Journal of Lie Theory
Volume **31** (2021) 833–869

Tempered Homogeneous Spaces III

Yves Benoist and Toshiyuki Kobayashi

Communicated by S. Gindikin

Abstract. Let G be a real semisimple algebraic Lie group and H a real reductive algebraic subgroup. We describe the pairs (G, H) for which the representation of G in $L^2(G/H)$ is tempered. The proof gives the complete list of pairs (G, H) for which $L^2(G/H)$ is not tempered.
When G and H are complex Lie groups, the temperedness condition is characterized by the fact that *the stabilizer in H of a generic point on G/H is virtually abelian.*

Mathematics Subject Classification: Primary 22E46; secondary 43A85, 22F30.

Key Words: Lie groups, homogeneous spaces, tempered representations, unitary representations, matrix coefficients, symmetric spaces.

1. Introduction

1.1. Main results

This paper is the third in a series of papers that also include [2], [3] and [4]. In this series we study homogeneous spaces G/H where G is a real semisimple Lie group and H is an algebraic subgroup. More precisely, we study the natural unitary representation of the group G on the Hilbert space $L^2(G/H)$ of square integrable functions on G/H. In the present paper, as in [2], we focus on the case where H is reductive.

We will give a characterization of those homogeneous spaces G/H for which this representation is tempered. We refer to the introduction of both [2] and [3] for motivations and perspectives on this question. In [2] we discussed the analytic and dynamical part of our method. In this paper we focus on the algebraic part of our method. Our main result is the following.

Theorem 1.1. *Let G be a real semisimple algebraic group and H a real reductive algebraic subgroup. One has the implications:*

(1) *If $L^2(G/H)$ is tempered, then the set of points in G/H with amenable stabilizer in H is dense.*

(2) *If the set of points in G/H with virtually abelian stabilizer in H is dense then $L^2(G/H)$ is tempered.*

The proof will also give a complete list of pairs (G, H) of real reductive algebraic groups for which $L^2(G/H)$ is not tempered.

ISSN 0949–5932 / $2.50

This list reduces to the case where G is simple and H is semisimple thanks to Proposition 2.16 and Lemma 2.15. When G is simple and H is semisimple, this list is given in Tables 1 and 2 for the complex case, and in Theorem 1.4 for the general case.

We recall some terminologies in Theorem 1.1. A unitary representation of a locally compact group G is said to be *tempered* if it is weakly contained in the regular representation in $L^2(G)$, see *e.g.,* [1, Appendix F]. An algebraic real Lie group is said to be *amenable* if it is a compact extension of a solvable group. A group is said to be *virtually abelian* if it contains a finite-index abelian subgroup.

We will see in Section 8.5 that in general neither of the converse of these implications in Theorem 1.1 holds. However, when G and H are complex Lie groups, our implications become an equivalence, since a reductive amenable complex algebraic Lie group is always virtually abelian.

Theorem 1.2. *Let G be a complex semisimple algebraic group and H a complex reductive subgroup. Then the unitary representation of G in $L^2(G/H)$ is tempered if and only if the set of points in G/H with virtually abelian stabilizer in H is dense.*

We recall that a semisimple Lie group is said to be *quasisplit* if its minimal parabolic subgroups are solvable. Then the following result is a particular case of Theorem 1.2.

Example 1.3. Let G_1 be a connected real semisimple algebraic Lie group, K_1 a maximal compact subgroup, and $G_{1,\mathbb{C}}$ and $K_{1,\mathbb{C}}$ their complexifications. Then the regular representation of $G_{1,\mathbb{C}}$ in $L^2(G_{1,\mathbb{C}}/K_{1,\mathbb{C}})$ is tempered if and only if G_1 is quasisplit.

Theorem 1.2 will allow us to give not only a complete description of the pairs (G, H) of *complex* reductive algebraic Lie groups for which $L^2(G/H)$ is tempered, but also a complete description of the pairs (G, H) of *real* reductive algebraic Lie groups for which $L^2(G/H)$ is tempered. The description is as follows.

Thanks to Propositions 8.3 and 8.4, we can assume that G is a real simple Lie group and H is a real semisimple Lie subgroup of G. The following Theorem 1.4 tells us then that Theorem 1.2 is still true for real Lie groups except for one list of classical homogeneous spaces and three exceptional homogeneous spaces. We will use Cartan's notation, see [11, p.518], for real simple Lie algebras.

Theorem 1.4. *Let G be a real simple Lie group and $H \subset G$ a real semisimple Lie subgroup without compact factor, $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras. Then the regular representation of G in $L^2(G/H)$ is tempered if and only if one of the following holds:*

(i) *the set of points in G/H with virtually abelian stabilizer in H is dense;*

(ii) $\mathfrak{g} = \mathfrak{sl}(2m-1, \mathbb{H})$ *and* $\mathfrak{h} = \mathfrak{sl}(m, \mathbb{H}) \oplus \mathfrak{h}_2$ *with* $m \geq 1$ *and* $\mathfrak{h}_2 \subset \mathfrak{sl}(m-1, \mathbb{H})$*;*

(iii) $\mathfrak{g} = \mathfrak{e}_{6(-26)}$ *and* $\mathfrak{h} = \mathfrak{so}(9, 1)$ *or its subalgebra* $\mathfrak{h} = \mathfrak{so}(8, 1)$*;*

(iv) $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ *and* $\mathfrak{h} = \mathfrak{so}(8, 1)$.

In other words, the regular representation of G in $L^2(G/H)$ is tempered if and only if either the representation of the complexified Lie group $G_{\mathbb{C}}$ in $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is tempered or the pair $(\mathfrak{g}, \mathfrak{h})$ is one of the examples (ii), (iii) or (iv).

We point out that, in examples (ii), (iii) and (iv), the Lie algebra $\mathfrak{h}$ is included in a reductive subalgebra $\widetilde{\mathfrak{h}}$ such that $(\mathfrak{g}, \widetilde{\mathfrak{h}})$ is a symmetric pair. We also point out that, in examples (iii) and (iv), the real rank of G is 2 and the real rank of H is 1.

1.2. What has already been proven in [2]

We need some notations. Let G be a real semisimple algebraic group, H a reductive algebraic subgroup, $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras and $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$. Let $\mathfrak{a} \equiv \mathfrak{a}_{\mathfrak{h}}$ be a maximal split abelian real Lie subalgebra in $\mathfrak{h}$. Let V be a finite-dimensional representation of $\mathfrak{h}$: for instance $V = \mathfrak{h}$ or $V = \mathfrak{q}$ via the adjoint representation. Let Y be an element in $\mathfrak{a}$, we denote by $V_{Y,+}$ the sum of eigenspaces in V of Y having positive eigenvalues, and set

$$\rho_V(Y) := \mathrm{Trace}_{V_{Y,+}}(Y).$$

According to the temperedness criterion given in [2, Thm. 4.1], one has the equivalence

$$L^2(G/H) \text{ is tempered} \Longleftrightarrow \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}. \tag{1}$$

Here the inequality $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$ means $\rho_{\mathfrak{h}}(Y) \leq \rho_{\mathfrak{q}}(Y)$, for all Y in $\mathfrak{a}$.
The generic stabilizers of G/H will be related to those of $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$ in Section 8.1. Thus Theorem 1.1 is a consequence of the following Theorem 1.5.

Theorem 1.5. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a pair of real semisimple Lie algebras and $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$. One has the implications:*

(1) $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}} \Longrightarrow$ *the set of points in $\mathfrak{q}$ with amenable stabilizer in H is dense;*

(2) *the set of points in $\mathfrak{q}$ with abelian stabilizer in $\mathfrak{h}$ is dense* $\Longrightarrow \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.

The first implication of Theorem 1.5 is proven in Proposition 2.8 by a short argument based on a *slice theorem* near a generic orbit. The proof of the converse implication is much longer. We will reduce it in Lemma 2.12 to the case where $\mathfrak{g}$ and $\mathfrak{h}$ are complex and semisimple Lie algebras, *i.e.* we will have to prove the following Theorem 1.6 which is a special case of Theorem 1.5.

Theorem 1.6. *Let $\mathfrak{h} \subset \mathfrak{g}$ be two complex semisimple Lie algebras and $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$. One has the equivalence:*
$\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}} \Longleftrightarrow$ *the set of points in $\mathfrak{q}$ with abelian stabilizer in $\mathfrak{h}$ is dense.*

Similarly, we can deduce Theorem 1.4 from the following Theorem 1.7 by the criterion (1).

Theorem 1.7. *Let $\mathfrak{g}$ be a real simple Lie algebra and $\mathfrak{h}$ a semisimple Lie subalgebra such that the adjoint group of $\mathfrak{h}$ has no compact factor. Then $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$ if and only if one of the following holds:*

(i) *the set of points in $\mathfrak{q}$ with abelian stabilizer in $\mathfrak{h}$ is dense;*

(ii) $\mathfrak{g} = \mathfrak{sl}(2m-1, \mathbb{H})$ *and* $\mathfrak{h} = \mathfrak{sl}(m, \mathbb{H}) \oplus \mathfrak{h}_2$ *with* $m \geq 1$ *and* $\mathfrak{h}_2 \subset \mathfrak{sl}(m-1, \mathbb{H})$;

(iii) $\mathfrak{g} = \mathfrak{e}_{6(-26)}$ *and* $\mathfrak{h} = \mathfrak{so}(9,1)$ *or its subalgebra* $\mathfrak{h} = \mathfrak{so}(8,1)$;

(iv) $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ *and* $\mathfrak{h} = \mathfrak{so}(8,1)$.

1.3. Strategy of proof of Theorem 1.6 for complex $\mathfrak{g}$

As we have already mentioned, we give a short proof for the implication $\Rightarrow$ of Theorem 1.6 in Chapter 2, and only the converse implication remains to be proven.

We reduce our analysis in Proposition 2.16 to the case where $\mathfrak{g}$ is simple. When $\mathfrak{g}$ is simple we give a complete classification of the semisimple Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ for which $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$ in Tables 1 and 2 and we compute in each case the generic stabilizer. The proof lasts from Chapter 3 to Chapter 6.

When $\mathfrak{g}$ is simple and classical, i.e. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ or $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$, the list of such pairs $(\mathfrak{g}, \mathfrak{h})$ is given in Table 1 in Chapter 2. In order to check this list, we first deal with the case when the standard representation of $\mathfrak{g}$ (*i.e.* $\mathbb{C}^n$ for $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{so}(n, \mathbb{C})$, and $\mathbb{C}^{2n}$ for $\mathfrak{sp}(n, \mathbb{C})$) remains irreducible as a representation of the subalgebra $\mathfrak{h}$ in Section 3.3, then we deal with the case where it is reducible in Section 3.4.

When $\mathfrak{g}$ is exceptional, the list of such pairs $(\mathfrak{g}, \mathfrak{h})$ is given in Table 2 in Chapter 3. In order to check this list, we use Dynkin's list (Tables 3 and 4) in [10] of maximal semisimple Lie subalgebras $\mathfrak{h}$ in $\mathfrak{g}$ (up to conjugacy). We extract, in Section 4.4, from Dynkin's classification those $\mathfrak{h}$ for which $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$. Then using this first list, we give, in Section 4.5, the list of the semisimple Lie algebras $\mathfrak{h}$ with $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$ which are maximal in one of the Lie algebras of the first list. We prove then that there are no other possibilities for $\mathfrak{h}$ (Lemma 4.9).

All this analysis relies on explicit upper bounds for an invariant p_V associated to any finite-dimensional representation V of $\mathfrak{h}$ (see Equation (6)). The proof of these upper bounds are given in Chapter 5 when $\mathfrak{h}$ is simple and in Chapter 6 when $\mathfrak{h}$ is not simple.

1.4. Strategy of the proof of Theorem 1.7 for real $\mathfrak{g}$

The proof occupies Chapter 7. The implication $(i) \Rightarrow \rho_\mathfrak{h} \leq \rho_\mathfrak{q}$ reduces to the complex case (Theorem 1.6) by Proposition 2.2 and Lemma 2.12, whereas the implication (ii), (iii) or (iv) $\Rightarrow \rho_\mathfrak{h} \leq \rho_\mathfrak{q}$ is straightforward. To see the converse implication, let $\mathfrak{g}$ be a real simple Lie algebra and $\mathfrak{h} \subsetneq \mathfrak{g}$ be a real semisimple Lie subalgebra for which the group $\mathrm{Aut}(\mathfrak{h})$ of automorphisms has no compact factor. We assume that this pair $(\mathfrak{g}, \mathfrak{h})$ does not satisfy (i), or equivalently, $\rho_{\mathfrak{h}_\mathbb{C}} \not\leq \rho_{\mathfrak{q}_\mathbb{C}}$ by Theorem 1.7, and we want to check that, except for cases (ii), (iii) and (iv), one has also $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$.

When the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ is not simple, equivalently when the Lie algebra $\mathfrak{g}$ has a complex structure, we prove in Proposition 7.1 that $\mathfrak{h}$ contains a complex semisimple Lie subalgebra $\mathfrak{h}_0$ such that the pair $(\mathfrak{g}_0, \mathfrak{h}_0)$ already satisfies $\rho_{\mathfrak{h}_0} \not\leq \rho_{\mathfrak{g}/\mathfrak{h}_0}$.

When $\mathfrak{g}_\mathbb{C}$ is simple, we know that the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is in Tables 1 or 2. For each pair in these tables, we know that there exists a *witness vector* X in the Cartan subalgebra $\mathfrak{j}_\mathbb{C}$ of $\mathfrak{h}_\mathbb{C}$, *i.e.* an element X such that $\rho_{\mathfrak{h}_\mathbb{C}}(X) > \rho_{\mathfrak{q}_\mathbb{C}}(X)$ (Definition 7.2). The main point is to find this witness X in a maximal split abelian subalgebra $\mathfrak{a}_\mathfrak{h}$ of $\mathfrak{h}$. Using the Satake diagram of $\mathfrak{h}$ which describes the embedding $\mathfrak{a}_\mathfrak{h} \subset \mathfrak{j}_\mathbb{C} \cap \mathfrak{h}$, we will check, based on Tables 9 and 10, that it is always possible to find a witness vector X in $\mathfrak{a}_\mathfrak{h}$, except in cases (ii), (iii) and (iv).

1.5. Comments on the proof

This text was written more than five years ago. Indeed we believe that there should exist a shorter proof for the implication $\Leftarrow$ in Theorem 1.6 which does not rely on a case-by-case analysis. This is why we delayed its publication trying to find such a

simpler proof. This is also why we present in this text only the main structure and ideas of our long proof leaving the lengthy calculations to the reader.

Relying on Theorem 1.2, we found recently in [4] various temperedness criteria for $L^2(G/H)$ valid for complex algebraic subgroups H of complex semisimple Lie groups G.

Acknowledgments. The authors are grateful to the IHES and to The University of Tokyo for their support. The second author was partially supported by JSPS Kakenhi Grant Number JP18H03669.

2. Notations and preliminary reductions

In this chapter, we prove the first assertion of Theorem 1.5, and explain how the second assertion of Theorem 1.5 can be deduced from Theorem 1.6. Then the proof of Theorem 1.6 is reduced to the case where $\mathfrak{g}$ is simple, for which we shall discuss in Chapters 3–6.

2.1. Reductive generic stabilizer

Let $\mathfrak{h}$ be a real semisimple Lie algebra, and V a finite-dimensional representation of $\mathfrak{h}$ over $\mathbb{R}$. For v in V, we denote by $\mathfrak{h}_v \equiv \mathrm{Stab}_{\mathfrak{h}}(v)$ the stabilizer of v in $\mathfrak{h}$. We recall that $\mathfrak{h}_v$ is said to be *reductive* if the adjoint representation of $\mathfrak{h}_v$ on $\mathfrak{h}$ is semisimple, or equivalently, if the action of $\mathfrak{h}_v$ on V is semisimple.

Definition 2.1. We say that V has RGS in $\mathfrak{h}$ if the set $\{v \in V \mid \mathfrak{h}_v \text{ is reductive}\}$ is dense in V.

Here, "dense" means dense for the locally compact topology. We can equivalently replace in this definitions dense by Zariski dense.

We say that the representation of $\mathfrak{h}$ in V is *self-dual* if it is equivalent to the contragredient representation in the dual space V^*. We say that the representation of $\mathfrak{h}$ in V is *orthogonal* (resp. *symplectic*) if it preserves a nondegenerate symmetric (resp. skew-symmetric) bilinear form on V.

For instance, when $\mathfrak{h}$ is a semisimple Lie subalgebra of a semisimple Lie algebra $\mathfrak{g}$, then the representation of $\mathfrak{h}$ in $\mathfrak{g}/\mathfrak{h}$ is orthogonal. Indeed since the restriction to $\mathfrak{h}$ of the Killing form of $\mathfrak{g}$ is nondegenerate, one can identify $\mathfrak{g}/\mathfrak{h}$ with the orthogonal complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$. On this space the action of $\mathfrak{h}$ preserves the restriction of the Killing form.

Proposition 2.2. *Let $\mathfrak{h}$ be a real semisimple Lie algebra, and V an orthogonal finite-dimensional representation of $\mathfrak{h}$. Then V has RGS in $\mathfrak{h}$.*

This will follow from the general lemmas below.

We denote by $\mathfrak{h}_{\mathbb{C}}$ the complexified Lie algebra of $\mathfrak{h}$, and by $V_{\mathbb{C}}$ the complexified representation of $\mathfrak{h}_{\mathbb{C}}$. We say that the representation of $\mathfrak{h}_{\mathbb{C}}$ in $V_{\mathbb{C}}$ is orthogonal (resp. symplectic) if it preserves a nondegenerate symmetric (resp. skew-symmetric) complex bilinear form.

Lemma 2.3. *Let $\mathfrak{g}$ be a complex semisimple Lie subalgebra of $\mathfrak{so}(n,\mathbb{C})$. Then there exists a real Lie subalgebra $\mathfrak{h}$ of $\mathfrak{so}(n,\mathbb{R})$ such that its complexification $\mathfrak{h}_{\mathbb{C}}$ is $\mathrm{SO}(n,\mathbb{C})$-conjugate to $\mathfrak{g}$.*

Proof. Let G be the connected algebraic subgroup of $\mathrm{SO}(n,\mathbb{C})$ with Lie algebra $\mathfrak{g}$, K a maximal compact subgroup of G and $\mathfrak{k}$ its Lie algebra. Then we can find $g \in \mathrm{SO}(n,\mathbb{C})$ such that $H := gKg^{-1}$ is contained in the maximal compact subgroup $\mathrm{SO}(n,\mathbb{R})$ of $\mathrm{SO}(n,\mathbb{C})$. Since $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$, we are done. ∎

Lemma 2.4. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and W a finite-dimensional representation of $\mathfrak{g}$. Assume that W has RGS in $\mathfrak{g}$. Then there exists a reductive Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that the set of w in W whose stabilizer $\mathfrak{g}_w$ is conjugate to $\mathfrak{m}$ contains a non-empty Zariski open subset of W.*

"Conjugate" means a "conjugate by the adjoint group G of $\mathfrak{g}$". We say that the Lie algebra $\mathfrak{m}$ in Lemma 2.4 is the *generic stabilizer* of V. It is well defined only up to conjugacy.

Proof of Lemma 2.4. For w in W, we denote by $\mathfrak{u}_w$ the unipotent radical of its stabilizer $\mathfrak{g}_w$ in $\mathfrak{g}$, that is $\mathfrak{u}_w$ is the largest nilpotent ideal of $\mathfrak{g}_w$ all of whose elements are nilpotent. Let $d := \min_{w\in W} \dim \mathfrak{g}_w$. We introduce the Zariski open subsets, $W' := \{w \in W \mid \dim \mathfrak{g}_w = d\}$ and $W'' := \{w \in W' \mid \dim \mathfrak{u}_w = 0\}$. By assumption the set W'' is a non-empty Zariski open set. In particular it is connected. Since the set of conjugacy classes of reductive algebraic Lie subalgebras of $\mathfrak{g}$ is countable, the map $w \mapsto \mathfrak{g}_w$ must be constant modulo conjugation on W''. ∎

Lemma 2.5. *Let $\mathfrak{h}$ be a real semisimple Lie algebra, and V a finite-dimensional representation of $\mathfrak{h}$ over $\mathbb{R}$. One has the equivalences:*

(1) *V is orthogonal $\iff$ $V_{\mathbb{C}}$ is orthogonal.*

(2) *V has RGS in $\mathfrak{h}$ $\iff$ $V_{\mathbb{C}}$ has RGS in $\mathfrak{h}_{\mathbb{C}}$.*

Proof. (1) The implication $\Rightarrow$ is obvious. Conversely, suppose that the representation of $\mathfrak{h}_{\mathbb{C}}$ in $V_{\mathbb{C}}$ is orthogonal. Then one has two $\mathfrak{h}$-invariant symmetric bilinear forms $A, B\colon V \times V \to \mathbb{R}$ such that $A + \sqrt{-1}B$ is nondegenerate. In turn, one can find $t \in \mathbb{R}$ such that $A + tB$ is nondegenerate, showing that the representation of $\mathfrak{h}$ in V is orthogonal.

(2) As in the proof of Lemma 2.4, for v in $V_{\mathbb{C}}$, we denote by $\mathfrak{u}_{\mathbb{C},v}$ the unipotent radical of its stabilizer $\mathfrak{h}_{\mathbb{C},v}$ in $\mathfrak{h}_{\mathbb{C}}$.

Let $d := \min_{v\in V_{\mathbb{C}}} \dim \mathfrak{h}_{\mathbb{C},v}$ and $V'_{\mathbb{C}} := \{v \in V_{\mathbb{C}} \mid \dim \mathfrak{h}_{\mathbb{C},v} = d\}$. Since $V'_{\mathbb{C}}$ is Zariski open, it meets V and one has $d = \min_{v\in V} \dim \mathfrak{h}_v$. One then introduces $\delta := \min_{v\in V'_{\mathbb{C}}} \dim \mathfrak{u}_{\mathbb{C},v}$ and $V''_{\mathbb{C}} := \{v \in V'_{\mathbb{C}} \mid \dim \mathfrak{u}_{\mathbb{C},v} = \delta\}$. Since $V''_{\mathbb{C}}$ is Zariski open, it meets V and one has the equivalences:

$$V \text{ has RGS in } \mathfrak{h} \iff \delta = 0 \iff V_{\mathbb{C}} \text{ has RGS in } \mathfrak{h}_{\mathbb{C}}.$$ ∎

In Lemma 2.5 (2), there exist finitely many reductive Lie subalgebras $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ of $\mathfrak{h}$ such that the set of w in V whose stabilizer $\mathfrak{h}_w$ is conjugate to one of the $\mathfrak{m}_i$ contains a non-empty Zariski open subset of V.

