)(x) &= L_x(S * \varphi_\epsilon)(x) = \int_{\mathbb{R}^n} L_x(S(y)) \varphi_\epsilon(x, y) \frac{dy}{(1 + |y|^2)^{\frac{n+1}{2}}} \\ &= \int_{\mathbb{R}^n} (\bar{\nabla}_i \bar{\nabla}_j S + \bar{g}_{ij} S)(y) \frac{\varphi_\epsilon(x, y) dy}{(1 + |y|^2)^{\frac{n+1}{2}}} + \int_{\mathbb{R}^n} (L_x(S(y)) - L_y(S(y))) \frac{\varphi_\epsilon(x, y) dy}{(1 + |y|^2)^{\frac{n+1}{2}}} \\ &\geq \eta \bar{g}_{ij} - |L_x(S(y)) - L_y(S(y))| \bar{g}_{ij}. \end{aligned} \quad (5.4)$$

Notice that $S * \varphi_\epsilon \rightarrow S$ uniformly on $C^1(S^n)$ by Proposition 5.3, and $|\partial^2 S|$ is bounded by (5.2) and (5.3). Also, we have $L_x \rightarrow L_y$ as $x \rightarrow y$ by (5.3). Hence, $|L_x(S(y)) - L_y(S(y))|$ converges to zero. Hence,

$$\liminf_{\epsilon \rightarrow 0} (\bar{\nabla}_i \bar{\nabla}_j S * \varphi_\epsilon + \bar{g}_{ij} S * \varphi_\epsilon)(x) \geq \eta \bar{g}_{ij}.$$

So, there exist some α such that for each $\epsilon \in (0, \alpha)$, there is a strictly convex hypersurface Σ_ϵ whose the support function is $S * \varphi_\epsilon$. Similarly, we can derive from (5.4) that

$$\limsup_{\epsilon \rightarrow 0} (\bar{\nabla}_i \bar{\nabla}_j S * \varphi_\epsilon + \bar{g}_{ij} S * \varphi_\epsilon)(x_\epsilon) \leq (\lambda_{\min}^{\text{loc}}(\Sigma^\eta)(X))^{-1} \bar{g}_{ij}$$

where x_ϵ converge to a point x such that $\xi(x)$ is the outer normal vector of Σ^η at X . \square

We will now give the proof of our long time existence result, Theorem 1.1.

Proof of Theorem 1.1 Let u_0 , Σ_0 and Ω_0 be as in the statement of the theorem and assume without loss of generality that $\inf_{\Omega} u_0 = 0$. We will obtain a solution $\Sigma_t := \{(x, u(\cdot, t)) : x \in \Omega_t \subset \mathbb{R}^n\}$ as a limit

$$\Sigma_t := \lim_{j \rightarrow +\infty} \Sigma_t^j$$

where Σ_t^j denotes the lower half of a strictly convex closed hypersurface N_t which is symmetric with respect to the hyperplane $\{x_{n+1} = j\}$ and also evolves by the Q_k -flow $(*_k^n)$. The symmetry guarantees that $\Sigma_t^j := N_t^j \cap \{x_{n+1} \leq j\}$ is a graph. Thus, our local a priori estimates shown in Sects. 2–4 on compact subsets of \mathbb{R}^{n+1} , give us the uniform C^∞ bounds on Σ_t^j necessary to pass to the limit.

Now, given the initial data u_0 , Σ_0 , and Ω_0 , we define Σ_t as follows.

Step 1 : The construction of the approximating sequence of closed hypersurfaces N_t^j . Let u_0 , Σ_0 and Ω_0 be as in Theorem 1.1 and assume that $\inf_{\Omega} u_0 = u_0(0) = 0$. Since Σ_0 is not necessarily strictly convex we consider the strictly convex rotationally symmetric non-negative entire function $\varphi(x) = \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ of class $C^\infty(\mathbb{R}^n)$ defined by

$$\varphi(x) = \int_0^{|x|} \arctan r \, dr$$

and for each $j \in \mathbb{N}$, we define the approximate strictly convex smooth function $\tilde{u}_0^j : \Omega_0 \rightarrow \mathbb{R}$ with the corresponding graph \tilde{N}_0^j by

$$\tilde{u}_0^j(x) = u_0(x) + \varphi(x)/j, \quad \tilde{\Sigma}_0^j = \{(x, \tilde{u}_0^j(x)) : x \in \Omega_0\}.$$

Then, we reflect $\tilde{\Sigma}_0^j \cap (\mathbb{R}^n \times [0, j])$ over the j -level hyperplane $\mathbb{R}^n \times \{j\} := \{(x, j) : x \in \mathbb{R}^n\}$ to obtain a strictly convex closed hypersurface \tilde{N}_0^j defined by

$$\tilde{N}_0^j = \{(x, h) \in \mathbb{R}^{n+1} : x \in \Omega_0, \tilde{u}_0^j(x) \leq j, h \in \{\tilde{u}_0^j(x), 2j - \tilde{u}_0^j(x)\}\}.$$

Since \tilde{N}_0^j fails to be smooth at its intersection with the hyperplane $\mathbb{R}^n \times \{j\}$, we again approximate \tilde{N}_0^j by a strictly convex closed $C^{1,1}$ hypersurface N_0^j which is the $(1/j)$ -envelope of \tilde{N}_0^j , namely $\bar{N}_0^j := (\tilde{N}_0^j)^{1/j}$.

