SCATTERING FOR THE RADIAL DEFOCUSING CUBIC NONLINEAR WAVE EQUATION WITH INITIAL DATA IN THE CRITICAL SOBOLEV SPACE

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Abstract

We prove global well-posedness and scattering for the defocusing cubic nonlinear wave equation on $\mathbf{R}^{1+3}$ *with radial initial data lying in the critical Sobolev space* $\dot{H}^{1/2}(\mathbf{R}^3) \times \dot{H}^{-1/2}(\mathbf{R}^3)$. *This result is sharp for radial initial data.*

Contents

1. Introduction

In this paper we study the defocusing cubic nonlinear wave equation

$$u_{tt} - \Delta u + u^3 = 0, \qquad u(0,x) = u_0, \qquad u_t(0,x) = u_1. \tag{1.1}$$

This problem is $\dot{H}^{1/2}$-critical, since the equation (1.1) is invariant under the scaling symmetry

$$u(t,x) \mapsto \lambda u(\lambda t, \lambda x). \tag{1.2}$$

This scaling symmetry completely determines the local well-posedness theory for (1.1). Positively, [15] proved the following.

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THEOREM 1.1
The equation (1.1) is locally well-posed for initial data in $u_0 \in \dot{H}^{1/2}(\mathbf{R}^3)$ *and* $u_1 \in \dot{H}^{-1/2}(\mathbf{R}^3)$ *on some interval* $[-T(u_0, u_1), T(u_0, u_1)]$. *The time of well-posedness* $T(u_0, u_1)$ *depends on the profile of the initial data* (u_0, u_1), *not just its size.*

Additional regularity is enough to give a lower bound on the time of well-posedness. Therefore, there exists some $T(\|u_0\|_{\dot{H}^s}, \|u_1\|_{\dot{H}^{s-1}}) > 0$ *for any* $\frac{1}{2} < s < \frac{3}{2}$.

Negatively, [15] proved the following.

THEOREM 1.2
The equation (1.1) is ill-posed for $u_0 \in \dot{H}^s(\mathbf{R}^3)$ *and* $u_1 \in \dot{H}^{s-1}(\mathbf{R}^3)$ *when* $s < \frac{1}{2}$.

Local well-posedness is defined in the usual way.

Definition 1.1 (Locally well-posed)
The initial value problem (1.1) is said to be *locally well-posed* if there exists an open interval $I \subset \mathbf{R}$ containing 0 such that

(1) a unique solution $u \in L_t^\infty \dot{H}^{1/2}(I \times \mathbf{R}^3) \cap L_{t,\mathrm{loc}}^4 L_x^4(I \times \mathbf{R}^3)$, $u_t \in L_t^\infty \times \dot{H}^{-1/2}(I \times \mathbf{R}^3)$ exists;
(2) the solution u is continuous in time, $u \in C(I; \dot{H}^{1/2}(\mathbf{R}^3))$, $u_t \in C(I; \dot{H}^{-1/2}(\mathbf{R}^3))$;
(3) the solution u depends continuously on the initial data in the topology of item (1).

Given this fact, it is natural to inquire as to the long-time behavior of solutions to (1.1) with initial data at the $\dot{H}^{1/2}$-critical regularity. Do they continue for all time, and if they do, what is their behavior at large times?

Global well-posedness for initial data in $\dot{H}^{1/2} \cap \dot{H}^1(\mathbf{R}^3) \times \dot{H}^{-1/2} \cap L^2(\mathbf{R}^3)$ follows from conservation of the energy

$$E\big(u(t)\big) = \frac{1}{2}\int u_t(t,x)^2\,dx + \frac{1}{2}\int \big|\nabla u(t,x)\big|^2\,dx + \frac{1}{4}\int u(t,x)^4\,dx. \tag{1.3}$$

By the Sobolev embedding theorem and Hölder's inequality,

$$\begin{aligned}\|u(0)\|_{L_x^4(\mathbf{R}^3)}^4 &\lesssim \|u(0)\|_{L_x^3(\mathbf{R}^3)}^2 \|u(0)\|_{L_x^6(\mathbf{R}^3)}^2 \\ &\lesssim \|u(0)\|_{\dot{H}^{1/2}(\mathbf{R}^3)}^2 \|u(0)\|_{\dot{H}^1(\mathbf{R}^3)}^2,\end{aligned} \tag{1.4}$$

and therefore,

$$E\big(u(0)\big) \lesssim_{\|u_0\|_{\dot{H}^{1/2}}} \|u_0\|_{\dot{H}^1(\mathbf{R}^3)}^2 + \|u_1\|_{L^2(\mathbf{R}^3)}^2. \tag{1.5}$$

By (1.3), $E(u(t)) = E(u(0))$ controls the size of $\|u(t)\|_{\dot{H}^1} + \|u_t(t)\|_{L^2}$, which by Theorem 1.1 gives global well-posedness.

Comparing (1.1) to the quintic wave equation in three dimensions,

$$u_{tt} - \Delta u + u^5 = 0, \qquad u(0,x) = u_0, \qquad u_t(0,x) = u_1, \tag{1.6}$$

a solution to (1.6) is invariant under the scaling symmetry $u(t,x) \mapsto \lambda^{1/2} u(\lambda t, \lambda x)$, a symmetry that preserves the $(\dot{H}^1 \times L^2)$-norm of (u_0, u_1). Observe that the conserved energy for (1.6)

$$E\big(u(t)\big) = \frac{1}{2}\int u_t(t,x)^2\,dx + \frac{1}{2}\int \big|\nabla u(t,x)\big|^2\,dx + \frac{1}{6}\int u(t,x)^6\,dx \tag{1.7}$$

is also invariant under the scaling symmetry. For this reason, (1.6) is called *energy-critical*, and it is possible to prove a result in the same vein as Theorems 1.1 and 1.2 at the critical regularity $\dot{H}^1 \times L^2$.

This fact combined with conservation of the energy (1.7) is insufficient to prove global well-posedness for (1.6). The reason is because the time of local well-posedness depends on the profile of the initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$, and not just its size. Instead, the proof of global well-posedness for the quintic problem uses a nonconcentration of energy argument. This result has been completely worked out, proving both global well-posedness and scattering, for both the radial (see [9], [26]) and the nonradial case (see [2], [11], [19], [20]).

Definition 1.2 (Scattering)
A solution to (1.6) is said to be *scattering* in some $\dot{H}^s(\mathbf{R}^3) \times \dot{H}^{s-1}(\mathbf{R}^3)$ if there exist $(u_0^+, u_1^+), (u_0^-, u_1^-) \in \dot{H}^s \times \dot{H}^{s-1}$ such that

$$\lim_{t\to+\infty} \big\|\big(u(t), u_t(t)\big) - S(t)(u_0^+, u_1^+)\big\|_{\dot{H}^s \times \dot{H}^{s-1}} = 0 \tag{1.8}$$

and

$$\lim_{t\to-\infty} \big\|\big(u(t), u_t(t)\big) - S(t)(u_0^-, u_1^-)\big\|_{\dot{H}^s \times \dot{H}^{s-1}} = 0, \tag{1.9}$$

where $S(t)(f,g)$ is the solution operator to the linear wave equation. That is, if $(u(t), u_t(t)) = S(t)(f,g)$, then

$$u_{tt} - \Delta u = 0, \qquad u(0,x) = f, \qquad u_t(0,x) = g. \tag{1.10}$$

Similar results for (1.1) may also be obtained if one assumes a uniform bound over $\|u\|_{\dot{H}^{1/2}(\mathbf{R}^3)} + \|u_t\|_{\dot{H}^{-1/2}(\mathbf{R}^3)}$ for the entire time of existence of the solution.

THEOREM 1.3
Suppose that $u_0 \in \dot{H}^{1/2}(\mathbf{R}^3)$ *and* $u_1 \in \dot{H}^{-1/2}(\mathbf{R}^3)$ *are radial functions and that* u *solves (1.1) on a maximal interval* $0 \in I \subset \mathbf{R}$, *with*

$$\sup_{t \in I} \|u(t)\|_{\dot{H}^{1/2}(\mathbf{R}^3)} + \|u_t(t)\|_{\dot{H}^{-1/2}(\mathbf{R}^3)} < \infty. \tag{1.11}$$

Then $I = \mathbf{R}$ *and the solution* u *scatters both forward and backward in time.*

Proof
See [7]. □

However, unlike the energy-critical problem, there is no a priori reason to believe that the critical Sobolev norm will remain bounded for the entire time of its existence. We remove this assumption on uniform boundedness of the critical norm in (1.11), proving the following result.

THEOREM 1.4
The initial value problem (1.1) is globally well-posed and scattering for radial initial data $u_0 \in \dot{H}^{1/2}(\mathbf{R}^3)$ *and* $u_1 \in \dot{H}^{-1/2}(\mathbf{R}^3)$. *Moreover, there exists a function* $f : [0,\infty) \to [0,\infty)$ *such that if* u *solves (1.1) with initial data* $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, *then*

$$\|u\|_{L^4_{t,x}(\mathbf{R} \times \mathbf{R}^3)} \leq f\big(\|u_0\|_{\dot{H}^{1/2}(\mathbf{R}^3)} + \|u_1\|_{\dot{H}^{-1/2}(\mathbf{R}^3)}\big). \tag{1.12}$$

The proof of Theorem 1.4 combines the Fourier truncation method and hyperbolic coordinates. Previously, [13] applied the Fourier truncation method to the cubic wave equation (1.1), proving global well-posedness of (1.1) with initial data lying in the inhomogeneous Sobolev spaces $H^s_x(\mathbf{R}^3) \times H^{s-1}_x(\mathbf{R}^3)$ for $s > \frac{3}{4}$. This argument was improved and modified in many subsequent papers, for both radial and nonradial data. In particular, see [6] for a proof of global well-posedness for (1.1) with radial initial data lying in

$$\big(\dot{H}^s(\mathbf{R}^3) \cap \dot{H}^{1/2}(\mathbf{R}^3)\big) \times \big(\dot{H}^{s-1}(\mathbf{R}^3) \cap \dot{H}^{-1/2}(\mathbf{R}^3)\big), \tag{1.13}$$

for any $s > \frac{1}{2}$, as well as for a description of other results along this line.

Remark
The method used in [6] was the I-method, a modification of the Fourier truncation method.

