



Global well-posedness for the defocusing, cubic nonlinear Schrödinger equation with initial data in a critical space

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Abstract. In this note we prove global well-posedness for the defocusing, cubic nonlinear Schrödinger equation with initial data lying in a critical Sobolev space.

1. Introduction

In this note, we discuss the defocusing, cubic, nonlinear Schrödinger equation in three dimensions,

$$(1.1) \quad iu_t + \Delta u = F(u) = |u|^2 u, \quad u(0, x) = u_0 \in \dot{H}^{1/2}(\mathbb{R}^3).$$

Equation (1.1) has a scaling symmetry. For any $\lambda > 0$, if u solves (1.1), then

$$(1.2) \quad u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x),$$

also solves (1.1). The initial data $\lambda u_0(\lambda x)$ has $\dot{H}^{1/2}(\mathbb{R}^3)$ norm that is invariant under the scaling (1.2).

The local theory for initial data lying in $\dot{H}^{1/2}(\mathbb{R}^3)$ has been completely worked out, and the scaling symmetry has been shown to control the local well-posedness theory.

Theorem 1.1. *Assume $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, $\|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq A$. Then there exists $\delta = \delta(A)$ such that if $\|e^{it\Delta} u_0\|_{L_{t,x}^5(I \times \mathbb{R}^3)} < \delta$, then there exists a unique solution to (1.1) on $I \times \mathbb{R}^3$ with $u \in C(I; \dot{H}^{1/2}(\mathbb{R}^3))$, and*

$$\|u\|_{L_{t,x}^5(I \times \mathbb{R}^3)} \leq 2\delta.$$

Moreover, if $u_{0,k} \rightarrow u_0$ in $\dot{H}^{1/2}(\mathbb{R}^3)$, then the corresponding solutions $u_k \rightarrow u$ in $C(I; \dot{H}^{1/2}(\mathbb{R}^3))$.

This theorem was proved in [3].

From this, it is straightforward to show that local well-posedness holds for (1.1) for any initial data $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$. Indeed, by the dominated convergence principle combined

with Strichartz estimates, for any $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$,

$$(1.3) \quad \lim_{T \searrow 0} \|e^{it\Delta} u_0\|_{L^5_{t,x}([-T, T] \times \mathbb{R}^3)} = 0.$$

Since $\delta(A)$ is decreasing as $A \nearrow +\infty$, Strichartz estimates imply that there exists $\delta_0 > 0$ such that if $\|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \delta_0$, (1.1) has a global solution that scatters. By scattering, we mean that there exist u_0^+, u_0^- so that

$$\lim_{t \rightarrow +\infty} \|u(t) - e^{it\Delta} u_0^+\|_{\dot{H}^{1/2}} = 0,$$

and

$$\lim_{t \rightarrow -\infty} \|u(t) - e^{it\Delta} u_0^-\|_{\dot{H}^{1/2}} = 0.$$

However, it is important to note that while (1.3) holds for any fixed $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, the convergence is not uniform, even for $\|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq A < \infty$. Thus, one cannot conclude directly from [3] that a uniform bound for $\|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ on the entire time of the existence of the solution to (1.1) implies that the solution is global. This result was instead proved in [9], using concentration compactness methods.

Theorem 1.2. *Suppose that u is a solution of (1.1) with initial data $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$ and a maximal interval of existence $I = (T_-, T_+)$. Also assume that $\sup_{t \in (T_-, T_+)} \|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = A < \infty$. Then $T_+(u_0) = +\infty$, $T_-(u_0) = -\infty$, and the solution u scatters.*

It is conjectured that (1.1) is globally well-posed and scattering for any $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, without the a priori assumption of a universal bound on the $\dot{H}^{1/2}$ norm of the solution $u(t)$. Partial progress has been made in this direction.

A solution to (1.1) has the conserved quantities mass,

$$M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)),$$

and energy,

$$(1.4) \quad E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{4} \int |u(t, x)|^4 dx.$$

This fact implies global well-posedness for (1.1) with $u_0 \in H_x^1(\mathbb{R}^3)$, where $H_x^1(\mathbb{R}^3)$ is the inhomogeneous Sobolev space of order one. In this case, one could also prove bounds on the scattering size directly, using the interaction Morawetz estimate of [5].

Theorem 1.3. *If u is a solution to (1.1), on an interval I , then*

$$(1.5) \quad \|u\|_{L^4_{t,x}(I \times \mathbb{R}^3)}^4 \lesssim \|u\|_{L_t^\infty L_x^2(I \times \mathbb{R}^3)}^2 \|u\|_{L_t^\infty \dot{H}^{1/2}(I \times \mathbb{R}^3)}^2 \lesssim E(u)^{1/2} M(u)^{3/2}.$$

Interpolating (1.4) and (1.5) then implies

$$(1.6) \quad \|u\|_{L_t^8 L_x^4(I \times \mathbb{R}^3)}^4 \lesssim M(u)^{3/4} E(u)^{3/4},$$

with bounds independent of $I \subset \mathbb{R}$. Combining Strichartz estimates and local well-posedness theory, a uniform bound on (1.6) for any $I \subset \mathbb{R}$ directly implies a uniform bound on

$$\|u\|_{L^5_{t,x}(I \times \mathbb{R}^3)}.$$

The argument from [3] implies that proving scattering is equivalent to proving

$$(1.7) \quad \|u\|_{L^5_{t,x}(\mathbb{R} \times \mathbb{R}^3)} < \infty.$$

Indeed, assuming that (1.7) is true, the interval $\mathbb{R}$ may be partitioned into finitely many pieces J_k such that

$$\|u\|_{L^5_{t,x}(J_k \times \mathbb{R}^3)} \leq \delta.$$

Then iterate the argument over the intervals J_k , which proves scattering.

