On Carrasco Piaggio's theorem characterizing quasisymmetric maps from compact doubling spaces to Ahlfors regular spaces

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Dedicated to the memory of Professor David R. Adams.

Abstract. In this note we deconstruct and explore the components of a theorem of Carrasco Piaggio, which relates Ahlfors regular conformal gauge of a compact doubling metric space to weights on Gromov-hyperbolic fillings of the metric space. We consider a construction of hyperbolic filling that is simpler than the one considered by Carrasco Piaggio, and we determine the effect of each of the four properties postulated by Carrasco Piaggio on the induced metric on the compact metric space.

Key words and phrases: Gromov hyperbolic filling, uniformization, metric space, quasisymmetry, Ahlfors regular, uniformly perfect, conformal change in metric.

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1. Introduction

Within the class of metric spaces, those that are Gromov hyperbolic possess the properties of negative curvature at large scale but are not concerned with small-scale behavior; and as such, Gromov hyperbolicity is stable under biLipschitz changes in the metric (unlike Alexandrov curvature conditions). First proposed as a structure useful in the study of Cayley graphs of hyperbolic groups [**13**], the study of Gromov hyperbolic spaces was subsequently found to be useful in the study of potential theory [**4**]. It is also connected to the study of metric geometry, as there is a close connection between Gromov hyperbolic spaces and uniform domains [**5**], and between rough quasiisometries between Gromov hyperbolic spaces and quasisymmetries between their visual boundaries. It is

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this latter connection that is explored further in [**11**], and is based on the fact that every compact doubling metric space is the boundary of a Gromov hyperbolic space, called hyperbolic filling, of the space. Now there is extensive literature on various uses of hyperbolic filling, dating back to the seminal paper of Gromov [**13**, page 95], and made explicit in [**3, 6, 7, 8, 9, 11, 17, 18, 19, 20, 21, 23**]; these are merely a sampling of current literature on the topic of Gromov hyperbolicity and hyperbolic filling.

During the author's stay at MSRI, there was an extensive discussion of the paper [**11**] characterizing metrics on a compact space that are quasisymmetrically equivalent and at least one of them an Ahlfors regular metric. The results of [**11**] were of great interest to many participants at MSRI. However, the complicated system of parameters used there made it difficult to see the underlying beautiful ideas in [**11**]. The goal of the current note is to deconstruct the role of some of the parameters in used there, and to eliminate others, thus providing a simplified expository discourse on parts of [**11**]. The focus is on [**11**, Theorem 1.1]. The following theorem is the result of exploring the role of each of the conditions (H1)–(H4) assumed in [**11**].

THEOREM 1.1. *Let (Z, d_Z) be a compact doubling metric space. Fixing $\alpha \geq 2$, and $\tau \geq 2\alpha^2 + 1$, we choose a hyperbolic filling X of Z associated with the parameters α and τ as in Definition 2.1.*

- **I.** *Suppose that $\rho : X \to (0, 1)$, and consider the function d_ρ on $X \times X$ associated with ρ as in Definition 3.4.*
  - (a) *If ρ satisfies Condition (H1) of Definition 3.1, then d_ρ is a metric on X, with (X, d_ρ) a locally compact, non-complete metric space. Let $\partial_\rho X := \overline{X} \setminus X$, with $\overline{X}$ the completion of X with respect to the metric d_ρ.*
  - (b) *If ρ satisfies Conditions (H1) and (H3) of Definition 3.1, then there is a homeomorphism $\Phi : Z \to \partial_\rho X$ and positive constants c, C such that for every $x, y \in Z$ we have*

$$c\, d_Z(x, y)^{\tau_-} \leq d_\rho(\Phi(x), \Phi(y)) \leq C\, d_Z(x, y)^{\tau_+}$$

  with

$$\tau_- := \frac{\log(\eta_-)}{\log(1/\alpha)}, \qquad \tau_+ := \frac{\log(\eta_+)}{\log(1/\alpha)}.$$

  - (c) *If ρ satisfies Conditions (H1), (H2), and (H3) of Definition 3.1, then the map Φ is a quasisymmetry.*
  - (d) *If ρ satisfies Conditions (H1), (H2), and (H3) of Definition 3.1 and Condition (H4) of Definition 6.1, then $(\partial_\rho X, d_\rho)$ is Ahlfors p-regular.*
- **II.** *Conversely, suppose that Z is C_U-uniformly perfect for some $C_U > 2$, and $\alpha > C_U^3$ with $\tau \geq \max\{\alpha^2 + 1, 2C_U^3(C_U^2 - 4)^{-1}\}$. If θ is any metric on Z for which (Z, θ) is Ahlfors p-regular and is quasisymmetric to (Z, d_Z), then there exists a function $\rho : X \to (0, 1)$ that satisfies Conditions (H1), (H2), (H3), and (H4).*

REMARK 1.2. Note in the above theorem that in Part **I.** we *do not* require (Z, d_Z) to be uniformly perfect; then, Conditions (H1)—(H3) do not imply uniform perfectness

either (and indeed, the choice of ρ as the constant function $\rho(x) = 1/\alpha$ satisfies Conditions (H1)—(H3) with the resulting quasisymmetry a biLipschitz map, see **[3]**, claim **I.**(b) of Theorem 1.1 above, or Theorem 4.2 below); however, Conditions (H1)—(H4) *together* imply that (Z, d_Z) must be uniformly perfect. Thus **I.**(a)—(c) on their own are not explicitly covered in **[11]**, for Carrasco Piaggio **[11]** does explicitly require Z to be uniformly perfect (see **[11**, Section 2.1**]**), that assumption is also implicit in the four conditions together (see Lemma 6.2 below), and conversely, if (Z, d) is quasisymmetric to (Z, θ) with θ Ahlfors d-regular, then necessarily (Z, d) is uniformly perfect as well. Interestingly also, in **[11**, page 507, (2.8)**]**, Carrasco Piaggio requires $\tau \geq 32$ (there τ is denoted λ) and then require $\alpha \geq 6\kappa^2 \max\{\tau, C_U\}$ (with α denoted as a and C_U denoted as K_P in **[11]**). The parameter κ is an additional one associated with the construction of hyperbolic filling as given in **[11]**; with the simplified construction as considered in this note and in **[3]**, we have $\kappa = 1$. Thus, in **[11]** the parameter α depends on the choice of λ and C_U, but in our note τ depends on the choice of α while in Part **II.**, both α and τ depend on C_U as well.

As pointed out above, when considering only the conditions (H1)—(H3), the metric space (Z, d) need not be uniformly perfect, but still the quasisymmetry Φ obtained in Section 5 is necessarily a power quasisymmetry. Since there are compact doubling metric spaces and quasisymmetries on them that are *not* power quasisymmetries (see for example the discussion in **[14]**), it follows that not all quasisymmetries on a doubling space are obtained using the method of Carrasco Piaggio **[11]**.

Section 2 is devoted to describing the construction of hyperbolic filling, and the last five sections of this note are devoted to the proof of the claims of the theorem. We choose to use the construction of hyperbolic filling from **[3]** for its simplicity in relation to the one used in **[11]**. While the construction in **[11]** (see also **[21]**) gives greater flexibility to the choice of sets and vertices, it is perhaps this very flexibility that makes it difficult to see what the effect of the conditions (H1)–(H4) are, and so we chose the simpler version given in **[3]**. However, the ideas and basic premises are as in **[11]**.

In Section 3 the conditions (H1)—(H3) are discussed and **I.**(a) of Theorem 1.1 is proved, while in Section 4 the claim **I.**(b) of the theorem is verified. Section 5 is devoted to the proof of **I.**(c) of Theorem 1.1, and the discussion in Section 6 completes the proof of the part **I.** of Theorem 1.1. The focus of Section 7 is to prove part **II.** of Theorem 1.1. In Section 8 we list a set of four conditions that parallel the conditions of Carrasco Piaggio **[11]**, but couched from the perspective of densities on a metric space that lead to conformal changes in the metric. We end that section by posing a query regarding an Adams-type inequality **[1, 2, 24]**, which is known to hold in the case that the function ρ is the constant function $\rho(x) = 1/\alpha$.

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2. Construction of hyperbolic filling

Recall that a metric space (Z, d_Z) is *metric doubling* if there is a positive integer N such that for each $z \in Z$ and $r > 0$, if $A \subset B(z, r)$ such that $d_Z(x, y) \geq r/2$ whenever $x, y \in A$ with $x \neq y$, then there are at most N number of elements in A.

In this note, (Z, d_Z) is a compact metric space, such that it is a metric doubling space. Later we will also assume that Z is uniformly perfect, that is, there is some $C_U > 1$ such that for each $z \in Z$ and $0 < r < \operatorname{diam}(Z)/2$, the annulus $B_{d_Z}(z, r) \setminus B_{d_Z}(z, r/C_U)$ is non-empty; however, for now we do not need this assumption. We will, however, also assume that $0 < \operatorname{diam}(Z) < 1$ without loss of generality (as we are not interested in singleton metric spaces).

Constructions of hyperbolic fillings of compact doubling metric spaces can be found for example in **[8, 6, 7, 9, 11, 3]**. The version we give here is that of **[3]**. The obtained graph in this construction, when equipped with the path metric, is Gromov hyperbolic; however, this fact is not essential for the discussion in this note, as we turn the graph into a metric graph by adding unit interval edges to connect neighboring pairs of vertices and then use path integrals to directly obtain a metric on the graph; hence its boundary can be realized via a metric completion rather than as the visual boundary of a Gromov hyperbolic space. For this reason, we do not devote space to discussing Gromov hyperbolicity here. We refer the interested reader to the discussion in **[3**, Section 3**]**.