In this paper, using the Fourier truncation method, global well-posedness is proved for (1.1) with radial initial data lying in $\dot{H}^{1/2}(\mathbf{R}^3) \times \dot{H}^{-1/2}(\mathbf{R}^3)$. The idea behind the proof is that at low frequencies, the initial data has finite energy, and a solution to (1.1) with finite energy is global. Meanwhile, at high frequencies, the $(\dot{H}^{1/2} \times \dot{H}^{-1/2})$-norm is small, and for such initial data, (1.1) may be treated using perturbative arguments (see, e.g., [24]). The mixed terms in the nonlinearity are then shown to have finite energy, proving global well-posedness.

Proof of scattering utilizes hyperbolic coordinates. Scattering for smooth data with sufficiently rapid decay was proved in [23] using conservation of a conformal energy. Hyperbolic coordinates were used in [27] to prove weighted Strichartz estimates that were proved in [8] for compactly supported data. More recently, Shen [21], working in hyperbolic coordinates, was able to prove a scattering result for data lying in a weighted energy space. Later, [4] combined the result of [21] with the I-method argument in [6] to prove scattering data lying in the subspace of $\dot{H}^{1/2} \times \dot{H}^{-1/2}$,

$$\begin{aligned} &\|u_0\|_{\dot{H}^{1/2+\epsilon}(\mathbf{R}^3)} + \big\||x|^{2\epsilon} u_0\big\|_{\dot{H}^{1/2+\epsilon}(\mathbf{R}^3)} + \|u_1\|_{\dot{H}^{-1/2+\epsilon}(\mathbf{R}^3)} \\ &\quad + \big\||x|^{2\epsilon} u_1\big\|_{\dot{H}^{-1/2+\epsilon}(\mathbf{R}^3)}. \end{aligned} \tag{1.14}$$

Here the Fourier truncation global well-posedness argument in hyperbolic coordinates shows that (1.1) is globally well-posed and scattering for any $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$.

This fact still falls short of (1.12), since the proof does not give any uniform control over the $\|u\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)}$-norm. To remedy this deficiency, and complete the proof of Theorem 1.4, a profile decomposition is used (see [1], [18]). The profile decomposition shows that for any bounded sequence of initial data

$$\|u_0^n\|_{\dot{H}^{1/2}(\mathbf{R}^3)} + \|u_1^n\|_{\dot{H}^{-1/2}(\mathbf{R}^3)} \leq A, \tag{1.15}$$

and if $u^n(t)$ is the global solution to (1.1) with initial data (u_0^n, u_1^n), then

$$\|u^n\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} < \infty \tag{1.16}$$

is uniformly bounded. Then by Zorn's lemma, the proof of Theorem 1.4 is complete.

The author believes this to be the first unconditional global well-posedness and scattering result for a nonlinear wave equation with initial data lying in the critical Sobolev space, with no conserved quantity that controls the critical norm. Previously, [5] proved global well-posedness and scattering for (1.1) with radial initial data lying in the Besov space $B^2_{1,1} \times B^1_{1,1}$. These spaces are also invariant under the scaling (1.2). Later, [16] proved a similar result in five dimensions.

There are two main improvements for this result over the results of [5] and [16]. The first is that, while scale-invariant, the Besov spaces are only subsets of the critical Sobolev spaces. The second improvement is that the $(\dot{H}^{1/2} \times \dot{H}^{-1/2})$-norm is invariant under the free evolution of the linear wave equation. Whereas, for initial data lying in a Besov space, the proof of scattering simply meant that the solution scattered in the $(\dot{H}^{1/2} \times \dot{H}^{-1/2})$-norm.

The main obstacle to extending the Besov space result to scattering in the critical Sobolev space lies in that the dispersive estimates cannot be easily applied in this setting. For data in $B^2_{1,1} \times B^1_{1,1}$, if $\tilde{u}$ solves the linear wave equation with the same initial data (u_0, u_1), then the dispersive estimate implies that

$$\|\tilde{u}\|_{L^\infty} \lesssim \frac{1}{t} \|(u_0, u_1)\|_{B^2_{1,1} \times B^1_{1,1}}. \tag{1.17}$$

This gives some time integrability for $\tilde{u}$ that proves quite useful in [5].

Additionally, for radial initial data, the space $B^2_{1,1} \times B^1_{1,1}$ is contained in the energy space if the initial data is supported away from the origin. Thus, [5] was able to split the initial data into a finite energy piece, and a piece whose linear solution must travel along the light cone. However, for generic $u_0 \in \dot{H}^{1/2}$ with radial symmetry, there is no reason to think that u_0 has a derivative that lies in any L^p space. There is no reason to think that u_1 lies in any Lebesgue space either.

2. Local well-posedness

The local well-posedness result of [15] may be proved via the Strichartz estimates of [25].

THEOREM 2.1

Let $I \subset \mathbf{R}$, $t_0 \in I$, be an interval, and let $u : I \times \mathbf{R}^3 \to \mathbf{R}$ be a solution to the linear wave equation

$$u_{tt} - \Delta u = F, \qquad u(t_0) = u_0, \qquad u_t(t_0) = u_1. \tag{2.1}$$

Then u satisfies the estimates

$$\begin{aligned} &\|u\|_{L^p_t L^q_x(I \times \mathbf{R}^3)} + \|u\|_{L^\infty_t \dot{H}^s(I \times \mathbf{R}^3)} + \|u_t\|_{L^\infty_t \dot{H}^{s-1}(I \times \mathbf{R}^3)} \\ &\qquad \lesssim_{p,q,s,\tilde{p},\tilde{q}} \|u_0\|_{\dot{H}^s(\mathbf{R}^3)} + \|u_1\|_{\dot{H}^{s-1}(\mathbf{R}^3)} + \|F\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}_x(I \times \mathbf{R}^3)}, \end{aligned} \tag{2.2}$$

whenever $s \geq 0$, $2 \leq p, \tilde{p} \leq \infty$, $2 \leq q, \tilde{q} < \infty$,

$$\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - s = \frac{1}{\tilde{p}'} + \frac{3}{\tilde{q}'} - 2, \tag{2.3}$$

and

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \qquad \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \leq \frac{1}{2}. \tag{2.4}$$

Proof

Theorem 2.1 was proved for $p = q = 4$ and $\tilde{p} = \tilde{q} = 4$ in [25] and then in [10] for a general choice of (p, q). □

To prove local well-posedness of (1.1), it will suffice to use (2.2) when $p = q = 4$. Indeed, (2.2) implies that for any I,

$$\|u\|_{L^4_{t,x}(I\times\mathbf{R}^3)} \lesssim \big\|S(t)(u_0,u_1)\big\|_{L^4_{t,x}(I\times\mathbf{R}^3)} + \|u\|^3_{L^4_{t,x}(I\times\mathbf{R}^3)}. \tag{2.5}$$

If $\|S(t)(u_0,u_1)\|_{L^4_{t,x}(I\times\mathbf{R}^3)} \le \epsilon$, then small data arguments imply that (1.1) is locally well-posed on the interval I.

Therefore, for $\|u_0\|_{\dot{H}^{1/2}} + \|u_1\|_{\dot{H}^{-1/2}}$ sufficiently small, (2.2) and (2.5) imply that (1.1) is well-posed on $I = \mathbf{R}$. For generic $(u_0,u_1) \in \dot{H}^{1/2}\times\dot{H}^{-1/2}$, the dominated convergence theorem and (2.2) imply that for any fixed $(u_0,u_1) \in \dot{H}^{1/2}\times\dot{H}^{-1/2}$,

$$\lim_{T\searrow 0}\big\|S(t)(u_0,u_1)\big\|_{L^4_{t,x}([-T,T]\times\mathbf{R}^3)} = 0, \tag{2.6}$$

which implies local well-posedness on some open interval I, where $0 \in I$.

Equation (2.5) also implies that (1.1) is locally well-posed on an interval I on which an a priori bound $\|u\|_{L^4_{t,x}(I\times\mathbf{R}^3)} < \infty$ is obtained. This may be seen by partitioning I into finitely many pieces I_j on which $\|u\|_{L^4_{t,x}(I_j\times\mathbf{R}^3)}$ is small, and then iterating local well-posedness arguments on each interval. This argument also shows that scattering is equivalent to $\|u\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} < \infty$.

Strichartz estimates also yield perturbative results.

LEMMA 2.2 (Perturbation lemma)

Let $I \subset \mathbf{R}$ be a time interval. Let $t_0 \in I$, $(u_0,u_1) \in \dot{H}^{1/2}\times\dot{H}^{-1/2}$ and some constants M, A, $A' > 0$. Let $\tilde{u}$ solve the equation

$$(\partial_{tt} - \Delta)\tilde{u} + \tilde{u}^3 = e \tag{2.7}$$

on $I \times \mathbf{R}^3$, and also suppose that $\sup_{t\in I}\|(\tilde{u}(t),\partial_t\tilde{u}(t))\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \le A$, $\|\tilde{u}\|_{L^4_{t,x}(I\times\mathbf{R}^3)} \le M$,

$$\big\|\big(u_0 - \tilde{u}(t_0), u_1 - \partial_t\tilde{u}(t_0)\big)\big\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \le A', \tag{2.8}$$

and

$$\|e\|_{L^{4/3}_{t,x}(I\times\mathbf{R}^3)} + \big\|S(t-t_0)\big(u_0 - \tilde{u}(t_0), u_1 - \partial_t\tilde{u}(t_0)\big)\big\|_{L^4_{t,x}(I\times\mathbf{R}^3)} \le \epsilon. \tag{2.9}$$

Then there exists $\epsilon_0(M, A, A')$ *such that if* $0 < \epsilon < \epsilon_0$*, then there exists a solution to* (1.1) *on* I *with* $(u(t_0), \partial_t u(t_0)) = (u_0, u_1)$, $\|u\|_{L^4_{t,x}(I\times\mathbf{R}^3)} \leq C(M, A, A')$*, and for all* $t \in I$*,*

$$\big\|\big(u(t), \partial_t u(t)\big) - \big(\tilde{u}(t), \partial_t \tilde{u}(t)\big)\big\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \leq C(A, A', M)(A' + \epsilon). \tag{2.10}$$

Proof

The method of proof is by now fairly well-known (see, e.g., Theorem 2.20 of [12]). □

Remark

The constant A' will typically be small. In fact, in Section 6, $A' = 0$. Since A' is small, we could probably replace $C(A, A', M)$ with $C(A, M)$; however, we will keep the same notation as [12] here to avoid any unnecessary confusion.