This argument also shows that a solution to (1.1) blowing up at a finite time $T_0 < \infty$ is equivalent to

$$\|u\|_{L^5_{t,x}([0, T_0) \times \mathbb{R}^3)} = \infty.$$

Remark. Prior to [5], [8] and [10] proved scattering using the standard Morawetz estimate. See [12] for more details on Strichartz estimates.

Many have attempted to lower the regularity needed in order to prove global well-posedness. For any $s > 1/2$, the inhomogeneous Sobolev space $H^s_x(\mathbb{R}^3) \subset \dot{H}^{1/2}(\mathbb{R}^3)$. Therefore, if $u_0 \in H^s_x(\mathbb{R}^3)$, then it would be conjectured that the solution to (1.1) with initial data u_0 is global and scatters.

Proving a uniform bound on the $H^s_x(\mathbb{R}^3)$ norm would be enough, since by interpolation this would guarantee a uniform bound on the $\dot{H}^{1/2}_x(\mathbb{R}^3)$ norm. The difficulty is that there does not exist a conserved quantity at regularity s that controls the $\dot{H}^s$ norm for $1/2 < s < 1$.

Instead, [2] used the Fourier truncation method (see also [1] for the cubic problem in two dimensions). Decompose the initial data

$$u_0 = P_{\leq N} u_0 + P_{> N} u_0 = v_0 + w_0.$$

Then $v_0 \in H^1(\mathbb{R}^3)$, and $\|w_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ is small. Thus, (1.1) has a global solution for initial data v_0 or w_0 , call them v and w . Since (1.1) is a nonlinear equation, it is necessary to also estimate the interaction between v and w in the nonlinearity of (1.1). Then, [2] proved global well-posedness for (1.1) with initial data $u_0 \in H^s_x(\mathbb{R}^3)$ when $s > 11/13$. Moreover, [2] proved that the solution is of the form

$$e^{it\Delta} u_0 + v(t), \quad \text{where } v(t) \in H^1_x(\mathbb{R}^3).$$

The results from the Fourier truncation method for (1.1) were improved using the I-method. First, [4] improved the regularity necessary for global well-posedness to $s > 5/6$. Then, [5] improved the necessary regularity to $s > 4/5$. To the author's best knowledge, the best known regularity result is the result of [11], proving global well-posedness and scattering for regularity $s > 5/7$. For radial initial data, [6] proved global well-posedness and scattering for any $s > 1/2$. This result is almost sharp at high frequencies.

In this paper, we study the cubic nonlinear Schrödinger equation (1.1) with initial data lying in the Sobolev space $W_x^{7/6, 11/7}(\mathbb{R}^3)$. That is,

$$\| |\nabla|^{11/7} u_0 \|_{L^{7/6}(\mathbb{R}^3)} < \infty.$$

Remark. This norm is well-defined using the Littlewood–Paley decomposition. See for example [13].

This norm is preserved under the scaling (1.2), and is therefore a critical Sobolev norm. Moreover, $W_x^{7/6, 11/7}(\mathbb{R}^3) \subset \dot{H}^{1/2}(\mathbb{R}^3)$, so (1.1) has a local solution for this initial data. We prove global well-posedness for (1.1) with this initial data.

Theorem 1.4. *The cubic nonlinear Schrödinger equation is globally well-posed for initial data $u_0 \in W_x^{7/6, 11/7}(\mathbb{R}^3)$.*

The proof of this theorem will heavily utilize dispersive estimates. Interpolating between the fact that $e^{it\Delta}$ is a unitary operator,

$$\| e^{it\Delta} u_0 \|_{L^2(\mathbb{R}^3)} = \| u_0 \|_{L^2(\mathbb{R}^3)},$$

and the dispersive estimate,

$$\| e^{it\Delta} u_0 \|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t^{3/2}} \| u_0 \|_{L^1(\mathbb{R}^3)},$$

gives the estimate

$$(1.8) \quad \| e^{it\Delta} u_0 \|_{L^7(\mathbb{R}^3)} \lesssim \frac{1}{t^{15/14}} \| u_0 \|_{L^{7/6}(\mathbb{R}^3)}.$$

This implies that the linear solution $e^{it\Delta} u_0$ has very good behavior when $t > 1$, in fact it is integrable in time. We then rescale so that u_0 has a local solution on an interval $[-1, 1]$. We prove that this solution may be decomposed into

$$u(t) = e^{it\Delta} u_0 + v(t) + w(t).$$

In particular,

$$u(1) = e^{i\Delta} u_0 + v(1) + w(1).$$

The term

$$e^{i(t-1)\Delta} e^{i\Delta} u_0 = e^{it\Delta} u_0$$

has good properties when $t > 1$. We can also show that

$$\| \nabla e^{i(t-1)\Delta} v(1) \|_{L^\infty} \lesssim \frac{1}{t^{3/2}},$$

which also has good properties when $t > 1$. Finally, $w(1) \in H_x^1$ and has finite energy. Making a Gronwall argument shows that

$$\| u(t) - e^{it\Delta} u_0 - e^{i(t-1)\Delta} v(1) \|_{\dot{H}^1},$$

is uniformly bounded on $[1, \infty)$. This is enough to give global well-posedness, but not scattering.

This result could be compared to the result in [7] for the nonlinear wave equation. There, the author proved global well-posedness and scattering for the cubic wave equation with initial radial data in the Besov space $B_{1,1}^2 \times B_{1,1}^1$. Here, we do not require radial symmetry, however, we only prove global well-posedness. We are unable to prove scattering at this time due to the lack of a scale invariant conformal symmetry.