"Conjugate" means a "conjugate by the adjoint group H of $\mathfrak{h}$". We say that the Lie algebras $\mathfrak{m}_i$ which cannot be removed from this list in Lemma 2.5 are the *generic stabilizers* of V. They are well-defined only up to conjugacy and permutation.

Proof of Proposition 2.2. We extend the quadratic form on V to a complex quadratic form on $V_{\mathbb{C}}$. Applying Lemma 2.3 to the complexified representation of $\mathfrak{h}_{\mathbb{C}}$ in $V_{\mathbb{C}}$, one finds a real form $\mathfrak{k}$ of the Lie algebra $\mathfrak{h}_{\mathbb{C}}$ and a $\mathfrak{k}$-invariant real form V_0 of $V_{\mathbb{C}}$ such that the restriction of the quadratic form to V_0 is positive. Since all the Lie subalgebras of $\mathfrak{so}(n,\mathbb{R})$ are reductive, V_0 has RGS in $\mathfrak{k}$. Applying twice Lemma 2.5 (2), we deduce successively that $V_{\mathbb{C}}$ has RGS in $\mathfrak{h}_{\mathbb{C}}$ and that V has RGS in $\mathfrak{h}$. ∎

2.2. Function ρ_V and invariant p_V

Let $\mathfrak{h}$ be a real Lie algebra, and V a finite-dimensional representation of $\mathfrak{h}$ over $\mathbb{R}$. For an element Y in $\mathfrak{h}$, we consider eigenvalues of Y in the complexification $V_{\mathbb{C}}$, and write $V_{\mathbb{C}} = V_+ \oplus V_0 \oplus V_-$ for the direct sum decomposition into the largest vector subspaces of $V_{\mathbb{C}}$ on which the real part of all the (generalized) eigenvalues of Y are positive, zero, and negative, respectively. We define the non-negative functions ρ_V^+ and ρ_V on $\mathfrak{h}$ by

$$\rho_V^+(Y) := \text{the real part of } \mathrm{Trace}(Y|_{V_+}),$$
$$\rho_V(Y) := \frac{1}{2}(\rho_V^+(Y) + \rho_V^+(-Y)),$$

where Trace denotes the trace of an endomorphism of a vector space.
By definition, one always has the equality $\rho_V(-Y) = \rho_V(Y)$. Moreover, when the action of $\mathfrak{h}$ on V is trace-free, one has the equality

$$\rho_V(Y) = \rho_V^+(Y) \qquad \text{for all } Y \in \mathfrak{h}.$$

The function denoted by ρ_V in [2, Sect. 3.1] is what we call now ρ_V^+.

Suppose $\mathfrak{h}$ is a real reductive Lie algebra and V is a semisimple representation. Let $\mathfrak{a} \equiv \mathfrak{a}_{\mathfrak{h}}$ be a maximal split abelian real Lie subalgebra in $\mathfrak{h}$. This subalgebra is a real vector space whose dimension ℓ is the real rank of $\mathfrak{h}$, to be denoted by $\mathrm{rank}_{\mathbb{R}}\,\mathfrak{h}$. Then ρ_V is determined completely by its restriction to $\mathfrak{a}$, and actual computations of ρ_V in Chapters 5–7 will be carried out by using the weight decomposition of V with respect to $\mathfrak{a}$, which we explain now. Let $P(V,\mathfrak{a})$ be the set of weights of $\mathfrak{a}$ in V, i.e. the set of linear forms $\alpha \in \mathfrak{a}^*$ for which the weight space

$$V_\alpha := \{v \in V \mid Yv = \alpha(Y)v \ ,\ \forall Y \in \mathfrak{a}\}$$

is nonzero. For such a weight α we set $m_\alpha := \dim V_\alpha$ and $|\alpha| := \max(\alpha, -\alpha)$.

By definition the restriction of ρ_V to the subspace $\mathfrak{a}$ is given by the formula

$$\rho_V = \tfrac{1}{2}\textstyle\sum m_\alpha |\alpha| \ , \tag{2}$$

where the sum is taken over all the weights $\alpha \in P(V,\mathfrak{a})$.

Since this function $\rho_V \colon \mathfrak{a} \to \mathbb{R}_{\geq 0}$ is very important in our analysis, we begin with a few elementary but useful comments. This function ρ_V is invariant under the Weyl group W of the (restricted) root system $\Sigma(\mathfrak{h},\mathfrak{a})$. Moreover the function ρ_V is convex, continuous and is piecewise linear in the sense that there exist finitely many convex polyhedral cones which cover $\mathfrak{a}$ and on which ρ_V is linear.

For two real semisimple representations V', V'' of $\mathfrak{h}$, one has

$$\rho_{V' \oplus V''} = \rho_{V'} + \rho_{V''}. \tag{3}$$

We denote by V^* the contragredient representation of V. Then one has

$$\rho_{V^*} = \rho_V, \tag{4}$$

$$\rho_{V\oplus V^*} = 2\rho_V. \tag{5}$$

When V is self-dual, *i.e.*, when V^* is isomorphic to V as an $\mathfrak{h}$-module, each nonzero weight α occurs in pair with its opposite $-\alpha$ and ρ_V is equal to

$$\rho_V^+ = \sum_{\alpha\in P(V,\mathfrak{a})} m_\alpha \alpha_+$$

where $\alpha_+ := \max(\alpha, 0)$. When $V = \mathfrak{h}$ is the adjoint representation, this function $\rho_\mathfrak{h}$ coincides with twice the "usual ρ" on a positive Weyl chamber $\mathfrak{a}_+$ with respect to the positive system $\Sigma^+(\mathfrak{h}, \mathfrak{a})$. For other representations V, the maximal convex polyhedral cones on which ρ_V is linear are most often much smaller than the Weyl chambers.

We introduce the invariant of an $\mathfrak{h}$-module V by

$$p_V := \inf\{t > 0 \mid \rho_\mathfrak{h} \le t\,\rho_V\}. \tag{6}$$

By definition, $p_V = \infty$ if V has nonzero fixed vectors of $\mathfrak{h}$. In general, for a finite-dimensional representation of $\mathfrak{h}$ on a real vector space V, one has the equivalences:

$$\rho_\mathfrak{h} \le \rho_V \Longleftrightarrow p_V \le 1\,, \tag{7}$$

$$\rho_\mathfrak{h} \le \rho_{V\oplus V^*} \Longleftrightarrow p_V \le 2\,. \tag{8}$$

Let us explain why this invariant p_V is relevant. Indeed, the main results of [2] may be reformulated as follows. We recall that a unitary representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ is called *almost* L^p $(p \ge 2)$ if there exists a dense subset $D \subset \mathcal{H}$ for which the matrix coefficients $g \mapsto (\pi(g)u, v)$ are in $L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$ and all u, v in D. If G is a semisimple Lie group, π is almost L^2 if and only if π is tempered [9]. Suppose H is a real reductive algebraic Lie group with Lie algebra $\mathfrak{h}$. For a positive even integer p and for an algebraic representation $H \to SL(V)$, one has the following equivalences ([2, Thm. 3.2])

$$L^2(V) \text{ is } H\text{-tempered} \Longleftrightarrow p_V \le 2,$$

$$L^2(V) \text{ is } H\text{-almost } L^p \Longleftrightarrow p_V \le p.$$

Moreover, for a real semisimple algebraic group G and a real reductive algebraic subgroup H, one has the following equivalences ([2, Thm. 4.1]):

$$L^2(G/H) \text{ is } G\text{-tempered} \Longleftrightarrow p_{\mathfrak{g}/\mathfrak{h}} \le 1, \tag{9}$$

$$L^2(G/H) \text{ is } G\text{-almost } L^p \Longleftrightarrow p_{\mathfrak{g}/\mathfrak{h}} \le p - 1.$$

The inequality $\rho_{\mathfrak{g}/\mathfrak{h}} \le 1$ in (9) is nothing but the criterion (1) by (7).

Hence, we would like to describe all the orthogonal representations V such that $p_V \le 1$, i.e., $\rho_\mathfrak{h} \le \rho_V$. In particular, we would like to describe all the representations V such that $p_V \le 2$, or equivalently, $\rho_\mathfrak{h} \le \rho_{V\oplus V^*}$.

We end this section by a useful remark. We note that, when V is a direct sum of two subrepresentations $V = V' \oplus V''$, then one has the inequality

$$p_V^{-1} \geq p_{V'}^{-1} + p_{V''}^{-1} \tag{10}$$

as one sees from (3) and from the following equivalent definition of p_V:

$$p_V^{-1} = \min_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_V(Y)}{\rho_{\mathfrak{h}}(Y)}. \tag{11}$$

In general, the equality in (10) may not hold, but if V is of the form

$$V = \underbrace{V' \oplus \cdots \oplus V'}_{m} \oplus \underbrace{(V')^* \oplus \cdots \oplus (V')^*}_{n},$$

then one has

$$p_V = \frac{1}{m+n}\, p_{V'}. \tag{12}$$

2.3. Abelian and amenable generic stabilizer

Let $\mathfrak{h}$ be a real reductive Lie algebra. We say that a subalgebra $\mathfrak{l}$ is *amenable reductive* if it is reductive and if the restriction of the Killing form of $\mathfrak{h}$ to $[\mathfrak{l}, \mathfrak{l}]$ is negative. Let V be a finite-dimensional representation of $\mathfrak{h}$ and define for every $v \in V$: $\mathfrak{h}_v := \{X \in \mathfrak{h} \mid X \cdot v = 0\}$.

Definition 2.6 (AGS and AmGS)**.** We say that

(1) V has AGS in $\mathfrak{h}$ if the set $\{v \in V \mid \mathfrak{h}_v$ is abelian reductive$\}$ is dense in V.

(2) V has AmGS in $\mathfrak{h}$ if the set $\{v \in V \mid \mathfrak{h}_v$ is amenable reductive$\}$ is dense in V.

Remark 2.7. In the first definition, it is equivalent to say Zariski dense instead of dense. However in the second definition, it is not equivalent to say Zariski dense instead of dense. Indeed, in the natural representation $\mathbb{R}^4$ of $\mathfrak{so}(3,1)$, the set of points v with reductive amenable stabilizer is Zariski dense but is not dense.

The statement (1) in Theorem 1.5 is a special case of the following proposition.

Proposition 2.8. *Let $\mathfrak{h}$ be a real semisimple Lie algebra and V an orthogonal representation of $\mathfrak{h}$. One has the implication:*

$$\rho_{\mathfrak{h}} \leq \rho_V \implies V \text{ has AmGS in } \mathfrak{h}. \tag{13}$$

Moreover, if one of the generic stabilizers $\mathfrak{m}$ of V has the same real rank as $\mathfrak{h}$, then the converse is true.

Proof. By Proposition 2.2, $\mathfrak{m}$ is reductive. Then Proposition 2.8 follows from Lemma 2.9 below and from the following equivalence for a reductive Lie algebra $\mathfrak{m}$: $\rho_{\mathfrak{m}} = 0 \Longleftrightarrow \mathfrak{m}$ is amenable. ∎

Lemma 2.9. *Let $\mathfrak{h}$ be a real semisimple Lie algebra, V an orthogonal representation of $\mathfrak{h}$, and $\mathfrak{m}$ one among the finitely many generic stabilizers $\mathfrak{m}_i$ of V. Let $t \geq 1$. One has the implication:*

$$\rho_{\mathfrak{h}} \leq t\rho_V \implies \rho_{\mathfrak{m}} \leq (t-1)\rho_{\mathfrak{h}/\mathfrak{m}}. \tag{14}$$

Moreover, if $\mathfrak{m}$ and $\mathfrak{h}$ have the same real rank, the converse is true.

Let $\mathfrak{a}_\mathfrak{m} \subset \mathfrak{a}_\mathfrak{h}$ be Cartan subspaces of $\mathfrak{m}$ and $\mathfrak{h}$. We recall that $\rho_\mathfrak{h}$ and ρ_V are functions on $\mathfrak{a}_\mathfrak{h}$ while $\rho_\mathfrak{m}$ and $\rho_{\mathfrak{h}/\mathfrak{m}}$ are functions on $\mathfrak{a}_\mathfrak{m}$.

Proof. We can assume the representation of $\mathfrak{h}$ to be faithful. Let H be an algebraic subgroup of $\mathrm{GL}(V)$ with Lie algebra $\mathfrak{h}$. Since V has RGS in $\mathfrak{h}$, we can find a slice Σ of points v of V whose stabilizer M in H has Lie algebra $\mathfrak{m}$ and an open neighborhood of v foliated by H-orbits. The tangent space at v to the orbit Hv is isomorphic as a representation of M to $\mathfrak{h}/\mathfrak{m}$. Since M preserves the leaves of the foliation the quotient $V/(\mathfrak{h}/\mathfrak{m})$ is a trivial representation of $\mathfrak{m}$. Hence, for X in $\mathfrak{a}_\mathfrak{m}$, one has the equivalences :

$$\begin{aligned} \rho_\mathfrak{h}(X) \le t\,\rho_V(X) &\iff \rho_\mathfrak{h}(X) \le t\,\rho_{\mathfrak{h}/\mathfrak{m}}(X) \\ &\iff \rho_\mathfrak{m}(X) \le (t-1)\,\rho_{\mathfrak{h}/\mathfrak{m}}(X). \end{aligned}$$

Our claims follow since, if $\mathfrak{h}$ and $\mathfrak{m}$ have the same real rank, one has $\mathfrak{a}_\mathfrak{m} = \mathfrak{a}_\mathfrak{h}$. ∎

The converse to Proposition 2.8 is not true, but we conjecture that a kind of converse is true:

Conjecture 2.10. Let $\mathfrak{h}$ be a real semisimple Lie algebra and V an orthogonal representation of $\mathfrak{h}$. One has the implications:

$$V \text{ has AGS in } \mathfrak{h} \implies \rho_\mathfrak{h} \le \rho_V. \tag{15}$$

Remark 2.11. We shall see that Conjecture 2.10 holds in the following settings:
(1) $\mathfrak{h}$ is simple (Corollary 4.4);
(2) there is a semisimple Lie algebra $\mathfrak{g}$ containing $\mathfrak{h}$ as a subalgebra such that $V = \mathfrak{g}/\mathfrak{h}$ (Theorem 1.7).

2.4. Real and complex Lie algebras

We see from Lemma 2.12 below that the second statement of Theorem 1.5 follows from Theorem 1.6. We recall that a real semisimple Lie algebra is split if its real rank and complex rank coincide.

Lemma 2.12. *Let $\mathfrak{h}$ be a real semisimple Lie algebra, V a finite-dimensional representation of $\mathfrak{h}$, and $V_\mathbb{C}$ the complexification of V.*
(1) *Assume that V has RGS in $\mathfrak{h}$ (Definition 2.1). Then one has the equivalence:*

$$V \textit{ has } \mathrm{AGS} \textit{ in } \mathfrak{h} \iff V_\mathbb{C} \textit{ has } \mathrm{AGS} \textit{ in } \mathfrak{h}_\mathbb{C}.$$

(2) *One has the implication:* $\rho_{\mathfrak{h}_\mathbb{C}} \le \rho_{V_\mathbb{C}} \Longrightarrow \rho_\mathfrak{h} \le \rho_V$. (16)

Moreover, the converse is true when $\mathfrak{h}$ is a split real semisimple Lie algebra.

The proof of Lemm3 2.12 is straightforward and is left to the reader.
According to Proposition 2.8, and to Lemma 2.12, the following Conjecture 2.13 is equivalent to Conjecture 2.10.

Conjecture 2.13. Let $\mathfrak{h}$ be a complex semisimple Lie algebra and V an orthogonal representation of $\mathfrak{h}$ over $\mathbb{C}$. One has the equivalence:

$$\rho_\mathfrak{h} \le \rho_V \iff V \text{ has AGS in } \mathfrak{h}. \tag{17}$$

Since the direct implication $\Longrightarrow$ follows from Proposition 2.8, one only has to understand the converse implication $\Longleftarrow$.

2.5. Representations of nonsimple Lie algebras

The following Lemma 2.14 gives useful upper bounds for the invariant p_V when the semisimple Lie algebra $\mathfrak{h}$ is not simple. We collect some basic properties of the function ρ_V (2) and the invariant p_V (6) for representation V of $\mathfrak{h}$.

Lemma 2.14. *Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ be a real semisimple Lie algebra, which is the direct sum of two ideals $\mathfrak{h}_1$, $\mathfrak{h}_2$ and V a finite-dimensional representation of $\mathfrak{h}$.*

(1) *For all X_1 in $\mathfrak{h}_1$ and X_2 in $\mathfrak{h}_2$, one has*

$$\rho_V(X_1) \;\leq\; \rho_V(X_1 + X_2). \tag{18}$$

(2) *Assume that $V = V_1 \otimes V_2$ where, for $i = 1, 2$, V_i are representations of $\mathfrak{h}_i$ of dimension d_i. Then one has*

$$p_V \leq \tfrac{p_{V_1}}{d_2} + \tfrac{p_{V_2}}{d_1}. \tag{19}$$

(3) *Assume now that $V = V_1 \oplus V_2$ where, for $i = 1, 2$, V_i are representations of $\mathfrak{h}$ such that $\rho_{\mathfrak{h}_i} \leq \rho_{V_i}$. Then one has $\rho_{\mathfrak{h}} \leq \rho_V$.*

Proof. (1) Let a be an eigenvalue of X_1 in V, and $b_1, \ldots, b_r$ eigenvalues of X_2 in $\operatorname{Ker}(X_1 - a\,\mathrm{id}_V)$ counted with multiplicities. Since $\mathfrak{h}_2$ is semisimple, one has $\sum_{j=1}^r b_j = 0$. In turn, $ra = \sum_{j=1}^r (a + b_j)$, yielding

$$\dim \operatorname{Ker}(X_1 - a\,\mathrm{id}_V) |\operatorname{Re} a| \leq \sum_{j=1}^r |\operatorname{Re}(a + b_j)|.$$

Hence $\rho_V(X_1) \leq \rho_V(X_1 + X_2)$.

(2) Take any $X = X_1 + X_2 \in \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. By the first statement, one has

$$\begin{aligned} \rho_{\mathfrak{h}}(X) \;&=\; \rho_{\mathfrak{h}_1}(X_1) + \rho_{\mathfrak{h}_2}(X_2) \;\leq\; p_{V_1}\rho_{V_1}(X_1) + p_{V_2}\rho_{V_2}(X_2) \\ &\leq\; \tfrac{p_{V_1}}{d_2}\rho_V(X_1) + \tfrac{p_{V_2}}{d_1}\rho_V(X_2) \;\leq\; \left(\tfrac{p_{V_1}}{d_2} + \tfrac{p_{V_2}}{d_1}\right)\rho_V(X). \end{aligned}$$

Hence the second statement follows.

(3) Take any $X = X_1 + X_2 \in \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. By the first statement, one has

$$\rho_V(X_1 + X_2) = \rho_{V_1}(X_1 + X_2) + \rho_{V_2}(X_1 + X_2) \geq \rho_{V_1}(X_1) + \rho_{V_2}(X_2),$$

whereas $\rho_{\mathfrak{h}}(X_1 + X_2) = \rho_{\mathfrak{h}_1}(X_1) + \rho_{\mathfrak{h}_2}(X_2)$. Hence the third statement follows. ∎

2.6. Reduction to simple Lie algebra

A real semisimple Lie algebra $\mathfrak{h}$ is said to be Ad-*compact* if the group of automorphisms $\operatorname{Aut}(\mathfrak{h})$ is compact. We denote by $\mathfrak{h}_{nc}$ the sum of the ideals of $\mathfrak{h}$ which are not Ad-compact.

The following Lemma 2.15 allows us to assume the reductive Lie subalgebra to be semisimple without Ad-compact ideals.

Lemma 2.15. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{h}$ a reductive Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{s}$ the semisimple Lie algebra $\mathfrak{s} := [\mathfrak{h}, \mathfrak{h}]_{nc}$. One has the equivalences:*

$$\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}} \iff \rho_{\mathfrak{s}} \leq \rho_{\mathfrak{g}/\mathfrak{s}},$$

$$\mathfrak{g}/\mathfrak{h} \text{ has AGS in } \mathfrak{h} \iff \mathfrak{g}/\mathfrak{s} \text{ has AGS in } \mathfrak{s},$$

$$\mathfrak{g}/\mathfrak{h} \text{ has AmGS in } \mathfrak{h} \iff \mathfrak{g}/\mathfrak{s} \text{ has AmGS in } \mathfrak{s}.$$

The proof of Lemma 2.15 is left to the reader.

The following Proposition 2.16 tells us that, in order to prove Theorem 1.6, we can assume $\mathfrak{g}$ to be simple.

Proposition 2.16. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{h}$ a semisimple Lie subalgebra of $\mathfrak{g}$, $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ a decomposition into ideals $\mathfrak{g}_j$, and, for $1 \le i \le r$, $\mathfrak{h}_i := \mathfrak{h} \cap \mathfrak{g}_i$ and $\mathfrak{q}_i := \mathfrak{g}_i/\mathfrak{h}_i$. One has the equivalences:*

(1) $\rho_{\mathfrak{h}} \le \rho_{\mathfrak{q}}$ *on* $\mathfrak{h} \iff \rho_{\mathfrak{h}_i} \le \rho_{\mathfrak{q}_i}$ *on* $\mathfrak{h}_i$ *for all* $1 \le i \le r$;

(2) $\mathfrak{q}$ *has* AGS *in* $\mathfrak{h} \iff \mathfrak{q}_i$ *has* AGS *in* $\mathfrak{h}_i$, *for all* $1 \le i \le r$;

(3) $\mathfrak{q}$ *has* AmGS *in* $\mathfrak{h} \iff \mathfrak{q}_i$ *has* AmGS *in* $\mathfrak{h}_i$, *for all* $1 \le i \le r$.