In Theorem 2.20 of [12], ϵ in (2.10) is replaced by ϵ^β for some $\beta > 0$. This is a consequence of having a nonlinearity of the form $|u|^{\frac{4}{N-2}}u$, where N can be arbitrarily large. Since we are only concerned with the cubic nonlinear wave equation, we can get $\beta = 1$ here.

The proof of Theorem 1.4 also utilizes some additional Strichartz estimates for radially symmetric data. First, [14] proved that the endpoint case of Theorem 2.1 also holds. This estimate fails for nonradial initial data by [17].

THEOREM 2.3

For (u_0, u_1) *radially symmetric, and* u *solves* (2.1) *with* $F = 0$,

$$\|u\|_{L^2_t L^\infty_x(\mathbf{R}\times\mathbf{R}^3)} \lesssim \|u_0\|_{\dot{H}^1(\mathbf{R}^3)} + \|u_1\|_{L^2(\mathbf{R}^3)}. \tag{2.11}$$

The results of [22] subsequently extended the range of (p, q).

THEOREM 2.4

Let (u_0, u_1) *be spherically symmetric, and suppose that* u *solves* (2.1) *with* $F = 0$. *Then, if* $q > 4$ *and*

$$\frac{1}{2} + \frac{3}{q} = \frac{3}{2} - s, \tag{2.12}$$

we obtain

$$\|u\|_{L^2_t L^q_x(\mathbf{R}\times\mathbf{R}^3)} \lesssim \|u_0\|_{\dot{H}^s(\mathbf{R}^3)} + \|u_1\|_{\dot{H}^{s-1}(\mathbf{R}^3)}. \tag{2.13}$$

3. Virial identities for the wave equation

The proof of Theorem 1.4 will also use some weighted Strichartz-type estimates. These estimates could actually be proved using Proposition 3.5 of [22] after making a Bessel function-type reduction from three dimensions to two dimensions using radial symmetry.

Here, these estimates will be proved using virial identities. There are at least two reasons for doing this. The first is that, in the author's opinion, the exposition is cleaner and more readable using virial identities. The second reason is that many of the computations may be applied equally well to defocusing problems as to linear problems.

Suppose that u solves the equation

$$u_{tt} - \Delta u + \mu u^3 = 0, \qquad u(0,x) = u_0, \qquad u_t(0,x) = u_1, \tag{3.1}$$

where $\mu = 0, 1$. The case when $\mu = 0$ is a solution to the linear wave equation, and $\mu = 1$ is the defocusing nonlinear wave equation (1.1).

THEOREM 3.1

If u solves (1.1) on an interval $[0, T]$, then

$$\int_0^T \int \frac{\mu}{|x|} u^4 \, dx \, dt \lesssim \|u\|_{L_t^\infty \dot{H}^1([0,T]\times\mathbf{R}^3)} \|u_t\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^3)}, \tag{3.2}$$

$$\sup_{R>0} \frac{1}{R^3} \int_0^T \int_{|x|\le R} u^2 \, dx \, dt \lesssim \|u\|_{L_t^\infty \dot{H}^1([0,T]\times\mathbf{R}^3)} \|u_t\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^3)}, \tag{3.3}$$

and

$$\sup_{R>0} \frac{1}{R} \int_0^T \int_{|x|\le R} \big[|\nabla u|^2 + u_t^2\big] \, dx \, dt \lesssim \|u\|_{L_t^\infty \dot{H}^1([0,T]\times\mathbf{R}^3)} \|u_t\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^3)}. \tag{3.4}$$

Remark

The implicit constants in (3.2)–(3.4) are independent of T.

Proof

Define the generic Morawetz potential, where $a(x) = a(|x|)$ is radially symmetric,

$$M(t) = \int u_t a\big(|x|\big) x \cdot \nabla u + \int u_t a\big(|x|\big) u. \tag{3.5}$$

Computing the time derivative, by (3.1),

$$\frac{d}{dt}M(t) = \int u_t a\big(|x|\big)x \cdot \nabla u_t + \int u_t^2 a\big(|x|\big)$$
$$+ \int \Delta u a\big(|x|\big)x \cdot \nabla u + \int \Delta u a\big(|x|\big)u$$
$$- \mu \int u^3 a\big(|x|\big)x \cdot \nabla u - \mu \int u^3 a\big(|x|\big)u. \tag{3.6}$$

Integrating by parts,

$$\frac{d}{dt}M(t) = -\frac{1}{2}\int \big[a\big(|x|\big) + a'\big(|x|\big)|x|\big]u_t^2 - \frac{1}{2}\int \big[a\big(|x|\big) + a'\big(|x|\big)|x|\big]|\nabla u|^2$$
$$+ \int a'\big(|x|\big)|x|\big[|\nabla u|^2 - |\partial_r u|^2\big] + \frac{1}{2}\int u^2 \Delta a\big(|x|\big)$$
$$- \frac{\mu}{4}\int a\big(|x|\big)u^4 + \frac{\mu}{4}\int a'\big(|x|\big)|x|u^4. \tag{3.7}$$

Choosing $a(|x|) = \frac{1}{|x|}$,

$$a\big(|x|\big) + a'\big(|x|\big)|x| = 0. \tag{3.8}$$

When u is radial, $|\nabla u|^2 - |\partial_r u|^2 = 0$. For a general u,

$$|\nabla u|^2 - |\partial_r u|^2 \geq 0, \tag{3.9}$$

so since $a'(|x|) \leq 0$,

$$a'\big(|x|\big)|x|\big[|\nabla u|^2 - |\partial_r u|^2\big] \leq 0. \tag{3.10}$$

Also, by direct calculation, $\Delta \frac{1}{|x|} = -2\pi\delta(x)$, so when $a(|x|) = \frac{1}{|x|}$,

$$\frac{d}{dt}M(t) \leq -\pi u(t,0)^2 - \frac{\mu}{2}\int \frac{1}{|x|}u^4 \, dx. \tag{3.11}$$

Now by Hardy's inequality, when $a(x) = \frac{1}{|x|}$,

$$\big|M(t)\big| \lesssim \|u_t\|_{L^2}\|\nabla u\|_{L^2}. \tag{3.12}$$

Therefore,

$$\int_0^T u(t,0)^2 \, dt + \int_0^T \int \frac{\mu}{|x|}u^4 \, dx \, dt \lesssim \|u_t\|_{L_t^\infty L_x^2}\|\nabla u\|_{L_t^\infty L_x^2}. \tag{3.13}$$

This takes care of (3.2).

Replacing $a(|x|)$ by $a(|x-y|)$ and x with $x-y$, (3.13) implies that

$$\frac{1}{R^3}\int_0^T\int_{|y|\leq R} u(t,y)^2\,dy\,dt + \frac{1}{R^3}\int_{|y|\leq R}\int \frac{\mu}{|x-y|}u(t,x)^4\,dx\,dy \lesssim \|u_t\|_{L_t^\infty L_x^2}\|\nabla u\|_{L_t^\infty L_x^2}, \tag{3.14}$$

which takes care of (3.3).

To prove (3.4), choose a smooth function $\chi : [0,\infty) \to [0,\infty)$ satisfying $\chi(|x|) = 1$ for $|x| \leq 1$, $\chi(|x|) = \frac{3}{2|x|}$ for $|x| \geq 2$, $\chi(|x|)$ is decreasing as a function of $|x|$, and such that

$$\chi(|x|) + \chi'(|x|)|x| = \phi(|x|) \tag{3.15}$$

is a smooth function, $\phi(|x|) \geq 0$, $\phi(|x|) = 1$ for $|x| \leq 1$, and $\phi(|x|)$ is supported on $|x| \leq 2$. Take $a(|x|) = \frac{1}{R}\chi(\frac{|x|}{R})$. Then,

$$a(|x|) + a'(|x|)|x| = \frac{1}{R}\chi\Big(\frac{|x|}{R}\Big) + \frac{1}{R}\chi'\Big(\frac{|x|}{R}\Big)\frac{|x|}{R} = \frac{1}{R}\phi\Big(\frac{|x|}{R}\Big). \tag{3.16}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}M(t) = &-\frac{1}{2R}\int \phi\Big(\frac{|x|}{R}\Big)\big[u_t^2 + |\nabla u|^2\big]\,dx \\ &+ \int a'(|x|)|x|\big[|\nabla u|^2 - (\partial_r u)^2\big]\,dx \\ &- \frac{\mu}{4}\int a(|x|)u^4\,dx + \frac{\mu}{4}\int a'(|x|)|x|u^4\,dx \\ &+ \frac{1}{2}\int u^2\Delta a(|x|)\,dx.\end{aligned} \tag{3.17}$$

Now, since $a(|x|) = \frac{3}{2|x|}$ when $|x| \geq 2R$, $\Delta a(|x|)$ is supported on $|x| \leq 2R$. Therefore,

$$\frac{1}{2}\int u^2\Delta a(|x|) \lesssim \sup_{R>0}\frac{1}{R^3}\int_{|x|\leq R} u^2. \tag{3.18}$$