We prove a local well-posedness result in section two, and a global result in section three. This argument could be generalized to many intercritical, defocusing nonlinear Schrödinger equations.

2. Local well-posedness

The Sobolev embedding theorem implies that $W_x^{7/6, 11/7}(\mathbb{R}^3)$ is embedded into $\dot{H}^{1/2}(\mathbb{R}^3)$. Therefore, (1.1) is locally well-posed, and there exists some $T(u_0) > 0$ such that (1.1) has a solution on $[-T, T]$ and $\|u\|_{L_t^5([-T, T] \times \mathbb{R}^3)} = \epsilon_0$, for some $\epsilon_0(\|u_0\|_{\dot{H}^{1/2}})$ small. After rescaling using (1.2), suppose

$$(2.1) \quad \|u\|_{L_{t,x}^5([-1, 1] \times \mathbb{R}^3)} = \epsilon_0.$$

Since $(3, 18/5)$ is an admissible pair, Strichartz estimates imply

$$(2.2) \quad \begin{aligned} & \| |\nabla|^{1/2} u \|_{L_t^\infty L_x^2 \cap L_t^2 L_x^6([-1, 1] \times \mathbb{R}^3)} \\ & \lesssim \| |\nabla|^{1/2} u_0 \|_{L_x^2(\mathbb{R}^3)} + \| |\nabla|^{1/2} u \|_{L_t^3 L_x^{18/5}([-1, 1] \times \mathbb{R}^3)} \|u\|_{L_{t,x}^5([-1, 1] \times \mathbb{R}^3)}^2. \end{aligned}$$

Therefore,

$$(2.3) \quad \| |\nabla|^{1/2} u \|_{L_t^\infty L_x^2 \cap L_t^2 L_x^6([-1, 1] \times \mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^{1/2}}.$$

Also, by Duhamel's principle, for any $t \in [-1, 1]$,

$$(2.4) \quad u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau = u_l(t) + u_{nl}(t).$$

Remark. Recall from (1.1) that $F(u) = |u|^2 u$.

We begin with a technical lemma. This lemma allows us to make a Littlewood–Paley decomposition of u_{nl} , treat each $P_j u_{nl}$ separately, and then sum up. It also implies that u_{nl} retains all the properties of a solution to the linear Schrödinger equation with initial data in a Besov space.

Remark. In this section, all implicit constants depend on the norm $\|u_0\|_{W^{7/6, 11/7}}$.

Remark. Throughout this section we rely very heavily on the bilinear Strichartz estimate

$$\|(e^{it\Delta} P_j u_0)(e^{it\Delta} P_k v_0)\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim 2^{-j/2} 2^k \|P_j u_0\|_{L^2} \|P_k v_0\|_{L^2}.$$

See [1] for a proof.

Lemma 2.1. *Let P_j be the customary Littlewood–Paley projection operator. Also suppose that u is a solution to (1.1) satisfying (2.1). Then*

$$(2.5) \quad \sum_j 2^{j/2} \|P_j F(u)\|_{L_t^1 L_x^2([-1,1] \times \mathbb{R}^3)} \lesssim 1.$$

Proof. Decompose the nonlinearity,

$$P_j F(u) = P_j F(P_{\geq j-3} u) + 3P_j((P_{\geq j-3} u)^2 (P_{\leq j-3} u)) + 3P_j((P_{j-3 \leq \cdot \leq j+3} u)(P_{\leq j-3} u)^2).$$

By Bernstein's inequality, and (2.2),

$$(2.6) \quad 2^{j/2} \|P_j F(P_{\geq j-3} u)\|_{L_t^1 L_x^2([-1,1] \times \mathbb{R}^3)} \\ \lesssim 2^{j/2} \|P_{\geq j-3} u\|_{L_t^3 L_x^6([-1,1] \times \mathbb{R}^3)}^3 \lesssim 2^{j/2} \left(\sum_{l \geq j-3} 2^{-l/6} \|\nabla\|^{1/6} P_l u \|_{L_t^3 L_x^6} \right)^3.$$

Next,

$$(2.7) \quad 2^{j/2} \|P_j((P_{\geq j-3} u)^2 (P_{\leq j-3} u))\|_{L_t^1 L_x^2([-1,1] \times \mathbb{R}^3)} \\ \lesssim 2^{j/2} \left(\sum_{l \geq j-3} 2^{-l/4} \|\nabla\|^{1/4} P_l u \|_{L_t^3 L_x^{36/7}} \right)^2 \|u\|_{L_t^3 L_x^9}.$$

Finally, by the bilinear Strichartz estimate

$$(2.8) \quad \|(e^{it\Delta} P_j u_0)(e^{it\Delta} P_{l_1} u_0)\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim 2^{-j/2} 2^{l_1} \|P_j u_0\|_{L^2} \|P_{l_1} u_0\|_{L^2},$$

combined with the principle of superposition and (2.4),

$$(2.9) \quad \|(P_j u)(P_{l_1} u)\|_{L_{t,x}^2} \\ \lesssim 2^{-j/2} 2^{l_1} (\|P_j u_0\|_{L^2} + \|P_j F(u)\|_{L_t^1 L_x^2}) (\|P_{l_1} u_0\|_{L^2} + \|P_{l_1} F(u)\|_{L_t^1 L_x^2}),$$

and the Sobolev embedding properties of Littlewood–Paley projections,

$$(2.10) \quad 2^{j/2} \|(P_{j-3 \leq \cdot \leq j+3} u)(P_{\leq j-3} u)^2\|_{L_t^1 L_x^2([-1,1] \times \mathbb{R}^3)} \\ \lesssim 2^{j/2} \sum_{l_1 \leq j-3} \|(P_{l_1} u)(P_{j-3 \leq \cdot \leq j+3} u)\|_{L_{t,x}^2} \sum_{l_1 \leq l_2 \leq j-3} \|P_{l_2} u\|_{L_t^2 L_x^\infty} \\ \lesssim \|\nabla\|^{1/2} u \|_{L_t^2 L_x^6} \sum_{l_1 \leq j-3} 2^{l_1} (j - l_1) (\|P_{j-3 \leq \cdot \leq j+3} u_0\|_{L^2} \\ + \|P_{j-3 \leq \cdot \leq j+3} F(u)\|_{L_t^1 L_x^2}) \cdot (\|P_{l_1} u_0\|_{L^2} + \|P_{l_1} F(u)\|_{L_t^1 L_x^2}).$$