Before giving a proof of Proposition 2.16, we set up some notation. We write $\pi_i \colon \mathfrak{g} \to \mathfrak{g}_i$ for the i-th projection $(1 \le i \le r)$. Given a subspace V in $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$, we define the "hull of V" by $\widetilde{V} := \pi_1(V) \oplus \cdots \oplus \pi_r(V)$.

For each $\sigma \in \mathrm{Map}(\{1, 2, \dots, r\}, \{+, -\})$, we define a vector space V^σ by

$$V^\sigma := \{(\sigma(1)v_1, \dots, \sigma(r)v_r) \,|\, (v_1, \dots, v_r) \in V\}.$$

Then $\widetilde{V} = \sum_\sigma V^\sigma$ where the sum is taken over all σ. We note that $V \subsetneq \widetilde{V}$ if and only if $V \cap \mathfrak{g}_i \subsetneq \pi_i(V)$ for some $1 \le i \le r$, or equivalently, $V \ne V^\sigma$ for some σ.

If V is a semisimple Lie algebra, then so is $\widetilde{V}$ because $\pi_i(V)$s are semisimple ideals. If $[V', V''] = \{0\}$, then $[\widetilde{V'}, \widetilde{V''}] = \{0\}$. In particular, if the Lie algebra V is a direct sum of two semisimple ideals V' and V'', then its hull $\widetilde{V}$ is also a direct sum of semisimple ideals $\widetilde{V'}$ and $\widetilde{V''}$,

$$\widetilde{V} = \widetilde{V'} \oplus \widetilde{V''}. \tag{20}$$

Proof of Proposition 2.16. For a nonempty set $I \subset \{1, \dots, r\}$, we define an ideal $\mathfrak{h}_I$ of $\mathfrak{h}$ inductively on the cardinality $\#I$ of I by

$$\mathfrak{h}_I := \mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i \quad \text{when } I = \{i\} \ \ (1 \le i \le r)$$

and by the following characterization:

$$\mathfrak{h} \cap (\bigoplus_{i \in I} \mathfrak{g}_i) = \mathfrak{h}_I \oplus (\bigoplus_{J \subsetneq I} \mathfrak{h}_J) \text{ when } \#I \ge 2.$$

Then one sees readily from the definition of $\mathfrak{h}_I$:

$$\mathfrak{h} = \bigoplus_I \mathfrak{h}_I \qquad \text{(direct sum of semisimple ideals)}, \tag{21}$$

$$\mathfrak{h}_I \cap (\bigoplus_{j \in J} \mathfrak{g}_j) = \{0\} \quad \text{if } I \not\subset J. \tag{22}$$

$$\text{In particular,} \qquad \mathfrak{h}_I \cap (\mathfrak{h}_I)^\sigma = \{0\} \quad \text{for any } \sigma \text{ with } \sigma|_I \ne \pm \mathbf{1}_I. \tag{23}$$

Here $\sigma|_I \ne \pm \mathbf{1}_I$ means that $\sigma(i) \ne \sigma(j)$ for some $i, j \in I$.

We choose an $\mathfrak{h}_I$-submodule $\mathfrak{q}_I$ in $\widetilde{\mathfrak{h}_I}$ with a direct sum decomposition

$$\widetilde{\mathfrak{h}_I} = \mathfrak{h}_I \oplus \mathfrak{q}_I.$$

We note that $\mathfrak{q}_I = \{0\}$ when $\#I = 1$. By (23), if $\#I \geq 2$, then we may and do take $\mathfrak{q}_I$ to contain the $\mathfrak{h}_I$-submodule $(\mathfrak{h}_I)^\sigma$ for some σ. Since $(\mathfrak{h}_I)^\sigma \simeq \mathfrak{h}_I$ as $\mathfrak{h}_I$-modules, this implies that if $\#I \geq 2$,

$$\bullet \quad \rho_{\mathfrak{h}_I} \leq \rho_{\mathfrak{q}_I} \quad \text{on } \mathfrak{h}_I, \tag{24}$$

$$\bullet \quad \mathfrak{q}_I \text{ has AGS in } \mathfrak{h}_I. \tag{25}$$

Moreover, if $i \in I$, one has $\pi_i(\mathfrak{h}_I) = \begin{cases} \mathfrak{h}_i & \text{when } \#I = 1, \\ \pi_i(\mathfrak{q}_I) & \text{when } \#I \geq 2. \end{cases}$.

By (20) and (21), one has

$$\widetilde{\mathfrak{h}} = \bigoplus_I \widetilde{\mathfrak{h}_I} \quad \text{(direct sum of semisimple ideals)}.$$

Taking the projection to the i-th component, one obtains $\pi_i(\mathfrak{h}) = \oplus_I \pi_i(\mathfrak{h}_I)$, hence

$$\pi_i(\mathfrak{h}) = \mathfrak{h}_i \oplus \pi_i(\bigoplus_{\#I \geq 2} \mathfrak{q}_I). \tag{26}$$

For each i $(1 \leq i \leq r)$, we write $\mathfrak{g}_i = \pi_i(\mathfrak{h}) \oplus \mathfrak{s}_i$ by taking a $\pi_i(\mathfrak{h})$-invariant subspace $\mathfrak{s}_i$ in $\mathfrak{g}_i$, and set

$$\mathfrak{s} := \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_r, \quad \text{and} \quad \mathfrak{q} := \mathfrak{s} \oplus (\bigoplus_{\#I \geq 2} \mathfrak{q}_I). \tag{27}$$

Then $\mathfrak{q} \simeq \mathfrak{g}/\mathfrak{h}$ as an $\mathfrak{h}$-module because one has the following direct sum decompositions:

$$\mathfrak{g} = \widetilde{\mathfrak{h}} \oplus \mathfrak{s} = (\bigoplus_I \widetilde{\mathfrak{h}_I}) \oplus \mathfrak{s} = \bigoplus_I \mathfrak{h}_I \oplus \bigoplus_I \mathfrak{q}_I \oplus \mathfrak{s} = \mathfrak{h} \oplus \mathfrak{q}.$$

Moreover, (26) tells

$$\mathfrak{g}_i = \pi_i(\mathfrak{h}) \oplus \mathfrak{s}_i = \mathfrak{h}_i \oplus \mathfrak{s}_i \oplus \pi_i(\bigoplus_{\#I \geq 2} \mathfrak{q}_I) = \mathfrak{h}_i \oplus \pi_i(\mathfrak{q}). \tag{28}$$

In particular, $\mathfrak{q}_i = \mathfrak{g}_i/\mathfrak{h}_i$ is expressed as an $\mathfrak{h}_i$-module:

$$\mathfrak{q}_i \simeq \mathfrak{s}_i \oplus (\text{trivial } \mathfrak{h}_i\text{-module}). \tag{29}$$

(1) Suppose $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$. Then for any $H \in \mathfrak{h}_i$,

$$\rho_{\mathfrak{h}_i}(H) \leq \rho_{\mathfrak{h}}(H) \leq \rho_{\mathfrak{q}}(H) = \rho_{\mathfrak{s}}(H) + \sum_{\#I \geq 2} \rho_{\mathfrak{q}_I}(H) = \rho_{\mathfrak{s}_i}(H)$$

because $\mathfrak{h}_i$ acts trivially on all $\mathfrak{s}_j$ with $j \neq i$ and $\mathfrak{q}_I$ with $\#I \geq 2$.
Conversely, suppose $\rho_{\mathfrak{h}_i} \leq \rho_{\mathfrak{q}_i}$ holds for all $1 \leq i \leq r$. Take any $H \in \mathfrak{h}$, and write

$$H = \sum_I H_I = \sum_{i=1}^r H_i + \sum_{\#I \geq 2} H_I \in \mathfrak{h} = \bigoplus_{i=1}^r \mathfrak{h}_i \oplus \bigoplus_{\#I \geq 2} \mathfrak{h}_I.$$

Then $\rho_{\mathfrak{h}}(H) = \sum_I \rho_{\mathfrak{h}_I}(H) = \sum_I \rho_{\mathfrak{h}_I}(H_I) = \sum_{i=1}^r \rho_{\mathfrak{h}_i}(H_i) + \sum_{\#I \geq 2} \rho_{\mathfrak{h}_I}(H_I)$.

By the assumption $\rho_{\mathfrak{h}_i}(H_i) \leq \rho_{\mathfrak{q}_i}(H_i)$ and by (29) and (24), one obtains

$$\rho_{\mathfrak{h}}(H) \leq \sum_{i=1}^{r} \rho_{\mathfrak{s}_i}(H_i) + \sum_{\#I\geq 2} \rho_{\mathfrak{q}_I}(H_I) = \rho_{\mathfrak{s}}(\sum_{i=1}^{r} H_i) + \sum_{\#I\geq 2} \rho_{\mathfrak{q}_I}(H).$$

By Lemma 2.14 (1), one has $\rho_{\mathfrak{s}}(\sum_{i=1}^{r} H_i) \leq \rho_{\mathfrak{s}}(H)$, hence $\rho_{\mathfrak{h}}(H) \leq \rho_{\mathfrak{q}}(H)$.

(2) Suppose $\mathfrak{q}$ has AGS in $\mathfrak{h}$. Let U be a dense subset of $\mathfrak{q}$ such that $\mathrm{Stab}_{\mathfrak{h}}(x) \equiv \mathfrak{h}_x$ is abelian and reductive for all $x \in U$. Then for all $1 \leq i \leq r$, $\mathrm{Stab}_{\mathfrak{h}_i}(\pi_i(x)) = \mathrm{Stab}_{\mathfrak{h}_i}(x)$ is abelian and reductive. Since $\pi_i(U)$ is dense in $\mathfrak{q}_i = \mathfrak{g}_i/\mathfrak{h}_i \simeq \pi_i(\mathfrak{q})$ by (28), $\mathfrak{q}_i$ has AGS in $\mathfrak{h}_i$.

Conversely, suppose $\mathfrak{q}_i$ has AGS in $\mathfrak{h}_i$ for all $1 \leq i \leq r$. By (29), $\mathfrak{s}_i$ has also AGS in $\mathfrak{h}_i$. By (25), one can find a dense subset W of $\mathfrak{q} = \bigoplus_{i=1}^{r} \mathfrak{s}_i \oplus \bigoplus_{\#I\geq 2} \mathfrak{q}_I$, see (27), such that if $x = \sum_{i=1}^{r} y_i + \sum_{\#I\geq 2} z_I \in W$ then $\mathrm{Stab}_{\mathfrak{h}_i}(y_i)$ $(1 \leq i \leq r)$ and $\mathrm{Stab}_{\mathfrak{h}_I}(z_I)$ $(\#I \geq 2)$ are all abelian and reductive. We now observe

$$\begin{aligned}\mathrm{Stab}_{\mathfrak{h}}(x) &= \bigcap_{i=1}^{r} \mathrm{Stab}_{\mathfrak{h}}(y_i) \cap \bigcap_{\#I\geq 2} \mathrm{Stab}_{\mathfrak{h}}(z_I)\\ &= \bigcap_{i=1}^{r} \mathrm{Stab}_{\mathfrak{h}}(y_i) \cap (\bigoplus_{i=1}^{r} \mathfrak{h}_i \oplus \bigoplus_{\#I\geq 2} \mathrm{Stab}_{\mathfrak{h}_I}(z_I)).\end{aligned}$$

Therefore the (splitting) exact sequence $0 \to \oplus_{i=1}^{r} \mathfrak{h}_i \to \mathfrak{h} \to \oplus_{\#I\geq 2} \mathfrak{h}_I \to 0$ induces an exact sequence of Lie algebras:

$$0 \to \bigoplus_{i=1}^{r} \mathrm{Stab}_{\mathfrak{h}_i}(y_i) \to \mathrm{Stab}_{\mathfrak{h}}(x) \to \bigoplus_{\#I\geq 2} \mathrm{Stab}_{\mathfrak{h}_I}(z_I).$$

By Proposition 2.2, the Lie algebra $\mathrm{Stab}_{\mathfrak{h}}(x)$ is reductive for x in an open dense subset of $\mathfrak{q}$. The above exact sequence tells us that it is also abelian.

(3) The proof parallels to that of (2). ∎

3. Classical simple Lie algebras

In this chapter we give a classification of the pairs $(\mathfrak{g}, \mathfrak{h})$ of complex semisimple Lie algebras satisfying $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{g}/\mathfrak{h}}$ in the case where $\mathfrak{g}$ is classical simple, and in particular, prove Theorems 1.6 and 1.2 for $\mathfrak{g}$ classical simple.

Throughout this chapter, $\mathfrak{g}$ is a complex classical simple Lie algebra, $\mathfrak{h}$ is a complex semisimple Lie subalgebra $\{0\} \neq \mathfrak{h} \subsetneq \mathfrak{g}$, $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$, $\widetilde{\mathfrak{h}}$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{m}$ is the generic stabilizer of $\mathfrak{q}$ in $\mathfrak{h}$, and $\mathfrak{m}_s := [\mathfrak{m}, \mathfrak{m}]$. "Classical" means that $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^n)$, $\mathfrak{so}(\mathbb{C}^n)$ or $\mathfrak{sp}(\mathbb{C}^{2n})$. We will denote by V the standard representation of $\mathfrak{g}$ in $\mathbb{C}^n$, $\mathbb{C}^n$ or $\mathbb{C}^{2n}$ respectively.

3.1. Main list for classical Lie algebras

We will use the notations $\mathfrak{sl}_n$, $\mathfrak{so}_n$ and $\mathfrak{sp}_n$ for $\mathfrak{sl}(\mathbb{C}^n)$, $\mathfrak{so}(\mathbb{C}^n)$ and $\mathfrak{sp}(\mathbb{C}^{2n})$ and also $\mathfrak{a}_\ell$, $\mathfrak{b}_\ell$, $\mathfrak{c}_\ell$, $\mathfrak{d}_\ell$ for $\mathfrak{sl}_{\ell+1}$, $\mathfrak{so}_{2\ell+1}$, $\mathfrak{sp}_\ell$, $\mathfrak{so}_{2\ell}$, and $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ for the five exceptional simple Lie algebras.

Theorem 3.1. *Let $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$ or $\mathfrak{sp}_n$ be a complex classical simple Lie algebra. The complex semisimple Lie subalgebras $\mathfrak{h} \subsetneq \mathfrak{g}$ satisfying $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$ form the list in Table 1. In this list, $\mathfrak{q}$ does not have* AGS *in $\mathfrak{h}$.*

Case	$\mathfrak{g}$	max. $\mathfrak{h}$	$\mathfrak{m}$	non max.$\mathfrak{h}$	$\mathfrak{m}$	parameters $p \geq q \geq 1$
$A1$	$\mathfrak{sl}_{p+q}$	$\mathfrak{sl}_p \oplus \mathfrak{sl}_q$	$\mathfrak{sl}_{p-q} \oplus \mathbb{C}^{q-1}$	$\mathfrak{sl}_p \oplus \mathfrak{h}_2$	$\supset \mathfrak{sl}_{p-q}$	$p \geq q+2$ $\mathfrak{h}_2 \subset \mathfrak{sl}_q$
$A2$	$\mathfrak{sl}_{2p}$	$\mathfrak{sp}_p$	$(\mathfrak{sl}_2)^p$			$p \geq 2$
$BD1$	$\mathfrak{so}_{p+q}$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$	$\mathfrak{so}_{p-q}$	$\mathfrak{so}_p \oplus \mathfrak{h}_2$	$\mathfrak{so}_{p-q}$	$p \geq q+3$, $q \neq 2$ $\mathfrak{h}_2 \subset \mathfrak{so}_q$
	$\mathfrak{so}_{p+2}$			$\mathfrak{so}_p$	$\mathfrak{so}_{p-2}$	$p \geq 5$
$D4$	$\mathfrak{so}_{7+1}$			$\mathfrak{g}_2$	$\mathfrak{sl}_2$	$\mathfrak{g}_2 \overset{irr}{\hookrightarrow} \mathfrak{so}_7$
$B4$	$\mathfrak{so}_{8+1}$			$\mathfrak{so}_7$	$\mathfrak{sl}_3$	$\mathfrak{so}_7 \overset{irr}{\hookrightarrow} \mathfrak{so}_8$
$D5$	$\mathfrak{so}_{8+2}$			$\mathfrak{so}_7$	$\mathfrak{sl}_2$	$\mathfrak{so}_7 \overset{irr}{\hookrightarrow} \mathfrak{so}_8$
$D2$	$\mathfrak{so}_{2p}$	$\mathfrak{sl}_p$	$\supset (\mathfrak{sl}_2)^{[p/2]}$			$p \geq 3$
$B3$	$\mathfrak{so}_7$	$\mathfrak{g}_2$	$\mathfrak{sl}_3$			$\mathfrak{g}_2 \overset{irr}{\hookrightarrow} \mathfrak{so}_7$
$C1$	$\mathfrak{sp}_{p+q}$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$	$\mathfrak{sp}_{p-q} \oplus (\mathfrak{sp}_1)^p$	$\mathfrak{sp}_p \oplus \mathfrak{h}_2$	$\supset \mathfrak{sp}_{p-q}$	$p \geq q+1$ $\mathfrak{h}_2 \subset \mathfrak{sp}_q$
$C2$	$\mathfrak{sp}_{2p}$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_p$	$(\mathfrak{sp}_1)^p$			$p \geq 1$

Table 1: Pairs $(\mathfrak{g}, \mathfrak{h})$ with $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$ for $\mathfrak{g}$ classical simple

The left-hand side of Table 1 lists the semisimple Lie subalgebras $\mathfrak{h} \subsetneq \mathfrak{g}$ which are maximal (among the semisimple Lie subalgebras of $\mathfrak{g}$), while the right-hand side lists non-maximal ones. Note that when a maximal $\mathfrak{h}$ does not contain a proper semisimple subalgebra $\mathfrak{h}'$ with $\rho_{\mathfrak{h}'} \not\leq \rho_{\mathfrak{q}'}$, one has a blank in the right-hand side (A2, D2, B3, C2). The blanks on the left-hand side (the second case of BD1, D4, B4, D5) means that the non-maximal $\mathfrak{h}$ is a subalgebra of a maximal semisimple subalgebra $\mathfrak{h}'$ which already occurred in another row (BD1 with $q = 1$).

Note that in Table 1, the pair $(\mathfrak{so}_7, \mathfrak{g}_2)$ is the only one for which $\mathfrak{h}$ is maximal and $(\mathfrak{g}, \widetilde{\mathfrak{h}})$ is not a symmetric pair.

In case $D2$, the morphisms $\mathfrak{sl}_p \hookrightarrow \mathfrak{so}_{2p}$ are those for which $\widetilde{\mathfrak{h}}$ are the stabilizers of a pair of transversal isotropic p-planes in $\mathbb{C}^{2p}$.

In the Cases $B3$ and $D4$, the morphisms $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}_n$ $(n = 7, 8)$ are given by the 7-dimensional irreducible representation $\mathfrak{g}_2 \overset{irr}{\hookrightarrow} \mathfrak{so}_7$, plus $n-7$ copies of the trivial one-dimensional representation.

In the Cases $B4$ and $D5$, the morphisms $\mathfrak{so}_7 \hookrightarrow \mathfrak{so}_n$ $(n = 9, 10)$ are given by the 8-dimensional irreducible representation $\mathfrak{so}_7 \overset{irr}{\hookrightarrow} \mathfrak{so}_8$, called the spin representation, plus $n-8$ copies of the trivial one-dimensional representation. Note that the pair $\mathfrak{so}_7 \overset{irr}{\hookrightarrow} \mathfrak{so}_8$ itself is not included in the left-hand side of Table 1, because it is isomorphic to the standard pair $\mathfrak{so}_7 \subset \mathfrak{so}_8$ by an outer automorphism of $\mathfrak{so}_8$.

The strategy of the proof of Theorem 3.1 is to deal first with natural examples of pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h}$ is maximal in $\mathfrak{g}$: for symmetric pairs in Section 3.2, for irreducible

representations in Section 3.3, and for reducible representations in Section 3.4. The proof of Theorem 3.1 is given in Section 3.5, except that most of the technical estimates are postponed to Chapter 6.

3.2. Classical symmetric pairs

We first deal with the seven families of pairs $(\mathfrak{g},\mathfrak{h})$ such that $(\mathfrak{g},\widetilde{\mathfrak{h}})$ is a classical symmetric pair, where $\widetilde{\mathfrak{h}}$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. We give a necessary and sufficient condition for $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$. We also list the generic stabilizer $\mathfrak{m}$, which is readily computed by using the Satake diagram of the structure theory of symmetric pairs $(\mathfrak{g},\widetilde{\mathfrak{h}})$, see [11, Chap. 10] for instance.

Proposition 3.2. *Let $p \geq q \geq 1$.*

- *If $\mathfrak{g} = \mathfrak{sl}_{p+q} \supset \mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{sl}_q$, then $\mathfrak{m} \simeq \mathfrak{sl}_{p-q} \oplus \mathbb{C}^q$ and $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}} \Leftrightarrow |p-q| \geq 2$.*
- *If $\mathfrak{g} = \mathfrak{so}_{p+q} \supset \mathfrak{h} = \mathfrak{so}_p \oplus \mathfrak{so}_q$, then $\mathfrak{m} \simeq \mathfrak{so}_{p-q}$ and $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}} \Leftrightarrow |p-q| \geq 3$.*
- *If $\mathfrak{g} = \mathfrak{sp}_{p+q} \supset \mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{sp}_q$, then $\mathfrak{m} \simeq \mathfrak{sp}_{p-q} \oplus (\mathfrak{sp}_1)^q$ and $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$.*

Proof. This follows from Proposition 6.6 because in these examples, one has respectively $\mathfrak{q} = \mathbb{C} \oplus (V \oplus V^*)$, $\mathfrak{q} = V$ and $\mathfrak{q} = V$ where $V = \mathbb{C}^p \otimes \mathbb{C}^q$ or $\mathbb{C}^{2p} \otimes \mathbb{C}^{2q}$. ∎

Proposition 3.3. *Let $p \geq 1$ and set $\ell := [\frac{p}{2}]$, $\varepsilon := p - 2\ell \in \{0,1\}$.*

- *If $\mathfrak{g} = \mathfrak{sl}_p \supset \mathfrak{h} = \mathfrak{so}_p$, then $\mathfrak{m} = \{0\}$ and $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.*
- *If $\mathfrak{g} = \mathfrak{sl}_{2p} \supset \mathfrak{h} = \mathfrak{sp}_p$, then $\mathfrak{m} \simeq (\mathfrak{sl}_2)^p$ and $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$.*
- *If $\mathfrak{g} = \mathfrak{so}_{2p} \supset \mathfrak{h} = \mathfrak{sl}_p$, then $\mathfrak{m} \simeq (\mathfrak{sl}_2)^\ell \oplus \mathbb{C}^\varepsilon$ and $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$.*
- *If $\mathfrak{g} = \mathfrak{sp}_p \supset \mathfrak{h} = \mathfrak{sl}_p$, then $\mathfrak{m} = \{0\}$ and $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.*

Proof. This follows from Propositions 5.1, 5.2, 5.3 and 5.4 because in these examples, one has respectively $\mathfrak{q} = S^2_0\mathbb{C}^p \simeq S^2\mathbb{C}^p/\mathbb{C}$, $\mathfrak{q} = \Lambda^2_0\mathbb{C}^{2p} \simeq \Lambda^2\mathbb{C}^{2p}/\mathbb{C}$, $\mathfrak{q} = \mathbb{C} \oplus (\Lambda^2\mathbb{C}^p \oplus \text{dual})$, and $\mathfrak{q} = \mathbb{C} \oplus (S^2\mathbb{C}^p \oplus \text{dual})$. ∎

3.3. Irreducible representations

In this section we deal with semisimple Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$ or $\mathfrak{sp}_n$ whose action on $V = \mathbb{C}^n$, $\mathbb{C}^n$ or $\mathbb{C}^{2n}$ is irreducible.