Also, $a(|x|) \lesssim \frac{1}{|x|}$ for any x, so again by Hardy's inequality,

$$\big|M(t)\big| \lesssim \|v_t\|_{L^2}\|\nabla v\|_{L^2}. \tag{3.19}$$

Plugging (3.9), (3.14), and (3.18) into (3.17) proves (3.4). □

COROLLARY 3.2

If u is an approximate solution to the cubic wave equation

$$u_{tt} - \Delta u + u^3 = F, \tag{3.20}$$

then

$$\frac{d}{dt}\Big[\int u_t \frac{x}{|x|}\cdot\nabla u + \int u_t \frac{1}{|x|}u\Big] \leq -\pi u(t,0)^2 - \frac{1}{2}\int \frac{1}{|x|}u^4 + \int F\frac{x}{|x|}\cdot\nabla u + \int F\frac{1}{|x|}u, \tag{3.21}$$

and

$$\begin{aligned}\frac{d}{dt}\frac{1}{R^3}\Big[&\int_{|y|\leq R}\int u_t(t,x)\frac{x-y}{|x-y|}\cdot\nabla u(t,x)\,dx\,dy \\ &+ \int_{|y|\leq R}\int u_t(t,x)\frac{1}{|x-y|}u(t,x)\,dx\,dy\Big] \\ \leq{}& -\pi\frac{1}{R^3}\int_{|y|\leq R} u(t,y)^2\,dy - \frac{1}{2}\frac{1}{R^3}\int_{|y|\leq R}\int\frac{1}{|x-y|}u(t,x)^4\,dx\,dy \\ &+ \frac{1}{R^3}\int_{|y|\leq R}\int F(t,x)\frac{x-y}{|x-y|}\cdot\nabla u(t,x)\,dx\,dy \\ &+ \frac{1}{R^3}\int_{|y|\leq R}\int F(t,x)\frac{1}{|x-y|}u(t,x)\,dx\,dy,\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\frac{d}{dt}\Big[&\frac{1}{R}\int u_t\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla u + \frac{1}{R}\int u_t\chi\Big(\frac{|x|}{R}\Big)u\Big] \\ \leq{}& -\frac{1}{2R}\int\phi\Big(\frac{|x|}{R}\Big)[u_t^2+|\nabla u|^2] - \frac{1}{4R}\int\chi\Big(\frac{|x|}{R}\Big)u^4 \\ &+ \frac{1}{4R}\int\chi'\Big(\frac{|x|}{R}\Big)\frac{|x|}{R}u^4 + \frac{1}{2R^3}\int u^2(\Delta\chi)\Big(\frac{|x|}{R}\Big) \\ &+ \frac{1}{R}\int F\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla u + \frac{1}{R}\int F\chi\Big(\frac{|x|}{R}\Big)u.\end{aligned} \tag{3.23}$$

Theorem 3.1 also gives some nice estimates for the linear wave equation ($\mu = 0$). Let P_j denote the usual Littlewood–Paley partition of unity operators. That is,

$$P_j f = \mathcal{F}^{-1}\big(\phi(2^{-j}\xi)\mathcal{F}f(\xi)\big), \tag{3.24}$$

where $\mathcal{F}$ denotes the usual Fourier transform, $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and $\phi(\xi)$ is a smooth, radially symmetric, compactly supported function satisfying

$$\sum_{j\in\mathbf{Z}}\phi(2^{-j}\xi) = 1, \quad \text{for all } \xi \neq 0. \tag{3.25}$$

COROLLARY 3.3

For any $j \in \mathbf{Z}$, let w be the solution to the linear wave equation

$$\partial_{tt} w - \Delta w = 0, \qquad w(0,x) = P_j u_0, \qquad w_t(0,x) = P_j u_1. \tag{3.26}$$

Then for any $2 < p < \infty$,

$$\big\| |x|^{1/2} w \big\|_{L_t^p L_x^\infty(\mathbf{R}\times\mathbf{R}^3)} \lesssim \|P_j u_0\|_{\dot{H}^{1/p'}(\mathbf{R}^3)} + \|P_j u_1\|_{\dot{H}^{1/p'-1}(\mathbf{R}^3)}, \tag{3.27}$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$ is the Lebesgue dual of p. Also, for $p = 2$, for any $0 < R < 1$ and $1 < R_1 < \infty$,

$$\begin{aligned} &\big\| |x|^{1/2} w \big\|^2_{L^2_{t,x}(\mathbf{R}\times\{x: R\le|x|\le R_1\})} \\ &\quad\lesssim \big(\ln(R_1) - \ln(R) + 1\big)\big[\|P_j u_0\|^2_{\dot{H}^{1/2}(\mathbf{R}^3)} + \|P_j u_1\|^2_{\dot{H}^{-1/2}(\mathbf{R}^3)}\big]. \end{aligned} \tag{3.28}$$

Proof

Let ψ be a smooth radial function supported on an annulus, $\psi(r) = 1$ for $1 \le r \le 2$, and $\psi(r)$ is supported on $\frac{1}{2} \le r \le 4$. By Bernstein's inequality and the product rule,

$$\Big\| P_k\Big(\psi\Big(\frac{r}{R}\Big)w\Big)\Big\|_{L^2} \lesssim 2^{-k}\Big\|\psi\Big(\frac{r}{R}\Big)w_r\Big\|_{L^2} + 2^{-k}R^{-1}\Big\|\psi'\Big(\frac{r}{R}\Big)w\Big\|_{L^2}. \tag{3.29}$$

Therefore, by (3.3), (3.4), and the radial Sobolev embedding theorem,

$$\sum_{k\ge j-3}\Big\| P_k\Big(\psi\Big(\frac{r}{R}\Big)w\Big)\Big\|_{L_t^2L_x^\infty} \lesssim 2^{-j/2}R^{-1/2}\big(\|P_j u_0\|_{\dot{H}^1} + \|P_j u_1\|_{L^2}\big). \tag{3.30}$$

Next, by the Fourier support properties of w,

$$\Big\| P_{\le j-3}\Big(\psi\Big(\frac{r}{R}\Big)w\Big)\Big\|_{L^\infty} \lesssim 2^{-j}R^{-1}\|w\|_{L^\infty}. \tag{3.31}$$

Combining (3.31) with (2.11),

$$\Big\| P_{\le j-3}\Big(\psi\Big(\frac{r}{R}\Big)w\Big)\Big\|_{L_t^2L_x^\infty} \lesssim 2^{-j}R^{-1}\big(\|P_j u_0\|_{\dot{H}^1} + \|P_j u_1\|_{L^2}\big). \tag{3.32}$$

Then when $R \ge 2^{-j}$,

$$\Big\| P_{\le j-3}\Big(\psi\Big(\frac{r}{R}\Big)w\Big)\Big\|_{L_t^2L_x^\infty} \lesssim 2^{-j/2}R^{-1/2}\big(\|P_j u_0\|_{\dot{H}^1} + \|P_j u_1\|_{L^2}\big), \tag{3.33}$$

and when $R \le 2^{-j}$, a straightforward application of the endpoint Strichartz estimate yields

$$\begin{aligned} \Big\|\psi\Big(\frac{r}{R}\Big)w\Big\|_{L_t^2L_x^\infty} &\lesssim \big(\|P_j u_0\|_{\dot{H}^1} + \|P_j u_1\|_{L^2}\big) \\ &\lesssim R^{-1/2}2^{-j/2}\big(\|P_j u_0\|_{\dot{H}^1} + \|P_j u_1\|_{L^2}\big). \end{aligned} \tag{3.34}$$

Since there are $\lesssim \ln(R_1) - \ln(R) + 1$ dyadic annuli overlapping $R \leq |x| \leq R_1$, (3.30)–(3.34) directly yields (3.28).

To prove (3.27), interpolating (3.34) with the radial Sobolev embedding theorem, for any $2 < p < \infty$,

$$\begin{aligned}\Big\|\psi\Big(\frac{r}{R}\Big)w\Big\|_{L_t^p L_x^\infty} &\lesssim \Big\|\psi\Big(\frac{r}{R}\Big)w\Big\|_{L_t^2 L_x^\infty}^{2/p}\Big\|\psi\Big(\frac{r}{R}\Big)w\Big\|_{L_{t,x}^\infty}^{1-2/p} \\ &\lesssim R^{-1/2}R^{-\frac{1}{2}(1-\frac{2}{p})}\big(\|P_j u_0\|_{\dot{H}^{1/2}} + \|P_j u_1\|_{\dot{H}^{-1/2}}\big), \qquad (3.35)\end{aligned}$$

which directly implies that

$$\big\||x|^{1/2}w\big\|_{L_t^p L_x^\infty(\mathbf{R}\times\{x:|x|\geq 2^{-j}\})} \lesssim \big(\|P_j u_0\|_{\dot{H}^{1/p'}} + \|P_j u_1\|_{\dot{H}^{1/p'-1}}\big). \qquad (3.36)$$

Meanwhile, by (2.11) and the Sobolev embedding theorem,

$$\begin{aligned}\big\||x|^{1/2}w\big\|_{L_t^p L_x^\infty(\mathbf{R}\times\{|x|\leq 2^{-j}\})} &\lesssim 2^{-j/2}\|w\|_{L_t^2 L_x^\infty}^{2/p}\|w\|_{L_{t,x}^\infty}^{1-2/p} \\ &\lesssim \big(\|P_j u_0\|_{\dot{H}^{1/p}} + \|P_j u_1\|_{\dot{H}^{1/p-1}}\big). \qquad (3.37)\end{aligned}$$

This finally proves the theorem. □

Remark
Also observe that by the radial Sobolev embedding theorem, Corollary 3.3 implies that

$$\begin{aligned}&\|w\|_{L_t^2 L_x^\infty([0,T]\times\{|x|\geq R\})}^2 \\ &\quad\lesssim \big(1 + \ln(T) - \ln(R)\big)\big[\|P_j u_0\|_{\dot{H}^{1/2}} + \|P_j u_1\|_{\dot{H}^{-1/2}}\big]. \qquad (3.38)\end{aligned}$$

The virial identities in Theorem 3.1 commute very well with Littlewood–Paley projections.