By Strichartz estimates, (2.3), Plancherel's theorem, and the fractional product rule,

$$\sum_j 2^j \|P_j u_0\|_{L^2}^2 + \sum_j 2^j \|P_j F(u)\|_{L_t^1 L_x^2([-1,1] \times \mathbb{R}^3)}^2 \lesssim \|u_0\|_{\dot{H}^{1/2}}^2 + \|\nabla\|^{1/2} F(u) \|_{L_t^1 L_x^2}^2 \\ \lesssim \|u_0\|_{\dot{H}^{1/2}}^2 + \|\nabla\|^{1/2} u \|_{L_t^3 L_x^{18/5}}^2 \|u\|_{L_t^3 L_x^9}^4 \lesssim 1.$$

Combining (2.6)–(2.10) with the Cauchy–Schwarz inequality implies

$$(2.11) \quad \sum_j 2^{j/2} \|P_j F(u)\|_{L_t^1 L_x^2([-1,1] \times \mathbb{R}^3)} \lesssim 1,$$

which proves the lemma. ■

Next, decompose u_{nl} in the following manner:

$$u_{nl}(t) = -i \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} F(u(\tau)) d\tau - i \int_{(1-\delta)t}^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau = v(t) + w(t),$$

for some $\delta > 0$ sufficiently small, to be specified later.

Lemma 2.2. *For any $t \in [0, 1]$,*

$$(2.12) \quad \|v(t)\|_{L^\infty} \lesssim \frac{1}{\delta^{1/2} t^{1/2}},$$

and

$$(2.13) \quad \|\nabla v(t)\|_{L^\infty} \lesssim \frac{1}{\delta t}.$$

Proof. By the dispersive estimate, since $\|u\|_{L^3} \lesssim \|u\|_{\dot{H}^{1/2}}$ is uniformly bounded on $[0, 1]$,

$$\|v(t)\|_{L^\infty} \lesssim \left\| \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} F(u) d\tau \right\|_{L^\infty} \lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \|u\|_{L^3}^3 d\tau \lesssim \frac{1}{\delta^{1/2} t^{1/2}}.$$

To prove (2.13), observe that by the product rule,

$$\nabla F(u) = 2|u|^2 \nabla u + u^2 \nabla \bar{u}.$$

Interpolating,

$$(2.14) \quad \| |\nabla|^{1/2} u_l \|_{L^2} \lesssim \| |\nabla|^{1/2} u_0 \|_{L^2} \lesssim 1,$$

with

$$(2.15) \quad t^{15/14} \| |\nabla|^{11/7} u_l \|_{L^7} \lesssim \| |\nabla|^{11/7} u_0 \|_{L^{7/6}} \lesssim 1,$$

we have

$$(2.16) \quad t^{1/2} \|\nabla u_l\|_{L^3} \lesssim 1.$$

Making a dispersive estimate and using (2.16),

$$\begin{aligned} \left\| \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} |u|^2 \nabla u_l(\tau) d\tau \right\|_{L^\infty} &\lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \|\nabla u_l(\tau)\|_{L^3} \|u\|_{L^3}^2 d\tau \\ &\lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \frac{1}{|\tau|^{1/2}} d\tau \lesssim \frac{1}{\delta t}. \end{aligned}$$

The same computation may also be made for $u^2 \nabla \bar{u}_l$.

Next, consider the contribution of $|u|^2 \nabla u_{nl}$. By (2.5), we can, without loss of generality, consider only one P_j Littlewood–Paley multiplier, provided the estimate is uniform in $2^{j/2} \|P_j F(u)\|_{L_t^1 L_x^2}$:

$$\begin{aligned} |u|^2 (\nabla P_j u_{nl}) &= |P_{\leq j} u|^2 (\nabla P_j u_{nl}) + 2\operatorname{Re}((P_{>j} \bar{u})(P_{\leq j} \bar{u}))(\nabla P_j u_{nl}) \\ &\quad + |P_{>j} u|^2 (\nabla P_j u_{nl}). \end{aligned}$$

Using the bilinear Strichartz estimate in (2.9), as well as (2.11) and the Cauchy–Schwartz inequality,

$$\begin{aligned} (2.17) \quad \| |u_{\leq j}|^2 (\nabla P_j u_{nl}) \|_{L_t^2 L_x^1([0,1] \times \mathbb{R}^3)} &\lesssim \sum_{j_1 \leq j_2 \leq j} \|(P_{j_1} u)(P_{j_2} \nabla u_{nl})\|_{L_{t,x}^2} \|P_{j_2} u\|_{L_t^\infty L_x^2} \\ &\lesssim \sum_{j_1 \leq j_2 \leq j} 2^{j_1/2} 2^{-j_2/2} 2^{j/2} \|P_j F(u)\|_{L_t^1 L_x^2} (\|\nabla|^{1/2} P_{j_1} u_0\|_{L^2} + \|\nabla|^{1/2} P_{j_1} F(u)\|_{L_t^1 L_x^2}) \\ &\quad \times (\|\nabla|^{1/2} P_{j_2} u_0\|_{L^2} + \|\nabla|^{1/2} P_{j_2} F(u)\|_{L_t^1 L_x^2}) \lesssim 1. \end{aligned}$$