Definition 2.1. By a rescaling of the metric if necessary, we may assume without loss of generality that $0 < \operatorname{diam}(Z) < 1$. We fix $\alpha \geq 2$ and $\tau > 1$, and for each non-negative integer n we set A_n to be a maximal α^{-n}-separated subset of Z, that is, if $z, w \in Z$ with $z \neq w$, then $d_Z(z, w) \geq \alpha^{-n}$, and $Z = \bigcup_{w \in A_n} B_{d_Z}(w, \alpha^{-n})$. We can, via an inductive construction, ensure that $A_n \subset A_{n+1}$ for each non-negative integer n. We set $V = \bigcup_{n=0}^{\infty} A_n \times \{n\}$. The set V is the vertex set of the metric graph X to be constructed next. We do this construction as follows. The vertex $w_0 = (x_0, 0)$, with $x_0 \in A_0$, will play the role of a root of the graph.

(1) Two vertices $v_1 = (z_1, n_1), v_2 = (z_2, n_2) \in V$ are neighbors, denoted $v_1 \sim v_2$, if $v_1 \neq v_2$ and either $n_1 = n_2$ with $B_{d_Z}(z_1, \tau\alpha^{-n_1}) \cap B_{d_Z}(z_2, \tau\alpha^{-n_2}) \neq \emptyset$, or else $n_1 = n_2 \pm 1$ and $B_{d_Z}(z_1, \alpha^{-n_1}) \cap B_{d_Z}(z_2, \alpha^{-n_2}) \neq \emptyset$.
(2) We turn V into a metric graph X by gluing a unit-length interval to each pair of neighboring vertices.
(3) We call a vertex $v_2 = (z_2, n_2)$ a child of a vertex $v_1 = (z_1, n_1)$ if $v_1 \sim v_2$ and $n_2 = n_1 + 1$; we also then say that the edge $[v_1, v_2]$ is a vertical edge. If $[v_1, v_2]$ is a vertical edge, then necessarily $d_Z(z_1, z_2) < \alpha^{-n_1} + \alpha^{-n_2}$, and so with $n = \min\{n_1, n_2\}$, we have that $d_Z(z_1, z_2) < \alpha^{1-n}$ (we use our choice of $\alpha \geq 2$ here).
(4) If $v_1 \sim v_2$ with $n_1 = n_2$, then we say that the edge $[v_1, v_2]$ is a horizontal edge. In this case we have that $d_Z(z_1, z_2) < \tau\alpha^{1-n_1}$.
(5) We say that a point $x \in X$ is a descendant of a point $y \in X$ if there is a vertically descending path *from* y to x.

(6) A vertex v is said to be a *common ancestor* of two points $x, y \in X$ if there are two vertically descending paths, one from v to x and the other from v to y.
(7) Also, given a vertex $v = (z, n) \in V$, we set

$$\Pi_1(v) = z \quad \text{and} \quad \Pi_2(v) = n.$$

(8) If $\tau \geq 1 + 1/\alpha$ and $(z, n), (x_1, n-1), (x_2, n-1) \in V$ such that $(z, n) \sim (x_i, n-1)$ for $i = 1, 2$, then $(x_1, n-1) \sim (x_2, n-1)$.
(9) Thanks to the doubling property, there is a constant $C \geq 1$, depending only on the doubling constant related to the metric doubling property of (Z, d_Z) and the choice of α, τ, such that for each positive integer n we have $\sum_{x \in A_n} \chi_{B_{d_Z}(x, \tau\alpha^{-n})} \leq C$ pointwise everywhere on Z.
(10) Suppose that $\cdots \sim (x_{n+1}, n+1) \sim (x_n, n) \sim (y_n, n) \sim (y_{n+1}, n+1) \sim \cdots$ is a path in the graph, allowing for the possibility that $x_n = y_n$ by a slight abuse of notation above, we see that for each $k \geq n$, $d_Z(x_k, x_{k+1}) \leq \alpha^{-k} + \alpha^{-k-1} \leq \alpha^{1-k}$ (we use the choice $\alpha \geq 2$ here). With similar estimates holding for $d(y_k, y_{k+1})$, we see that the two sequences $(x_k)_{k \geq n}$ and $(y_k)_{k \geq n}$ are Cauchy sequences in Z, converging to points denoted x and y respectively. We see that then for each $j \geq n$,

$$d_Z(x, x_j) \leq \sum_{n=j}^{\infty} \alpha^{1-n} = \frac{\alpha^{2-j}}{\alpha - 1},$$

with a similar estimate holding for $d_Z(y, y_j)$. Suppose that $x \neq y$. With n_{xy} a non-negative integer such that $\alpha^{-n_{xy}} < d_Z(x, y) \leq \alpha^{1-n_{xy}}$, and j_0 a non-negative integer such that $\alpha^{-j_0} < \tau - 1 \leq \alpha^{1-j_0}$, we have that

$$\alpha^{-n_{xy}} < d_Z(x, y) \leq d_Z(x, x_n) + d_Z(x_n, y_n) + d_Z(y_n, y) \leq \frac{2\alpha^{2-j}}{\alpha - 1} + 2\tau\alpha^{-n} \leq \alpha^{3+j_0-n}.$$

It follows that

$$n \leq 3 + j_0 + n_{xy}. \tag{2.2}$$

(11) Given a vertex $v = (x, n) \in V$, there is a vertically descending geodesic ray $w_0 = v_0 \sim v_1 \sim \cdots \sim v_k \sim \cdots$ with $v_k = v$ for each $k \geq n$. This is done by choosing $v_k = (x_k, k)$ for $k = 1, \cdots, n-1$ such that $x_k \in A_k$ with $d_Z(x, x_k) \leq \alpha^{-k}$.

Note that A_0 has only one point by our hypothesis that $\operatorname{diam}(Z) < 1$. The vertex $w_0 = (x_0, 0)$ plays a distinguished role in the graph corresponding to $x_0 \in A_0$. If $z \in A_{n+1} \setminus A_n$, then by the maximality of A_n there is a point $w_z \in A_n$ such that $d_Z(z, w_z) < \alpha^{-n}$, and so $(z, n+1) \sim (w_z, n)$; therefore it is easy to see that X is path-connected. While this construction is not exactly the one considered in **[11]**, it is in the spirit of **[11]** and is the one used in **[3]**. From **[3**, Theorem 3.4] we know that X is Gromov hyperbolic, with hyperbolicity constant depending solely on α and τ.

Larger the choice of τ is, the greater the number of horizontal edges. Since Z is doubling, each vertex $v \in V$ has a uniformly bounded degree, with the upper bound on the degree depending solely on the doubling constant associated with ν and the parameters α and τ.

Henceforth, we will fix $\alpha \ge 2$ and $\tau \ge 1+\frac{1}{\alpha}$. The condition on τ ensures that the conclusion of (8) above holds.

3. Weighted uniformization metric and three conditions

Since $\operatorname{diam}(Z) > 0$, the graph X, equipped with the path metric d_X, is necessarily unbounded. In this section we consider a family of uniformizations, each dampening the metric d_X at locations far from the root vertex w_0, so that the dampened metric on X turns X into a bounded non-complete metric space. The principal object of study in this note is the boundary of the damped space, as it is in [**11**].

DEFINITION 3.1. We consider a function $\rho : V \to \mathbb{R}$ that satisfies the following conditions (using the labels from [**11**]):

(H1) There exist $0 < \eta_- \le \eta_+ < 1$ such that $\rho : X \to [\eta_-, \eta_+]$.

(H2) There is a constant $K_0 > 0$ so that if $v_1, v_2 \in V$ with $v_1 \sim v_2$, and if $w_0 \sim w_1 \sim \cdots \sim w_k = v_1$ and $w_0 = u_0 \sim u_1 \sim \cdots \sim u_n = v_2$ are vertical edges, then

$$\pi(v_1) := \prod_{j=0}^{k} \rho(w_j) \le K_0 \prod_{j=0}^{n} \rho(u_j) =: K_0\, \pi(v_2).$$

This also defines $\pi : V \to (0, \infty)$. We extend π to all of X by setting $\pi(x) = t\pi(v_1) + (1-t)\pi(v_2)$ when x is a non-vertex point in the edge $[v_1, v_2]$, and t denotes the distance from x to the vertex v_1.

(H3) There is a constant $K_1 > 0$ satisfying the following condition. Whenever $x, y \in X$ with x, y belonging to different edges of X, there are two vertically descending paths $w_0 = v_0 \sim v_1 \sim \cdots \sim v_k$, $w_0 = u_0 \sim u_1 \sim \cdots \sim u_n$ with $x \in [v_{k-1}, v_k]$, $y \in [u_{k-1}, u_k]$. Let v_{xy} denote the vertex in the path $w_0 = v_0 \sim v_1 \sim \cdots \sim v_k$ with largest possible value of $\Pi_2(v_{xy})$ such that either $v_{xy} = u_{\Pi_2(v_{xy})}$ or else $v_{xy} \sim u_{\Pi_2(v_{xy})}$. For every path γ in X with end points x and y, we must have

$$\int_\gamma \pi(\gamma(t))\, dt \ge K_1^{-1} \pi(v_{xy}).$$

REMARK 3.2. Note that in Condition (H2), if we have $v_2 = v_1$ instead of $v_2 \sim v_1 = (x_v, n)$, then $k = n$ and necessarily

$$\begin{aligned} d_Z(\Pi_1(w_{n-1}), \Pi_2(u_{n-1})) &\le d_Z(\Pi_1(w_{n-1}), x_v) + d_Z(x_v, \Pi_2(u_{n-1})) \\ &\le 2\left[\alpha^{-(n-1)} + \alpha^{-n}\right] \le 4\alpha^{1-n}. \end{aligned}$$

It follows that if $\alpha \ge 2$ and $\tau \ge 2\alpha^2 + 1 > 4$, then $w_{n-1} \sim u_{n-1}$. Hence from (H2) we have that $\pi(v)$, up to the ambiguity of the multiplicative constant K_0, is well-defined in that the choice of the descending path used to define $\pi(v)$ is not crucial.