LEMMA 3.4
For any j,

$$\int \frac{1}{|x|}|P_{\leq j}v|^4\,dx + \int \frac{1}{|x|}|P_{\geq j}v|^4\,dx \lesssim \int \frac{1}{|x|}|v|^4\,dx. \qquad (3.39)$$

Proof
Let ψ be the Littlewood–Paley kernel. That is,

$$\frac{1}{|x|^{1/4}}P_{\leq j}v(x) = \frac{1}{|x|^{1/4}}\int 2^{3j}\psi\big(2^j(x-y)\big)v(y)\,dy. \qquad (3.40)$$

When $|y| \lesssim |x|$,

$$\frac{1}{|x|^{1/4}} 2^{3j} \psi\big(2^j(x-y)\big) \lesssim 2^{3j} \psi\big(2^j(x-y)\big) \frac{1}{|y|^{1/4}}. \tag{3.41}$$

When $|y| \gg |x|$ and $|x| \geq 2^{-j}$, since ψ is rapidly decreasing, for any N,

$$\begin{aligned} \frac{1}{|x|^{1/4}} 2^{3j} \psi\big(2^j(x-y)\big) &\lesssim_N \frac{1}{|x|^{1/4}} \frac{2^{3j}}{(1+2^j|x-y|)^N} \\ &\lesssim \frac{1}{|x|^{1/4} 2^j |y|} \frac{2^{3j}}{(1+2^j|x-y|)^{N-1}} \\ &\lesssim \frac{1}{|y|^{1/4}} \frac{2^{3j}}{(1+2^j|x-y|)^{N-1}}. \end{aligned} \tag{3.42}$$

Combining (3.41) and (3.42),

$$\Big\| \frac{1}{|x|^{1/4}} |P_{\leq j} v| \Big\|_{L^4(|x| \geq 2^{-j})} \lesssim \Big\| \frac{1}{|x|^{1/4}} v \Big\|_{L^4(\mathbf{R}^3)}. \tag{3.43}$$

When $|y| \gg |x|$ and $|x| \leq 2^{-j}$, since ψ is rapidly decreasing, for any N,

$$\begin{aligned} \frac{1}{|x|^{1/4}} 2^{3j} \psi\big(2^j(x-y)\big) &\lesssim_N \frac{1}{|x|^{1/4}} \frac{2^{3j}}{(1+2^j|x-y|)^N} \\ &\lesssim \frac{1}{|x|^{1/4}} \frac{2^{3j}}{(1+2^j|x-y|)^{N-1/4}} \frac{1}{2^{j/4}|y|^{1/4}}, \end{aligned} \tag{3.44}$$

$$\Big\| \frac{2^{11j/4}}{(1+2^j|x-y|)^N} \Big\|_{L^{4/3}(\mathbf{R}^3)} \lesssim 2^{j/2}, \tag{3.45}$$

so by (3.41), (3.45), Young's inequality, and Hölder's inequality,

$$\Big\| \frac{1}{|x|^{1/4}} |P_{\leq j} v| \Big\|_{L^4(|x| \leq 2^{-j})} \lesssim \Big\| \frac{1}{|x|^{1/4}} v \Big\|_{L^4(\mathbf{R}^3)}. \tag{3.46}$$

This proves (3.39). □

LEMMA 3.5
We have

$$\begin{aligned} \|P_{\geq j} v\|^2_{L^4(|x| \leq \frac{R}{2})} \lesssim \|P_{\geq j} v\|_{L^3} \Big[\|\nabla v\|_{L^2(|x| \leq R)} + \frac{1}{R} \|v\|_{L^2(|x| \leq R)} \Big] \\ + 2^{-j/2} \Big(\int \frac{1}{|x|} v^4 \Big)^{1/2}. \end{aligned} \tag{3.47}$$

Proof
Let $\phi \in C_0^\infty(\mathbf{R}^3)$ be supported on $|x| \leq 1$ and $\phi(x) = 1$ for $|x| \leq \frac{1}{2}$. By Hölder's inequality,

$$\|P_{\geq j} v\|^2_{L^4(|x| \leq \frac{R}{2})} \leq \Big\| \phi\Big(\frac{x}{R}\Big)(P_{\geq j} v)\Big\|^2_{L^4(\mathbf{R}^3)}. \tag{3.48}$$

Then, by the triangle inequality, Hölder's inequality, and the Cauchy–Schwarz inequality,

$$\begin{aligned}
\Big\| \phi\Big(\frac{x}{R}\Big)(P_{\geq j} v)\Big\|^2_{L^4(\mathbf{R}^3)} &\leq \Big\| \phi\Big(\frac{x}{R}\Big)(P_{\geq j} v) \cdot P_{\geq j}\Big(\phi\Big(\frac{x}{R}\Big)v\Big)\Big\|_{L^2(\mathbf{R}^3)} \\
&\quad + \Big\| \phi\Big(\frac{x}{R}\Big)(P_{\geq j} v) \cdot \Big[\phi\Big(\frac{x}{R}\Big), P_{\geq j}\Big]v\Big\|_{L^2(\mathbf{R}^3)} \\
&\leq \Big\| P_{\geq j}\Big(\phi\Big(\frac{x}{R}\Big)v\Big)\Big\|_{L^6(\mathbf{R}^3)} \|P_{\geq j} v\|_{L^3(\mathbf{R}^3)} \\
&\quad + \frac{1}{2}\Big\| \phi\Big(\frac{x}{R}\Big)(P_{\geq j} v)\Big\|^2_{L^4(\mathbf{R}^3)} \\
&\quad + \frac{1}{2}\Big\| \Big[\phi\Big(\frac{x}{R}\Big), P_{\geq j}\Big]v\Big\|^2_{L^4(\mathbf{R}^3)},
\end{aligned} \tag{3.49}$$

where

$$\Big[\phi\Big(\frac{x}{R}\Big), P_{\geq j}\Big]v = \phi\Big(\frac{x}{R}\Big)(P_{\geq j} v) - P_{\geq j}\Big(\phi\Big(\frac{x}{R}\Big)v\Big). \tag{3.50}$$

Then by the Littlewood–Paley theorem,

$$\begin{aligned}
\Big\| \phi\Big(\frac{x}{R}\Big)(P_{\geq j} v)\Big\|^2_{L^4(\mathbf{R}^3)} &\lesssim \Big\| \phi\Big(\frac{x}{R}\Big)v\Big\|_{L^6(\mathbf{R}^3)} \|P_{\geq j} v\|_{L^3(\mathbf{R}^3)} \\
&\quad + \Big\| \Big[\phi\Big(\frac{x}{R}\Big), P_{\geq j}\Big]v\Big\|^2_{L^4(\mathbf{R}^3)},
\end{aligned} \tag{3.51}$$

and by the Sobolev embedding theorem,

$$\begin{aligned}
\Big\| \phi\Big(\frac{x}{R}\Big)v\Big\|_{L^6(\mathbf{R}^3)} &\lesssim \Big\| \nabla\Big(\phi\Big(\frac{x}{R}\Big)v\Big)\Big\|_{L^2(\mathbf{R}^3)} \\
&\lesssim \frac{1}{R}\|v\|_{L^2(|x| \leq R)} + \|\nabla v\|_{L^2(|x| \leq R)}.
\end{aligned} \tag{3.52}$$

This is bounded by the right-hand side of (3.47).

To handle the commutator, observe that

$$\Big[\phi\Big(\frac{x}{R}\Big), P_{\geq j}\Big] = -\Big[P_{\leq j}, \phi\Big(\frac{x}{R}\Big)\Big]. \tag{3.53}$$

Then compute

$$\Big[P_{\leq j}, \phi\Big(\frac{x}{R}\Big)\Big]v = 2^{3j} \int \psi\big(2^j(x-y)\big)\Big[\phi\Big(\frac{y}{R}\Big) - \phi\Big(\frac{x}{R}\Big)\Big]v(y)\,dy. \tag{3.54}$$

When $|y| \gg |x|$, we have the kernel estimate

$$\begin{aligned} &2^{3j}\psi\big(2^j(x-y)\big)\Big[\phi\Big(\frac{y}{R}\Big) - \phi\Big(\frac{x}{R}\Big)\Big] \\ &\quad \lesssim_N \frac{2^{3j}}{(1+2^j|x-y|)^N} \\ &\quad \lesssim 2^{-j/4}\frac{2^{3j}}{(1+2^j|x-y|)^{N-1/4}}\frac{1}{|y|^{1/4}}. \end{aligned} \tag{3.55}$$

When $|y| \lesssim |x|$ and $|x| \leq R$, by the fundamental theorem of calculus,

$$\begin{aligned} &2^{3j}\psi\big(2^j(x-y)\big)\Big[\phi\Big(\frac{y}{R}\Big) - \phi\Big(\frac{x}{R}\Big)\Big] \\ &\quad \lesssim_N \frac{2^{3j}}{(1+2^j|x-y|)^N}\frac{|x-y|^{1/4}}{R^{1/4}} \\ &\quad \lesssim 2^{-j/4}\frac{2^{3j}}{(1+2^j|x-y|)^{N-1/4}} \cdot \frac{1}{|y|^{1/4}}. \end{aligned} \tag{3.56}$$

When $|y| \lesssim |x|$ and $|x| > R$, interpolating

$$\begin{aligned} 2^{3j}\psi\big(2^j(x-y)\big)\Big[\phi\Big(\frac{y}{R}\Big) - \phi\Big(\frac{x}{R}\Big)\Big] &= 2^{3j}\psi\big(2^j(x-y)\big)\phi\Big(\frac{y}{R}\Big) \\ &\lesssim_N \frac{2^{3j}}{(1+2^j|x-y|)^N}\frac{R^{1/2}}{|y|^{1/2}} \end{aligned} \tag{3.57}$$

with the fact that

$$\begin{aligned} &2^{3j}\psi\big(2^j(x-y)\big)\Big[\phi\Big(\frac{y}{R}\Big) - \phi\Big(\frac{x}{R}\Big)\Big] \\ &\quad \lesssim_N \frac{2^{3j}}{(1+2^j|x-y|)^N}\frac{|x-y|^{1/2}}{R^{1/2}} \\ &\quad \lesssim 2^{-j/4}\frac{2^{3j}}{(1+2^j|x-y|)^{N-1/2}} \cdot \frac{1}{2^{j/2}R^{1/2}} \end{aligned} \tag{3.58}$$

implies that

$$2^{3j}\psi\big(2^j(x-y)\big)\Big[\phi\Big(\frac{y}{R}\Big) - \phi\Big(\frac{x}{R}\Big)\Big] \lesssim_N 2^{-j/4}\frac{2^{3j}}{(1+2^j|x-y|)^N}\frac{1}{|y|^{1/4}}. \tag{3.59}$$

The kernel estimates (3.55), (3.56), and (3.59) imply that

$$\Big\| \Big[\phi\Big(\frac{x}{R}\Big), P_{\geq j} \Big] v \Big\|_{L^4(\mathbf{R}^3)} \lesssim 2^{-j/4} \Big\| \frac{1}{|x|^{1/4}} v \Big\|_{L^4(\mathbf{R}^3)}, \tag{3.60}$$

proving Lemma 3.5. □

4. Global well-posedness

The global well-posedness of (1.1) is proved using the Fourier truncation method, a method introduced in [3] for the nonlinear Schrödinger equation and used in [13] for the cubic wave equation.