Given \bar{N}_0^j , we consider $(0, j)$ as the origin and let S^j denote the support function of \bar{N}_0^j with respect to our new origin $(0, j)$. We define S_ϵ^j by the convolution $S_\epsilon^j = S^j * \varphi_\epsilon$ with the

mollifier φ_ϵ given in Proposition 5.3. By Proposition 5.4, for a small enough $\epsilon_j \leq \frac{1}{2}\epsilon_{j-1} \ll 1$, there is a strictly convex smooth closed hypersurface N_0^j whose support function is $S_{\epsilon_j}^j$.

Then, we denote by N_t^j the unique closed strictly convex solution of the Q_k -flow $(*_k^n)$ defined for $t \in [0, T_j)$, where T_j is its maximal existing time (c.f. in [1]). Finally, we set

$$\Sigma_t^j := N_t^j \cap \{x_{n+1} \leq j\}$$

and

$$\Sigma_t := \partial \left\{ \bigcup_{j \in \mathbb{N}} \text{Conv}(N_t^j) \right\}, \quad T = \liminf_{j \in \mathbb{N}} T_j.$$

Step 2 : Passing Σ_t^j to the limit Σ_t . We apply the estimates in Sects. 2–4 for the solution Σ_t^j . We choose any constant $M_0 > 0$, and for each large enough $j \gg M_0 + 2$ satisfying

$$\begin{aligned} \sup_{\bar{u} \leq M_0} v(\Sigma_0^j) &\leq \sup_{\bar{u} \leq M_0+1} 2v(\Sigma_0), \quad \sup_{\bar{u} \leq M_0} \lambda_{\max}(\Sigma_0^j) \leq \sup_{\bar{u} \leq M_0+1} 2\lambda_{\max}(\Sigma_0), \\ \inf_{\bar{u} \leq M_0} Q_k(\Sigma_0^j) &\geq \inf_{\bar{u} \leq M_0+1} \frac{1}{2} Q_k(\Sigma_0). \end{aligned}$$

We apply the estimates in Sects. 2–4 in the following order

- (i) Theorem 2.4 and Theorem 2.5 for Σ_t^j with $M = M_0$.
- (ii) Theorem 3.1 for Σ_t^j with $M = M_0 - 1$.
- (iii) Theorem 4.2 for Σ_t^j with $M = M_0 - 2$.

Thus, we obtain uniform bounds for $\sup v$, $\inf Q_k$, and $\sup \lambda_{\max}$ of Σ_t^j in $\{x_{n+1} \leq M_0 - 2\} \subset \mathbb{R}^{n+1}$.

Next, we consider the functions $u(\cdot, t)$ and $u^j(\cdot, t)$ whose graphs are Σ_t and the lower part of Σ_t^j , respectively. Given a point (x_0, t_0) satisfying $0 < t_0 < T$ and $(x_0, u(x_0, t_0)) \in \Sigma_{t_0}$, we set $M_0 = u(x_0, t_0) + 4$. There exists a sequence of sufficiently large j such that $u^j(x_0, t_0) \leq M_0 - 3$. Also, there exists a small constant $r < t_0$ depending on $\sup v$ and $\sup \lambda_{\max}$ such that $u^j(x, t_0) \leq M_0 - 2$ holds in $B_r(x_0)$. Then, we can apply Proposition 5.2 to u^j in Q_r , which yields uniform interior $C^{2,\alpha}$ estimates. Hence, by passing j to the limit, we have the uniform interior $C^{2,\alpha}$ estimates for $u(x, t)$, and therefore Σ_t is a solution to the Q_k -flow. The standard regularity theory yields the smoothness of u and Σ_t .

Step 3 : All time existence in the case $d_W \geq n - k + 1$. We recall the unique type II closed ancient solution Φ_t to the curve shortening flow in [7], which exists for $t \in (-\infty, 0)$. Then, we obtain a family of strictly convex closed hypersurfaces $\bar{\Phi}_t$ in \mathbb{R}^{n+1} with $O(k) \times O(n+1-k)$ symmetry by rotating the curve Φ_t about its long axis and the space in \mathbb{R}^{k+1} and next by rotating the k -dimensional surface of revolution about its short axis in \mathbb{R}^{n+1} .