REMARK 3.3. If $x \in X$, we can find paths $= w_0 = v_0 \sim v_1 \sim \cdots$ in X so that for each positive integer n we have that $\Pi_1(v_n) \in A_n$ with $d_Z(x, \Pi_1(v_n)) < \alpha^{-n}$. Let $w_0 \sim w_1 \sim \cdots$ be another such path associated with a point $y \in X$, and let v_{xy} be the

vertex point in the path $\{v_n : n = 0, 1, \cdots\}$ that is a neighbor of $w_{\Pi_2(v_{xy})}$ such that $\Pi_2(v_{xy})$ be the largest possible (i.e., the latest common ancestor). Then from (H3) above, when γ is the concatenation of the curves from v_{xy} to x and to y respectively via the sequences $(v_n)_{n\geq\Pi_2(v_{xy})}$, $v_{xy} \sim w_{\Pi_2(v_{xy})}$, and $(w_n)_{n\geq\Pi_2(v_{xy})}$, we have that

$$\int_\gamma \pi(\gamma(t))\, dt \geq K_1^{-1}\, \pi(v_{xy}),$$

see point (10) of Definition 2.1 above.

We will use the weight π as the conformal density, to modify the metric on the graph X from the path metric to the metric d_ρ.

In [**11**] a fourth condition is also required, but we will not consider that condition until the penultimate section of this note. We postpone its definition to that section, see Definition 6.1 below.

Definition 3.4. Let $d_\rho : X \times X \to [0, \infty)$ be given as follows. For $x, y \in X$, we set

$$d_\rho(x, y) = \inf_\gamma \int_\gamma \pi(\gamma(t))\, dt,$$

where the infimum is over all paths γ in X with end points x and y. We only consider paths that are arc-length parametrized with respect to the graph metric d_X.

Lemma 3.5. *Suppose that ρ satisfies Condition (H1). Then d_ρ is a metric on X. Moreover, (X, d_ρ) is locally compact, non-complete metric space.*

Proof. Let $x, y \in X$ with $x \neq y$ and γ be a curve in X with end points x and y. If x and y belong to the same edge $[v_1, v_2]$ in X, then any curve γ connecting x to y has to contain a subcurve of d_X-length at least $d_X(x, y)$ that lies in the subgraph obtained by adding the edges that have either v_1 or v_2 as a vertex-endpoint. Hence, with $n = \max\{\Pi_2(v_1), \Pi_2(v_2)\}$, we have that

$$\int_\gamma \pi(\gamma(t))\, dt \geq \eta_-^{n+1}\, d_X(x, y) > 0,$$

and taking the infimum over all curves γ gives $d_\rho(x, y) \geq \eta_-^{n+1} d_X(x, y) > 0$.

Next, suppose that x and y belong to different edges. Then any curve γ connecting x to y has to have a sub-curve of positive d_X-length that passes through a vertex $v \neq x$ such that v is a neighbor of one of the two vertices that make up the edge x lies in. It follows that $\Pi_2(v) \leq n_x + 1$, with n_x a positive integer that depends solely on x. Hence

$$\int_\gamma \pi(\gamma(t))\, dt \geq \pi(v) d_X(x, v) \geq \eta_-^{n_x+1}\, d_X(x, v) > 0.$$

Taking the infimum over all γ gives $d_\rho(x, y) \geq \eta_-^{n_x+1}\, d_X(x, v) > 0$. Thus, in both cases we have that if $x \neq y$ then $d_\rho(x, y) > 0$. The triangle inequality and symmetry follow immediately from the definition of d_ρ, and so d_ρ is a metric on X.

From the first paragraph of this proof, we know that for each vertex v, the subgraph made up of all the edges that have v as an end-point is a compact subset of (X, d_ρ), and moreover, v is in the d_ρ-interior of this subgraph. Hence (X, d_ρ) is locally compact.

Finally, for each non-negative integer n we set $w_n = (x_0, n)$. Then $w_0 \sim w_1 \sim \cdots \sim w_n \sim w_{n+1} \sim \cdots$, and as the edge $[w_n, w_{n+1}]$ is a path connecting the two vertices w_n and w_{n+1}, we see that

$$d_\rho(w_n, w_{n+1}) \le \eta_+^n. \tag{3.6}$$

As $0 < \eta_+ < 1$, it follows that $(w_n)_n$ is a Cauchy sequence in (X, d_ρ). This sequence does not converge to any element in X. Therefore (X, d_ρ) is non-complete. □

If ρ is the constant function $\rho(x) = 1/\alpha$, where α (together with τ) is the parameter used in constructing the hyperbolic filling X of Z, then $\pi(v) \approx \alpha^{-n}$ where $n = \Pi_2(v)$. Therefore, from [**3**, Proposition 4.4] we know that $\partial_\rho X =: \overline{X} \setminus X$ is biLipschitz equivalent to Z. Here, the completion $\overline{X}$ is taken with respect to the metric d_ρ. *We will not need this information for our discussion in this note, and so we do not elaborate on this further* but refer the interested reader to [**3**].

In Lemma 3.5 only (H1) played a role. In the next section Conditions (H1) and (H3) together will play a key role, but Condition (H2) will not.

4. Bi-Hölder property

Recall from the pervious section that (X, d_ρ) is locally compact but not complete. We set $\partial_\rho X := \overline{X} \setminus X$, where $\overline{X}$ is the completion of X with respect to d_ρ. As X is locally compact with respect to d_ρ (see Lemma 3.5), it follows that X is an open subset of $\overline{X}$.

As shown in [**3**, Proposition 4.1], if $\eta_- < 1/\alpha$, then there is no guarantee that $\partial_\rho X$ is even homeomorphic to Z; hence if $\eta_- < 1/\alpha$, then Condition (H3) becomes vital in obtaining that $\partial_\rho X$ is homeomorphic to Z.

We now construct a natural map $\Phi : Z \to \partial_\rho X$ as follows.

Definition 4.1. For $z \in Z$ and for each positive integer n we can find $v_n \in V$ such that with $x_n = \Pi_1(v_n) \in A_n$ and $\Pi_2(v_n) = n$, with $d_Z(x_n, z) < \alpha^{-n}$. Note that then $z \in B_{d_Z}(x_n, \alpha^{-n}) \cap B_{d_Z}(x_{n+1}, \alpha^{-(n+1)})$, and so $v_n \sim v_{n+1}$, and hence $w_0 = v_0 \sim v_1 \sim \cdots \sim v_n \sim v_{n+1} \sim \cdots$ is a vertically descending path in X, with $\pi(v_n) \le \eta_+^n$. Hence the sequence $(v_n)_n$ is a Cauchy sequence in (X, d_ρ), for we have that $d_\rho(v_n, v_{n+1}) \le 2\eta_+^n$, see (3.6). We set $\Phi(x)$ to be the class of all Cauchy sequences in (X, d_ρ) that are equivalent to this Cauchy sequence.

To see that Φ is well-defined, suppose that $y_n \in A_n$ for each positive integer n such that $d_Z(x, y_n) < \alpha^{-n}$. Then $(y_n, n) \sim v_n$, because $z \in B_{d_Z}(x_n, \alpha^{-n}) \cap B_{d_Z}(y_n, \alpha^{-n})$. As above, the sequence $((y_n, n))_n$ is also Cauchy with respect to the metric d_ρ, but also $d_\rho(v_n, (y_n, n)) \le \eta_+^n + \eta_+^{n+1}$, and so the two Cauchy sequences are equivalent with respect to the metric d_ρ. Thus, $\Phi : Z \to \partial_\rho X$ is well-defined.

Theorem 4.2. *Suppose that ρ satisfies Conditions (H1) and (H3). Then Φ is a homeomorphism with*

$$C^{-1}\, d_Z(x, y)^{\tau_-} \le d_\rho(\Phi(x), \Phi(y)) \le C\, d_Z(x, y)^{\tau_+} \tag{4.3}$$

for each $x, y \in Z$, where

$$\tau_- := \frac{\log(\eta_-)}{\log(1/\alpha)}, \qquad \tau_+ := \frac{\log(\eta_+)}{\log(1/\alpha)}.$$

Moreover, $d_\rho(\Phi(x), \Phi(y)) \approx \pi(v_{xy})$.

We remind the reader that the root of X is denoted $w_0 = (x_0, 0)$.

PROOF. Let j_0 be the unique integer such that $\alpha^{-j_0} < \tau - 1 \leq \alpha^{1-j_0}$.