Decompose the initial data into a finite energy piece and a small data piece, that is, $u_0 = v_0 + w_0$ and $u_1 = v_1 + w_1$, where

$$E(v_0, v_1) = \frac{1}{2} \int |\nabla v_0|^2 \, dx + \frac{1}{2} \int |v_1|^2 \, dx + \frac{1}{4} \int |v_0|^4 \, dx < \infty \tag{4.1}$$

and

$$\|w_0\|_{\dot{H}^{1/2}} + \|w_1\|_{\dot{H}^{-1/2}} \ll 1. \tag{4.2}$$

A local solution u to (1.1) may then be decomposed into $u = w + v$, where w solves

$$w_{tt} - \Delta w + w^3 = 0, \qquad w(0, x) = w_0, \qquad w_t(0, x) = w_1, \tag{4.3}$$

and v solves

$$v_{tt} - \Delta v + v^3 + 3v^2 w + 3v w^2 = 0, \qquad v(0, x) = v_0, \qquad v_t(0, x) = v_1. \tag{4.4}$$

If $\|w_0\|_{\dot{H}^{1/2}} + \|w_1\|_{\dot{H}^{-1/2}} < \epsilon$ for some $\epsilon > 0$ sufficiently small, then (4.3) is globally well-posed by small data arguments. Moreover, by Theorems 2.1 and 2.4, the Sobolev embedding $L^{3/2} \subset \dot{H}^{-1/2}$, and the principle of superposition,

$$\begin{aligned} \|w\|_{L_t^2 L_x^6(\mathbf{R} \times \mathbf{R}^3)} &+ \big\| |\nabla|^{1/10} w \big\|_{L_t^2 L_x^5(\mathbf{R} \times \mathbf{R}^3)} + \big\| |\nabla|^{1/6} w \big\|_{L_t^6 L_x^3(\mathbf{R} \times \mathbf{R}^3)} \\ &+ \|w\|_{L_{t,x}^4(\mathbf{R} \times \mathbf{R}^3)} \lesssim \big\| (w_0, w_1) \big\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} + \|w^3\|_{L_t^1 L_x^{3/2}}, \end{aligned} \tag{4.5}$$

and therefore, by small data arguments,

$$\begin{aligned} &\|w\|_{L_t^2 L_x^6(\mathbf{R} \times \mathbf{R}^3)} + \big\| |\nabla|^{1/10} w \big\|_{L_t^2 L_x^5(\mathbf{R} \times \mathbf{R}^3)} \\ &\qquad + \big\| |\nabla|^{1/6} w \big\|_{L_t^6 L_x^3(\mathbf{R} \times \mathbf{R}^3)} + \|w\|_{L_{t,x}^4(\mathbf{R} \times \mathbf{R}^3)} \lesssim \epsilon, \\ &\|w^3\|_{L_t^1 L_x^{3/2}(\mathbf{R} \times \mathbf{R}^3)} \lesssim \epsilon^3. \end{aligned} \tag{4.6}$$

Remark
The precise $\epsilon > 0$ will be chosen later.

Remark
Plugging $q = 5$ into (2.12),

$$\|w\|_{L_t^2 L_x^5} \lesssim \|(w_0, w_1)\|_{\dot{H}^{2/5} \times \dot{H}^{-3/5}} + \|w^3\|_{L_t^1 \dot{H}^{-3/5}}, \tag{4.7}$$

and therefore,

$$\||\nabla|^{1/10} w\|_{L_t^2 L_x^5} \lesssim \|(w_0, w_1)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} + \|w^3\|_{L_t^1 \dot{H}^{-1/2}}. \tag{4.8}$$

For the solution to (4.4), following (1.3), let $E(t)$ denote the energy of v:

$$E(t) = \frac{1}{2}\int v_t(t,x)^2\,dx + \frac{1}{2}\int |\nabla v(t,x)|^2\,dx + \frac{1}{4}\int v(t,x)^4\,dx. \tag{4.9}$$

To prove global well-posedness it suffices to prove that $E(t) < \infty$ for all $t \in \mathbf{R}$. Indeed, we have the following result.

THEOREM 4.1
Suppose that $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$ has the decomposition $u_0 = v_0 + w_0$ and $u_1 = v_1 + w_1$, where (v_0, v_1) has the finite energy $E(0) < \infty$, where E is given by (4.9), and $\|w_0\|_{\dot{H}^{1/2}} + \|w_1\|_{\dot{H}^{-1/2}} \ll 1$. Then for some $c > 0$ sufficiently small and independent of $E(0)$, the initial value problem (1.1) with initial data (u_0, u_1) is locally well-posed in $L_t^\infty \dot{H}^{1/2} \cap L_{t,x}^4$ on the time interval $[-\frac{c}{E(0)}, \frac{c}{E(0)}]$.

Proof
To simplify notation, let $I = [-\frac{c}{E(0)}, \frac{c}{E(0)}]$. By Theorem 1.1, (1.1) has a solution for initial data (v_0, v_1), and, moreover, by conservation of energy,

$$\|v\|_{L_{t,x}^4(I \times \mathbf{R}^3)}^4 \lesssim |I| E(0) \leq c. \tag{4.10}$$

Therefore, for $c > 0$ sufficiently small, independent of $E(0)$, the perturbation lemma (Lemma 2.2) and (4.5) proves Theorem 4.1. □

THEOREM 4.2
Equation (1.1) is globally well-posed for radial $(u_0, u_1) \in \dot{H}^{1/2}(\mathbf{R}^3) \times \dot{H}^{-1/2}(\mathbf{R}^3)$.

Proof
Computing the time derivative of $E(t)$, by Hölder's inequality,

$$\frac{d}{dt}E(t) = -3\int v_t v^2 w - 3\int v_t v w^2$$
$$\lesssim \|v_t\|_{L^2}\|v\|_{L^6}^2\|w\|_{L^6} + \|v_t\|_{L^2}\|v\|_{L^6}\|w\|_{L^6}^2. \tag{4.11}$$

Therefore, by the Cauchy–Schwarz inequality,

$$\left|\frac{d}{dt}E(t)\right| \lesssim E(t)^2 + \|w\|_{L^6}^2 E(t). \tag{4.12}$$

If only the second term on the right-hand side of (4.12) were present, then global boundedness of $E(t)$ would be an easy consequence of (4.5) and Gronwall's inequality.

However, the bound $|\frac{d}{dt}E(t)| \lesssim E(t)^2$ is not enough to exclude blowup in finite time. Instead, we will use a modification of $E(t)$, $\mathcal{E}(t)$, which has much better global derivative bounds and satisfies $\mathcal{E}(t) \sim E(t)$.

To simplify notation, rescale by (1.2) so that

$$\|P_{\geq 1}u_0\|_{\dot{H}^{1/2}(\mathbf{R}^3)} + \|P_{\geq 1}u_1\|_{\dot{H}^{-1/2}(\mathbf{R}^3)} < \epsilon, \tag{4.13}$$

and then let $v_0 = P_{\leq 1}u_0$ and $v_1 = P_{\leq 1}u_1$.

Remark
The $\lambda > 0$ in (1.2) depends on the profile of the initial data, not just its size.

Following (3.21), (3.22), and (3.23), let

$$\begin{aligned} M_1(t) &= c_1\int v_t \frac{x}{|x|}\cdot\nabla v + c_1\int v_t\frac{1}{|x|}v, \\ M_2(t) &= \frac{c_2}{8R^3}\int_{|y|\leq 2R}\int v_t\frac{(x-y)}{|x-y|}\cdot\nabla v\,dx\,dy \\ &\quad + \frac{c_2}{8R^3}\int_{|y|\leq 2R}\int v_t\frac{1}{|x-y|}v\,dx\,dy, \\ M_3(t) &= \frac{c_3}{R}\int v_t\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v + \frac{c_3}{R}\int v_t\chi\Big(\frac{|x|}{R}\Big)v, \end{aligned} \tag{4.14}$$

where $c_1, c_2, c_3 > 0$ are small constants, and let

$$\mathcal{E}(t) = E(t) + M_1(t) + M_2(t) + M_3(t) + \int v^3 w\,dx. \tag{4.15}$$

The Sobolev embedding theorem implies that

$$\int v^3 w\,dx \lesssim \|v\|_{L^6}\|w\|_{L^3}\|v\|_{L^4}^2 \lesssim \epsilon E(t). \tag{4.16}$$

Next, by (3.12), (3.14), and (3.19),

$$\big|M_1(t)\big| + \big|M_2(t)\big| + \big|M_3(t)\big| \lesssim c_1 E(t) + c_2 E(t) + c_3 E(t). \tag{4.17}$$

Therefore, choosing c_1, c_2, and $c_3 > 0$ to be sufficiently small, determined only by the constant in Hardy's inequality in three dimensions and the volume of the unit sphere in $\mathbf{R}^3$,

$$\mathcal{E}(t) \sim E(t). \tag{4.18}$$

We will also require that $c_3 \ll c_2$; $c_3 \leq \frac{1}{100} c_2$ will do.

By (3.21), (3.22), (3.23), and (4.11),

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) \leq{}& -2c_1 \pi v(t,0)^2 - \frac{c_2 \pi}{8R^3} \int_{|y| \leq 2R} v(t,y)^2 \\
& - \frac{c_1}{2} \int \frac{1}{|x|} v^4 - \frac{c_2}{16R^3} \int_{|y| \leq 2R} \int \frac{1}{|x-y|} v^4 \\
& - \frac{c_3}{2R} \int \phi\Big(\frac{|x|}{R}\Big)\big[v_t^2 + |\nabla v|^2\big] + \frac{c_3}{2R^3} \int v^2 \Delta\chi\Big(\frac{|x|}{R}\Big) \\
& - \frac{c_3}{4R} \int \chi\Big(\frac{|x|}{R}\Big) v^4 + \frac{c_3}{4R} \int \chi'\Big(\frac{|x|}{R}\Big)\frac{|x|}{R} v^4 \\
& + \frac{d}{dt} \int v^3 w \, dx + \int F v_t + c_1 \int F \frac{x}{|x|} \cdot \nabla v + c_1 \int F \frac{1}{|x|} v \\
& + \frac{c_2}{8R^3} \int_{|y| \leq 2R} \int F \frac{(x-y)}{|x-y|} \cdot \nabla v + \frac{c_2}{8R^3} \int_{|y| \leq 2R} \int F \frac{1}{|x-y|} v \\
& + \frac{c_3}{R} \int F \chi\Big(\frac{|x|}{R}\Big) x \cdot \nabla v + \frac{c_3}{R} \int F \chi\Big(\frac{|x|}{R}\Big) v,
\end{aligned} \tag{4.19}$$

where $F = -3v^2 w - 3vw^2$.