Also, by Bernstein’s inequality and Lemma 2.1,

$$\begin{aligned} \|\nabla P_j u_{nl}\|_{P_{>j} u} &(\|P_{\leq j} u\| + \|P_{>j} u\|)_{L_t^2 L_x^1} \\ &\lesssim \|\nabla|^{1/2} P_j u_{nl}\|_{L_t^2 L_x^5} \|\nabla|^{1/2} P_{>j} u\|_{L_t^\infty L_x^2} \|u\|_{L_t^\infty L_x^3} \lesssim 1. \end{aligned}$$

Therefore,

$$\begin{aligned} (2.18) \quad \left\| \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} |u|^2 \nabla u_{nl}(\tau) d\tau \right\|_{L^\infty} &\lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \| |u|^2 \nabla u_{nl} \|_{L^1} d\tau \\ &\lesssim \frac{1}{\delta t} \| |u|^2 \nabla u_{nl} \|_{L_t^2 L_x^1} \lesssim \frac{1}{\delta t}. \end{aligned}$$

The same computation can be also be made for $u^2 \nabla \bar{u}_{nl}$. This completes the proof of Lemma 2.2. $\blacksquare$

Lemma 2.3. *For any $t \in [0, 1]$,*

$$(2.19) \quad \|\nabla|^{1/2} w(t)\|_{L^3} \lesssim \frac{1}{\delta^{1/4} t^{1/4}}.$$

Proof. First observe that by interpolation, Bernstein’s inequality, and (2.16),

$$(2.20) \quad \|\nabla|^{1/2} e^{it\Delta} u_0\|_{L^3} \lesssim t^{1/4} \|\nabla e^{it\Delta} P_{\geq t^{-1/2}} u_0\|_{L^3} + t^{-1/4} \|P_{\leq t^{-1/2}} u_0\|_{\dot{H}^{1/2}} \lesssim t^{-1/4}.$$

Also since $e^{it\Delta}$ is unitary in L^2 , by (2.1) and (2.2),

$$(2.21) \quad \|v(t)\|_{\dot{H}^{1/2}} = \|u_{nl}((1-\delta)t)\|_{\dot{H}^{1/2}} \lesssim \|\nabla|^{1/2} u\|_{L_t^3 L_x^{18/5}} \|u\|_{L_{t,x}^5}^2 \lesssim \epsilon_0^2,$$

so interpolating (2.12), (2.13), and (2.21),

$$(2.22) \quad \|\nabla|^{1/2} v\|_{L^3} \lesssim \|\nabla|^{1/2} v\|_{L^\infty}^{1/3} \|\nabla|^{1/2} v\|_{L^2}^{2/3} \lesssim \frac{\epsilon_0^{4/3}}{\delta^{1/4} t^{1/4}}.$$

Finally, making a dispersive estimate, for any $t \in [0, 1]$, by (2.20) and (2.22), if $\delta^{1/4} \ll \epsilon_0$,

$$\begin{aligned}
 & \delta^{1/4} t^{1/4} \left\| \int_{(1-\delta)t}^t e^{i(t-\tau)\Delta} |\nabla|^{1/2} F(u) d\tau \right\|_{L^3} \\
 (2.23) \quad & \lesssim \delta^{1/4} t^{1/4} \int_{(1-\delta)t}^t \frac{1}{|t-\tau|^{1/2}} \| |\nabla|^{1/2} u(\tau) \|_{L^3} \|u(\tau)\|_{L^6}^2 d\tau \\
 & \lesssim \left(\sup_{t \in [0,1]} \delta^{1/4} t^{1/4} \| |\nabla|^{1/2} u \|_{L^3} \right)^3 \lesssim \epsilon_0^4 + \left(\sup_{t \in [0,1]} \delta^{1/4} t^{1/4} \| |\nabla|^{1/2} w \|_{L^3} \right)^3.
 \end{aligned}$$

Thus, absorbing the second term on the right-hand side into the left-hand side of (2.23) proves (2.19):

$$\| |\nabla|^{1/2} w(t) \|_{L^3} \lesssim \frac{\epsilon_0^4}{\delta^{1/4} t^{1/4}}. \quad \blacksquare$$

Remark. To make the proofs of Lemmas 2.2 and 2.3 completely rigorous, truncate u_0 in frequency. Then the bounds (2.12), (2.13), and (2.19) all hold on some open subset of $[0, 1]$ that contains 0. Making the bootstrap argument using the proof of Lemma 2.3 gives bounds on all of $[0, 1]$ that do not depend on the frequency truncation of u_0 . Standard perturbation arguments then give the lemmas.

Lemma 2.3 can be strengthened to an estimate on the $\dot{H}^1$ norm of w .