The first proposition deals with the case when $\mathfrak{h}$ is not simple, *i.e.* $\mathfrak{h}$ is the sum of two non-zero ideals $\mathfrak{h}_1$ and $\mathfrak{h}_2$, *i.e.,* $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Then $\mathfrak{h}$ is realized as a subalgebra of $\mathfrak{g}$ via the outer tensor product of the natural representations of $\mathfrak{h}_1$ and $\mathfrak{h}_2$.

Proposition 3.4. *Suppose $p > 1$ and $q > 1$.*

- *If $\mathfrak{g} = \mathfrak{sl}_{pq} \supset \mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{sl}_q$, then $\mathfrak{m} = \{0\}$ and $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.*
- *If $\mathfrak{g} = \mathfrak{so}_{pq} \supset \mathfrak{h} = \mathfrak{so}_p \oplus \mathfrak{so}_q$, then $\mathfrak{m} = \{0\}$ and $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.*

Suppose $p \geq 1$ and $q > 1$.

- *If $\mathfrak{g} = \mathfrak{so}_{4pq} \supset \mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{sp}_q$, then $\mathfrak{m} = \{0\}$ and $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.*
- *If $\mathfrak{g} = \mathfrak{sp}_{pq} \supset \mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{so}_q$, then $\mathfrak{m} = \{0\}$ and $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.*

The computation of the generic stabilizers $\mathfrak{m}$ is straightforward, and the inequality $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$ in Proposition 3.4 follows from Proposition 6.7.

Proposition 3.5. *Let $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$ or $\mathfrak{sp}_n$ and $\mathfrak{h} \subsetneq \mathfrak{g}$ a simple Lie subalgebra which is irreducible on V and satisfies $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$.*

- *If $\mathfrak{g} = \mathfrak{sl}_n$, then $n = 2p$, $\mathfrak{h} = \mathfrak{sp}_p$ and $\mathfrak{m} \simeq (\mathfrak{sl}_2)^p$.*
- *If $\mathfrak{g} = \mathfrak{so}_n$, then $n = 7$, $\mathfrak{h} = \mathfrak{g}_2$, $\mathfrak{m} \simeq \mathfrak{sl}_3$ or $n = 8$, $\mathfrak{h} = \mathfrak{so}_7$, $\mathfrak{m} \simeq \mathfrak{so}_6$.*
- *If $\mathfrak{g} = \mathfrak{sp}_n$ then such an $\mathfrak{h}$ does not exist.*

The proof of Proposition 3.5 relies on explicit computations of $\rho_\mathfrak{h}$ and $\rho_\mathfrak{q}$.

3.4. Example of reducible representations

In this section, we deal with semisimple Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$ or $\mathfrak{sp}_n$ whose action on $V = \mathbb{C}^n$, $\mathbb{C}^n$ or $\mathbb{C}^{2n}$ is reducible. We have already discussed those subalgebras $\mathfrak{h}$ which are maximal in $\mathfrak{g}$ in Propositions 3.2 and 3.3. We focus on the most important examples for which $\mathfrak{h}$ is not maximal.
The first proposition deals mainly with the case where the vector space V has more than two irreducible components.

Proposition 3.6. *Let $r \geq 1$, $n \geq n_1 + \cdots + n_r$ with $n_1 \geq \cdots \geq n_r \geq 1$.*

- *If $\mathfrak{g} = \mathfrak{sl}_n \supset \mathfrak{h} = \mathfrak{sl}_{n_1} \oplus \cdots \oplus \mathfrak{sl}_{n_r}$, then*
  $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \Leftrightarrow 2n_1 \geq n + 2$. In this case, one has $\mathfrak{m}_s \simeq \mathfrak{sl}_{2n_1-n}$.
- *If $\mathfrak{g} = \mathfrak{so}_n \supset \mathfrak{h} = \mathfrak{so}_{n_1} \oplus \cdots \oplus \mathfrak{so}_{n_r}$, then*
  $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \Leftrightarrow 2n_1 \geq n + 3$. In this case, one has $\mathfrak{m} \simeq \mathfrak{so}_{2n_1-n}$.
- *If $\mathfrak{g} = \mathfrak{sp}_n \supset \mathfrak{h} = \mathfrak{sp}_{n_1} \oplus \cdots \oplus \mathfrak{sp}_{n_r}$, then*
  $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \Leftrightarrow 2n_1 \geq n + 1$ or $n = 2n_1 = 2n_2$.
  In this case, one has $\mathfrak{m} \supset \mathfrak{sp}_{2n_1-n}$ or $\mathfrak{m} \simeq (\mathfrak{sp}_1)^{n_1}$ respectively.

When $r = 2$, Proposition 3.6 is Proposition 3.2.
When $r = 3$, Proposition 3.6 follows from Proposition 6.10.
When $r \geq 4$, the proof for the implication $\Rightarrow$ is by induction on r replacing the last two integers n_{r-1} and n_r by their sum $n_{r-1}+n_r$ and reordering.
The opposite implication $\Leftarrow$ is easier. To see this, let $\mathfrak{h}_1$ be the first factor of $\mathfrak{h}$, and we set $c = 2, 3$, and 1 for $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$, respectively. Then one computes

$$p_{\mathfrak{g}/\mathfrak{h}_1} = (n_1 + 1 - c) \,/\, (n - n_1),$$

by using (12) and Propositions 5.1, 5.2, 5.4, and 5.3, respectively. Hence

$$\rho_{\mathfrak{h}_1} \not\leq \rho_{\mathfrak{g}/\mathfrak{h}_1} \quad \text{if } 2n_1 \geq n + c, \tag{30}$$

and thus the sufficiency of the inequality in Proposition 3.6 is shown.
The second proposition deals mainly with the case where the vector space V has two irreducible components.

Proposition 3.7. *Let $p \geq 1$, $q \geq 1$.*

- *If $\mathfrak{g} = \mathfrak{sl}_{2p+q} \supset \mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{sl}_q$, then $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \Leftrightarrow q \geq 2p + 2$.*
  In this case, one has $\mathfrak{m}_s \simeq \mathfrak{sl}_{q-2p}$.
- *If $\mathfrak{g} = \mathfrak{so}_{2p+q} \supset \mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{so}_q$, then $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \Leftrightarrow q \geq 2p + 3$.*
  In this case, one has $\mathfrak{m} \simeq \mathfrak{so}_{q-2p}$.
- *If $\mathfrak{g} = \mathfrak{sp}_{p+q} \supset \mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{sp}_q$, then $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \Leftrightarrow q \geq p + 1$.*
  In this case, one has $\mathfrak{m}_s \simeq \mathfrak{sp}_{q-p}$.
- *If $\mathfrak{g} = \mathfrak{so}_{4p} \supset \mathfrak{h}' = \mathfrak{sl}_{2p} \supset \mathfrak{h} = \mathfrak{sp}_p$ and $p \geq 2$, then one has $\rho_\mathfrak{h} \leq \rho_\mathfrak{q}$.*

Proposition 3.7 follows from Proposition 6.8. Alternatively, the implication $\Leftarrow$ in Proposition 3.7 follows readily from (30).

The second proposition deals mainly with the case where the vector space V has two irreducible components.

Proposition 3.8. *Let $q \geq 1$.*

- *If $\mathfrak{g} = \mathfrak{so}_{7+q} \supset \mathfrak{h} = \mathfrak{g}_2 \oplus \mathfrak{so}_q$, then $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \Leftrightarrow q = 1$ or $q \geq 10$.*
  In this case, one has $\mathfrak{m} \simeq \mathfrak{sl}_2$ or $\mathfrak{m} \simeq \mathfrak{so}_{q-7}$.
- *If $\mathfrak{g} = \mathfrak{so}_{8+q} \supset \mathfrak{h} = \mathfrak{so}_7 \oplus \mathfrak{so}_q$, then $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \Leftrightarrow q = 1$, $q = 2$, or $q \geq 11$.*
  In this case, one has $\mathfrak{m} \simeq \mathfrak{sl}_3$, $\mathfrak{m} \simeq \mathfrak{sl}_2$ or $\mathfrak{m} \simeq \mathfrak{so}_{q-8}$.

Proposition 3.8 follows from Proposition 6.9.

3.5. Checking Theorem 3.1

In this section, we check Theorem 3.1. Let V be $\mathbb{C}^n$ or $\mathbb{C}^{2n}$ when $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{so}_n$ or $\mathfrak{g} = \mathfrak{sp}_n$, respectively. We have to deal now with pairs $(\mathfrak{g}, \mathfrak{h})$ for which the action of $\mathfrak{h}$ on V is reducible.

Proposition 3.9. *Let $p \geq q \geq 1$*

- *Let $\mathfrak{g} = \mathfrak{sl}_{p+q}$, and $\mathfrak{h} \subset \mathfrak{g}$ a semisimple Lie subalgebra included in $\mathfrak{sl}_p \oplus \mathfrak{sl}_q$ irreducible on $\mathbb{C}^p$. One has the equivalence:*
$$\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \iff p \geq q+2 \text{ and } \mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{h}' \text{ with } \mathfrak{h}' \subset \mathfrak{sl}_q. \tag{31}$$
- *Let $\mathfrak{g} = \mathfrak{so}_{p+q}$, and $\mathfrak{h} \subset \mathfrak{g}$ a semisimple Lie subalgebra included in $\mathfrak{so}_p \oplus \mathfrak{so}_q$, irreducible on $\mathbb{C}^p$. One has the equivalence:*
$$\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \iff \begin{cases} \text{either } p \geq q+3 \text{ and } \mathfrak{h} = \mathfrak{so}_p \oplus \mathfrak{h}' \text{ with } \mathfrak{h}' \subset \mathfrak{so}_q; \\ \text{or } p = 7,\ q = 1 \text{ and } \mathfrak{h} = \mathfrak{g}_2 \overset{irr}{\hookrightarrow} \mathfrak{so}_7; \\ \text{or } p = 8,\ q \leq 2 \text{ and } \mathfrak{h} = \mathfrak{so}_7 \overset{irr}{\hookrightarrow} \mathfrak{so}_8. \end{cases} \tag{32}$$
- *Let $\mathfrak{g} = \mathfrak{sp}_{p+q}$, and $\mathfrak{h} \subset \mathfrak{g}$ a semisimple Lie subalgebra included in $\mathfrak{sp}_p \oplus \mathfrak{sp}_q$, irreducible on $\mathbb{C}^{2p}$. One has the equivalence :*
$$\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q} \iff \begin{cases} \text{either } p \geq q+1 \text{ and } \mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{h}' \text{ with } \mathfrak{h}' \subset \mathfrak{sp}_q; \\ \text{or } p = q \text{ and } \mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{sp}_p. \end{cases} \tag{33}$$

The implication $\Leftarrow$ is straightforward. To see the nontrivial implication $\Rightarrow$, we observe that $\rho_{\mathfrak{k}_2} \leq \rho_{\mathfrak{g}/\mathfrak{k}}$ on $\mathfrak{k}_2$ as in (30), where $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_{p+q}, \mathfrak{sl}_p \oplus \mathfrak{sl}_q)$, $(\mathfrak{so}_{p+q}, \mathfrak{so}_p \oplus \mathfrak{so}_q)$, or $(\mathfrak{sp}_{p+q}, \mathfrak{sp}_p \oplus \mathfrak{sp}_q)$ with $p \geq q$. Then the implication $\Rightarrow$ in Proposition 3.9 follows from Lemma 3.10 below and from the three previous Propositions 3.5, 3.7 and 3.8.

Lemma 3.10. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{k} \subset \mathfrak{g}$ a semisimple Lie subalgebra which is a direct sum $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ of two ideals of $\mathfrak{k}$, $\mathfrak{h}_1 \subset \mathfrak{k}_1$ a semisimple Lie subalgebra and $\mathfrak{h} := \mathfrak{h}_1 \oplus \mathfrak{k}_2$. Assume that $\rho_{\mathfrak{h}_1} \leq \rho_{\mathfrak{k}_1/\mathfrak{h}_1}$ and $\rho_{\mathfrak{k}_2} \leq \rho_{\mathfrak{g}/\mathfrak{k}}$, then one has $\rho_\mathfrak{h} \leq \rho_\mathfrak{q}$.*

Lemma 3.10 is a special case of Lemma 2.14 (3).

Theorem 3.1 follows from Dynkin's classification of maximal semisimple Lie algebras in the classical Lie algebras by using the eight previous propositions.

4. Exceptional simple Lie algebras

In this chapter we give a classification of the pairs $(\mathfrak{g},\mathfrak{h})$ of complex semisimple Lie algebras satisfying $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{g}/\mathfrak{h}}$ in the case where $\mathfrak{g}$ is exceptional simple, and in particular, prove Theorems 1.6 and 1.2 for $\mathfrak{g}$ exceptional simple.

Throughout this chapter, $\mathfrak{g}$ is a complex exceptional simple Lie algebra, $\mathfrak{h}$ is a complex semisimple Lie subalgebra $\{0\} \neq \mathfrak{h} \neq \mathfrak{g}$, $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$, $\widetilde{\mathfrak{h}}$ the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{m}$ is the generic stabilizer of $\mathfrak{q}$ in $\mathfrak{h}$, and $\mathfrak{m}_s := [\mathfrak{m},\mathfrak{m}]$.

4.1. Main list for exceptional Lie algebras

Theorem 4.1. *Let $\mathfrak{g} = \mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ or $\mathfrak{e}_8$ be a complex exceptional simple Lie algebra. The complex semisimple Lie subalgebras $\mathfrak{h} \subsetneq \mathfrak{g}$ satisfying $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$ form the list in Table 2. In this list $\mathfrak{q}$ does not have* AGS *in $\mathfrak{h}$.*

Case	$\mathfrak{g}$	maximal $\mathfrak{h}$	$\mathfrak{m}$	non maximal $\mathfrak{h}$	$\mathfrak{m}$
$G2$	$\mathfrak{g}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_1 \oplus \mathbb{C}$		
$F4$	$\mathfrak{f}_4$	$\mathfrak{b}_4$	$\mathfrak{b}_3$	$\mathfrak{d}_4$	$\mathfrak{a}_2$
$E6.1$	$\mathfrak{e}_6$	$\mathfrak{d}_5$	$\mathfrak{d}_3 \oplus \mathbb{C}$	$\mathfrak{b}_4$	$\mathfrak{a}_1$
$E6.2$	$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{d}_4$	$\mathfrak{b}_4$	$\mathfrak{a}_1$
$E7.1$	$\mathfrak{e}_7$	$\mathfrak{d}_6 \oplus \mathfrak{a}_1$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$	$\mathfrak{d}_6$	$\supset \mathfrak{a}_1$
$E7.2$	$\mathfrak{e}_7$	$\mathfrak{e}_6$	$\mathfrak{d}_4$		
$E8$	$\mathfrak{e}_8$	$\mathfrak{e}_7 \oplus \mathfrak{a}_1$	$\mathfrak{d}_4$	$\mathfrak{e}_7$	$\mathfrak{d}_4$

Table 2: Pairs $(\mathfrak{g},\mathfrak{h})$ with $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$ for $\mathfrak{g}$ exceptional simple

Here are some comments on the pairs $(\mathfrak{g},\mathfrak{h})$ in this list with $\mathfrak{h}$ maximal.

The pair $(\mathfrak{g}_2,\mathfrak{a}_2)$ is the only one for which $(\mathfrak{g},\widetilde{\mathfrak{h}})$ is not a symmetric pair.

The pair $(\mathfrak{e}_6,\mathfrak{f}_4)$ is the only one with $\operatorname{rank}\widetilde{\mathfrak{h}} < \operatorname{rank}\mathfrak{g}$.

The pairs $(\mathfrak{f}_4,\mathfrak{b}_4)$, $(\mathfrak{e}_7,\mathfrak{a}_1\oplus\mathfrak{d}_6)$ and $(\mathfrak{e}_8,\mathfrak{a}_1\oplus\mathfrak{e}_7)$ are equal rank symmetric pairs.

The pairs $(\mathfrak{e}_6,\widetilde{\mathfrak{d}}_5)$ and $(\mathfrak{e}_7,\widetilde{\mathfrak{e}}_6)$, are equal rank Hermitian symmetric pairs.

Once we find the list of the pairs $(\mathfrak{g},\mathfrak{h})$ in Table 2, it is straightforward to verify $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$ for such $(\mathfrak{g},\mathfrak{h})$ by finding a witness (Definition 7.2), or alternatively, by using Proposition 2.8 and checking that the generic stabilizer $\mathfrak{m}$ is nonabelian as indicated in Table 2. Thus the nontrivial part of Theorem 4.1 is to prove that Table 2 exhausts all the pairs $(\mathfrak{g},\mathfrak{h})$ satisfying $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$.

The strategy of this proof is to deal first with pairs $(\mathfrak{g},\mathfrak{h})$ where $\mathfrak{h}$ is maximal in $\mathfrak{g}$. Dynkin's list of all these pairs is given in Section 4.2. These pairs are studied one by one in Section 4.4 using upper bounds for the invariants p_V which are stated in Section 4.3 and explained in Chapter 5. We find that there are exactly 7 pairs $(\mathfrak{g},\mathfrak{h})$ with $\mathfrak{g}$ simple exceptional and $\mathfrak{h}$ semisimple maximal for which $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$: they form the left-hand side of Table 2.

Then for each of these 7 pairs $(\mathfrak{g},\mathfrak{h}')$ we describe in Section 4.5, the semisimple Lie subalgebras $\mathfrak{h} \subset \mathfrak{h}'$ for which one still has $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$. This uses also the bounds for the p_Vs proven in Chapter 5.

4.2. Dynkin classification

For all complex simple Lie algebras $\mathfrak{g}$, Dynkin [10] has classified maximal semisimple Lie subalgebras $\mathfrak{h}$.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{q}$	i	$d_{\mathfrak{g}} = d_{\mathfrak{h}} + d_{\mathfrak{q}}$	AGS
$\mathfrak{g}_2$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$S^3\mathbb{C}^2 \otimes \mathbb{C}^2$	2	$14 = 6 + 8$	Y
	$\mathfrak{a}_2$	$\mathbb{C}^3 \oplus$ dual	3	$14 = 8 + 6$	N
$\mathfrak{f}_4$	$\mathfrak{b}_4$	$\mathbb{C}^{16}$	2	$52 = 36 + 16$	N
	$\mathfrak{a}_1 \oplus \mathfrak{c}_3$	$\mathbb{C}^2 \otimes \Lambda^3_0\mathbb{C}^6$	2	$52 = 24 + 28$	Y
	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$S^2\mathbb{C}^3 \otimes \mathbb{C}^3 \oplus$ dual	3	$52 = 16 + 36$	Y
$\mathfrak{e}_6$	$\mathfrak{d}_5$	$\mathbb{C} \oplus (\mathbb{C}^{16} \oplus$ dual$)$	1	$78 = 45 + 33$	N
	$\mathfrak{a}_1 \oplus \mathfrak{a}_5$	$\mathbb{C}^2 \otimes \Lambda^3\mathbb{C}^6$	2	$78 = 38 + 40$	Y
	$\mathfrak{a}_2 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \oplus$ dual	3	$78 = 24 + 54$	Y
$\mathfrak{e}_7$	$\mathfrak{e}_6$	$\mathbb{C} \oplus (\mathbb{C}^{27} \oplus$ dual$)$	1	$133 = 78 + 55$	N
	$\mathfrak{a}_7$	$\Lambda^4\mathbb{C}^8$	2	$133 = 63 + 70$	Y
	$\mathfrak{a}_1 \oplus \mathfrak{d}_6$	$\mathbb{C}^2 \otimes \mathbb{C}^{32}$	2	$133 = 69 + 64$	N
	$\mathfrak{a}_2 \oplus \mathfrak{a}_5$	$\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^6 \oplus$ dual	3	$133 = 43 + 90$	Y
$\mathfrak{e}_8$	$\mathfrak{d}_8$	$\mathbb{C}^{128}$	2	$248 = 120 + 128$	Y
	$\mathfrak{a}_1 \oplus \mathfrak{e}_7$	$\mathbb{C}^2 \otimes \mathbb{C}^{56}$	2	$248 = 136 + 112$	N
	$\mathfrak{a}_8$	$\Lambda^3\mathbb{C}^9 \oplus$ dual	3	$248 = 80 + 168$	Y
	$\mathfrak{a}_2 \oplus \mathfrak{e}_6$	$\mathbb{C}^3 \otimes \mathbb{C}^{27} \oplus$ dual	3	$248 = 86 + 162$	Y
	$\mathfrak{a}_4 \oplus \mathfrak{a}_4$	$(\Lambda^2\mathbb{C}^5 \otimes \mathbb{C}^5 \oplus \mathbb{C}^5 \otimes \Lambda^2\mathbb{C}^5) \oplus$ dual	5	$248 = 48 + 200$	Y

Table 3: R-subalgebras of exceptional simple Lie algebras

Definition 4.2. A maximal semisimple Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called an *R-subalgebra* if $\operatorname{rank} \widetilde{\mathfrak{h}} = \operatorname{rank} \mathfrak{g}$ and called an *S-subalgebra* if $\operatorname{rank} \widetilde{\mathfrak{h}} < \operatorname{rank} \mathfrak{g}$.