We first aim to prove (4.3). Let $x, y \in Z$, and choose a positive integer n_{xy} such that $\alpha^{-n_{xy}} < d_Z(x, y) \leq \alpha^{1-n_{xy}}$. We fix a path $w_0 = v_0 \sim v_1 \sim \cdots$ such that for each non-negative integer n we have that $\Pi_1(v_n) \in A_n$ with $d_Z(\Pi_1(v_n), x) \leq \alpha^{-n}$. Let $v_0 = w_0 \sim w_1 \sim \cdots$ be a corresponding choice of descending sequence with respect to y. We claim that for each non-negative integer n with $n \leq n_{xy} - j_0 - 1$, either $v_n = w_n$ or $v_n \sim w_n$. To this end, we assume that $v_n \neq w_n$ and $1 \leq n \leq n_{xy} - j_0 - 1$, for otherwise there is nothing to prove. Since $d_Z(x, \Pi_1(v_n)) \leq \alpha^{-n}$ and $d_Z(y, \Pi_1(w_n)) \leq \alpha^{-n}$, and as $n \leq n_{xy} - j_0 - 1$, it follows that

$$d_Z(x, \Pi_1(w_n)) \leq \alpha^{-n} + \alpha^{1-n_{xy}} \leq \alpha^{-n}(1 + \alpha^{-j_0}) < \tau\alpha^{-n}.$$

It follows that $x \in B_{d_Z}(\Pi_1(v_n), \alpha^{-n}) \cap B_{d_Z}(\Pi_1(w_n), \tau\alpha^{-n})$, and so $v_n \sim w_n$. Next we claim that if n is a positive integer with $v_n \sim w_n$, then $n \leq n_{xy} + (-j_0)_+ + 3$. Indeed, we have that

$$\alpha^{-n_{xy}} < d(x, y) \leq d(x, \Pi_1(v_n)) + d(y, \Pi_1(w_n)) + 2\tau\alpha^{-n} \leq 2(1 + \tau)\alpha^{-n} \leq \alpha^{1-n}(1 + \tau),$$

with $1 + \tau \leq 2 \leq 1 + \alpha^{1-j_0} \leq \alpha^{3-n}$ if $j_0 \geq 0$, and $1 + \tau \leq \alpha^{3-n-j_0}$ if $j_0 < 0$. From this we obtain $n + (-j_0)_+ - 3 < n_{xy}$. As n and n_{xy} are integers, it follows that $n \leq n_{xy} + (-j_0)_+ + 3$.

We now fix a choice of sequences v_n, w_n, $n = 0, 1, \cdots$ as above corresponding to the points $x, y \in Z$, and let $F[x, y]$ denote the collection of all vertices v_n for which $v_n \sim w_n$ or $v_n = w_n$. Let v_{xy} be the vertex in F for which $\Pi_2(v_{xy}) = \max\{\Pi_2(v) : v \in F[x, y]\}$. For symmetry's sake, we also set w_{xy} to be from the sequence corresponding to y such that $w_{xy} = w_{\Pi_2(v_{xy})}$.

Recall that j_0 is the integer such that $\alpha^{-j_0} < \tau - 1 \leq \alpha^{1-j_0}$. From the above argument, we see that

$$n_{xy} - |j_0| - 1 \leq \Pi_2(v_{xy}) = \Pi_2(w_{xy}) \leq n_{xy} + |j_0| + 1, \tag{4.4}$$

and that either $w_{xy} = v_{xy}$ or $w_{xy} \sim v_{xy}$. The curve β given by the path $\cdots \sim v_n \sim v_{n-1} \sim \cdots \sim v_{xy} \sim w_{xy} \sim \cdots \sim w_{n-1} \sim w_n \sim \cdots$ has $\Phi(x)$ and $\Phi(y)$ as its end points, and so

$$d_\rho(\Phi(x), \Phi(y)) \leq \int_\beta \pi(\beta(t))\, dt = \pi(v_{xy}) \sum_{j=\Pi_2(v_{xy})}^{\infty} \left[\frac{\pi(v_i)}{\pi(v_{xy})} + \frac{\pi(w_i)}{\pi(w_{xy})} \right].$$

Note that for $j \geq \Pi_2(v_{xy})$,

$$\eta_-^{j-\Pi_2(v_{xy})} \leq \frac{\pi(v_i)}{\pi(v_{xy})} \leq \eta_+^{j-\Pi_2(v_{xy})}, \quad \text{and} \quad \eta_-^{j-\Pi_2(v_{xy})} \leq \frac{\pi(w_i)}{\pi(v_{xy})} \leq \eta_+^{j-\Pi_2(v_{xy})}.$$

Therefore

$$d_\rho(\Phi(x),\Phi(y)) \le \frac{2}{1-\eta_+}\,\pi(v_{xy}).$$

On the other hand, by (H3) we have that for all curves γ in X that have $\Phi(x)$ and $\Phi(y)$ as their endpoints (with respect to the metric d_ρ),

$$\int_\gamma \pi(\gamma(t))\,dt \ge K_1^{-1}\,\pi(v_{xy}).$$

It follows that

$$K_1^{-1}\pi(v_{xy}) \le d_\rho(\Phi(x),\Phi(y)) \le \frac{2}{1-\eta_+}\,\pi(v_{xy}). \tag{4.5}$$

Finally, we note from (4.4) that

$$\eta_-^{n_{xy}} \le \pi(v_{xy}) \le \eta_+^{n_{xy}-1-|j_0|}.$$

Recall that we choose n_{xy} so that $\alpha^{-n_{xy}} < d(x,y) \le \alpha^{1-n_{xy}}$. Now the definition of τ_+ and τ_-, together with the choice of n_{xy} above, gives us the validity of (4.3) with constant C depending only on η_-, η_+, and j_0 (which in turn depends only on τ and α). The last claim of the theorem follows from (4.5).

Note that Z is compact. Therefore, to prove that Φ is a homeomorphism, it now suffices to prove surjectivity of Φ. Let $(w_k)_k$ be a Cauchy sequence in (X,d_ρ) that is *not* convergent in (X,d_ρ). By replacing w_k with its nearest vertex if necessary, we may assume without loss of generality that each w_k is in the vertex set V (for this change in the sequence gives us a Cauchy sequence that is equivalent to the original sequence). By passing to a subsequence if necessary, we may also assume that for each positive integer k,

- $d_\rho(w_k,w_{k+1}) < (K_1^2\alpha)^{-k}$,
- $\Pi_2(w_k) < \Pi_2(w_{k+1})$.

Indeed, if there is some positive integer n_0 such that $\Pi_2(v_k) \le n_0$ for each positive integer k, then the sequence lies in the d_X-ball $\{w \in X : d_X(w,w_0) \le n_0\}$ where d_X is the graph metric on X (obtained by considering path metric with each edge in X to be of unit length). In this case, we would have from the proof of Lemma 3.5 that d and d_ρ are biLipschitz on this ball and hence $(w_k)_k$ would be convergent to a point in this d_X-ball with respect to d_X and hence with respect to d_ρ, violating our assumption that the sequence is not convergent in (X,d_ρ). Thus the above two conditions can be met by choosing a subsequence.

For positive integers k we set $x_k = \Pi_1(w_k)$. Then by the compactness of Z we have that there is some $x_\infty \in Z$ and a subsequence of the sequence $(x_k)_k$, also denoted $(x_k)_k$, such that $x_k \to x_\infty$ with respect to the metric d_Z. For each positive integer n we choose $v_n \in V$ such that $d_Z(\Pi_1(v_n),x_\infty) < \alpha^{-n}$. As in the construction of Φ we know that $(v_k)_k$ is a Cauchy sequence with respect to d_ρ, and that $\Phi(x_\infty) = [(v_n)_n]_\rho$ (where $[(v_n)_n]_\rho$ denotes the collection of all Cauchy sequences in (X,d_ρ) that are equivalent to the Cauchy sequence $(v_n)_n$). We now show that $(w_k)_k \in [(v_n)_n]_\rho$, for this would conclude the proof of surjectivity of Φ. Since $d_Z(x_k,x_\infty) \to 0$ as $k \to \infty$, for each positive integer n we can find $k_n > n$ such that $d_Z(x_\infty,x_{k_n}) < \alpha^{-n-1}$. Then by the choice of v_k we have that $d_Z(\Pi_2(v_n),x_{k_n}) < \alpha^{1-n}$.

Then with u_n a common ancestor of v_n and w_{k_n} with the largest value of $\Pi_2(u_n)$, we have from Condition (H3) and (H1) that

$$d_\rho(v_n, w_{k_n}) \approx \pi(u_n) \leq \eta_+^{\Pi_2(u_n)} \to 0 \text{ as } n \to \infty,$$

the last assertion above following from (4.4). It follows that $(w_{k_n})_n$ and $(v_n)_n$ are equivalent Cauchy sequences in (X, d_ρ), completing the proof of surjectivity of Φ. □

5. Quasisymmetry

Recall that a homeomorphism $\Psi : W \to Y$, with (W, d_W) and (Y, d_Y) two metric spaces, is quasisymmetric if there is a homeomorphism $\eta : (0, \infty) \to (0, \infty)$ with $\lim_{t\to 0^+} \eta(t) = 0$ such that for every triple of distinct points $x_1, x_2, x_3 \in W$ we have

$$\frac{d_Y(\Psi(x_1), \Psi(x_2))}{d_Y(\Psi(x_1), \Psi(x_3))} \leq \eta\left(\frac{d_W(x_1, x_2)}{d_W(x_1, x_3)}\right).$$

Given that η is a homeomorphism, it can be seen that Ψ^{-1} is also a quasisymmetry if Ψ is. In the event that W (and hence Y) is uniformly perfect, then η can be chosen to be a power function; there are constants $C \geq 1$ and $0 < \Theta \leq 1$ such that the following choice of η works:

$$\eta(t) = C \max\{t^\Theta, t^{1/\Theta}\}.$$

We refer the interested reader to the discussion on quasisymmetric and quasiconformal maps found in [**14**]. We will see in Lemma 6.2 in the next section that when ρ satisfies Conditions (H1) through (H4), Z is necessarily uniformly perfect. We do not assume Condition (H4) here, and so the space Z need not be uniformly perfect; however, the quasisymmetric maps we obtain are still of the above-mentioned power function format.

In this section we will focus on quasisymmetric aspects of the map Φ defined in the previous section. Here Condition (H2) plays a vital role.