Lemma 2.4. For any $t \in [0, 1]$,

$$\| \nabla w(t) \|_{L^2} \lesssim \frac{1}{\delta^{1/4} t^{1/4}}.$$

Proof. Once again make use of the bilinear Strichartz estimate. Again by the product rule,

$$\nabla F(u) = 2|u|^2 \nabla u + u^2 \nabla \bar{u}.$$

First, by Strichartz estimates, (2.16), Lemma 2.3, and the Sobolev embedding theorem,

$$\begin{aligned}
 & \left\| \int_{(1-\delta)t}^t e^{i(t-\tau)\Delta} [2|u|^2 \nabla u_l + u^2 \nabla \bar{u}_l] d\tau \right\|_{L^2} \lesssim \| 2|u|^2 \nabla u_l + u^2 \nabla \bar{u}_l \|_{L_t^2 L_x^{6/5}} \\
 & \lesssim \delta^{1/2} t^{1/2} \| \nabla u_l \|_{L_t^\infty L_x^3((1-\delta)t, t] \times \mathbb{R}^3)} \| u \|_{L_t^\infty L_x^3((1-\delta)t, t] \times \mathbb{R}^3)} \| |\nabla|^{1/2} u \|_{L_t^\infty L_x^3((1-\delta)t, t] \times \mathbb{R}^3)} \\
 & \lesssim \frac{\delta^{1/4}}{t^{1/4}}.
 \end{aligned}$$

Next, by (2.19), bilinear Strichartz estimates in (2.9), and the Littlewood–Paley theorem,

$$\begin{aligned}
 & \| 2|u_{\leq j}|^2 (\nabla P_j u_{nl}) + (u_{\leq j})^2 (\nabla P_j \bar{u}_{nl}) \|_{L_t^2 L_x^{6/5}} \\
 & \lesssim \sum_{k \geq 0} 2^{-k/2} \left\| \left(\sum_{j_1 \leq j} 2^{j_1+k} |P_{j_1+k} u|^2 \right)^{1/2} \left(\sum_{j_1 \leq j} 2^{-j_1} 2^{2j} |P_{j_1} u|^2 |P_j u_{nl}|^2 \right)^{1/2} \right\|_{L_t^2 L_x^{6/5}} \\
 & \lesssim \sum_{k \geq 0} 2^{-k/2} \| |\nabla|^{1/2} u(t) \|_{L_t^\infty L_x^3((1-\delta)t, t] \times \mathbb{R}^3)} \\
 & \quad \times \left(\sum_{j_1 \leq j} \| P_{j_1} u_0 \|_{\dot{H}^{1/2}}^2 + \| P_{j_1} F(u) \|_{L_t^1 L_x^2}^2 \right)^{1/2} 2^{j/2} \| P_j F(u) \|_{L_t^1 L_x^2} \\
 & \lesssim \frac{1}{\delta^{1/4} t^{1/4}} \| |\nabla|^{1/2} P_j F(u) \|_{L_t^1 L_x^2}.
 \end{aligned}$$

Next, by Bernstein's inequality and (2.19)–(2.21),

$$\begin{aligned} & \|(\nabla P_j u_{nl})|u_{\geq j}||u|\|_{L_t^2 L_x^{6/5}} \\ & \lesssim \delta^{1/4} t^{1/4} \| |\nabla|^{1/2} u \|_{L_t^\infty L_x^3([(1-\delta)t, t] \times \mathbb{R}^3)}^2 \| |\nabla|^{1/2} P_j u_{nl} \|_{L_t^4 L_x^3([(1-\delta)t, t] \times \mathbb{R}^3)} \\ & \lesssim \frac{1}{\delta^{1/4} t^{1/4}} \| |\nabla|^{1/2} P_j F(u) \|_{L_t^1 L_x^2([0, 1] \times \mathbb{R}^3)}. \end{aligned}$$

Summing up in j using Lemma 2.1 completes the proof. $\blacksquare$

Remark. The above arguments would work equally well in the time interval $[-1, 0]$.

3. Global well-posedness

We are ready to prove Theorem 1.4. The proof will use conservation of the energy (1.4). Decompose

$$u(1) = \tilde{v}(1) + w(1),$$

where

$$(3.1) \quad \tilde{v}(1) = u_l(1) + v(1),$$

and $w(1)$ is the w in the previous section. Let $T_0 > 1$ be a time value for which we know that (1.1) has a solution on $[0, T_0)$. By standard local well-posedness arguments and we know that such a T_0 exists. Then on $[1, T_0)$, decompose

$$u(t) = \tilde{v}(t) + w(t),$$

where $\tilde{v}(t)$ is the solution to

$$(3.2) \quad (i\partial_t + \Delta)\tilde{v}(t) = 0, \quad \tilde{v}(1) = \tilde{v}(1, x),$$

and $w(t)$ is the solution to

$$(3.3) \quad (i\partial_t + \Delta)w = |u|^2 u, \quad w(1) = w(1, x).$$

Let $E(t)$ denote the energy of w ,

$$E(t) = \frac{1}{2} \int |\nabla w|^2 + \frac{1}{4} \int |w|^4.$$

First observe that Lemma 2.4 and $\|w(1)\|_{\dot{H}^{1/2}} \lesssim 1$ implies that $E(1) < \infty$. The estimate $\|w(1)\|_{\dot{H}^{1/2}}$ is a consequence of Lemma 2.1 and the definition of w . To prove Theorem 1.4, it suffices to prove that for any $T_0 > 1$ such that (1.1) has a solution on $[0, T_0)$,

$$(3.4) \quad \sup_{t \in [1, T_0)} E(t) < \infty.$$

Indeed, by interpolation and the Sobolev embedding theorem, $E(t) < \infty$ implies that $\|w(t)\|_{L^5} < \infty$. Meanwhile, by (2.14)–(2.16), (2.12), and (2.21), $\|\tilde{v}(t)\|_{L^5}$ is uniformly bounded on $\mathbb{R}$. Therefore, (3.4) implies

$$\|u\|_{L^5_{t,x}([0,T_0]\times\mathbb{R}^3)} < \infty.$$

To estimate the growth of $E(t)$, compute the derivative in time of the energy. By (3.3),

$$\frac{d}{dt}E(t) = -\langle \Delta w, w_t \rangle + \langle |w|^2 w, w_t \rangle = \langle |w|^2 w - |u|^2 u, w_t \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product