The classification of R-subalgebras goes back to Borel–Siebenthal. Later, Dynkin has given a nice interpretation of this list using the so called extended Dynkin diagram. This list is given in Table 3.

The classification of S-subalgebras is due to Dynkin (except for $\mathfrak{a}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_2$ in $\mathfrak{e}_8$ which is forgotten there, see [14]). The list is given in Table 4.

Here are a few comments on Tables 3 and 4. To each R-subalgebra, Dynkin associates an integer $i = 1,\ 2,\ 3$ or 5. When $i = 1$, $\widetilde{\mathfrak{h}}$ has a one-dimensional center and $(\mathfrak{g}, \widetilde{\mathfrak{h}})$ is the complexification of a Hermitian symmetric pair. When $i \geq 2$, one has $\widetilde{\mathfrak{h}} = \mathfrak{h}$ and $\mathfrak{h}$ is the set of fixed points of an automorphism of $\mathfrak{g}$ of order i.

In the Tables 3 and 4, the third column describes $\mathfrak{q}$ as a representation of $\mathfrak{h}$. For Table 3, it is obtained thanks to Dynkin's construction of the R-subalgebras using the extended Dynkin diagrams. For Table 4, it is based on the Tables 10.1 in [14, p. 214–215].

In this third column, the notation $\mathbb{C}^n$ means "one of irreducible representation of dimension n". The slight ambiguity does not affect our consequence. For instance $\mathbb{C}^{27}$ is one of the two 27-dimensional irreducible representations of $\mathfrak{e}_6$, which are

dual to each other. Similarly $\mathbb{C}^{16}$ is one of the two dual 16-dimensional irreducible representations of $\mathfrak{d}_5$ called the half-spin representation. As a representation of $\mathfrak{b}_4$, $\mathbb{C}^{16}$ is still irreducible and is orthogonal. The subscript 0 in notations like $\Lambda^3_0\mathbb{C}^6$ means the irreducible subrepresentation spanned by the highest weight vector *e.g.*, $\Lambda^3_0\mathbb{C}^6 \simeq \Lambda^3\mathbb{C}^6/\mathbb{C}^6$.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{q}$	$d_\mathfrak{g}$	=	$d_\mathfrak{h} + d_\mathfrak{q}$	AGS
$\mathfrak{g}_2$	$\mathfrak{a}_1$	$*$	14	=	$3 + 11$	Y
$\mathfrak{f}_4$	$\mathfrak{a}_1$	$*$	52	=	$3 + 49$	Y
	$\mathfrak{a}_1 \oplus \mathfrak{g}_2$	$*$	52	=	$17 + 35$	Y
$\mathfrak{e}_6$	$\mathfrak{a}_2$	$*$	78	=	$8 + 70$	Y
	$\mathfrak{g}_2$	$*$	78	=	$14 + 64$	Y
	$\mathfrak{c}_4$	$\Lambda^4_0\mathbb{C}^8$	78	=	$36 + 42$	Y
	$\mathfrak{f}_4$	$\mathbb{C}^{26}$	78	=	$52 + 26$	N
	$\mathfrak{a}_2 \oplus \mathfrak{g}_2$	$\mathbb{C}^8 \otimes \mathbb{C}^7$	78	=	$22 + 56$	Y
$\mathfrak{e}_7$	$\mathfrak{a}_1$ (twice)	$*$	133	=	$3 + 130$	Y
	$\mathfrak{a}_2$	$*$	133	=	$8 + 125$	Y
	$\mathfrak{a}_1 \oplus \mathfrak{a}_1$	$*$	133	=	$6 + 127$	Y
	$\mathfrak{a}_1 \oplus \mathfrak{g}_2$	$*$	133	=	$17 + 116$	Y
	$\mathfrak{a}_1 \oplus \mathfrak{f}_4$	$\mathbb{C}^3 \otimes \mathbb{C}^{26}$	133	=	$55 + 78$	Y
	$\mathfrak{g}_2 \oplus \mathfrak{c}_3$	$\mathbb{C}^7 \otimes \Lambda^2_0\mathbb{C}^6$	133	=	$35 + 98$	Y
$\mathfrak{e}_8$	$\mathfrak{a}_1$ (3 times)	$*$	248	=	$3 + 245$	Y
	$\mathfrak{b}_2$	$*$	248	=	$10 + 238$	Y
	$\mathfrak{a}_1 \oplus \mathfrak{a}_2$	$*$	248	=	$11 + 237$	Y
	$\mathfrak{a}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_2$	$\mathbb{C}^3 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7 \oplus \mathbb{C}^5 \otimes (\mathbb{C}^7 \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathbb{C}^7)$	248	=	$31 + 217$	Y
	$\mathfrak{g}_2 \oplus \mathfrak{f}_4$	$\mathbb{C}^7 \otimes \mathbb{C}^{26}$	248	=	$66 + 182$	Y

Table 4: S-subalgebras of exceptional simple Lie algebras

In Tables 3 and 4, the last column tells us (*Yes* or *No*) according to whether $\mathfrak{h}$ has AGS in $\mathfrak{q}$ or not. The answers *No* are deduced from the fact that those pairs $(\mathfrak{g}, \mathfrak{h})$ are equal to the complexification $(\mathfrak{g}_{1,\mathbb{C}}, \mathfrak{k}_{1,\mathbb{C}})$ of a Riemannian symmetric pair $(\mathfrak{g}_1, \mathfrak{k}_1)$ for which the real semisimple Lie algebra $\mathfrak{g}_1$ is not quasisplit (except for the pair $(\mathfrak{g}_2, \mathfrak{a}_2)$ for which one computes directly that the generic stabilizer is $\mathfrak{m} = \mathfrak{a}_1$). The answers *Yes* will be deduced from Proposition 2.8, once we will have checked the inequality $\rho_\mathfrak{h} \leq \rho_\mathfrak{q}$.

4.3. Irreducible representations of simple Lie algebras

In order to prove Theorem 4.1, we will need to compute accurately the real number p_V defined in (6) for many irreducible representations V of simple Lie algebras $\mathfrak{h}$. Most of the results that we will need are contained in Tables 5, 6, and 7 below.

Theorem 4.3. *Let $\mathfrak{h}$ be a complex simple Lie algebra and V be an irreducible faithful representation of $\mathfrak{h}$.*

If V is self-dual and $\rho_\mathfrak{h} \not\leq \rho_V$, then $(\mathfrak{h}, V)$ is in Table 5.

If V is not self-dual and $\rho_\mathfrak{h} \not\leq 2\,\rho_V$, then $(\mathfrak{h}, V)$ or $(\mathfrak{h}, V^*)$ is in Table 6.

Theorem 4.3 will be explained in Chapter 5. The following corollary tells us that Conjecture 2.10 is true when $\mathfrak{h}$ is simple.

$\mathfrak{h}$	V	p_V	parameter	duality	name
$\mathfrak{a}_1$	$\mathbb{C}^2$	2		sympl.	standard
$\mathfrak{a}_3$	$\Lambda^2\mathbb{C}^4$	4		orth.	V_{ω_2}
$\mathfrak{a}_5$	$\Lambda^3\mathbb{C}^6$	2		sympl.	V_{ω_3}
$\mathfrak{b}_\ell$	$\mathbb{C}^{2\ell+1}$	$2\ell-1$	$\ell \geq 2$	orth.	standard
$\mathfrak{b}_2$	$\mathbb{C}^4$	4		sympl.	spin
$\mathfrak{b}_3$	$\mathbb{C}^8$	4		orth.	spin
$\mathfrak{b}_4$	$\mathbb{C}^{16}$	3		orth.	spin
$\mathfrak{b}_5$	$\mathbb{C}^{32}$	2		sympl.	spin
$\mathfrak{b}_6$	$\mathbb{C}^{64}$	4/3		sympl.	spin
$\mathfrak{c}_\ell$	$\mathbb{C}^{2\ell}$	2ℓ	$\ell \geq 3$	sympl.	standard
	$\Lambda^2_0\mathbb{C}^{2\ell}$	$\frac{\ell+1}{\ell-1}$	$\ell \geq 3$	orth.	V_{ω_2}
$\mathfrak{c}_3$	$\Lambda^3_0\mathbb{C}^6$	5/3		sympl.	V_{ω_3}
$\mathfrak{d}_\ell$	$\mathbb{C}^{2\ell}$	$2\ell-2$	$\ell \geq 4$	orth.	standard
$\mathfrak{d}_4$	$\mathbb{C}^8$	6		orth.	half-spin
$\mathfrak{d}_6$	$\mathbb{C}^{32}$	5/2		sympl.	half-spin
$\mathfrak{e}_7$	$\mathbb{C}^{56}$	17/6		sympl.	V_{ω_7}
$\mathfrak{f}_4$	$\mathbb{C}^{26}$	8/3		orth.	V_{ω_4}
$\mathfrak{g}_2$	$\mathbb{C}^7$	3		orth.	V_{ω_1}

Table 5: Self-dual irreducible faithful representations V of simple Lie algebra $\mathfrak{h}$ with $p_V > 1$

$\mathfrak{h}$	V	p_V	parameter	name
$\mathfrak{a}_\ell$	$\mathbb{C}^{\ell+1}$	2ℓ	$\ell \geq 2$	standard
$\mathfrak{a}_\ell$	$\Lambda^2\mathbb{C}^{\ell+1}$	$2\,\frac{\ell+2}{\ell}$	$\ell \geq 4$, even	V_{ω_2}
		$2\,\frac{\ell+1}{\ell-1}$	$\ell \geq 5$, odd	V_{ω_2}
$\mathfrak{d}_5$	$\mathbb{C}^{16}$	7/2		half-spin
$\mathfrak{e}_6$	$\mathbb{C}^{27}$	7/2		V_{ω_1}

Table 6: Non-self-dual irreducible representations for $\mathfrak{h}$ simple with $p_V > 2$

Corollary 4.4. *Let $\mathfrak{h}$ be a complex simple Lie algebra and V a complex orthogonal representation of $\mathfrak{h}$. One has the equivalence:*

$$V \text{ has AGS in } \mathfrak{h} \iff \rho_{\mathfrak{h}} \leq \rho_V. \tag{34}$$

We already know from Proposition 2.8 the implication $\Leftarrow$ in Corollary 4.4 holds. The opposite implication of Corollary 4.4 is proven by decomposing V into irreducible components and by checking, using Tables 5 and 6, that when $\rho_{\mathfrak{h}} \not\leq \rho_V$ then the generic stabilizer is not abelian.

4.4. Checking Theorem 4.1 for $\mathfrak{h}$ maximal

We just have to check that all pairs $(\mathfrak{g}, \mathfrak{h})$ occurring in Dynkin's classification (Tables 3 and 4) with "Y" in the last column satisfy $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.

For the 12 cases with $\mathfrak{h}$ simple of rank 1 or 2, this follows from Corollary 5.8. We just notice that when $\mathfrak{h}$ is an S-subalgebra, the centralizer of $\mathfrak{h}$ is trivial.

For the 5 cases with $\mathfrak{h}$ product of $\mathfrak{a}_1$ by a simple Lie algebra $\mathfrak{h}_2$ of rank 1 or 2, this follows from Corollary 6.5. We just notice that when $\mathfrak{h}'$ is a non-zero ideal of an S-subalgebra, the centralizer of $\mathfrak{h}'$ is included in $\mathfrak{h}$.

For the 4 cases with $\mathfrak{h}$ simple of rank ≥ 3, this follows from Table 7 which is part of Theorem 4.3.

$\mathfrak{h}$	V	p_V	duality	name
$\mathfrak{a}_7$	$\Lambda^4\mathbb{C}^8$	$p_V \leq 1$	orth.	V_{ω_4}
$\mathfrak{a}_8$	$\Lambda^3\mathbb{C}^9$	$p_V \leq 2$	non-auto.	V_{ω_3}
$\mathfrak{c}_4$	$\Lambda^4_0\mathbb{C}^8$	$p_V \leq 1$	orth.	V_{ω_4}
$\mathfrak{d}_8$	$\mathbb{C}^{128}$	$p_V \leq 1$	orth.	half-spin

Table 7: "Useful" representations which are not in Tables 5 and 6

For the 7 remaining cases in Table 3, we conclude with Lemma 4.5.

For the 5 remaining cases in Table 4, we conclude with Lemma 4.6.

Lemma 4.5. (1) *Let* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{c}_3$ *and* $V = \mathbb{C}^2 \otimes \Lambda^3_0\mathbb{C}^6$. *Then one has* $\rho_{\mathfrak{h}} \leq \rho_V$.
(2) *Let* $\mathfrak{h} = \mathfrak{a}_2 \oplus \mathfrak{a}_2$ *and* $V = S^2\mathbb{C}^3 \otimes \mathbb{C}^3$. *Then one has* $\rho_{\mathfrak{h}} \leq 2\rho_V$.
(3) *Let* $\mathfrak{h} = \mathfrak{a}_2 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_2$ *and* $V = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. *Then one has* $\rho_{\mathfrak{h}} \leq 2\rho_V$.
(4) *Let* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_5$ *and* $V = \mathbb{C}^2 \otimes \Lambda^3\mathbb{C}^6$. *Then one has* $\rho_{\mathfrak{h}} \leq \rho_V$.
(5) *Let* $\mathfrak{h} = \mathfrak{a}_2 \oplus \mathfrak{a}_5$ *and* $V = \mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^6$. *Then one has* $\rho_{\mathfrak{h}} \leq 2\rho_V$.
(6) *Let* $\mathfrak{h} = \mathfrak{a}_2 \oplus \mathfrak{e}_6$ *and* $V = \mathbb{C}^3 \otimes \mathbb{C}^{27}$. *Then one has* $\rho_{\mathfrak{h}} \leq 2\rho_V$.
(7) *Let* $\mathfrak{h} = \mathfrak{a}_4 \oplus \mathfrak{a}_4$ *and* $V = \mathbb{C}^5 \otimes \Lambda^2\mathbb{C}^5$. *Then one has* $\rho_{\mathfrak{h}} \leq 4\rho_V$.

Lemma 4.6. (1) *Let* $\mathfrak{h} = \mathfrak{a}_2 \oplus \mathfrak{g}_2$ *and* $V = \mathbb{C}^8 \otimes \mathbb{C}^7$. *Then one has* $\rho_{\mathfrak{h}} \leq \rho_V$.
(2) *Let* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{f}_4$ *and* $V = S^2\mathbb{C}^2 \otimes \mathbb{C}^{26}$. *Then one has* $\rho_{\mathfrak{h}} \leq \rho_V$.
(3) *Let* $\mathfrak{h} = \mathfrak{g}_2 \oplus \mathfrak{c}_3$ *and* $V = \mathbb{C}^7 \otimes \Lambda^2_0\mathbb{C}^6$. *Then one has* $\rho_{\mathfrak{h}} \leq \rho_V$.
(4) *Let* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_2$ *and* $V = \mathbb{C}^3 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$. *Then one has* $\rho_{\mathfrak{h}} \leq \rho_V$.
(5) *Let* $\mathfrak{h} = \mathfrak{g}_2 \oplus \mathfrak{f}_4$ *and* $V = \mathbb{C}^7 \otimes \mathbb{C}^{26}$. *Then one has* $\rho_{\mathfrak{h}} \leq \rho_V$.

Checking Lemma 4.5. We will write $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_1$ simple and $V = V_1 \otimes V_2$. For $i = 1, 2$, we will write $p_i = p_{V_i}$, $d_i = \dim V_i$ and apply the bound $p_V \leq \frac{p_1}{d_2} + \frac{p_2}{d_1}$ from Lemma 2.14, and the values of p_i given in Tables 5 and 6.
(1) One has $d_1 = 2$, $p_1 = 2$, $d_2 = 14$, $p_2 = \frac{5}{3}$, hence $p_V \leq \frac{2}{14} + \frac{5}{6} \leq 1$.
(2) One has $d_1 = 6$, $p_1 = \frac{4}{3}$, $d_2 = 3$, $p_2 = 4$, hence $p_V \leq \frac{4}{9} + \frac{4}{6} \leq 2$.
(3) One has $d_1 = 3$, $p_1 = 4$, $d_2 = 9$, $p_2 \leq \frac{8}{3}$, hence $p_V \leq \frac{4}{9} + \frac{8}{9} \leq 2$.
(4) One has $d_1 = 2$, $p_1 = 2$, $d_2 = 20$, $p_2 = 2$. This is not enough to conclude. But a direct computation shows $\rho_{\mathfrak{h}} \leq \rho_V$.

(5) One has $d_1 = 3,\ p_1 = 4,\ d_2 = 15,\ p_2 = 3$, hence $p_V \leq \frac{4}{15} + \frac{3}{3} \leq 2$.
(6) One has $d_1 = 3,\ p_1 = 4,\ d_2 = 27,\ p_2 = \frac{7}{2}$, hence $p_V \leq \frac{4}{27} + \frac{7}{6} \leq 2$.
(7) One has $d_1 = 5,\ p_1 = 8,\ d_2 = 10,\ p_2 = 3$, hence $p_V \leq \frac{8}{10} + \frac{3}{5} \leq 4$. ∎

Checking Lemma 4.6. We use the same notations.
(1) One has $d_1 = 8,\ p_1 = 1,\ d_2 = \ 7,\ p_2 = 3$, hence $p_V \leq \frac{1}{7} + \frac{3}{8} \leq 1$.
(2) One has $d_1 = 3,\ p_1 = 1,\ d_2 = 26,\ p_2 = \frac{8}{3}$, hence $p_V \leq \frac{1}{26} + \frac{8}{9} \leq 1$.
(3) One has $d_1 = 7,\ p_1 = 3,\ d_2 = 14,\ p_2 = 2$, hence $p_V \leq \frac{3}{14} + \frac{2}{7} \leq 1$.
(4) One has $d_1 = 3,\ p_1 = 1,\ d_2 = 49,\ p_2 \leq \frac{6}{7}$, hence $p_V \leq \frac{1}{49} + \frac{6}{21} \leq 1$.
(5) One has $d_1 = 7,\ p_1 = 3,\ d_2 = 26,\ p_2 = \frac{8}{3}$, hence $p_V \leq \frac{3}{26} + \frac{8}{21} \leq 1$. ∎

4.5. Checking Theorem 4.1 for $\mathfrak{h}$ non-maximal

We first consider the case where $\mathfrak{h}$ is maximal in a maximal semisimple Lie algebra $\mathfrak{h}'$ of $\mathfrak{g}$. Taking an $\mathfrak{h}'$-invariant subspace $\mathfrak{q}'$ in $\mathfrak{g}$ and an $\mathfrak{h}$-invariant subspace $\mathfrak{q}''$ in $\mathfrak{h}'$, we write

$$\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{q}', \quad \mathfrak{h}' = \mathfrak{h} \oplus \mathfrak{q}'' \text{ and } \mathfrak{q} = \mathfrak{q}' \oplus \mathfrak{q}''$$

According to Section 4.4, the pair $(\mathfrak{g}, \mathfrak{h}')$ is among the 7 pairs in the left side of Table 2. Moreover the Lie algebra $\mathfrak{h}$ is also a maximal semisimple Lie subalgebra of $\mathfrak{h}'$ satisfying $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}''}$. One can find the list of such subalgebras $\mathfrak{h}$ from Theorem 3.1 and Table 2. Hence the triple $(\mathfrak{g}, \mathfrak{h}', \mathfrak{h})$ has to be in the following Table 8.