THEOREM 5.1. *Suppose that ρ satisfies all three of the conditions (H1), (H2), and (H3). Then the map*

$$\Phi\colon Z \to \partial_\rho X$$

constructed in Definition 4.1 is a quasisymmetric map.

PROOF. Let x, y, z be three distinct points in Z. Then by Theorem 4.2, and in particular, by (4.5), we have that

$$\frac{d_\rho(\Phi(x), \Phi(y))}{d_\rho(\Phi(x), \Phi(z))} \approx \frac{\pi(v_{xy})}{\pi(v_{xz})}.$$

Suppose first that $\Pi_2(v_{xy}) \geq \Pi_2(v_{xz})$. Let γ be a descending path from the root vertex w_0 to x, passing through v_{xy}, and let β be a descending path from w_0 to x, passing through v_{xz}. Then there is a vertex w in the path γ such that $\Pi_2(w) = \Pi_2(v_{xz})$; it follows that $x \in B_{d_Z}(\Pi_1(v_{xz}), \alpha^{-\Pi_2(v_{x,z})}) \cap B_{d_Z}(\Pi_1(w), \alpha^{-\Pi_2(w)})$, and so $w \sim v_{xz}$. Therefore, by

Condition (H2) and Remark 3.2, we have that $\pi(v_{xz}) \approx \pi(w)$ with comparison constant K_0. Let γ be the path $w_0 = v_0 \sim v_1 \sim \cdots \sim v_{xy}$. It follows that

$$\frac{d_\rho(\Phi(x),\Phi(y))}{d_\rho(\Phi(x),\Phi(z))} \approx \frac{\pi(v_{xy})}{\pi(w)} = \prod_{j=\Pi_2(v_{xz})}^{\Pi_2(v_{xy})} \rho(w_j) \le \eta_+^{\Pi_2(v_{xy})-\Pi_2(v_{xz})} = \alpha^{-\tau_+(\Pi_2(v_{xy})-\Pi_2(v_{xz}))}.$$

Now by (4.4), we see that

$$\frac{d_\rho(\Phi(x),\Phi(y))}{d_\rho(\Phi(x),\Phi(z))} \lesssim \left(\frac{d_Z(x,y)}{d_Z(x,z)}\right)^{\tau_+}.$$

Now suppose that $\Pi_2(v_{xy}) \le \Pi_2(v_{xz})$. Then, reversing the roles of y and z in the above argument gives us (with $\beta = (w_0 = u_0 \sim u_1 \sim \cdots)$ and u the vertex in β such that $\Pi_2(u) = \Pi_2(v_{xy})$),

$$\frac{d_\rho(\Phi(x),\Phi(z))}{d_\rho(\Phi(x),\Phi(y))} \approx \frac{\pi(v_{xz})}{\pi(u)} = \prod_{j=\Pi_2(v_{xy})}^{\Pi_2(v_{xz})} \rho(u_j) \gtrsim \eta_-^{\Pi_2(v_{xz})-\Pi_2(v_{xy})} = \alpha^{-\tau_-(\Pi_2(v_{xz})-\Pi_2(v_{xy}))}.$$

Invoking (4.4) again, we see that

$$\frac{d_\rho(\Phi(x),\Phi(z))}{d_\rho(\Phi(x),\Phi(y))} \gtrsim \left(\frac{d_Z(x,z)}{d_Z(x,y)}\right)^{\tau_-},$$

from whence we obtain

$$\frac{d_\rho(\Phi(x),\Phi(y))}{d_\rho(\Phi(x),\Phi(z))} \lesssim \left(\frac{d_Z(x,y)}{d_Z(x,z)}\right)^{\tau_-}.$$

Thus Φ is η-quasisymmetric with

$$\eta(t) \approx \max\{t^{\tau_+}, t^{\tau_-}\}.$$

□

Up to now we have made use of Conditions (H1), (H2), and (H3). In the next section we introduce and use Condition (H4).

6. Ahlfors regularity

For each non-negative integer m and $x \in A_m$, and for each positive integer n with $n > m$, we set $D_n((x,m))$ to be the collection of all vertices $(y,n) \in V$ such that there is a vertically descending path from the vertex (x,m) to (y,n). Observe that such a path is a sub-path of a vertically descending path from the root vertex w_0 to (y,n).

Definition 6.1. We say that $\rho : V \to \mathbb{R}$ satisfies Condition (H4) if there exist $p > 0$ and $K_2 > 0$ such that whenever $x \in A_m$ and $n > m$, we have

$$K_2^{-1}\,\pi((x,m))^p \le \sum_{v \in D_n(x,m)} \pi(v)^p \le K_2\,\pi((x,m))^p.$$

For the rest of this section we consider the Condition (H4) in addition to the three conditions given in Definition 3.1.

LEMMA 6.2. *Suppose that ρ satisfies Conditions (H1) through (H4). Then (Z, d_Z) is uniformly perfect, and for each $x \in A_n$, $\operatorname{diam}_{d_\rho} \Phi((B_{d_Z}(x, \alpha^{-n})) \approx \pi((x, n))$.*

PROOF. To prove uniform perfectness, it suffices to show that there is some positive integer $N > 1$ such that for each positive integer n and each $x \in A_n$, the annulus $B_{d_Z}(x, \alpha^{-n}) \setminus B_{d_Z}(x, \alpha^{-n-N})$ is non-empty. To this end, suppose that $N > 2$ is an integer and $x \in A_n$ such that the annulus $B_{d_Z}(x, \alpha^{-n}) \setminus B_{d_Z}(x, \alpha^{-n-N})$ is empty. Then from Condition (H4) we see that

$$\pi((x,n))^p \leq K_2 \pi((x, n+N))^p = K_2 \pi((x,n))^p \prod_{j=0}^{N-1} \rho((x, n+j))^p \leq K_2 \eta_+^{Np} \pi((x,n))^p.$$

It follows that $K_2 \eta_+^{Np} \geq 1$. Hence if

$$N > \frac{1}{p} \frac{\log(K_2)}{\log(1/\eta_+)},$$

then the annulus $B_{d_Z}(x, \alpha^{-n}) \setminus B_{d_Z}(x, \alpha^{-n-N})$ must be non-empty. It follows that (Z, d_Z) is uniformly perfect, with uniform perfectness constant $C_U = \alpha^N$ where N satisfies the above inequality.

The second claim now follows from the restriction on N given above as well. Indeed, we can find $z \in B_{d_Z}(x, \alpha^{-n})$ such that $d_Z(x, z) \geq \alpha^{-n-N}$. With v_{xz} as in Condition (H3), we see from (4.4) that $\alpha^{-\Pi_2(v_{xz})} \approx \alpha^{-n_{xz}} \approx \alpha^{-n}$ and that the graph-distance between the vertices v_{xy} and (x, n) is bounded by a constant that depends only on the constants η_+, η_-, K_0, and K_1. By (H2) and (H1) we have that $\pi((v, n)) \approx \pi(v_{xz})$. Now by the last claim of Theorem 4.2 we have that

$$\pi(v_{xz}) \approx d_\rho(\Phi(x), \Phi(z)) \leq \operatorname{diam}_{d_\rho}(\Phi(B_{d_Z}(x, \alpha^{-n})).$$

Now if we choose $w \in B_{d_Z}(x, \alpha^{-n})$ such that $\frac{1}{2} \operatorname{diam}_{d_\rho}(\Phi(B_{d_Z}(x, \alpha^{-n})) \leq d_\rho(\Phi(x), \Phi(w))$, then as there is a vertically descending path from the root w_0, through (x, n), ending at $\Phi(w)$, it follows that v_{xw} is a descendant of (x, n); it follows that $\pi(v_{xw}) \leq \pi((x, n))$, and so by Theorem 4.2 again,

$$\frac{1}{2} \operatorname{diam}_{d_\rho}(\Phi(B_{d_Z}(x, \alpha^{-n})) \leq d_\rho(\Phi(x), \Phi(w)) \approx \pi(v_{xw}) \leq \pi((x,n)) \approx \pi(v_{xz}).$$

The combination of the above two inequalities yields the final claim of this lemma. □

From now on we will denote $\Phi(x)$ by x as well whenever $x \in Z$; thus we will also conflate $\Phi(B_{d_Z}(x, \alpha^{-n})$ with $B_{d_Z}(x, \alpha^{-n})$, as this will not lead to confusion.

REMARK 6.3. We fix $0 < l \leq L < \infty$. A set $E \subset Z$ is said to be an (L, l)-quasi-ball in (Z, θ) with center $x \in E$ if there is some $\rho > 0$ such that $B_\theta(x, l\rho) \subset E \subset B_\theta(x, L\rho)$. Now, for $x \in A_n$, we set

$$r := \sup_{y \in B_{d_Z}(x, \alpha^{-n})} \theta(x, y), \qquad \tau := \inf_{y \in X \setminus B_{d_Z}(x, \alpha^{-n})} \theta(x, y).$$

By the quasisymmetry of (Z,θ) with respect to (Z,d_Z), we see that $r \le \eta(1)\,\tau$. If $\tau \ge r$, then we have that $B_{d_Z}(x,\alpha^{-n}) = B_\theta(x,r)$, and so we can take $l = L = 1$ and $\rho = r$. If $\tau < r$, then we have that $\tau < r \le \eta(1)\tau$, and so $B_\theta(x,\tau) \subset B_{d_Z}(x,\alpha^{-n}) \subset B_\theta(x,r) \subset B_\theta(x,\eta(1)\tau)$, and we can then take $\rho = \tau$ and $l = 1$, $L = \max\{1,\eta(1)\}$. Thus for each $x \in A_n$ we have that $B_{d_Z}(x,\alpha^{-n})$ is $(1,\max\{1,\eta(1)\})$-quasi-ball in (Z,θ) with center x.