By the support properties of $\Delta\chi(\frac{|x|}{R})$, for $c_2 \geq 100 c_3$,

$$-\frac{c_2 \pi}{8R^3} \int_{|y| \leq 2R} v(t,y)^2 + \frac{c_3}{2R^3} \int v^2 \Delta\chi\Big(\frac{|x|}{R}\Big) \leq -\frac{c_2}{16R^3} \int_{|y| \leq 2R} v(t,y)^2. \tag{4.20}$$

Also, since $\chi'(\frac{|x|}{R}) \leq 0$,

$$\frac{c_3}{4R} \int \chi'\Big(\frac{|x|}{R}\Big)\frac{|x|}{R} v^4 \leq 0. \tag{4.21}$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) + 2c_1\pi v(t,0)^2 + \frac{c_2\pi}{16R^3}\int_{|y|\le 2R} v(t,y)^2 \\
&\qquad + \frac{c_1}{2}\int \frac{1}{|x|}v^4 + \frac{c_2}{16R^3}\int_{|y|\le 2R}\int \frac{1}{|x-y|}v^4 \\
&\qquad + \frac{c_3}{2R}\int \phi\Big(\frac{|x|}{R}\Big)\big[v_t^2 + |\nabla v|^2\big] + \frac{c_3}{4R}\int \chi\Big(\frac{|x|}{R}\Big)v^4 \\
&\quad \le \frac{d}{dt}\int v^3 w + \int F v_t + c_1\int F\frac{x}{|x|}\cdot\nabla v + c_1\int F\frac{1}{|x|}v \\
&\qquad + \frac{c_2}{8R^3}\int_{|y|\le 2R}\int F\frac{(x-y)}{|x-y|}\cdot\nabla v + \frac{c_2}{8R^3}\int_{|y|\le 2R}\int F\frac{1}{|x-y|}v \\
&\qquad + \frac{c_3}{R}\int F\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v + \frac{c_3}{R}\int F\chi\Big(\frac{|x|}{R}\Big)v.
\end{aligned} \tag{4.22}
$$

Each of the terms on the right-hand side may be controlled using a combination of Strichartz estimates and terms on the left-hand side. The terms on the right-hand side may be grouped into three main categories: category-1 terms,

$$
\begin{aligned}
&c_1\int F\frac{1}{|x|}v\,dx + \frac{c_2}{8R^3}\int_{|y|\le 2R}\int F(t,x)\frac{1}{|x-y|}v(t,x)\,dx\,dy \\
&\qquad + \frac{c_3}{R}\int F(t,x)\chi\Big(\frac{|x|}{R}\Big)v(t,x)\,dx,
\end{aligned} \tag{4.23}
$$

category-2 terms,

$$
\begin{aligned}
&-3\int v w^2 v_t\,dx - 3c_1\int v w^2\frac{x}{|x|}\cdot\nabla v\,dx \\
&\qquad - \frac{3c_2}{8R^3}\int_{|y|\le 2R}\int v w^2\frac{(x-y)}{|x-y|}\cdot\nabla v\,dx \\
&\qquad - \frac{3c_3}{R}\int v w^2\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v\,dx,
\end{aligned} \tag{4.24}
$$

and category-3 terms,

$$
\begin{aligned}
&\int v^3 w_t\,dx - 3c_1\int v^2 w\frac{x}{|x|}\cdot\nabla v\,dx \\
&\qquad - \frac{3c_2}{8R^3}\int_{|y|\le 2R}\int v^2 w\frac{(x-y)}{|x-y|}\cdot\nabla v\,dx \\
&\qquad - \frac{3c_3}{R}\int v^2 w\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v\,dx.
\end{aligned} \tag{4.25}
$$

Estimating each group of terms separately,

$$\int_0^T (4.23)\,dt \lesssim \delta\Big(\int_0^T \int \frac{1}{|x|} v^4 \,dx\,dt\Big) + \frac{1}{\delta}\int_0^T E(t)\|w(t)\|_{L^6}^2\,dt, \quad (4.26)$$

$$\int_0^T (4.24)\,dt \lesssim \int_0^T E(t)\|w(t)\|_{L^6}^2\,dt \quad (4.27)$$

and

$$\begin{aligned}\int_0^T (4.25)\,dt \lesssim \delta\Big(\int_0^T \int \frac{1}{|x|} v^4 \,dx\,dt\Big) \\ &+ \delta R \int_0^T E(t)\Big(\frac{1}{R}\int_{|x|\le R} |\nabla v|^2\,dx + \frac{1}{R^3}\int_{|x|\le R} v^2\,dx\Big)\,dt \\ &+ \frac{1}{\delta}\int_0^T E(t)\Big(\sum_j 2^{-2j}\,\||x|^{1/2}\partial_t w_j\|_{L^\infty(|x|\ge \frac{R}{2})}^2\Big)\,dt \\ &+ \frac{1}{\delta}\int_0^T E(t)\Big(\sum_j 2^{-2j}\|\partial_t w_j\|_{L^6}^2\Big)\,dt \\ &+ \frac{1}{\delta}\int_0^T E(t)\Big(\sum_j 2^{j/5}\|w_j(t)\|_{L^5}^2\Big)\,dt. \end{aligned} \quad (4.28)$$

Theorem 4.2 then proves to be a direct consequence of (4.26)–(4.28).

Category-1 *terms*
By Hardy's inequality, the Sobolev embedding theorem, and the Cauchy–Schwarz inequality, for $\delta > 0$ small,

$$\begin{aligned}\int v^2 w \frac{1}{|x|} v\,dx &\lesssim \Big(\int \frac{1}{|x|} v^4\,dx\Big)^{1/2} \Big\|\frac{1}{|x|^{1/2}} v\Big\|_{L^3}\|w(t)\|_{L^6} \\ &\lesssim \delta\Big(\int \frac{1}{|x|} v^4\,dx\Big) + \frac{1}{\delta} E(t)\|w(t)\|_{L^6}^2. \end{aligned} \quad (4.29)$$

Also, by Hölder's inequality and Hardy's inequality,

$$\int v w^2 \frac{1}{|x|} v \lesssim \|w\|_{L^6}^2 \|\nabla v\|_{L^2}\|v\|_{L^6} \lesssim E(t)\|w\|_{L^6}^2. \quad (4.30)$$

Therefore,

$$\int F \frac{1}{|x|} v\,dx \lesssim \delta\Big(\int \frac{1}{|x|} v^4\,dx\Big) + \frac{1}{\delta} E(t)\|w(t)\|_{L^6}^2. \quad (4.31)$$

Because $\chi(|x|) \lesssim \frac{1}{|x|}$, the same argument also implies that

$$\frac{1}{R}\int F(t,x)\chi\Big(\frac{|x|}{R}\Big)v(t,x)\,dx \lesssim \delta\Big(\int \frac{1}{|x|}v^4\,dx\Big) + \frac{1}{\delta}E(t)\big\|w(t)\big\|_{L^6}^2. \tag{4.32}$$

Finally, since $\frac{1}{R^3}\int_{|y|\leq 2R}\frac{1}{|x-y|}\,dy \lesssim \frac{1}{|x|}$, with bound independent of R,

$$\begin{aligned}
&\frac{1}{8R^3}\int_{|y|\leq 2R}\int F(t,x)\frac{1}{|x-y|}v(t,x)\,dx\,dy \\
&\quad\lesssim \int \frac{1}{|x|}F(t,x)v(t,x)\,dx \\
&\quad\lesssim \delta\Big(\int \frac{1}{|x|}v^4\,dx\Big) + \frac{1}{\delta}E(t)\big\|w(t)\big\|_{L^6}^2.
\end{aligned} \tag{4.33}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) + c_1\pi v(t,0)^2 + \frac{c_2\pi}{16R^3}\int_{|y|\leq 2R} v(t,y)^2 \\
&\quad + \frac{c_1}{2}\int \frac{1}{|x|}v^4 + \frac{c_2}{16R^3}\int_{|y|\leq 2R}\int \frac{1}{|x-y|}v^4 \\
&\quad + \frac{c_3}{2R}\int \phi\Big(\frac{|x|}{R}\Big)\big[v_t^2 + |\nabla v|^2\big] + \frac{c_3}{4R}\int \chi\Big(\frac{|x|}{R}\Big)v^4 \\
&\quad - \frac{d}{dt}\int v^3 w - \int F v_t - c_1\int F\frac{x}{|x|}\cdot\nabla v \\
&\quad - \frac{c_2}{8R^3}\int_{|y|\leq 2R}\int F\frac{(x-y)}{|x-y|}\cdot\nabla v - \frac{c_3}{R}\int F\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v \\
&\quad\lesssim \delta\Big(\int \frac{1}{|x|}v^4\Big) + \frac{1}{\delta}E(t)\|w\|_{L^6}^2.
\end{aligned} \tag{4.34}$$

Category-2 terms

The Sobolev embedding theorem implies that

$$-3\int v_t v w^2\,dx \lesssim \|w\|_{L_x^6(\mathbf{R}^3)}^2\|v\|_{L_x^6(\mathbf{R}^3)}\|v_t\|_{L_x^2(\mathbf{R}^3)} \lesssim E(t)\big\|w(t)\big\|_{L_x^6(\mathbf{R}^3)}^2. \tag{4.35}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) + c_1\pi v(t,0)^2 + \frac{c_2\pi}{16R^3}\int_{|y|\leq 2R} v(t,y)^2 \\
&\quad + \frac{c_1}{2}\int \frac{1}{|x|}v^4 + \frac{c_2}{16R^3}\int_{|y|\leq 2R}\int \frac{1}{|x-y|}v^4 \\
&\quad + \frac{c_3}{2R}\int \phi\Big(\frac{|x|}{R}\Big)\big[v_t^2 + |\nabla v|^2\big] + \frac{c_3}{4R}\int \chi\Big(\frac{|x|}{R}\Big)v^4
\end{aligned}$$

$$
\begin{aligned}
&-\frac{d}{dt}\int v^3 w + 3\int v^2 w v_t - c_1 \int F \frac{x}{|x|}\cdot\nabla v \\
&-\frac{c_2}{8R^3}\int_{|y|\le 2R}\int F\frac{(x-y)}{|x-y|}\cdot\nabla v - \frac{c_3}{R}\int F\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v \\
\lesssim\ & \delta\Big(\int \frac{1}{|x|}v^4\Big) + \frac{1}{\delta}E(t)\|w\|_{L^6}^2.
\end{aligned} \tag{4.36}
$$