$\mathfrak{g}$	$\mathfrak{h}'$	$\mathfrak{h}$	$\mathfrak{q}' \simeq \mathfrak{g}/\mathfrak{h}'$	$\mathfrak{q}'' \simeq \mathfrak{h}'/\mathfrak{h}$	dimension $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$	AGS
$\mathfrak{f}_4$	$\mathfrak{b}_4$	$\mathfrak{d}_4$	$\mathbb{C}^8 \oplus \mathbb{C}^8$	$\mathbb{C}^8$	52=28+24	N
		$\mathfrak{b}_1\mathfrak{d}_3$	$(\mathbb{C}^2 \otimes \mathbb{C}^4) \oplus$ dual	$\mathbb{C}^3 \otimes \mathbb{C}^6$	52=18+34	Y
$\mathfrak{e}_6$	$\mathfrak{d}_5$	$\mathfrak{b}_4$	$\mathbb{C} \oplus (\mathbb{C}^{16} \oplus \mathbb{C}^{16})$	$\mathbb{C}^9$	78=36+42	N
		$\mathfrak{b}_1\mathfrak{b}_3$	$\mathbb{C} \oplus (\mathbb{C}^2 \otimes \mathbb{C}^8 \oplus$ dual)	$\mathbb{C}^3 \otimes \mathbb{C}^7$	78=24+54	Y
		$\mathfrak{a}_4$	$\mathbb{C}\oplus((\mathbb{C}\oplus\mathbb{C}^5\oplus\Lambda^2\mathbb{C}^5)\oplus$dual)	$\mathbb{C}\oplus(\Lambda^2\mathbb{C}^5\oplus$dual)	78=24+54	Y
	$\mathfrak{f}_4$	$\mathfrak{b}_4$	$\mathbb{C} \oplus \mathbb{C}^9 \oplus \mathbb{C}^{16}$	$\mathbb{C}^{16}$	78=36+42	N
$\mathfrak{e}_7$	$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}^{26}\oplus\mathbb{C}^{26}$	$\mathbb{C}^{26}$	133=45+88	Y
		$\mathfrak{d}_5$	$\mathbb{C}\oplus((\mathbb{C}\oplus\mathbb{C}^{10}\oplus\mathbb{C}^{16})\oplus$dual)	$\mathbb{C}\oplus(\mathbb{C}^{16}\oplus$dual)	133=52+81	Y
	$\mathfrak{a}_1\mathfrak{d}_6$	$\mathfrak{d}_6$	$\mathbb{C}^{32} \oplus \mathbb{C}^{32}$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	133=66+67	N
		$\mathfrak{a}_1\mathfrak{b}_5$	$\mathbb{C}^2 \otimes \mathbb{C}^{32}$	$\mathbb{C} \otimes \mathbb{C}^{11}$	133=58+75	Y
		$\mathfrak{a}_1\mathfrak{b}_1\mathfrak{b}_4$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{16}$	$\mathbb{C}\otimes\mathbb{C}^3\otimes\mathbb{C}^9$	133=42+91	Y
		$\mathfrak{a}_1\mathfrak{d}_2\mathfrak{d}_4$	$\mathbb{C}^2\otimes(\mathbb{C}^2\otimes\mathbb{C}^8\oplus\mathbb{C}^2\otimes\mathbb{C}^8)$	$\mathbb{C}\otimes\mathbb{C}^4\otimes\mathbb{C}^8$	133=37+96	Y
		$\mathfrak{a}_1\mathfrak{a}_5$	$\mathbb{C}^2\otimes((\mathbb{C}\oplus\Lambda^2\mathbb{C}^6)\oplus$dual)	$\mathbb{C}\oplus((\mathbb{C}\otimes\Lambda^2\mathbb{C}^6)\oplus$dual)	133=38+95	Y
		$\mathfrak{a}_1\mathfrak{a}_5$	$\mathbb{C}^2\otimes(\mathbb{C}^6\oplus\mathbb{C}^6\oplus\Lambda^3\mathbb{C}^6)$	$\mathbb{C}\oplus((\mathbb{C}\otimes\Lambda^2\mathbb{C}^6)\oplus$dual)	133=38+95	Y
$\mathfrak{e}_8$	$\mathfrak{a}_1\mathfrak{e}_7$	$\mathfrak{e}_7$	$\mathbb{C}^{56} \oplus \mathbb{C}^{56}$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	248=133+115	N
		$\mathfrak{a}_1\mathfrak{e}_6$	$\mathbb{C}^2\otimes((\mathbb{C}\oplus\mathbb{C}^{27})\oplus$dual)	$\mathbb{C}\oplus(\mathbb{C}^{27}\oplus$dual)	248=81+167	Y
		$\mathfrak{a}_1\mathfrak{a}_1\mathfrak{d}_6$	$\mathbb{C}^2\otimes(\mathbb{C}\otimes\mathbb{C}^{32}\oplus\mathbb{C}^2\otimes\mathbb{C}^{12})$	$\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{32}$	248=72+176	Y

Table 8: Triples $(\mathfrak{g}, \mathfrak{h}', \mathfrak{h})$ to be studied

Note that triples $(\mathfrak{g}, \mathfrak{h}', \mathfrak{h})$ like $(\mathfrak{f}_4, \mathfrak{b}_4, \mathfrak{b}_3)$ or $(\mathfrak{e}_6, \mathfrak{d}_5, \mathfrak{d}_4)$ do not occur in Table 8 because in these examples $\mathfrak{h}$ is not maximal in $\mathfrak{h}'$. Such examples will be taken care

of in Lemma 4.9. Similarly, the triple $(\mathfrak{g}_2, \mathfrak{a}_2, \mathfrak{a}_1)$ does not occur in Table 8 because in this example $\rho_\mathfrak{h} \leq \rho_{\mathfrak{q}''}$.

In Table 8 we describe $\mathfrak{q}'$ and $\mathfrak{q}''$ as a representation of $\mathfrak{h}$, using Tables 3 and 4, which describes $\mathfrak{q}'$ as a representation of $\mathfrak{h}'$ and decomposing this representation as a sum of irreducible representations of $\mathfrak{h}$, i.e., the branching law for $\mathfrak{h}' \downarrow \mathfrak{h}$. There are two realizations of $\mathfrak{a}_5 = \mathfrak{sl}_6$ in $\mathfrak{d}_6 = \mathfrak{so}_{12}$, which is conjugate by an outer automorphism of $\mathfrak{d}_6$, but this automorphism does not extend to $\mathfrak{g} = \mathfrak{e}_7$. Accordingly, we have needed to list two different structures of $\mathfrak{q}' = \mathfrak{g}/\mathfrak{h}'$ as $\mathfrak{h}$-modules for the triple $(\mathfrak{g}, \mathfrak{h}', \mathfrak{h}) = (\mathfrak{e}_7, \mathfrak{a}_1\mathfrak{d}_6, \mathfrak{a}_1\mathfrak{a}_5)$. The Lie algebra $\mathfrak{d}_4 = \mathfrak{so}_8$ has three 8-dimensional irreducible representations V_{ω_1}, V_{ω_3}, V_{ω_4}, we have noted all of them as $\mathbb{C}^8$ since we will not need to know which is which.

We already know that for all triples $(\mathfrak{g}, \mathfrak{h}', \mathfrak{h})$ occurring in this Table 8 with "N" in the last column where $\mathfrak{q}$ does not have AGS in $\mathfrak{h}$, hence, by Proposition 2.8, they satisfy $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$.

It remains to check that all triples $(\mathfrak{g}, \mathfrak{h}', \mathfrak{h})$ occurring in this Table 8 with "Y" in the last column satisfy $\rho_\mathfrak{h} \leq \rho_\mathfrak{q}$.

For the 3 cases where $\mathfrak{h}$ is simple, this follows directly from inequality (10) and Lemma 4.7.

For the 9 cases where $\mathfrak{h}$ is not simple, this follows directly from inequality (10) and Lemma 4.8.

Lemma 4.7. (1) *Let $\mathfrak{h} = \mathfrak{a}_4$ and $V = \Lambda^2\mathbb{C}^5$. Then one has $p_V \leq 3$.*

(2) *Let $\mathfrak{h} = \mathfrak{f}_4$ and $V = \mathbb{C}^{26}$. Then one has $p_V \leq 3$.*

(3) *Let $\mathfrak{h} = \mathfrak{d}_5$ and $V = \mathbb{C}^{16}$. Then one has $p_V \leq 4$.*

Checking Lemma 4.7. These values are obtained from Tables 5 and 6. ∎

Lemma 4.8. (1) *Let $\mathfrak{h} = \mathfrak{b}_1 \oplus \mathfrak{d}_3$, $V' = \mathbb{C}^2 \otimes \mathbb{C}^4$, and $V'' = \mathbb{C}^3 \otimes \mathbb{C}^6$. Then one has $p_{V'} \leq 4$ and $p_{V''} \leq 2$.*

(2) *Let $\mathfrak{h} = \mathfrak{b}_1 \oplus \mathfrak{b}_3$, $V' = \mathbb{C}^2 \otimes \mathbb{C}^8$ and $V'' = \mathbb{C}^3 \otimes \mathbb{C}^7$. Then one has $p_{V'} \leq 4$ and $p_{V''} \leq 2$.*

(3) *Let $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{b}_5$ and $V = \mathbb{C}^2 \otimes \mathbb{C}^{32} \oplus \mathbb{C} \otimes \mathbb{C}^{11}$. Then one has $p_V \leq 1$.*

(4) *Let $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_4$ and $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{16}$. Then one has $p_V \leq 1$.*

(5) *Let $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{d}_2 \oplus \mathfrak{d}_4$ and $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^8 \oplus \mathbb{C} \otimes \mathbb{C}^4 \otimes \mathbb{C}^8$. Then one has $p_V \leq 1$.*

(6) *Let $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_5$ and $V = \mathbb{C}^2 \otimes \Lambda^2\mathbb{C}^6$. Then one has $p_V \leq 2$.*

(7) *Let $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_5$ and $V = \mathbb{C}^2 \otimes (\mathbb{C}^6 \oplus \Lambda^3\mathbb{C}^6 \oplus \mathbb{C}^6)$. Then one has $p_V \leq 1$.*

(8) *Let $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{e}_6$ and $V = \mathbb{C}^2 \otimes \mathbb{C}^{27}$. Then one has $p_V \leq 2$.*

(9) *Let $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{d}_6$ and $V = (\mathbb{C}^2 \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathbb{C}^2) \otimes \mathbb{C}^{32}$. Then one has $p_V \leq 1$.*

Checking Lemma 4.8. In all proofs, we will write $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_1$ simple and $V = V_1 \otimes V_2$. For $i = 1, 2$, we will write $p_i = p_{V_i}$, $d_i = \dim V_i$ and apply the bound $p_V \leq \frac{p_1}{d_2} + \frac{p_2}{d_1}$ from Lemma 2.14 (2), and the values of p_i given in Tables 5 and 6. And similarly with primes and double primes.

(1) One has $d_1' = 2$, $p_1' = 2$, $d_2' = 4$, $p_2' = 6$. Hence $p_{V'} \leq \frac{2}{4} + \frac{6}{2} \leq 4$. One has $d_1'' = 3$, $p_1'' = 1$, $d_2'' = 6$, $p_2'' = 4$. Hence $p_{V''} \leq \frac{1}{6} + \frac{4}{3} \leq 2$.

(2) One has $d_1' = 2$, $p_1' = 2$, $d_2' = 8$, $p_2' = 4$. Hence $p_{V'} \leq \frac{2}{8} + \frac{4}{2} \leq 4$. One has $d_1'' = 3$, $p_1'' = 1$, $d_2'' = 7$, $p_2'' = 5$. Hence $p_{V''} \leq \frac{1}{7} + \frac{5}{3} \leq 2$.

(3) We write $V = V' \oplus V''$. One has $d_1' = 2$, $p_1' = 2$, $d_2' = 32$, $p_2' = 2$. This is not enough to conclude. But a direct computation shows $p_{V'} = 1$ for the irreducible $\mathfrak{h}$-module $V' = \mathbb{C}^2 \otimes \mathbb{C}^{36}$.

(4) One has $d_1 = 4$, $p_1 = 2$, $d_2 = 16$, $p_2 = 3$ and hence $p_V \leq \frac{2}{16} + \frac{3}{4} \leq 1$.

(5) We first check as above that if $\mathfrak{h}_0 = \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{d}_4$ and $W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^8$, then $\rho_{\mathfrak{h}_0} \leq 2\rho_W$. Now, one has $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_4$ with $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}_3 = \mathfrak{a}_1$ and $\mathfrak{h}_4 = \mathfrak{d}_4$ and V is the sum of three irreducible components $V' \oplus V'' \oplus V'''$ with kernel, respectively $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}_3$. Hence one has the bound $\rho_{\mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_4} \leq 2\rho_{V'}$, and similarly for V'' and V'''. Adding these three inequalities gives $\rho_{\mathfrak{h}} \leq \rho_V$.

(6) One has $d_1 = 2$, $p_1 = 2$, $d_2 = 15$, $p_2 = 3$. Hence $p_V \leq \frac{2}{15} + \frac{3}{2} \leq 2$.

(7) We write $V = V' \oplus V'' \oplus V'''$. One has $d_1' = 2$, $p_1' = 2$, $d_2' = 6$, $p_2' = 10$, hence $p_{V'} \leq \frac{2}{6} + \frac{10}{2} = \frac{16}{3}$. One has $d_1'' = 2$, $p_1'' = 2$, $d_2'' = 20$, $p_2'' = 2$, hence $p_{V''} \leq \frac{2}{20} + \frac{2}{2} = \frac{11}{10}$. Thus $p_V^{-1} \geq p_{V'}^{-1} + p_{V''}^{-1} + p_{V'''}^{-1} = \frac{3}{16} + \frac{10}{11} + \frac{3}{16} \geq 1$.

(8) One has $d_1 = 2$, $p_1 = 2$, $d_2 = 27$, $p_2 = \frac{7}{2}$. Hence $p_V \leq \frac{2}{27} + \frac{7}{4} \leq 2$.

(9) One has $d_1 = 4$, $p_1 = 2$, $d_2 = 32$, $p_2 = \frac{5}{2}$. Hence $p_V \leq \frac{2}{32} + \frac{5}{8} \leq 1$. ∎

Ending the proof of Theorem 4.1. The following Lemma 4.9 tells us that we have already encountered all possible cases. ∎

Lemma 4.9. *Let $\mathfrak{g}$ be a simple exceptional complex Lie algebra and $\mathfrak{h} \subsetneq \mathfrak{g}$ a semisimple Lie subalgebra such that $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$. Then either $\mathfrak{h}$ is maximal in $\mathfrak{g}$, or $\mathfrak{h}$ is maximal in a maximal semisimple Lie algebra $\mathfrak{h}'$ of $\mathfrak{g}$.*

Checking Lemma 4.9. If this were not the case, one could find a sequence of semisimple Lie algebras $\mathfrak{h} \subsetneq \mathfrak{h}'' \subsetneq \mathfrak{h}' \subsetneq \mathfrak{g}$, each one being maximal in the next one, such that $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$. According to the previous discussion, the triple $(\mathfrak{g}, \mathfrak{h}', \mathfrak{h}'')$ has to be among the 5 cases in Table 8 with "N" in the last column, *i.e.* $(\mathfrak{f}_4, \mathfrak{b}_4, \mathfrak{d}_4)$, $(\mathfrak{e}_6, \mathfrak{d}_5, \mathfrak{b}_4)$, $(\mathfrak{e}_6, \mathfrak{f}_4, \mathfrak{b}_4)$, $(\mathfrak{e}_7, \mathfrak{a}_1\mathfrak{d}_6, \mathfrak{d}_6)$, or $(\mathfrak{e}_8, \mathfrak{a}_1\mathfrak{e}_7, \mathfrak{e}_7)$. Since, one also has $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{h}'/\mathfrak{h}}$, there are very few possibilities for such an $\mathfrak{h}$. Here is the list of quadruples $(\mathfrak{g}, \mathfrak{h}', \mathfrak{h}'', \mathfrak{h})$:

Case 1: $(\mathfrak{f}_4, \mathfrak{b}_4, \mathfrak{d}_4, \mathfrak{b}_3)$, **Case 2:** $(\mathfrak{e}_6, \mathfrak{d}_5, \mathfrak{b}_4, \mathfrak{d}_4)$, **Case 3:** $(\mathfrak{e}_6, \mathfrak{f}_4, \mathfrak{b}_4, \mathfrak{d}_4)$,
Case 4: $(\mathfrak{e}_7, \mathfrak{a}_1 \oplus \mathfrak{d}_6, \mathfrak{d}_6, \mathfrak{h})$, or **Case 5:** $(\mathfrak{e}_8, \mathfrak{a}_1 \oplus \mathfrak{e}_7, \mathfrak{e}_7, \mathfrak{h})$.

In Case 1, for all the possible embeddings $\mathfrak{b}_3 \hookrightarrow \mathfrak{d}_4$ the representation $\mathfrak{q}$ of $\mathfrak{h}$ are isomorphic, hence we can assume that this embedding is the standard embedding. But since $\mathfrak{h} = \mathfrak{b}_3$ and $\mathfrak{q} = \mathbb{C} \oplus V_1 \oplus V_1 \oplus V_2 \oplus V_2$ with $V_1 = \mathbb{C}^7$ and $V_2 = \mathbb{C}^8$ the spin representation, a direct computation gives $p_{V_1 \oplus V_2} \leq 2$, and hence $p_{\mathfrak{q}} \leq 1$. Contradiction.

In Cases 2 and 3, $\mathfrak{q}$ contains the sum of six 8-dimensional irreducible representations $\mathbb{C}^8$ of $\mathfrak{d}_4$. Since these representations V satisfy $p_V = 6$, one has $p_{\mathfrak{q}} \leq 1$. Contradiction.

Cases 4 and 5 are excluded because $\mathfrak{h}$ is included in $\mathfrak{h}''' = \mathfrak{a}_1 \oplus \mathfrak{h}$ which is already excluded in Table 8. ∎

5. Bounding p_V for simple Lie algebras

The aim of this chapter is to check Theorem 4.3 that we used in the proof of Theorem 4.1. This theorem 4.3 follows from the concatenation of Propositions 5.1 to 5.7.

We will use freely the notations of Bourbaki [7, 8], when describing the root system, simple roots α_j, fundamental weights ω_j, and irreducible representations of a complex simple Lie algebra $\mathfrak{h}$.

When $\mathfrak{h}$ is a complex semisimple Lie algebra and V a representation of $\mathfrak{h}$, the function ρ_V, as in Section 2.2, takes the form $\rho_V = \frac{1}{2}\sum m_\alpha|\alpha|$ on a maximally split abelian real subalgebra $\mathfrak{a}$ of $\mathfrak{h}$. From now on, we will choose m_α to be the complex dimension of V instead of the real dimension. This modification of both ρ_V and $\rho_\mathfrak{h}$ by a factor $\frac{1}{2}$ is harmless since it does not affect the inequality $\rho_\mathfrak{h} \leq \rho_V$ or the value of p_V.

The checking of the following seventeen propositions from 5.1 to 5.7 and from 6.1 to 6.10 relies on explicit and about thirty-pages-long calculations that we do not reproduce here.

5.1. Bounding p_V for $\mathfrak{a}_\ell$

In this section $\mathfrak{h}$ is the complex simple Lie algebra $\mathfrak{h} = \mathfrak{a}_\ell = \mathfrak{sl}_{\ell+1}$ with $\ell \geq 2$. The case $\ell = 1$ will be treated in Corollary 5.8 when V is not necessarily irreducible.

Proposition 5.1. *Let $\mathfrak{h} = \mathfrak{a}_\ell$ with $\ell \geq 2$, and V be an irreducible faithful representation of $\mathfrak{h}$ such that $p_V > 1$, equivalently, $\rho_\mathfrak{h} \not\leq \rho_V$, then V or V^* is either*
$V_{\omega_1} = \mathbb{C}^{\ell+1}$ and $p_V = 2\ell$, or
$V_{2\,\omega_1} = S^2\mathbb{C}^{\ell+1}$ and $p_V = 2\,\frac{\ell}{\ell+1} < 2$, or
$V_{\omega_2} = \Lambda^2\mathbb{C}^{\ell+1}$ and $p_V = 2\frac{\ell+2}{\ell}$ for ℓ even and $p_V = 2\frac{\ell+1}{\ell-1}$ for ℓ odd, or
$V_{\omega_3} = \Lambda^3\mathbb{C}^{\ell+1}$ when $\ell = 3,\ 4,\ 5,\ 6,\ 7$, and $p_V = 6,\ 3,\ 2,\ \frac{10}{7},\ \frac{10}{9}$, respectively.

5.2. Bounding p_V for $\mathfrak{b}_\ell$

In this section $\mathfrak{h}$ is the complex simple Lie algebra $\mathfrak{h} = \mathfrak{b}_\ell = \mathfrak{so}_{2\,\ell+1}$.

Proposition 5.2. *Let $\mathfrak{h} = \mathfrak{b}_\ell$ with $\ell \geq 2$, and V be an irreducible faithful representation of $\mathfrak{h}$ such that $\rho_\mathfrak{h} \not\leq \rho_V$, then V is either*
$V_{\omega_1} = \mathbb{C}^{2\,\ell+1}$ and $p_V = 2\,\ell - 1$, or
$V_{\omega_\ell} = \mathbb{C}^{2^\ell}$ when $\ell = 2,\ 3,\ 4,\ 5,\ 6$ and $p_V = 4,\ 4,\ 3,\ 2,\ \frac{4}{3}$ respectively.

5.3. Bounding p_V for $\mathfrak{c}_\ell$

In this section $\mathfrak{h}$ is the complex simple Lie algebra $\mathfrak{h} = \mathfrak{c}_\ell = \mathfrak{sp}_\ell$.

Proposition 5.3. *Let $\mathfrak{h} = \mathfrak{c}_\ell$ with $\ell \geq 3$, and V be an irreducible faithful representation of $\mathfrak{h}$ such that $\rho_\mathfrak{h} \not\leq \rho_V$, then V is either*
$V_{\omega_1} = \mathbb{C}^{2\,\ell}$ and $p_V = 2\,\ell$, or
$V_{\omega_2} = \Lambda^2_0\mathbb{C}^{2\,\ell}$ and $p_V = \frac{\ell+1}{\ell-1}$, or
$V_{\omega_\ell} = \Lambda^3_0\mathbb{C}^{2\ell}$ when $\ell = 3$ and $p_V = \frac{5}{3}$.

5.4. Bounding p_V for $\mathfrak{d}_\ell$

In this section $\mathfrak{h}$ is the complex simple Lie algebra $\mathfrak{h} = \mathfrak{d}_\ell = \mathfrak{so}_{2\,\ell}$.

Proposition 5.4. *Let $\mathfrak{h} = \mathfrak{d}_\ell$ with $\ell \geq 4$, and V be an irreducible faithful representation of $\mathfrak{h}$ such that $\rho_\mathfrak{h} \not\leq \rho_V$, then V is either*
$V_{\omega_1} = \mathbb{C}^{2\,\ell}$ and $p_V = 2\,\ell - 2$, or
$V_{\omega_{\ell-1}}$ or $V_{\omega_\ell} = \mathbb{C}^{2^{\ell-1}}$ when $\ell = 4,\ 5,\ 6,\ 7$ and $p_V = 6,\ \frac{7}{2},\ \frac{5}{2},\ \frac{3}{2}$ respectively.

5.5. Bounding p_V for $\mathfrak{e}_\ell$

In this section $\mathfrak{h}$ is the complex simple Lie algebra $\mathfrak{h} = \mathfrak{e}_\ell$.

Proposition 5.5. *Let $\mathfrak{h} = \mathfrak{e}_\ell$ with $\ell = 6,\ 7$ or 8 and V be an irreducible faithful representation of $\mathfrak{h}$ such that $\rho_\mathfrak{h} \not\leq \rho_V$, then V is either*
V_{ω_1} or $V_{\omega_6} = \mathbb{C}^{27}$ when $\ell = 6$ and $p_V = \frac{7}{2}$, or
$V_{\omega_7} = \mathbb{C}^{56}$ when $\ell = 7$ and $p_V = \frac{17}{6}$.

5.6. Bounding p_V for $\mathfrak{f}_4$

In this section $\mathfrak{h}$ is the complex simple Lie algebra $\mathfrak{h} = \mathfrak{f}_4$.

Proposition 5.6. *Let $\mathfrak{h} = \mathfrak{f}_4$ and V be an irreducible faithful representation of $\mathfrak{h}$ such that $\rho_\mathfrak{h} \not\leq \rho_V$, then $V = V_{\omega_4} = \mathbb{C}^{26}$ and $p_V = \frac{8}{3}$.*

5.7. Bounding p_V for $\mathfrak{g}_2$

In this section $\mathfrak{h}$ is the complex simple Lie algebra $\mathfrak{h} = \mathfrak{g}_2$.