$$\langle f, g \rangle = \operatorname{Re} \int f(x) \bar{g}(x) dx.$$

By the product rule,

$$\begin{aligned} \langle w_t, |u|^2 u - |w|^2 w \rangle &= \frac{d}{dt} \langle |w|^2 w, \tilde{v} \rangle + \frac{d}{dt} \langle |\tilde{v}|^2, |w|^2 \rangle \\ (3.5) \quad &+ \frac{1}{2} \frac{d}{dt} \operatorname{Re} \int \bar{w}^2 \tilde{v}^2 + \frac{d}{dt} \langle w, |\tilde{v}|^2 \tilde{v} \rangle - 2 \langle \tilde{v}_t \bar{\tilde{v}}, |w|^2 \rangle \\ &- \langle |w|^2 w, \tilde{v}_t \rangle - \operatorname{Re} \int w^2 \bar{\tilde{v}} \tilde{v}_t - 2 \langle w, |\tilde{v}|^2 \tilde{v}_t \rangle - \langle w, \tilde{v}^2 \bar{\tilde{v}}_t \rangle. \end{aligned}$$

Then define the modified energy,

$$\mathcal{E}(t) = E(t) - \langle |w|^2 w, \tilde{v} \rangle - \langle |\tilde{v}|^2, |w|^2 \rangle - \frac{1}{2} \operatorname{Re} \int w^2 \bar{\tilde{v}}^2 - \langle w, |\tilde{v}|^2 \tilde{v} \rangle.$$

By Hölder's inequality, and the fact that $\|\tilde{v}\|_{L^4} \lesssim_\delta 1$ for all $t \in [1, \infty)$ (again using (2.14)–(2.16), (2.12), and (2.21)),

$$\langle |w|^2 w, \tilde{v} \rangle + \langle |\tilde{v}|^2, |w|^2 \rangle + \frac{1}{2} \operatorname{Re} \int w^2 \bar{\tilde{v}}^2 + \langle w, |\tilde{v}|^2 \tilde{v} \rangle \lesssim E(t)^{3/4} + E(t)^{1/4}.$$

Therefore, when $E(t)$ is large, $E(t) \sim \mathcal{E}(t)$. Since we are attempting to prove a uniform bound for $E(t)$, it is enough to uniformly bound $\mathcal{E}(t)$.

Also, by (3.5),

$$\frac{d}{dt} \mathcal{E}(t) = -\langle |w|^2 w, \tilde{v}_t \rangle - 2 \langle \tilde{v}_t \bar{\tilde{v}}, |w|^2 \rangle - \operatorname{Re} \int w^2 \bar{\tilde{v}} \tilde{v}_t - 2 \langle w, |\tilde{v}|^2 \tilde{v}_t \rangle - \langle w, \tilde{v}^2 \bar{\tilde{v}}_t \rangle.$$

Since $\tilde{v}$ solves (3.2), $\tilde{v}_t = i \Delta \tilde{v} = i \Delta u_t + i \Delta v$.

Lemma 2.2 implies that for any $t > 1$,

$$(3.6) \quad \|v(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty} = \left\| \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} \langle \nabla \rangle F(u) d\tau \right\|_{L^\infty} \lesssim \frac{1}{\delta^{3/2} t^{3/2}}.$$

Therefore,

$$\langle |w|^2 w, i \Delta v \rangle = -\langle \nabla(|w|^2 w), i \nabla v \rangle \lesssim \|\nabla v\|_{L^\infty} \|\nabla w\|_{L^2} \|w\|_{L^4}^2 \lesssim_\delta \frac{1}{t^{3/2}} E(t).$$

Remark. Since $\delta > 0$ is fixed, we will ignore it from now on.

Also, by Hölder's inequality and (1.8),

$$\langle i \Delta(e^{it\Delta} u_0), |w|^2 w \rangle \lesssim \| |\nabla|^{11/7} u_t \|_{L^7} \| \nabla w \|_{L^2}^{3/7} \| w \|_{L^4}^{18/7} \lesssim \frac{1}{t^{15/14}} E(t)^{6/7}.$$

This takes care of the contribution of $\langle \tilde{v}_t, |w|^2 w \rangle$.

Next, integrating by parts,

$$(3.7) \quad 2 \langle i(\Delta \tilde{v}) \tilde{v}, |w|^2 \rangle = -2 \langle i |\nabla \tilde{v}|^2, |w|^2 \rangle - 2 \langle i(\nabla \tilde{v}) \tilde{v}, \nabla |w|^2 \rangle = -2 \langle i(\nabla \tilde{v}) \tilde{v}, \nabla |w|^2 \rangle.$$

Then by Hölder's inequality and (3.6), since $\|\tilde{v}\|_{L^4} \lesssim 1$,

$$\langle i(\nabla v) \tilde{v}, \nabla |w|^2 \rangle \lesssim \|\nabla v\|_{L^\infty} \|\tilde{v}\|_{L^4} \|w\|_{L^4} \|\nabla w\|_{L^2} \lesssim \frac{1}{t^{3/2}} E(t)^{3/4}.$$

Also, by Hölder's inequality and interpolation,

$$(3.8) \quad \langle i(\nabla u_t)(u_t), \nabla |w|^2 \rangle \lesssim \|\nabla u_t\|_{L_x^\infty} \|u_t\|_{L^4} \|\nabla w\|_{L^2} \|w\|_{L^4} \lesssim \frac{1}{t} \frac{1}{t^{1/8}} E(t)^{3/4}.$$