THEOREM 6.4. *Suppose that ρ satisfies Conditions (H1), (H2), (H3), and (H4). Then $(\partial_\rho X, d_\rho)$ is Ahlfors p-regular.*

PROOF. To prove the claim we construct an Ahlfors p-regular measure on Z as a weak limit of a sequence of measures on Z. We fix a positive integer n and set the measure μ_n on Z as follows: for each Borel set $E \subset Z$, we set

$$\mu_n(E) := \sum_{x \in A_n \cap E} \pi((x,n))^p.$$

Note that, thanks to Condition (H4), there is a relationship between μ_n and μ_m for $n > m$ given by

$$K_2^{-1}\mu_n(Z) \le \mu_m(Z) \le C\,K_2\mu_n(Z).$$

Here the constant C is the bounded overlap constant mentioned in Definition 2.1 (9). It follows that for each positive integer n,

$$0 < (CK_2)^{-1}\mu_1(Z) \le \mu_n(Z) \le K_2\mu_1(Z) < \infty,$$

and hence the sequence of measures $(\mu_n)_n$ is tight on Z, and so there is a subsequence $(\mu_{n_k})_k$ and a Radon measure μ on Z such that μ_{n_k} converges weakly to μ; moreover, $K_2^{-1}\mu_1(Z) \le \mu(Z) \le K_2\mu_1(Z)$. Thus μ is non-trivial on Z.

Note also that for each $x \in Z = \partial_\rho X$,

$$d_\rho(w_0,x) \le \sup_\gamma \int_\gamma \pi(\gamma(t))\,dt \le \sum_{n=0}^{\infty} \eta_+^n = \frac{1}{1-\eta_+} < \infty.$$

We now wish to show that μ is Ahlfors p-regular on Z with respect to the metric d_ρ. Since Φ is a quasisymmetric map from (Z,d_Z) to (Z,d_ρ), it follows that balls in the metric d_Z are quasi-balls in the metric d_ρ, see Remark 6.3 above. Hence it suffices to verify the regularity condition for d_Z-balls.

Note first that if n is a positive integer and $z \in A_n$, then $\mu_n(B_{d_Z}(z,\alpha^{-n})) = \pi((z,n))^p$. We fix $x \in Z$ and $0 < r < \frac{1}{2}\operatorname{diam}_{d_Z}(Z)$, and choose the unique positive integer n_r such that $\alpha^{-n_r-1} < r \le \alpha^{-n_r}$. Then, for integers $m > n_r + 3$, by the definition of μ_m we have

$$\mu_m(B_{d_Z}(x,r)) = \sum_{z \in A_m \cap B_{d_Z}(x,r)} \pi((z,m))^p.$$

With $z_0 \in A_{n_r+2}$ such that $d(x,z_0) < \alpha^{-n_r-2}$, note that necessarily $z_0 \in B_{d_Z}(x,r)$. Moreover, for $m \ge n_r+3$, if $z \in A_m$ such that there is a vertically descending path from (z_0,n_r+3) to (z,m), then $d(z,z_0) < \alpha^{-n_r-2}$, and so $d(x,z) < \alpha^{-n_r-2} + \alpha^{-n_r-2} < \alpha^{-n_r-1} \le r$, that

is, $z \in A_m \cap B_{d_Z}(x,r)$. Hence $\Pi_1(D_m(z_0, n_r+3)) \subset B_{d_Z}(x,r)$, whence we obtain from Condition (H4) that

$$\mu_m(B_{d_Z}(x,r)) \geq \sum_{v \in D_m(z_0, n_r+3)} \pi(v)^p \geq K_2^{-1}\, \pi(z_0, n_r+3)^p.$$

Next, let us consider two points $z, w \in A_m \cap B_{d_Z}(x,r)$. Then $d(z,w) < \alpha^{1-n_r}$. Let $v_0 \sim v_1 \sim \cdots \sim v_m = (z,m)$ and $v_0 \sim v_1' \sim \cdots \sim v_m' = (w,m)$ be two vertically descending paths from the root vertex w_0 to the vertices (z,m) and (w,m) respectively. Then as $m \geq n_r + 3$, we can find $(z', n_r - 1)$ in the first path and $(w', n_r - 1)$ in the second path. We will show that $(z', n_r - 1) \sim (w', n_r - 1)$. Indeed,

$$d(z', w) \leq d(z', z) + d(z, w) < \frac{\alpha + 1}{\alpha - 1}\alpha^{-n_r} + \alpha^{1-n_r} \leq \frac{2\alpha}{\alpha - 1}\alpha^{1-n_r},$$

and hence as $\tau \geq 2\alpha/(\alpha - 1)$, we conclude that w is in both $B_{d_Z}(w', \alpha^{1-n_r})$ and $B_{d_Z}(z', \tau\alpha^{1-n_r})$; that is, $(z', n_r - 1) \sim (w', n_r - 1)$. It follows that

$$A_m \cap B_{d_Z}(x,r) \subset \bigcup_{(a, n_r-1) \sim (z', n_r-1)} D_m(a, n_r - 1).$$

Hence

$$\mu_m(B_{d_Z}(x,r)) \leq \sum_{(a, n_r-1) \sim (z', n_r-1)} \pi((a, n_r - 1))^p.$$

Given that Z is doubling, the number of vertices $(a, n_r - 1)$ that are neighbors of $(z', n_r - 1)$ is at most the doubling constant, and so by Condition (H1) we have that

$$\mu_m(B_{d_Z}(x,r)) \leq C\, \pi((z', n_r - 1))^p.$$

As the graph-distance between $(z', n_r - 1)$ and $(z_0, n_r + 3)$ is uniformly bounded, (by the doubling property of Z again) it follows that

$$\mu_m(B_{d_Z}(x,r)) \lesssim \pi((z_0, n_r + 3))^p.$$

From the above arguments, we obtain that for each positive integer $m \geq n_r + 3$,

$$\mu_m(B_{d_Z}(x,r)) \approx \pi((z_0, n_r + 3))^p \approx \pi((z', n_r - 1))^p.$$

Now from Lemma 6.2 together with Condition (H1),

$$\operatorname{diam}_{d_\rho}(B_{d_Z}(x, \alpha^{-n_r})) \approx \pi((z_0, n_r + 3)),$$

which completes the proof. □

7. Obtaining ρ from quasisymmetric change in metric

The sections prior to this, together, completes the proof of part **I.** of Theorem 1.1. We now consider the converse directional claim given in part **II.** of Theorem 1.1. To do so, we consider a metric θ on Z such that (Z, d_Z) and (Z, θ) are quasisymmetric equivalent with quasisymmetry parametric function $\eta : [0, \infty) \to [0, \infty)$. In the prior sections we did not have to assume that (Z, d_Z) is uniformly perfect (with constant $C_U \geq 2$), and indeed, the uniform perfectness of (Z, d_Z) followed from Condition (H4) needed only in

establishing that d_ρ is Ahlfors regular. In this section, however, we seem to need uniform perfectness of (Z, d_Z) a priori, and then construct a choice of function ρ corresponding to the quasisymmetry. We will give this construction in Definition 7.5 below.

The standing assumptions in this section are that (Z, d_Z) is compact, doubling, and uniformly perfect, and that (Z, θ) is quasisymmetric to (Z, d_Z). We will also use the construction of hyperbolic filling from Section 2, with $\alpha \geq 2$ and $\tau \geq \max\{\alpha^2 + 1, C_U^3(C_U^2 - 4)^{-1}\}$, where C_U is the uniform perfectness constant of (Z, d_Z). We will soon also require $\alpha > C_U^3$, but this requirement is not needed in the first lemma below. We reserve the notation $B(x, r)$ to denote balls in Z, centered at $z \in Z$, of radius r with respect to the metric d_Z. Balls with respect to the metric θ will be denoted $B_\theta(x, r)$.

LEMMA 7.1. *Let $x, y \in A_n$ such that $(x, n) \sim (y, n)$. Then*

$$\operatorname{diam}_\theta(B(x, \alpha^{-n})) \approx \operatorname{diam}_\theta(B(y, \alpha^{-n}))$$

with the constant of comparison given by $2\eta(1)\eta(2\tau C_U)$.

Moreover, if $x, z \in Z$ and n_{xz} is the positive integer with $\alpha^{-n_{xz}} < d_Z(x, z) \leq \alpha^{1-n_{xz}}$, then

$$\operatorname{diam}_\theta(B(x, \alpha^{-n_{xz}})) \approx \theta(x, z),$$

with comparison constant $\max\{2\eta(1), \eta(C_U\alpha)\}$.

Finally, if $x \in A_n$, then there exists $y \in B(x, \alpha^{-n})$ such that

$$\operatorname{diam}_\theta(B(x, \alpha^{-n})) \approx \theta(x, y)$$

with comparison constant $2\eta(C_U)$.

PROOF. The claim in the first part of the lemma follows immediately if $x = y$, so we assume without loss of generality that $x \neq y$. Then we have that $\alpha^{-n} \leq d_Z(x, y) \leq 2\tau\alpha^{-n}$.