Proposition 5.7. *Let $\mathfrak{h} = \mathfrak{g}_2$ and V be an irreducible faithful representation of $\mathfrak{h}$ such that $\rho_\mathfrak{h} \not\leq \rho_V$, then $V = V_{\omega_1} = \mathbb{C}^7$ and $p_V = 3$.*

5.8. Bounding p_V for $\mathfrak{a}_1,\ \mathfrak{a}_2,\ \mathfrak{b}_2,\ \mathfrak{g}_2$

From the discussion in this chapter, we get from (10) the following bound for p_V when V is not assumed to be irreducible.

Corollary 5.8. *Let $\mathfrak{h}$ be a simple Lie algebra and V a representation of $\mathfrak{h}$ without nonzero $\mathfrak{h}$-invariant vector. Assume that either $\mathfrak{h} = \mathfrak{a}_1$ and $\dim V \geq 3$, or $\mathfrak{h} = \mathfrak{a}_2$ and $\dim V \geq 11$, or $\mathfrak{h} = \mathfrak{b}_2$ and $\dim V \geq 15$, or $\mathfrak{h} = \mathfrak{g}_2$ and $\dim V \geq 21$, then one has $\rho_\mathfrak{h} \leq \rho_V$.*

6. Bounding p_V for non-simple Lie algebras

In the previous chapters we used quite a few upper bounds for the invariant p_V of various representations V of semisimple Lie algebras $\mathfrak{h}$. The aim of this chapter is to state precisely these upper bounds.

6.1. Bounding p_V for $\mathfrak{a}_1 \oplus \mathfrak{h}_2$

In this section $\mathfrak{h}$ is a semisimple Lie algebra of the form $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_1 = \mathfrak{a}_1$ and $\operatorname{rank} \mathfrak{h}_2 \leq 2$. We want to bound p_V when V is a representation of $\mathfrak{h}$ such that, for $i = 1,\ 2$, the spaces $V^{\mathfrak{h}_i}$ of $\mathfrak{h}_i$-invariant vectors are 0.

Proposition 6.1. *Let* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_1$ *and* V *be an irreducible faithful representation of* $\mathfrak{h}$ *such that* $\rho_{\mathfrak{h}} \not\leq \rho_V$, *then* $V = \mathbb{C}^2 \otimes \mathbb{C}^2$ *and* $p_V = 2$.

Proposition 6.2. *Let* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ *and* V *be an irreducible faithful representation of* $\mathfrak{h}$ *such that* $\rho_{\mathfrak{h}} \not\leq \rho_V$, *then either*
$V = \mathbb{C}^2 \otimes \mathbb{C}^3$ *or* $\mathbb{C}^2 \otimes (\mathbb{C}^3)^*$ *and* $p_V = 2$, *or*
$V = S^2\mathbb{C}^2 \otimes \mathbb{C}^3$ *or* $S^2\mathbb{C}^2 \otimes (\mathbb{C}^3)^*$ *and* $p_V = \frac{4}{3}$.

Proposition 6.3. *Let* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{b}_2$ *and* V *be an irreducible faithful representation of* $\mathfrak{h}$ *such that* $\rho_{\mathfrak{h}} \not\leq \rho_V$, *then either*
$V = \mathbb{C}^2 \otimes \mathbb{C}^4$ *and* $p_V = 2$, *or*
$V = S^2\mathbb{C}^2 \otimes \mathbb{C}^4$ *and* $p_V = \frac{4}{3}$, *or*
$V = \mathbb{C}^2 \otimes \mathbb{C}^5$ *and* $p_V = \frac{3}{2}$.

Proposition 6.4. *Let* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{g}_2$ *and* V *be an irreducible faithful representation of* $\mathfrak{h}$ *such that* $\rho_{\mathfrak{h}} \not\leq \rho_V$, *then* $V = \mathbb{C}^2 \otimes \mathbb{C}^7$ *and* $p_V = \frac{3}{2}$.

From the discussion in this section, we get the following bound for p_V when V is not assumed to be irreducible.

Corollary 6.5. *Let* $\mathfrak{h}_1 = \mathfrak{a}_1$, $\mathfrak{h}_2$ *be a simple Lie algebra,* $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ *and* V *a representation of* $\mathfrak{h}$ *without* $\mathfrak{h}_1$*-invariant vector or* $\mathfrak{h}_2$*-invariant vector. Assume that either* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_1$ *and* $\dim V \geq 6$, *or* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ *and* $\dim V \geq 12$, *or* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{b}_2$ *and* $\dim V \geq 15$, *or* $\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{g}_2$ *and* $\dim V \geq 21$, *then one has* $\rho_{\mathfrak{h}} \leq \rho_V$.

6.2. Bounding p_V for $\mathfrak{h}_1 \oplus \mathfrak{h}_2$

The following proposition is a reformulation of Proposition 3.2.

Proposition 6.6. *Let* $p \geq 1$ *and* $q \geq 1$.

- *Let* $\mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{sl}_q$ *acts on* $V = \mathbb{C}^p \otimes \mathbb{C}^q$. *Then* $\rho_{\mathfrak{h}} \not\leq 2\rho_V \Leftrightarrow |p - q| \geq 2$.
- *Let* $\mathfrak{h} = \mathfrak{so}_p \oplus \mathfrak{so}_q$ *acts on* $V = \mathbb{C}^p \otimes \mathbb{C}^q$. *Then* $\rho_{\mathfrak{h}} \not\leq \rho_V \Leftrightarrow |p - q| \geq 3$.
- *Let* $\mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{sp}_q$ *acts on* $V = \mathbb{C}^{2p} \otimes \mathbb{C}^{2q}$. *Then one has* $\rho_{\mathfrak{h}} \not\leq \rho_V$.

6.3. Bounding p_V for tensor products

The following proposition is a reformulation of Proposition 3.4.

Proposition 6.7. *Suppose* $p > 1$ *and* $q > 1$.

- *If* $\mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{sl}_q$ *acts on* $V = \mathrm{End}_0\mathbb{C}^p \otimes \mathrm{End}_0\mathbb{C}^q$, *then* $\rho_{\mathfrak{h}} \leq \rho_V$.
- *If* $\mathfrak{h} = \mathfrak{so}_p \oplus \mathfrak{so}_q$ *acts on* $V = \Lambda^2\mathbb{C}^p \otimes S^2_0\mathbb{C}^q \oplus S^2_0\mathbb{C}^p \otimes \Lambda^2\mathbb{C}^q$, *then* $\rho_{\mathfrak{h}} \leq \rho_V$.
  Suppose $p \geq 1$ *and* $q > 1$.
- *If* $\mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{sp}_q$ *acts on* $V = S^2\mathbb{C}^{2p} \otimes \Lambda^2_0\mathbb{C}^{2q} \oplus \Lambda^2_0\mathbb{C}^{2p} \otimes S^2\mathbb{C}^{2q}$, *then* $\rho_{\mathfrak{h}} \leq \rho_V$.
- *If* $\mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{so}_q$ *acts on* $V = S^2\mathbb{C}^{2p} \otimes S^2_0\mathbb{C}^q \oplus \Lambda^2_0\mathbb{C}^{2p} \otimes \Lambda^2\mathbb{C}^q$, *then* $\rho_{\mathfrak{h}} \leq \rho_V$.

The following proposition is a reformulation of Proposition 3.7.

Proposition 6.8. *Let* $p \geq 1$, $q \geq 1$.

- *Let* $\mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{sl}_q$ *acts on* $V = \Lambda^2_0\mathbb{C}^{2p} \oplus (\mathbb{C}^{2p} \otimes \mathbb{C}^q \oplus \mathit{dual})$.
  Then $\rho_{\mathfrak{h}} \not\leq \rho_V \Leftrightarrow q \geq 2p+2$.

- *Let* $\mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{so}_q$ *acts on* $V = \Lambda^2\mathbb{C}^p \oplus \mathbb{C}^p \otimes \mathbb{C}^q$. *Then* $\rho_\mathfrak{h} \not\leq 2\rho_V \Leftrightarrow q \geq 2p+3$.
- *Let* $\mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{sp}_q$ *acts on* $V = S^2\mathbb{C}^p \oplus \mathbb{C}^p \otimes \mathbb{C}^{2q}$. *Then* $\rho_\mathfrak{h} \not\leq 2\rho_V \Leftrightarrow q \geq p+1$.
- *Let* $\mathfrak{h} = \mathfrak{sp}_p$ *acts on* $V = \Lambda^2_0\mathbb{C}^{2p}$ *and* $p \geq 2$. *Then* $\rho_\mathfrak{h} \leq 3\rho_V$.

The following proposition is a reformulation of Proposition 3.8.

Proposition 6.9. *Let* $q \geq 1$.

- *Let* $\mathfrak{h} = \mathfrak{g}_2 \oplus \mathfrak{so}_q$ *act on* $V = \mathbb{C}^7 \otimes (\mathbb{C} \oplus \mathbb{C}^q)$ *via* $\mathfrak{g}_2 \overset{irr}{\hookrightarrow} \mathfrak{so}_7$. *Then* $\rho_\mathfrak{h} \not\leq \rho_V \Leftrightarrow q = 1$ *or* $q \geq 10$.
- *Let* $\mathfrak{h} = \mathfrak{so}_7 \oplus \mathfrak{so}_q$ *act on* $V = \mathbb{C}^8 \otimes (\mathbb{C} \oplus \mathbb{C}^q)$ *via* $\mathfrak{so}_7 \overset{irr}{\hookrightarrow} \mathfrak{so}_8$. *Then* $\rho_\mathfrak{h} \not\leq \rho_V \Leftrightarrow q = 1$, $q = 2$, *or* $q \geq 11$.

6.4. Bounding p_V for $\mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$

The following proposition was used in the proof of Proposition 3.6.

Proposition 6.10. *Let* $p \geq q \geq r \geq 1$.

- *Let* $\mathfrak{h} = \mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathfrak{sl}_r$ *act on* $V = \mathbb{C}^p \otimes (\mathbb{C}^q)^* \oplus \mathbb{C}^q \otimes (\mathbb{C}^r)^* \oplus \mathbb{C}^p \otimes (\mathbb{C}^r)^*$. *Then* $\rho_\mathfrak{h} \leq 2\rho_V \Leftrightarrow p \leq q+r+1$.
- *Let* $\mathfrak{h} = \mathfrak{so}_p \oplus \mathfrak{so}_q \oplus \mathfrak{so}_r$ *act on* $V = \mathbb{C}^p \otimes \mathbb{C}^q \oplus \mathbb{C}^q \otimes \mathbb{C}^r \oplus \mathbb{C}^p \otimes \mathbb{C}^r$. *Then* $\rho_\mathfrak{h} \leq \rho_V \Leftrightarrow p \leq q+r+2$.
- *Let* $\mathfrak{h} = \mathfrak{sp}_p \oplus \mathfrak{sp}_q \oplus \mathfrak{sp}_r$ *act on* $V = \mathbb{C}^{2p} \otimes \mathbb{C}^{2q} \oplus \mathbb{C}^{2q} \otimes \mathbb{C}^{2r} \oplus \mathbb{C}^{2p} \otimes \mathbb{C}^{2r}$. *Then* $\rho_\mathfrak{h} \leq \rho_V \Leftrightarrow p \leq q+r$.

7. Real reductive Lie algebras

The aim of this chapter is to check Theorem 1.7. We note that Theorem 1.7 allows us to give a complete description of the pairs $G \supset H$ of real reductive algebraic Lie groups for which $L^2(G/H)$ is not tempered. In fact, let $\mathfrak{g}$ be a real reductive Lie algebra, $\mathfrak{h}$ a reductive Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$. By the criterion (1), we want to classify the pairs $(\mathfrak{g}, \mathfrak{h})$ such that $\rho_\mathfrak{h} \not\leq \rho_\mathfrak{q}$. According to Lemma 2.15 and Proposition 2.16, we can assume that $\mathfrak{h}$ is semisimple without Ad-compact ideals and that $\mathfrak{g}$ is simple.

To prove Theorem 1.7, we recall that either the simple Lie algebra $\mathfrak{g}$ has a complex structure or $\mathfrak{g}$ is absolutely simple *i.e.* the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ is simple. We deal the first case in Section 7.1, and the second case in Sections 7.2–7.4.

7.1. When $\mathfrak{g}$ is a complex Lie algebra

We first deal with the case where $\mathfrak{g}$ has a complex structure.

Proposition 7.1. *Let* $\mathfrak{g}$ *be a complex simple Lie algebra with complex structure* J, *and* $\mathfrak{h}$ *a real semisimple Lie subalgebra of* $\mathfrak{g}$ *such that* $\rho_\mathfrak{h} \not\leq \rho_{\mathfrak{g}/\mathfrak{h}}$. *Then the complex Lie subalgebra* $\mathfrak{h}_0 := \mathfrak{h} \cap J\mathfrak{h}$ *also satisfies* $\rho_{\mathfrak{h}_0} \not\leq \rho_{\mathfrak{g}/\mathfrak{h}_0}$.

Proof. The complex subspace $\mathfrak{h}_0$ is indeed an ideal of $\mathfrak{h}$. Since $\mathfrak{h}$ is semisimple, it decompose into the direct sum $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ of two ideals $\mathfrak{h}_0$ and $\mathfrak{h}_1$, where the semisimple ideal $\mathfrak{h}_1$ is totally real in $\mathfrak{h}$, *i.e.*, $\mathfrak{h}_1 \cap J\mathfrak{h}_1 = \{0\}$. We set $\widetilde{\mathfrak{h}} := \mathfrak{h} \oplus J\mathfrak{h}_1 = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus J\mathfrak{h}_1$. Then $\widetilde{\mathfrak{h}}$ is a complex subalgebra of $\mathfrak{g}$.

Assume there exists $X = X_0 + X_1 \in \mathfrak{h}$ such that $\rho_\mathfrak{h}(X) > \rho_{\mathfrak{g}/\mathfrak{h}}(X)$.

We claim $\rho_{\mathfrak{h}_0}(X_0) > \rho_{\mathfrak{g}/\mathfrak{h}_0}(X_0)$. Indeed, since $[\mathfrak{h}_0, \mathfrak{h}_1] = \{0\}$, one has

$$\rho_{\mathfrak{h}_0}(X_0) = \rho_{\mathfrak{h}}(X) - \rho_{\mathfrak{h}_1}(X_1) > (\rho_{\mathfrak{g}/\widetilde{\mathfrak{h}}}(X) + \rho_{J\mathfrak{h}_1}(X_1)) - \rho_{\mathfrak{h}_1}(X_1) = \rho_{\mathfrak{g}/\widetilde{\mathfrak{h}}}(X).$$

Using Lemma 2.14 (1), one goes on with $\rho_{\mathfrak{g}/\widetilde{\mathfrak{h}}}(X) \geq \rho_{\mathfrak{g}/\widetilde{\mathfrak{h}}}(X_0) = \rho_{\mathfrak{g}/\mathfrak{h}_0}(X_0)$. Therefore, one gets $\rho_{\mathfrak{h}_0} \not\leq \rho_{\mathfrak{g}/\mathfrak{h}_0}$. ∎

By Proposition 7.1, Theorem 1.7 in the case where $\mathfrak{g}$ has a complex structure is deduced from Theorem 1.6. Moreover, Proposition 7.1 implies that the list of such pairs $(\mathfrak{g}, \mathfrak{h})$ are given by Tables 1 and 2 with the following two modifications: In Table 1, one allows $\mathfrak{h}_2$ to be real Lie subalgebras, and, in Table 2, one allows pairs $(\mathfrak{e}_7, \mathfrak{d}_6 \oplus \mathfrak{h}_2)$ and $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{h}_2)$ with $\mathfrak{h}_2 \subset \mathfrak{sl}_2$.

7.2. Finding a witness in $\mathfrak{a}_{\mathfrak{h}}$

We assume in this section that $\mathfrak{g}$ is an absolutely simple Lie algebra, and that $\mathfrak{h}$ is a semisimple Lie subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{q}_{\mathbb{C}}$ the complexifications of $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{q}$. According to Lemma 2.12, one has $\rho_{\mathfrak{h}_{\mathbb{C}}} \not\leq \rho_{\mathfrak{q}_{\mathbb{C}}}$ if $\rho_{\mathfrak{h}} \not\leq \rho_{\mathfrak{q}}$. According to Theorems 3.1 and 4.1, the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ satisfying $\rho_{\mathfrak{h}_{\mathbb{C}}} \not\leq \rho_{\mathfrak{q}_{\mathbb{C}}}$ has to be in Tables 1 or 2.

We consider the $\mathfrak{h}$-module $V := [\mathfrak{h}, \mathfrak{q}]$ and its complexification $V_{\mathbb{C}}$. We note that $\rho_{\mathfrak{q}} = \rho_V$ and $\rho_{\mathfrak{q}_{\mathbb{C}}} = \rho_{V_{\mathbb{C}}}$, hence $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}} \Leftrightarrow \rho_{\mathfrak{h}} \leq \rho_V \Leftrightarrow 2\rho_{\mathfrak{h}} \leq \rho_{V_{\mathbb{C}}}$ on $\mathfrak{h}$ and $\rho_{\mathfrak{h}_{\mathbb{C}}} \leq \rho_{\mathfrak{q}_{\mathbb{C}}}$ $\Leftrightarrow$ $\rho_{\mathfrak{h}_{\mathbb{C}}} \leq \rho_{V_{\mathbb{C}}}$ on $\mathfrak{h}_{\mathbb{C}}$. For a while, we will forget $\mathfrak{g}$ and just remember the list of representations $(\mathfrak{h}_{\mathbb{C}}, V_{\mathbb{C}})$. For each case in Tables 1 and 2, we look for the minimal $\mathfrak{h}_{\mathbb{C}}$ and we report the corresponding representation in Tables 9 and 10.

For the representations $(\mathfrak{h}_{\mathbb{C}}, V_{\mathbb{C}})$ with $\rho_{\mathfrak{h}_{\mathbb{C}}} \not\leq \rho_{V_{\mathbb{C}}}$ in Tables 9 and 10, we want first to know whether one can find a real form $\mathfrak{h}$ of $\mathfrak{h}_{\mathbb{C}}$ and an $\mathfrak{h}$-invariant real form V of $V_{\mathbb{C}}$ such that $\rho_{\mathfrak{h}} \leq \rho_V$. The answer is most often *No* but there are a few exceptions. To see this, we introduce useful notion that helps us to find when $\rho_{\mathfrak{h}_{\mathbb{C}}} \not\leq \rho_{V_{\mathbb{C}}}$ implies $\rho_{\mathfrak{h}} \not\leq \rho_V$:

Definition 7.2. [witness] Let V be an $\mathfrak{h}$-module. We say a vector X in $\mathfrak{h}$ is a *witness* if $\rho_{\mathfrak{h}}(X) > \rho_V(X)$. We denote by $\mathrm{Wit}(\mathfrak{h}, V)$ the subset of $\mathfrak{h}$ consisting of witness vectors.

By definition, $\mathrm{Wit}(\mathfrak{h}, V) \neq \emptyset$ if and only if $\rho_{\mathfrak{h}} \not\leq \rho_V$. If $(\mathfrak{h}_{\mathbb{C}}, V_{\mathbb{C}})$ is the complexification of $(\mathfrak{h}, V)$, then one has

$$\mathrm{Wit}(\mathfrak{h}_{\mathbb{C}}, V_{\mathbb{C}}) \cap \mathfrak{h} = \mathrm{Wit}(\mathfrak{h}, V). \tag{35}$$

Back to our setting where $\rho_{\mathfrak{h}_{\mathbb{C}}} \not\leq \rho_{V_{\mathbb{C}}}$, we choose a Cartan subalgebra $\mathfrak{j}_{\mathbb{C}}$ of the complex semisimple Lie algebra $\mathfrak{h}_{\mathbb{C}}$ such that $\mathfrak{j}_{\mathbb{C}} \cap \mathfrak{h}$ contains a maximal split abelian subalgebra $\mathfrak{a}_{\mathfrak{h}}$ of a real form $\mathfrak{h}$ of $\mathfrak{h}_{\mathbb{C}}$. We know that there exists a witness X in $\mathfrak{j}_{\mathbb{C}}$ *i.e.* an element such that $\rho_{\mathfrak{h}_{\mathbb{C}}}(X) > \rho_{V_{\mathbb{C}}}(X)$. We shall see that we can find a witness X in $\mathfrak{a}_{\mathfrak{h}}$ for most of noncompact real forms $\mathfrak{h}$ of $\mathfrak{h}_{\mathbb{C}}$. More precisely, one has the following lemma:

Lemma 7.3. *Let $\mathfrak{h}$ be a real semisimple Lie algebra without* Ad*-compact ideals, and V a representation of $\mathfrak{h}$ over $\mathbb{R}$. Assume that the pair $(\mathfrak{h}_{\mathbb{C}}, V_{\mathbb{C}})$ is in Tables 9 or 10. Then one has $\rho_{\mathfrak{h}} \not\leq \rho_V$, except*

*in Case A*1 *with $p - q = 2$, $p = 2p'$ and $\mathfrak{h} = \mathfrak{sl}(p', \mathbb{H})$ or*

*in Cases D*5*, E*6.1*, E*6.2*, E*7.1 *with* $\mathrm{rank}_{\mathbb{R}}\, \mathfrak{h} = 1$.