The $O(k) \times O(n+1-k)$ symmetric surfaces $\bar{\Phi}_t$ have two principal curvatures $\kappa(\cdot, t)$ and $\lambda(\cdot, t)$, where κ is the curvature of Φ_t . Since $\lambda \leq C\kappa$ for some universal constant C , the Q_k curvature of $\bar{\Phi}_t$ satisfies $Q_k \leq C\kappa$ for some positive constant C depending on n, k . Hence, the family $\bar{\Phi}_{Ct}$ shrinks with the normal direction speed larger than Q_k .

For arbitrary large $\tilde{T} > 0$, there exists a point $Y \in \mathbb{R}^{n+1}$ such that Σ_0 encloses $\bar{\Phi}_{-\tilde{T}} + Y$ because of the condition $d_W \geq n+1-k$. So, for sufficiently large j , the closed solution Σ_0^j in Step 1 encloses $\bar{\Phi}_{-\tilde{T}} + Y$. Namely, Σ_t^j encloses $\bar{\Phi}_{-\tilde{T}+Ct} + Y$, which implies $T \geq T_j \geq C^{-1}\tilde{T}$. Therefore, the maximal existence time T is the infinity.

Step 4 : The maximal existence time in the case $d_W \leq n - k$. Without loss of generality, we assume $\sup |\langle F_0, e_i \rangle| = +\infty$ for each $n+2-d_W \leq i \leq n+1$. We recall the closed convex

viscosity solution Γ_t in \mathbb{R}^{n+1-d_w} such that Σ_0 converges to $\Gamma_0 \times \mathbb{R}^{d_w}$ as $|F_0| \rightarrow \infty$. We denote by $T_\Gamma < +\infty$ the maximal existence time of Γ_t .

We first show $T \leq T_\Gamma$. Since Γ_t is a viscosity solution, there are strictly convex closed smooth solutions Γ_t^j to the Q_k -flow in \mathbb{R}^{n+1-d_w} such that Γ_t^j encloses Γ_t^{j+1} , $\Gamma_t^j \rightarrow \Gamma_t$, and $T_{\Gamma_j} \rightarrow T_\Gamma$, where T_{Γ_j} is the maximal existence time of Γ_t^j . Since $\Gamma_0^j \times \mathbb{R}^{d_w}$ encloses $\Gamma_0 \times \mathbb{R}^{d_w}$, which encloses Σ_0 . Hence, $\Gamma_0^j \times \mathbb{R}^{d_w}$ encloses the closed hypersurface Σ_0^j in Step 1. Therefore, the complete solution $\Gamma_0^j \times \mathbb{R}^{d_w}$ encloses the closed solution Σ_t^j . Thus, we have $T_{\Gamma_j} \geq T_j$. Namely, $T_\Gamma \geq T$.

Next, we show $T \geq T_\Gamma$. From now on, we denote by Γ_t^j strictly convex closed smooth solutions to the Q_k -flow in \mathbb{R}^{n+1-d_w} such that Γ_t^{j+1} encloses Γ_t^j , $\Gamma_t^j \rightarrow \Gamma_t$, and $T_{\Gamma_j} \rightarrow T_\Gamma$. Then, $\Gamma_0^j \times \mathbb{R}^{d_w}$ satisfy $\lambda \leq Q_k$, $H \leq \Lambda$ for some constants λ , $\Lambda > 0$. So, for each j there are strictly convex closed smooth hypersurfaces $\Sigma_0^{j,i}$ in \mathbb{R}^{n+1} with symmetry over $\{x_{n+1} = 0\}$ and points $Y_{j,i} \in \mathbb{R}^{n+1}$ such that $\Sigma_0^{j,i} \rightarrow \Gamma_0^j \times \mathbb{R}^{d_w}$, $\Sigma_0^{j,i+1}$ encloses $\Sigma_0^{j,i}$, Σ_0 encloses $\Sigma_0^{j,i} + Y_{j,i}$, and $\Sigma_0^{j,i}$ satisfy $\frac{1}{2}\lambda \leq Q_k$, $H \leq 2\Lambda$. Now, we denote by $\Sigma_t^{j,i}$ the solutions to the Q_k -flow, and denote by $T_{\Gamma_{j,i}}$ their maximal existence time. Then, we can show that $\Sigma_t^{j,i}$ converges to the solution $\Gamma_t^j \times \mathbb{R}^{d_w}$ in C_{loc}^∞ topology. Thus, we have $T_{\Gamma_{j,i}} \rightarrow T_{\Gamma_j}$. So, $T \geq T_{j,i}$ implies $T \geq T_j$. Namely, $T \geq T_\Gamma$. \square

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