Let $x_1 \in B(x, \alpha^{-n})$ and $\widehat{y_1} \in B(y, \alpha^{-n})$ such that

$$\operatorname{diam}_\theta(B(x, \alpha^{-n})) \leq 2\,\theta(x, x_1), \qquad d_Z(\widehat{y_1}, y) \geq \alpha^{-n}/C_U.$$

Then by the quasisymmetry of the metric θ with respect to d_Z, we see that

$$\frac{\theta(x_1, x)}{\theta(y, x)} \leq \eta\left(\frac{d_Z(x_1, x)}{d_Z(y, x)}\right) \leq \eta\left(\frac{\alpha^{-n}}{\alpha^{-n}}\right) = \eta(1).$$

Therefore, by the choice of x_1, we have that

$$\operatorname{diam}_\theta(B(x, \alpha^{-n})) \leq 2\eta(1)\,\theta(y, x). \tag{7.2}$$

Again by the quasisymmetry,

$$\frac{\theta(y, x)}{\theta(\widehat{y_1}, y)} \leq \eta\left(\frac{2\tau\alpha^{-n}}{\alpha^{-n}/C_U}\right) = \eta(2\tau C_U),$$

and so

$$\theta(y, x) \leq \eta(2\tau C_U)\,\theta(\widehat{y_1}, y) \leq \eta(2\tau C_U)\,\operatorname{diam}_\theta(B(y, \alpha^{-n})). \tag{7.3}$$

Combining (7.2) with (7.3) gives

$$\operatorname{diam}_\theta(B(x, \alpha^{-n})) \leq 2\eta(1)\eta(2\tau C_U)\,\operatorname{diam}_\theta(B(y, \alpha^{-n})).$$

The first part of the lemma is now proved by reversing the roles of x and y in the above argument.

To prove the second part of the lemma, note that we can find $x_1, \widehat{x_1} \in B(x, \alpha^{-n_{xz}})$ such that

$$d_Z(x, \widehat{x_1}) \geq \alpha^{-n_{xz}}/C_U, \qquad \operatorname{diam}_\theta(B(x, \alpha^{-n_{xz}})) \leq 2\theta(x_1, x).$$

Then

$$\frac{\theta(x, x_1)}{\theta(x, z)} \leq \eta\left(\frac{\alpha^{-n_{xz}}}{\alpha^{-n_{xz}}}\right) = \eta(1),$$

and so $\operatorname{diam}_\theta(B(x, \alpha^{-n_{xz}})) \leq 2\eta(1)\,\theta(x, z)$. On the other hand,

$$\frac{\theta(x, z)}{\theta(x, \widehat{x_1})} \leq \eta\left(\frac{\alpha^{1-n_{xz}}}{\alpha^{-n_{xz}}/C_U}\right) = \eta(C_U\alpha).$$

It follows that

$$\frac{\theta(x, z)}{\eta(C_U\alpha)} \leq \theta(x, \widehat{x_1}) \leq \operatorname{diam}_\theta(B(x, \alpha^{-n_{xz}})).$$

The second claim of the lemma follows.

To prove the final claim of the lemma, by the use of uniform perfectness we can find a point $y \in B(x, \alpha^{-n}) \setminus B(x, \alpha^{-n}/C_U)$. Let $\widehat{x} \in B(x, \alpha^{-n})$ such that $\theta(\widehat{x}, x) \geq \operatorname{diam}_\theta(B(x, \alpha^{-n}))/2$. Then

$$\frac{\theta(\widehat{x}, x)}{\theta(y, x)} \leq \eta\left(\frac{\alpha^{-n}}{\alpha^{-n}/C_U}\right) = \eta(C_U).$$

It follows that

$$\frac{1}{2}\operatorname{diam}_\theta(B(x, \alpha^{-n})) \leq \eta(C_U)\,\theta(y, x) \leq \eta(C_U)\,\operatorname{diam}_\theta(B(x, \alpha^{-n})).$$

□

Throughout the rest of this section we also assume that

$$C_U > 2, \qquad \alpha > C_U^3, \qquad \tau \geq \max\left\{\alpha^2 + 1, \frac{2C_U^3}{C_U^2 - 4}\right\}. \tag{7.4}$$

The first of the above conditions can always be assumed without loss of generality, and the remaining two conditions merely give us control over the hyperbolic filling parameters α and τ in terms of C_U. Note that these assumptions are independent of the quasisymmetric metric θ. We are now ready to construct the function $\rho : V \to [0, \infty)$ as follows.

As in **[11]** and in the converse statement given in Theorem 1.1, *we assume that (Z, θ) is also an Ahlfors p-regular space for some $p > 0$, and set μ to be the p-dimensional Hausdorff measure on Z induced by the metric θ.*

Definition 7.5. We fix a maximal spanning tree $T \subset X$ of the graph X such that w_0 is the root of the spanning tree made up solely of vertical edges, and so that if $[v, w]$ is an

edge in T with w a child of v, then $d_Z(\Pi_1(v), \Pi_1(w)) < \alpha^{-\Pi_2(v)}$. We set $\rho(w_0) = \mu(Z)^{1/p}$, and for each positive integer n and $x \in A_n$ we set

$$\rho((x,n)) = \left(\frac{\mu(B(x, \tau\alpha^{-n}))}{\mu(B(z, \tau\alpha^{1-n}))} \right)^{1/p},$$

where $z \in A_{n-1}$ such that $(z, n-1)$ is the parent of (x,n) in T.

While the definition of ρ uses the measure μ associated with the metric θ, the balls $B(x, \tau\alpha^{-n})$ are with respect to the metric d_Z. However, note that as θ is quasisymmetric with respect to d_Z, balls with respect to the metric d_Z are quasi-balls with respect to the metric θ, as seen in Remark 6.3. The reason for using the balls with respect to d_Z is that we do not know what the θ-radius of corresponding balls should be.

In [**18**, (2.11)] and [**18**, Proof of Theorem 3.14(b)], (see [**18**, (2.17)] for yet another variant) the authors propose an alternative construction of ρ in the absence of Ahlfors regularity:

$$\rho((x,n)) := \frac{\operatorname{diam}_\theta(B(x, \alpha^{-n}))}{\operatorname{diam}_\theta(B(z, \alpha^{1-n}))},$$

where $z \in A_{n-1}$ such that (x,n) is a child of $(z, n-1)$ in T. However, their construction requires the choice of parameter α involved in the hyperbolic filling to be dependent on the quasisymmetry scaling function η in relation to the uniform perfectness constant C_U, and so gives a slightly different result than that of [**11**]; see the comment at the top of page 33 of [**18**], where the uniform perfectness constant C_U is denoted K_P, τ is denoted by λ, and α is denoted by a.

LEMMA 7.6. *The function ρ from Definition 7.5 satisfies Conditiions (H1) of Definition 3.1.*

PROOF. Let $(x,n), (z, n-1) \in V$ such that $(x,n) \sim (z, n-1)$ in the tree T. Now, if $w \in B(x, \tau\alpha^{-n})$, then by triangle inequality and (7.4),

$$d_Z(w,z) < \tau\alpha^{-n} + \alpha^{-n} + \alpha^{1-n} = \left(\frac{\tau+1}{\alpha} + 1 \right) \alpha^{1-n} < \frac{\tau}{2}\, \alpha^{1-n}.$$

That is, $B(x, \tau\alpha^{-n}) \subset B(z, [1 + \frac{1+\tau}{\alpha}]\alpha^{1-n}) \subset B(z, \tau\alpha^{1-n})$. By the uniform perfectness of (Z, d_Z) we can find a point $v_0 \in B(z, \frac{\tau}{2}\alpha^{1-n}) \setminus B(z, \frac{\tau}{2C_U}\alpha^{1-n})$. We now show that $B(v_0, \frac{\tau}{\alpha^3}\alpha^{1-n}) \subset B(z, \tau\alpha^{1-n}) \setminus B(x, \tau\alpha^{-n})$. Indeed, if $y \in B(v_0, \frac{\tau}{\alpha^3}\alpha^{1-n})$, then by triangle inequality we have

$$\begin{aligned} d_Z(y,z) &< \frac{\tau}{\alpha^3}\alpha^{1-n} + \frac{\tau}{2}\alpha^{1-n} < \tau\alpha^{1-n}, \\ d_Z(y,z) &\geq d_Z(z,v_0) - d_Z(v_0,y) \geq \left[\frac{\tau}{2C_U} - \frac{\tau}{\alpha^3} \right] \alpha^{1-n} \geq \left[1 + \frac{\tau}{\alpha} \right] \alpha^{1-n}, \end{aligned}$$

where we have used (7.4) again and the fact that $\alpha^3 > \alpha$. Hence

$$\frac{\mu(B(x, \tau\alpha^{-n}))}{\mu(B(z, \tau\alpha^{1-n}))} = 1 - \frac{\mu(B(z, \tau\alpha^{1-n}) \setminus B(x, \tau\alpha^{-n}))}{\mu(B(z, \tau\alpha^{1-n}))} \leq 1 - \frac{\mu(B(v_0, \frac{\tau}{\alpha^3}\alpha^{1-n}))}{\mu(B(z, \tau\alpha^{1-n}))}.$$

As (Z, d_Z) is quasisymmetric to (Z, θ), we know that balls in the metric d_Z are quasi-balls in the metric θ, see Remark 6.3 above. Hence, as in the proof of Lemma 7.1, by the Ahlfors regularity of μ with respect to theta, we have that

$$\frac{\mu(B(v_0, \frac{\tau}{\alpha^3}\alpha^{1-n}))}{\mu(B(z, \tau\alpha^{1-n}))} \geq c_0$$

for some constant $0 < c_0 < 1$ that depends on the quasisymmetry function η and the Ahlfors regularity constant of μ, but does not depend on v_0, x, z, n. Hence we have that

$$\rho((x, n)) \leq (1 - c_0)^{1/p}.$$

By the Ahlfors regularity of μ, we also have that $\rho(x, n) \geq c_1 > 0$ with c_1 depending only on the quasisymmetry parameter η and the Ahlfors regularity constant of μ. Thus we can take $\eta_- = c_1$ and $\eta_+ = (1 - c_0)^{1/p}$ to complete the proof. □