Case	$\mathfrak{h}_{\mathbb{C}}$	$V_{\mathbb{C}}$	parameters	witnesses	$\rho_{\mathfrak{h}} \leq \rho_V$
$A1$	$\mathfrak{sl}_p$	$(\mathbb{C}^p \oplus \text{dual})^q$	$p = q+2$	(1,0,...,0,−1)	$\mathfrak{h}=\mathfrak{sl}(m,\mathbb{H})$ $p=2m$
			$p \geq q+3$	(1,0,0,...,0,0,−1) and (1,1,0,...,0,−1,−1)	No
$BD1$	$\mathfrak{so}_p$	$(\mathbb{C}^p)^q$	$p = q+3$	(1,0,0,...,0)	No
			$p \geq q+4$	(1,0,0,...,0) and (1,1,0,...,0)	No
$C1$	$\mathfrak{sp}_p$	$(\mathbb{C}^{2p})^q$	$p \geq q+1$	(1,0,0,...) and (1,1,0,0,...)	No
$A2$	$\mathfrak{sp}_p$	$\Lambda^2_0\mathbb{C}^{2p}$	$p \geq 2$	any $X \neq 0$	No
$C2$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_p$	$\mathbb{C}^{2p} \otimes \mathbb{C}^{2p}$	$p \geq 2$	(Y,Y) for any $Y \neq 0$	No
$D2$	$\mathfrak{sl}_p$	$\Lambda^2\mathbb{C}^p \oplus \text{dual}$	$p = 3$	any $X \neq 0$	No
			$p \geq 4$	(1,0,0,...,0,0,−1) and (1,1,0,...,0,−1,−1)	No
$B3$	$\mathfrak{g}_2$	$\mathbb{C}^7$		any $X \neq 0$	No
$B4$	$\mathfrak{g}_2$	$\mathbb{C}^7 \oplus \mathbb{C}^7$		any $X \neq 0$	No
$D4$	$\mathfrak{so}_7$	$\mathbb{C}^7 \oplus \mathbb{C}^8$		any $X \neq 0$	No
$D5$	$\mathfrak{so}_7$	$\mathbb{C}^7 \oplus \mathbb{C}^8 \oplus \mathbb{C}^8$		(1,1,0)	$\text{rank}_{\mathbb{R}}\ \mathfrak{h} = 1$

Table 9: Representations $V_{\mathbb{C}}$ of $\mathfrak{h}_{\mathbb{C}}$ when $\mathfrak{g}_{\mathbb{C}}$ is classical

Case	$\mathfrak{h}_{\mathbb{C}}$	$V_{\mathbb{C}}$	witnesses	$\rho_{\mathfrak{h}} \leq \rho_V$
$G2$	$\mathfrak{sl}_3$	$\mathbb{C}^3 \oplus \text{dual}$	any $X \neq 0$	No
$F4$	$\mathfrak{so}_8$	$\mathbb{C}^8 \oplus \mathbb{C}^8 \oplus \mathbb{C}^8$	any $X \neq 0$	No
$E6.1.a$	$\mathfrak{so}_{10}$	$\mathbb{C}^{16} \oplus \text{dual}$	(1,1,0,0,0)	$\text{rank}_{\mathbb{R}}\ \mathfrak{h} = 1$
$E6.2.a$	$\mathfrak{f}_4$	$\mathbb{C}^{26}$	any $X \neq 0$	No
$E6.1.b$ $E6.2.b$	$\mathfrak{so}_9$	$\mathbb{C}^9 \oplus \mathbb{C}^{16} \oplus \mathbb{C}^{16}$	(1,1,0,0)	$\text{rank}_{\mathbb{R}}\ \mathfrak{h} = 1$
$E7.1$	$\mathfrak{so}_{12}$	$\mathbb{C}^{32} \oplus \mathbb{C}^{32}$	(1,1,0,0,0,0)	$\text{rank}_{\mathbb{R}}\ \mathfrak{h} = 1$
$E7.2$	$\mathfrak{e}_6$	$\mathbb{C}^{27} \oplus \text{dual}$	(0,0,0,0,1,−1,−1,1)	No
$E8$	$\mathfrak{e}_7$	$\mathbb{C}^{56} \oplus \mathbb{C}^{56}$	(0,0,0,0,1,1,−1,1)	No

Table 10: Representations $V_{\mathbb{C}}$ of $\mathfrak{h}_{\mathbb{C}}$ when $\mathfrak{g}_{\mathbb{C}}$ is exceptional

Here are a few comments on Tables 9 and 10:

- The name of the cases in the first column are those from Tables 1 and 2.
- In the third column each $\mathbb{C}^n$ stands for an irreducible representation of $\mathfrak{h}_{\mathbb{C}}$.
- In Case $F4$ the representations $\mathbb{C}^8$ are the three distinct 8-dimensional representations of $\mathfrak{so}_8$.
- In the last column we describe all the real form $\mathfrak{h}$ of $\mathfrak{h}_{\mathbb{C}}$ without Ad-compact ideal for which $\rho_{\mathfrak{h}} \leq \rho_V$.
- The answer "No" indicates that such $\mathfrak{h}$ does not exist.

- The notation for the witness in the Cartan subspace of $\mathfrak{h}_{\mathbb{C}}$ uses the standard basis with notation as in [7, Chap. 6].
- For most of the case, we only reported in Tables 9 and 10 the Lie subalgebra $\mathfrak{h}_{\mathbb{C}}$ which are minimal in the case, since when $\rho_{\mathfrak{h}} \leq \rho_V$ fails for $\mathfrak{h}$, so does it for any larger subalgebras.
- We will see that Cases $A1$, $E6.1.a$, $E6.1.b$ and $E6.2.b$ correspond to the Cases (ii), (iii) and (iv), respectively in Theorem 1.7.
- We will see that Cases $D5$ and $E7.1$ cannot occur from a pair $(\mathfrak{g}, \mathfrak{h})$ when $\mathrm{rank}_{\mathbb{R}}\mathfrak{h} = 1$.

Checking Lemma 7.3. By a direct case-by-case calculation one sees that the vectors in the fourth column are witnesses.

According to the classification of real forms of semisimple Lie algebras, for a given semisimple Lie algebra $\mathfrak{h}_{\mathbb{C}}$ the various Cartan subspaces $\mathfrak{a}_{\mathfrak{h}} \subset \mathfrak{j}_{\mathbb{C}}$ of real forms $\mathfrak{h}$ are described by the Satake diagrams (see [11, pp. 532–534]). For any real form $\mathfrak{h}$, one can often choose one of the witness in the fourth column to be in $\mathfrak{a}_{\mathfrak{h}}$. The only exceptions are the ones indicated in the last column or $BD1$ with $p-q=3$, $p=2p'$ and $\mathfrak{h} = \mathfrak{so}^*(2p')$.

In this latter case $BD1$ where $\mathfrak{h} = \mathfrak{so}^*(p)$ (p:even), one has $\mathrm{Wit}(\mathfrak{h}, V_{\mathbb{C}}) \cap \mathfrak{h} = \emptyset$. However, we can exclude this case because the $\mathfrak{h}_{\mathbb{C}}$-module $V_{\mathbb{C}} = (\mathbb{C}^p)^{p-3}$ is not defined over $\mathbb{R}$.

Finally we check that indeed the remaining Cases $A1$, $D5$, $E6.1$, $E6.2$ and $E7.1$ satisfy $\rho_{\mathfrak{h}} \leq \rho_V$. ∎

Now we want to detect whether these remaining Cases $A1$, $D5$, $E6.1$, $E6.2$ and $E7.1$ can arise from a pair $(\mathfrak{g}, \mathfrak{h})$ with $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{q}}$.

7.3. Subalgebras defined over $\mathbb{R}$

For all pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which occur in Tables 1 and 2, the Lie subalgebra $\mathfrak{h}_{\mathbb{C}}$ is included in a maximal semisimple subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Moreover, except for Case G_2, this maximal Lie subalgebra is the derived algebra of a symmetric Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We first need to know that all these Lie subalgebras are defined over $\mathbb{R}$. This will follow from the general lemmas below:

Lemma 7.4. *Let $\mathfrak{g}$ be a real simple Lie algebra and $\mathfrak{h}$ a maximal real semisimple Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}_{\mathbb{C}}$ is a maximal complex semisimple Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$.*

Lemma 7.5. *Let $\mathfrak{g}$ be a real simple Lie algebra and $\mathfrak{l}$ a symmetric Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. If the semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$ is defined over $\mathbb{R}$ and $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{C})$ then $\mathfrak{l}$ is also defined over $\mathbb{R}$.*

7.4. Checking Theorem 1.7

Two points remain to be checked when $\mathfrak{g}$ is absolutely simple.

It remains to check that Cases $A1$, $E6.1.a$, $E6.1.b$ and $E6.2.b$ correspond to the Cases (ii), (iii) and (iv), respectively, in Theorem 1.7. This follows from Lemmas 7.4 and 7.5 and from Berger's classification of irreducible real symmetric spaces [5].

It remains also to check that Cases $D5$ and $E7.1$ cannot occur from a pair $(\mathfrak{g},\mathfrak{h})$ when $\mathrm{rank}_{\mathbb{R}}\mathfrak{h}=1$.

In Case $D5$, one has $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}_{10}$ and $\mathfrak{h}=\mathfrak{so}(6,1)$. By repeated applications of Lemma 7.4, the representation of $\mathfrak{h}$ in $\mathfrak{q}$ must be a direct sum of irreducible representations $\mathfrak{q}=\mathbb{R}^7\oplus\mathbb{R}^8\oplus(\mathbb{R}\oplus\mathbb{R}^8)$. This contradicts the fact that, for $\mathfrak{h}=\mathfrak{so}(p,q)$ with $n=p+q$ odd, the spin representation of $\mathfrak{h}_{\mathbb{C}}$ can be defined over $\mathbb{R}$ only if $p-q=\pm1 \bmod 8$.

In Case $E7.1$, one has $\mathfrak{g}_{\mathbb{C}}=\mathfrak{e}_7$ and $\mathfrak{h}=\mathfrak{so}(11,1)$. According to Lemma 7.4, the Lie subalgebra $\mathfrak{h}$ is included in a subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$ such that $\mathfrak{h}'_{\mathbb{C}}=\mathfrak{d}_6\oplus\mathfrak{a}_1$ But according to Berger's classification of real symmetric spaces, the complex symmetric pair $(\mathfrak{e}_7,\mathfrak{d}_6\oplus\mathfrak{a}_1)$ has only four real forms $(\mathfrak{g},\mathfrak{h}')$ and none of the $\mathfrak{h}'$ contains $\mathfrak{so}(11,1)$.

8. Reductive homogeneous spaces

In this chapter, we come back to the point of view of Lie groups and their homogeneous spaces G/H.

We first relate in Section 8.1 the generic stabilizers of $\mathfrak{q}$ and of G/H. This will allow us to state in Sections 8.2, 8.3 and 8.4, a few direct consequences of what we have proven so far.

In Section 8.5, we give two delicate examples of real reductive homogeneous spaces that one shall have in mind when looking for a more precise converse of Theorem 1.1 (1).

8.1. Generic stabilizer in $\mathfrak{g}/\mathfrak{h}$ and in G/H

Let G be a semisimple algebraic Lie group, H a reductive subgroup, $\mathfrak{g}$, $\mathfrak{h}$ their Lie algebras and $\mathfrak{q}=\mathfrak{g}/\mathfrak{h}$.

For x in G/H, we denote by $\mathfrak{h}_x$ the stabilizer of x in $\mathfrak{h}$. As in Definitions 2.1 and 2.6, we say that G/H has RGS (resp. AGS, AmGS) in $\mathfrak{h}$ if the set $\{x\in G/H \mid \mathfrak{h}_x \text{ is reductive (resp. abelian reductive, amenable reductive)}\}$ is dense in G/H.

The following Lemmas 8.1 and 8.2 relate the generic stabilizers of G/H in $\mathfrak{h}$ and the generic stabilizers of $\mathfrak{q}$ in $\mathfrak{h}$. The first lemma should be compared with Lemma 2.5.

Lemma 8.1. *Let G be a real semisimple algebraic Lie group, and H a reductive algebraic subgroup. Then G/H has RGS in $\mathfrak{h}$. More precisely, there exists finitely many reductive Lie subalgebras $\mathfrak{m}_1,\dots,\mathfrak{m}_r$ of $\mathfrak{h}$ such that the set of x in G/H whose stabilizer $\mathfrak{h}_x$ in $\mathfrak{h}$ is conjugate by the adjoint group H of $\mathfrak{h}$ to one of the $\mathfrak{m}_i$ contains a non-empty Zariski open subset of G/H.*

The Lie algebras $\mathfrak{m}_i$ which cannot be removed from this list will be called the *generic stabilizers* of G/H. They are well defined only up to conjugacy and permutation.

Lemma 8.2. *Let G be a real semisimple algebraic Lie group, H be a reductive algebraic subgroup, and $\mathfrak{q}=\mathfrak{g}/\mathfrak{h}$. Then G/H and $\mathfrak{q}$ have the same set of generic stabilizers in $\mathfrak{h}$. In particular, one has the equivalences:*

(1) *G/H has* AGS *in $\mathfrak{h}$ $\iff$ $\mathfrak{q}$ has* AGS *in $\mathfrak{h}$.*

(2) *G/H has AmGS in $\mathfrak{h}$ $\iff$ $\mathfrak{q}$ has AmGS in $\mathfrak{h}$.*

Proof of Lemmas 8.1 and 8.2. By Chevalley's theorem, there exists a finite-dimensional representation V of G and a point v in V whose stabilizer in G is H. The tangent space at v to the G-orbit Gv is isomorphic to $\mathfrak{g}/\mathfrak{h}$ as a representation of H. Since H is reductive, there exists an H-invariant decomposition $V = T_v(G/H) \oplus W$. In particular, there is an H-equivariant projection $\pi\colon V \to \mathfrak{g}/\mathfrak{h}$, which induces an H-equivariant dominant map $\pi\colon G/H \to \mathfrak{g}/\mathfrak{h}$. In particular, there exists an open Zariski dense subset U of $\underline{G/H}$, such that, for all x in U, x and $\pi(x)$ have same stabilizer in $\mathfrak{h}$. Note that $\overline{\pi(U)}$ contains a neighborhood of 0, and that $\mathfrak{h}_v = \mathfrak{h}_{tv}$, for all v in V, t in $\mathbb{R} \setminus \{0\}$. Our claims follow. ∎

8.2. Reductive and semisimple subgroups

The following proposition reduces our classification to the case of a semisimple Lie subgroup H without compact factor.

Proposition 8.3. *Let G be a real semisimple algebraic Lie group, and $H_1 \supset H_2$ two unimodular subgroups.*

(1) *If $L^2(G/H_1)$ is tempered then $L^2(G/H_2)$ is tempered.*

(2) *The converse is true when H_2 is normal in H_1 and H_1/H_2 is amenable (for instance, finite, or compact, or abelian).*

The following proposition follows from Proposition 2.16, and reduces our classification to the case of a simple Lie group G.

Proposition 8.4. *Let G be a real semisimple algebraic Lie group, H a real reductive algebraic subgroup of G. Let G_i $(1 \leq i \leq r)$ be simple factors of G, and we set $H_i := H \cap G_i$. The representation of G in $L^2(G/H)$ is tempered if and only if, for all $i \leq r$, the representation of G_i in $L^2(G_i/H_i)$ is tempered.*

The following proposition is an easy corollary of our criterion (1).

Proposition 8.5. *Let G be a real semisimple algebraic Lie group, and H a real reductive algebraic subgroup.*

(1) *If the representation of $G_{\mathbb{C}}$ in $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is tempered, then the representation of G in $L^2(G/H)$ is tempered.*

(2) *The converse is true when H is a split group.*

8.3. Examples of complex homogeneous spaces

In this section we give a few examples of complex homogeneous spaces G/H where G and H are *complex* Lie groups. We recall that Theorem 1.6 together with the criterion (1) implies the following:

Corollary 8.6. *Suppose G is a complex semisimple algebraic group and H a complex reductive subgroup. Then the representation of G in $L^2(G/H)$ is tempered if and only if the set of points in $\mathfrak{g}/\mathfrak{q}$ with abelian stabilizer in $\mathfrak{h}$ is dense.*

Example 8.7. (1) $L^2(\mathrm{SL}(n,\mathbb{C})/\mathrm{SO}(n,\mathbb{C}))$ is always tempered.
(2) $L^2(\mathrm{SL}(2m,\mathbb{C})/\mathrm{Sp}(m,\mathbb{C}))$ is never tempered.
(3) $L^2(\mathrm{SO}(7,\mathbb{C})/G_2)$ is not tempered.

The first two cases above are symmetric spaces, see also Example 1.3. The next example is a consequence of Proposition 3.6.

Example 8.8. Let $n = n_1 + \cdots + n_r$ with $n_1 \geq \cdots \geq n_r \geq 1$, $r \geq 2$.
(1) $L^2(\mathrm{SL}(n,\mathbb{C})/\prod \mathrm{SL}(n_i,\mathbb{C}))$ is tempered iff $2n_1 \leq n+1$.
(2) $L^2(\mathrm{SO}(n,\mathbb{C})/\prod \mathrm{SO}(n_i,\mathbb{C}))$ is tempered iff $2n_1 \leq n+2$.
(3) $L^2(\mathrm{Sp}(n,\mathbb{C})/\prod \mathrm{Sp}(n_i,\mathbb{C}))$ is tempered iff $r \geq 3$ and $2n_1 \leq n$.

8.4. Examples of real homogeneous spaces

Here are a few examples of application of our criterion (1).

Example 8.9. Let G_1 be a real semisimple algebraic Lie group and K_1 a maximal compact subgroup.
(1) $L^2(G_1 \times G_1/\Delta(G_1))$ is always tempered.
(2) $L^2(G_{1,\mathbb{C}}/G_1)$ is always tempered.
The first statement is obvious from the definition of temperedness, and alternatively follows immediately from (1) and Proposition 2.16. The second statement follows from the first one as a special case of the example below.

Example 8.10. Let G/H be a symmetric space *i.e.* G is a real semisimple algebraic Lie group and H is the set of fixed points of an involution of G. Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the H-invariant decomposition of $\mathfrak{g}$. Let G^c be a semisimple algebraic Lie group with Lie algebra $\mathfrak{g}^c = \mathfrak{h} \oplus \sqrt{-1}\mathfrak{q}$, so that the $\mathfrak{h}$-modules $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{g}^c/\mathfrak{h}$ are isomorphic. Therefore, $L^2(G/H)$ is tempered iff $L^2(G^c/H)$ is tempered.

Example 8.11. (1) $L^2(\mathrm{SL}(p+q,\mathbb{R})/\mathrm{SO}(p,q))$ is always tempered.
(2) $L^2(\mathrm{SL}(2m,\mathbb{R})/\mathrm{Sp}(m,\mathbb{R}))$ is never tempered.
(3) $L^2(\mathrm{SL}(m+n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}))$ is tempered iff $|m-n| \leq 1$.

Example 8.12. Let $p_1 + \cdots + p_r \leq p$ and $q_1 + \cdots + q_r \leq q$.
$L^2(\mathrm{SO}(p,q)/\prod \mathrm{SO}(p_i,q_i))$ is tempered iff $2 \max_{p_i q_i \neq 0}(p_i + q_i) \leq p+q+2$.

The homogeneous spaces in Examples 8.8 and 8.12 are not symmetric spaces when $r \geq 3$. In most cases, they are not even real spherical ([13]), too.

8.5. About the converse of Theorem 1.1

Even when H has no compact factors and G/H is a reductive symmetric space, the converses of the implications in Theorem 1.1 are not always true. Here are two examples that follow from Theorem 1.4.

(1) Counterexample to the converse of Theorem 1.1(1).

$L^2(\mathrm{Sp}(p_1+p_2,q_1+q_2)/\mathrm{Sp}(p_1,q_1)\times\mathrm{Sp}(p_2,q_2))$ is not tempered when $p_1\geq 1$, $q_1\geq 1$ and $p_1+q_1=p_2+q_2+1$, even though the set of points in G/H with amenable stabilizer in H is dense.

(2) Counterexample to the converse of Theorem 1.1 (2).

$L^2(\mathrm{SL}(2m-1,\mathbb{H})/S(\mathrm{GL}(m,\mathbb{H})\times\mathrm{GL}(m-1,\mathbb{H})))$ is tempered when $m\geq 2$ even though the set of points in G/H with abelian stabilizer in $\mathfrak{h}$ is not dense.

References

[1] B. Bekka, P. de la Harpe, A. Valette: *Kazhdan's Property (T)*, New Mathematical Monographs 11, Cambridge University Press, Cambridge (2008).

[2] Y. Benoist, T. Kobayashi: *Tempered reductive homogeneous spaces*, J. Eur. Math. Soc. 17 (2015) 3015–3036.

[3] Y. Benoist, T. Kobayashi: *Tempered homogeneous spaces II*, in: *Dynamics, Geometry, Number Theory: The Impact of Margulis on Modern Mathematics*, D. Fisher, D. Kleinbock, G. Soifer (eds.), University of Chicago Press, Chicago (2021), arXiv: 1706.10131 (2017).

[4] Y. Benoist, T. Kobayashi: *Tempered homogeneous spaces IV*, arXiv:2009.10391 (2020).

[5] M. Berger: *Les espaces symétriques noncompacts*, Ann. Sci. École Norm. Sup. (3) 74 (1957) 85–177.

[6] A. Borel: *Linear Algebraic Groups*, 2nd edition, Graduate Texts in Mathematics 126, Springer, New York (1991).

[7] N. Bourbaki: *Éléments de Mathématique: Groupes et Algèbres de Lie*, Chap. 4, 5, 6, Hermann, Paris (1974).

[8] N. Bourbaki: *Éléments de Mathématique: Groupes et Algèbres de Lie*, Chap. 7, 8, Hermann, Paris (1975).

[9] M. Cowling, U. Haagerup, R. Howe: *Almost L^2 matrix coefficients*, J. Reine Angew. Math. 387 (1988) 97–110.

[10] E. Dynkin: *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sbornik 30 (1952) 349–462; translation in American Mathematical Society Translation Series (2) 6 (1957) 111–244.

[11] S. Helgason: *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York (1978).

[12] A. Knapp: *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton (1986).

[13] T. Kobayashi, T. Oshima: *Finite multiplicity theorems for induction and restriction*, Advances Math. 248 (2013) 921–944.

[14] M. Liebeck, G. Seitz: *The Maximal Subgroups of Positive Dimension in Exceptional Algebraic Groups*, Memoirs of the AMS 169, American Mathematical Society, Providence (2004).

Yves Benoist, CNRS - Université Paris-Sud Orsay, Paris, France; yves.benoist@math.u-psud.fr.

Toshiyuki Kobayashi, Graduate School of Mathematical Sciences and Kavli IPMU (WPI), The University of Tokyo, Komaba, Japan; toshi@ms.u-tokyo.ac.jp.

Received January 28, 2021
and in final form February 19, 2021