Finally, by (3.6), and Lemma 2.1, which by the Sobolev embedding theorem and the definition of v implies $\|v\|_{L^3} \lesssim 1$

$$(3.9) \quad \langle i(\nabla u_t)v, \nabla |w|^2 \rangle \lesssim \|\nabla u_t\|_{L_x^\infty} \|v\|_{L^3}^{3/4} \|v\|_{L^\infty}^{1/4} \|\nabla w\|_{L^2} \|w\|_{L^4} \lesssim \frac{1}{t} \frac{1}{t^{3/8}} E(t)^{3/4}.$$

In (3.8) and (3.9) we used:

Lemma 3.1. *For any $t \geq 0$,*

$$(3.10) \quad \|u_t\|_{L^4} \lesssim \frac{1}{t^{1/8}},$$

and

$$(3.11) \quad \|\nabla u_t\|_{L^\infty} \lesssim \frac{1}{t}.$$

Proof. This is proved by interpolating (2.14)–(2.16). By Bernstein's inequality, (2.15), (2.16), and the Sobolev embedding theorem,

$$(3.12) \quad \|\nabla P_{\leq t^{-1/2}} u_t\|_{L^\infty} + \|\nabla P_{\geq t^{-1/2}} u_t\|_{L^\infty} \lesssim \frac{1}{t}.$$

Also by the Bernstein inequality and the Sobolev embedding theorem, along with (2.16) and $u_t \in \dot{H}^{1/2}$,

$$(3.13) \quad \|P_{\geq t^{-1/2}} u_t\|_{L^4} + \|P_{\leq t^{-1/2}} u_t\|_{L^4} \lesssim \frac{1}{t^{1/8}}.$$

This proves the lemma. ■

The contribution of $2\operatorname{Re} \int w^2 \bar{v} \bar{v}_t$ may be estimated in a similar manner as the contribution of (3.7), except that there is an additional term to consider,

$$-2\operatorname{Re} \int i w^2 (\nabla \bar{v})^2.$$

Interpolating (3.11) with (2.16),

$$-2\operatorname{Re} \int i w^2 (\nabla \bar{u}_l)^2 \lesssim \|\nabla u_l\|_{L^4}^2 \|w\|_{L^4}^2 \lesssim \frac{1}{t^{5/4}} E(t)^{1/2}.$$

Meanwhile, following (2.17) and using Strichartz estimates,

$$\begin{aligned} \| |u_{\leq j}|^2 (\nabla P_j u_{nl}) \|_{L_t^1 L_x^{3/2}([0,1] \times \mathbf{R}^3)} &\lesssim \sum_{j_1 \leq j_2 \leq j} \|(P_{j_1} u)(P_{j_2} \nabla u_{nl})\|_{L_{t,x}^2} \|P_{j_2} u\|_{L_t^2 L_x^6} \\ &\lesssim \sum_{j_1 \leq j_2 \leq j} 2^{j_1/2} 2^{-j_2/2} 2^{j/2} \|P_j F(u)\|_{L_t^1 L_x^2} (\|\nabla\|^{1/2} P_{j_1} u_0\|_{L^2} + \|\nabla\|^{1/2} P_{j_1} F(u)\|_{L_t^1 L_x^2}) \\ &\quad \times (\|\nabla\|^{1/2} P_{j_2} u_0\|_{L^2} + \|\nabla\|^{1/2} P_{j_2} F(u)\|_{L_t^1 L_x^2}) \lesssim 1. \end{aligned}$$

Plugging this estimate into (2.18) implies that for $t > 1$,

$$\left\| \nabla \int_0^{(1-\delta)} e^{i(t-\tau)\Delta} F(u) \right\|_{L_x^3} \lesssim \frac{1}{t^{1/2}}.$$

Interpolating (3.6) with (3.10),

$$-2\operatorname{Re} \int i w^2 (\nabla \bar{v})^2 \lesssim \|\nabla \bar{v}\|_{L^4}^2 \|w\|_{L^4}^2 \lesssim \frac{1}{t^{3/2}} E(t)^{1/2}.$$

Now treat

$$(3.14) \quad 2\langle w, |\bar{v}|^2 \bar{v}_t \rangle + \langle w, \bar{v}^2 \bar{v}_t \rangle = 2\langle w, |\bar{v}|^2 (i\Delta \bar{v}) \rangle + \langle w, \bar{v}^2 (i\Delta \bar{v}) \rangle.$$

After integrating by parts, by (2.13) and (3.11),

$$\begin{aligned} (3.14) &\lesssim \langle |\nabla \bar{v}|^2, |v||w| \rangle + \langle |\nabla \bar{v}| |\nabla w|, |v|^2 \rangle \\ &\lesssim \|\nabla \bar{v}\|_{L^4}^2 \|\bar{v}\|_{L^4} \|w\|_{L^4} + \|\nabla w\|_{L^2} \|\nabla \bar{v}\|_{L^\infty} \|\bar{v}\|_{L^4}^2 \lesssim \frac{1}{t^{5/4}} E(t)^{1/4} + \frac{1}{t} E(t)^{1/2} \|\bar{v}(t)\|_{L^4}. \end{aligned}$$

Interpolating (3.6) with $\|v\|_{L^3} \lesssim 1$ implies $\|v\|_{L^4} \lesssim t^{-3/8}$. Meanwhile, (3.10) implies $\|u_l\|_{L^4} \lesssim t^{-1/8}$, so therefore, by (3.1), $\|\bar{v}\|_{L^4} \lesssim 1/t^{1/8}$. Therefore, we have proved

$$(3.15) \quad \frac{d}{dt} \mathcal{E}(t) \lesssim \frac{1}{t^{15/14}} (1 + \mathcal{E}(t)).$$

By Gronwall's inequality, (3.15) implies a uniform bound on $\mathcal{E}(t)$. This implies a uniform bound on $E(t)$, since $E(t) \sim \mathcal{E}(t)$ when $E(t)$ is large, which proves Theorem 1.4.

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