LEMMA 7.7. *The function ρ satisfies Conditions (H3) and (H2).*

PROOF. From the definition of ρ, for each $v \in V$ we have that

$$\pi(v) = \mu(B(\Pi_1(v), \alpha^{-\Pi_2(v)}))^{1/p}.$$

Since d_Z-balls are quasi-balls in the metric θ (see Remark 6.3), we have by the Ahlfors p-regularity of μ with respect to θ that

$$\pi(v) \approx \operatorname{diam}_\theta(B(\Pi_1(v), \alpha^{-\Pi_2(v)})).$$

Let $u, w \in V$ be two distinct vertices, and let $v_{uw} \in V$ be as described in Condition (H3), and let γ be any curve in X with end points u, w. Denoting the vertices in γ by $u = u_0 \sim u_1 \sim \cdots \sim u_k = w$, we have that

$$\int_\gamma \pi(\gamma(t))\, dt = \sum_{j=0}^{k-1} \pi(u_j) \approx \sum_{j=0}^{k-1} \operatorname{diam}_\theta \left(B(\Pi_1(u_j), \tau\alpha^{-\Pi_2(u_j)}) \right).$$

For $j = 0, \cdots, k-1$ we have that $B(\Pi_1(u_j), \tau\alpha^{-\Pi_2(u_j)})$ intersects $B(\Pi_1(u_{j+1}), \tau\alpha^{-\Pi_2(u_{j+1})})$, see the construction given in Definition 2.1. Hence by the triangle inequality and the second part of Lemma 7.1, we have that

$$\int_\gamma \pi(\gamma(t))\, dt \gtrsim \theta(\Pi_1(u), \Pi_1(w)) \approx \operatorname{diam}_\theta(B(\Pi_1(v_{uw}), \alpha^{-\Pi_2(v_{uw})})),$$

from which the first claim of the lemma follows.

Condition (H2) now follows from an application of the first part of Lemma 7.1. □

LEMMA 7.8. *The function ρ satisfies Condition (H4) given in Definition 6.1.*

PROOF. Let m be a positive integer and $x \in A$. We fix a positive integer n such that $n > m$. Note that for each $v \in D_n(x, m)$ we have that there is a vertically descending path $(x, m) = v_0 \sim v_1 \sim \cdots \sim v_{m-n} = v$, and so we have that

$$d_Z(x, \Pi_1(v)) \leq \sum_{j=1}^{m-n} d_Z(\Pi_1(v_{j-1}), \Pi_1(v_j)) \leq \sum_{j=1}^{m-n} \alpha^{-m-j+1} + \alpha^{-m-j} \leq \frac{2}{\alpha - 1}\alpha^{-m},$$

and so we have that $\Pi_1(D_n(x,m)) \subset B(x, A\alpha^{-m})$, where $A = 2/(\alpha-1)$. It follows from the pairwise disjointness property of the balls $B(\Pi_1(v), \alpha^{-n})$ that

$$\begin{aligned}\sum_{v\in D_n(x,m)} \pi(v)^p = \sum_{v\in D_n(x,m)} \mu(B(\Pi_1(v),\alpha^{-n})) &\le \mu(B(x,(A+1)\alpha^{-m})) \\ &\le C\,\mu(B(x,\alpha^{-m})) \\ &= C\,\pi((x,m))^p.\end{aligned}$$

Here we have used the fact that μ is Ahlfors regular and also from the construction of ρ, for each $z \in A_m$ we have $\mu(B(z,\alpha^{-m})) = \pi((z,m))^p$.

On the other hand, for each $y \in B(x,\alpha^{-m}/2)$ there exists $z \in A_n$ such that $d_Z(z,y) \le \alpha^{-n}$. It follows from the choice of $\alpha \ge 2$ that

$$d_Z(x,z) \le d_Z(x,y) + d_Z(y,z) < \frac{\alpha^{-m}}{2} + \alpha^{-n} \le \frac{\alpha^{-m}}{2} + \frac{\alpha^{-m}}{\alpha} \le \alpha^{-m}.$$

Therefore $(z,n) \in D_n(x,m)$. It follows that $B(x,\alpha^{-m}/2) \subset \bigcup_{v\in D_n(x,m)} \overline{B}(\Pi_1(v),\alpha^{-n})$, and so we have

$$\begin{aligned}\pi((x,m))^p = \mu(B(x,\alpha^{-m})) \le C\mu(B(x,\alpha^{-m}/2)) &\le C \sum_{v\in D_n(x,m)} \mu(B(\Pi_1(v),\alpha^{-n})) \\ &= C \sum_{v\in D_n(x,m)} \pi(v)^p,\end{aligned}$$

completing the proof. □

8. An alternative formulation of Conditions (H1)—(H4), and a query

A different perspective of the construction in [**11**] is to begin with a density function $\omega : V \to [0,1]$ on the vertex set V such that the following four conditions are satisfied:

(H1-a) There exist η_-, η_+ with $0 < \eta_- \le \eta_+ < 1$ such that for each $v, w \in V$ with $v \sim w$, we have

$$\eta_- \le \frac{\omega(v)}{\omega(w)} \le \eta_+.$$

(H2-a) There is a constant $K_0 > 0$ such that whenever $v, w \in V$ with $v \sim w$, we have

$$\omega(v) \le K_0\,\omega(w).$$

(H3-a) We extend ω to edges $v \sim w$ in X linearly by setting $\omega(tv + (1-t)w) = t\omega(v) + (1-t)\omega(w)$, where $tv + (1-t)w$ is the point on the edge $v \sim w$ that is a distance $t \in [0,1]$ away from v. There is a constant $K_1 > 0$ such that whenever γ is a curve in X connecting $x, y \in X$, then

$$\int_\gamma \omega(\gamma(t))\,ds \ge K_1\,\omega(v_{xy}).$$

(H4-a) There exist $p > 0$ and $K_2 > 0$ such that whenever $x \in A_m$ and $n > m$, we have

$$\frac{1}{K_2}\,\omega((x,m))^p \leq \sum_{v \in D_n(x,m)} \omega(v)^p \leq K_2\,\omega((x,m))^p.$$

It is not difficult to see that setting $\rho(v) = \frac{\omega(v)}{\omega(w)}$ where w is any ancestor of v with $w \sim v$, we have the original four conditions with $\pi \sim \omega$. Indeed, Condition (H1-a) corresponds to Condition (H1) of [**11**], Condition (H2-a) corresponds to Condition (H2) of [**11**], Condition (H3-a) corresponds to Condition (H3) of [**11**], and Condition (H4-a) corresponds to Condition (H4) of [**11**]. This perspective allows us to see that d_ρ is actually a conformal change in the path-metric on the graph X. In this note we chose to use the original formulation of the conditions as found in [**11**], see Definition 3.1 and Definition 6.1, as the purpose of this note is to provide an analysis of [**11**, Theorem 1.1]. However, this perspective helps bridge the gap between the construction proposed in [**11**] and the conformal changes in metrics associated with a Harnack density $\omega : X \to (0,\infty)$. A density ω is a Harnack density if there are constants $C, A \geq 1$ such thatfor $x, y \in X$ with $d(x,y) < A$ we have

$$\frac{1}{C} \leq \frac{\omega(x)}{\omega(y)} \leq C.$$

Given such a density ω, we can equip the (not complete, but locally complete) metric space (X,d) with the new metric (X,d_ω) given by

$$d_\omega(x,y) = \inf_\gamma \int_\gamma \omega\, ds,$$

where $x, y \in X$ and the infimum is over all rectifiable curves in X with end points x, y. The papers [**4, 5, 10, 12, 15, 16**] are some of the many papers in current literature using such transformations. Any density ω that satisfies the conditions listed at the beginning of this section is automatically a Harnack density, thanks to (H2-a).

Concluding remarks: The results of [**11**] link the quasisymmetric geometry of Z, the boundary of the hyperbolic filling, to the metrics on this filling. We note here that in potential theory as well there is a connection between nonlocal energy minimization problems on the boundary of compact doublings spaces and local energy minimization problems in the hyperbolic filling [**10**]; and this connection is given through the perspective of Adams inequality, on the compactification of the hyperbolic filling, via a measure supported on the boundary Z. In [**3**] it was shown that if Z is equipped with a doubling measure, then its hyperbolic filling, modified according to the density $\omega_\alpha(x) = \alpha^{-d_X(x,w_0)}$, yields a uniform domain which can be equipped with a lift μ_ω of the measure on Z so that the corresponding metric measure space $X_\alpha := (X, d_{\omega_\alpha}, \mu_{\omega_\alpha})$ is bounded, doubling and supports a 1-Poincaré inequality, as does its metric completion (with the zero-extension of the measure μ_{ω_α} to $\partial_{\omega_\alpha} X$). Moreover, the trace of the Sobolev classes on X_α are certain Besov classes on Z. This fact was exploited in [**10**] to study Neumann boundary value problem on X_α and link it to certain nonlocal fractional operators on Z, and one of the key motivating ideas behind that analysis was an Adams-type inequality [**1, 2, 24**], with the singular measure given by the doubling measure on $Z = \partial_{\omega_\alpha} X$. Such an inequality was possible because the measure

on Z has a co-dimensional relationship with the measure μ_{ω_α} on X. If X is equipped with the metric d_ω corresponding to a general density function ω satisfying Conditions (H1-a)—(H3-a) and Z is equipped with a doubling measure, then it would be interesting to know whether it is possible to lift the measure on Z to X so that a corresponding co-dimensional relationship between the lift and the measure on Z is valid and supports an Adams inequality, and would indicate a connection between the study done in [**11**] and nonlinear potential theory as in [**22**]. The author recently was able to prove the validity of Poincaré inequality, and as a consequence the Adams inequality (using [**24**]) under certain additional conditions on the parameters α and τ.

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