Analysis of the other terms involving $-3vw^2$ is similar:

$$
\int v w^2 \frac{x}{|x|}\cdot\nabla v \lesssim \|w\|_{L^6}^2\|\nabla v\|_{L^2}\|v\|_{L^6} \lesssim E(t)\|w\|_{L^6}^2 \tag{4.37}
$$

and

$$
\frac{1}{8R^3}\int_{|y|\le 2R}\int v w^2 \frac{(x-y)}{|x-y|}\cdot\nabla v \lesssim E(t)\|w\|_{L^6}^2. \tag{4.38}
$$

Since $\chi(\frac{|x|}{R})\frac{x}{R}$ is also uniformly bounded,

$$
\frac{1}{R}\int v w^2 \chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v \lesssim E(t)\|w\|_{L^6}^2. \tag{4.39}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) + c_1\pi v(t,0)^2 + \frac{c_2\pi}{16R^3}\int_{|y|\le 2R} v(t,y)^2 \\
&\quad + \frac{c_1}{2}\int \frac{1}{|x|}v^4 + \frac{c_2}{8R^3}\int_{|y|\le 2R}\int \frac{1}{|x-y|}v^4 \\
&\quad + \frac{c_3}{2R}\int \phi\Big(\frac{|x|}{R}\Big)\big[v_t^2 + |\nabla v|^2\big] + \frac{c_3}{4R}\int \chi\Big(\frac{|x|}{R}\Big)v^4 \\
&\quad - \int v^3 w_t + 3c_1\int v^2 w \frac{x}{|x|}\cdot\nabla v \\
&\quad + \frac{3c_2}{8R^3}\int_{|y|\le 2R}\int v^2 w\frac{(x-y)}{|x-y|}\cdot\nabla v + \frac{3c_3}{R}\int v^2 w\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v \\
&\lesssim \delta\Big(\int\frac{1}{|x|}v^4\Big) + \frac{1}{\delta}E(t)\|w\|_{L^6}^2.
\end{aligned} \tag{4.40}
$$

Category-3 *terms*
Making a Littlewood–Paley decomposition,

$$
\int v^3 w_t\,dx = \sum_j \int v^3 \partial_t w_j\,dx. \tag{4.41}
$$

By Fourier support properties,

$$\int v^3 \partial_t w_j \, dx = \int \big(v^3 - (P_{\le j-3} v)^3\big)(\partial_t w_j)\, dx$$
$$= \int (P_{\ge j-3} v)^3 (\partial_t w_j)\, dx$$
$$+ 3 \int (P_{\ge j-3} v)(P_{\le j-3} v) v \cdot \partial_t w_j \, dx. \tag{4.42}$$

Using Lemma 3.4,

$$\sum_j \int_{|x| \ge \frac{R}{2}} \big[v^3 - (P_{\le j-3} v)^3\big](\partial_t w_j)\, dx$$
$$\lesssim \sum_j \Big(\Big\| \frac{1}{|x|^{1/4}} |P_{\le j} v| \Big\|_{L^4}^2 + \Big\| \frac{1}{|x|^{1/4}} |P_{\ge j} v| \Big\|_{L^4}^2 \Big)$$
$$\times \| P_{\ge j-3} v \|_{L_x^2} \big\| |x|^{1/2} \partial_t w_j \big\|_{L_x^\infty(|x| \ge \frac{R}{2})}$$
$$\lesssim \Big(\int \frac{1}{|x|} v^4 \Big)^{1/2} \sum_j \| P_{\ge j-3} v \|_{L_x^2} \big\| |x|^{1/2} \partial_t w_j \big\|_{L_x^\infty(|x| \ge \frac{R}{2})}. \tag{4.43}$$

By the Cauchy–Schwarz inequality,

$$(4.43) \lesssim \delta \Big(\int \frac{1}{|x|} v^4 \Big) + \frac{1}{\delta} \Big(\sum_j \| P_{\ge j-3} v \|_{L^2} \big\| |x|^{1/2} \partial_t w_j \big\|_{L^\infty(|x| \ge \frac{R}{2})} \Big)^2. \tag{4.44}$$

By Bernstein's inequality and Young's inequality,

$$\Big(\sum_j \| P_{\ge j-3} v \|_{L^2} \big\| |x|^{1/2} \partial_t w_j \big\|_{L^\infty(|x| \ge \frac{R}{2})} \Big)^2$$
$$\le \Big(\sum_j \Big(\sum_{k \ge j-3} 2^k 2^{j-k} \| P_k v \|_{L^2} \Big) \cdot 2^{-j} \big\| |x|^{1/2} \partial_t w_j \big\|_{L^\infty(|x| \ge \frac{R}{2})} \Big)^2$$
$$\lesssim \Big(\sum_k 2^{2k} \| P_k v \|_{L^2}^2 \Big) \Big(\sum_j 2^{-2j} \big\| |x|^{1/2} \partial_t w_j \big\|_{L^\infty(|x| \ge \frac{R}{2})}^2 \Big)$$
$$\lesssim E(t) \Big(\sum_j 2^{-2j} \big\| |x|^{1/2} \partial_t w_j \big\|_{L^\infty(|x| \ge \frac{R}{2})}^2 \Big). \tag{4.45}$$

Therefore,

$$\frac{d}{dt} \mathcal{E}(t) + c_1 \pi v(t,0)^2 + \frac{c_2 \pi}{16 R^3} \int_{|y| \le 2R} v(t,y)^2$$
$$+ \frac{c_1}{2} \int \frac{1}{|x|} v^4 + \frac{c_2}{16 R^3} \int_{|y| \le 2R} \int \frac{1}{|x-y|} v^4$$

$$
\begin{aligned}
&+\frac{c_3}{2R}\int \phi\Big(\frac{|x|}{R}\Big)\big[v_t^2+|\nabla v|^2\big]+\frac{c_3}{4R}\int \chi\Big(\frac{|x|}{R}\Big)v^4\\
&-\sum_j\int_{|x|\le\frac{R}{2}}\big(v^3-(P_{\le j-3}v)^3\big)\cdot\partial_t w_j+3c_1\int v^2 w\frac{x}{|x|}\cdot\nabla v\\
&+\frac{3c_2}{8R^3}\int_{|y|\le 2R}\int v^2 w\frac{(x-y)}{|x-y|}\cdot\nabla v+\frac{3c_3}{R}\int v^2 w\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v\\
\lesssim\ &\delta\Big(\int\frac{1}{|x|}v^4\Big)+\frac{1}{\delta}E(t)\|w\|_{L^6}^2\\
&+\frac{1}{\delta}E(t)\Big(\sum_j 2^{-2j}\big\||x|^{1/2}\partial_t w_j\big\|^2_{L^\infty(|x|\ge\frac{R}{2})}\Big).
\end{aligned}\tag{4.46}
$$

By the Sobolev embedding theorem, Hölder's inequality, and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\sum_j\int_{|x|\le\frac{R}{2}} v(P_{\le j-3}v)(P_{\ge j-3}v)\cdot\partial_t w_j\,dx\\
&\quad\le\sum_j\|\partial_t w_j\|_{L^6}\|v\|_{L^6(|x|\le\frac{R}{2})}\|P_{\ge j-3}v\|_{L^2}\|P_{\le j-3}v\|_{L^6}\\
&\quad\lesssim\delta R E(t)\Big(\frac{1}{R}\int_{|x|\le R}|\nabla v|^2+\frac{1}{R^3}\int_{|x|\le R}v^2\Big)\\
&\qquad+\frac{1}{\delta}\Big(\sum_j\|P_{\ge j-3}v\|_{L^2}\|\partial_t w_j\|_{L^6}\Big)^2.
\end{aligned}\tag{4.47}
$$

Following (4.45),

$$
\Big(\sum_j\|P_{\ge j-3}v\|_{L^2}\|\partial_t w_j\|_{L^6}\Big)^2\lesssim E(t)\Big(\sum_j 2^{-2j}\|\partial_t w_j\|^2_{L^6}\Big).\tag{4.48}
$$

Next, following (4.45), by the Cauchy–Schwarz inequality and Lemma 3.5,

$$
\begin{aligned}
&\sum_j\int_{|x|\le\frac{R}{2}}(P_{\ge j-3}v)^3\cdot\partial_t w_j\,dx\\
&\quad\lesssim\sum_j\|P_{\ge j-3}v\|^2_{L^4(|x|\le\frac{R}{2})}\|P_{\ge j-3}v\|_{L^3(\mathbf{R}^3)}\|\partial_t w_j\|_{L^6(\mathbf{R}^3)}\\
&\quad\lesssim\delta R\|\nabla v(t)\|^2_{L^2}\Big[\frac{1}{R}\|\nabla v\|^2_{L^2(|x|\le R)}+\frac{1}{R^3}\|v\|^2_{L^2(|x|\le R)}\Big]\\
&\qquad+\delta\Big(\int\frac{1}{|x|}v^4\Big)+\frac{1}{\delta}\Big(\sum_j 2^{-j/2}\|P_{\ge j-3}v\|_{L^3}\|\partial_t w_j\|_{L^6}\Big)^2
\end{aligned}
$$

$$\lesssim \delta R E(t)\Big[\frac{1}{R}\|\nabla v\|_{L^2(|x|\leq R)}^2 + \frac{1}{R^3}\|v\|_{L^2(|x|\leq R)}^2\Big] + \delta\Big(\int \frac{1}{|x|}v^4\Big) + \frac{1}{\delta}E(t)\Big[\sum_j 2^{-2j}\|\partial_t w_j\|_{L^6}^2\Big]. \tag{4.49}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) + c_1\pi v(t,0)^2 + \frac{c_2\pi}{16R^3}\int_{|y|\leq 2R} v(t,y)^2 \\
&\quad + \frac{c_1}{2}\int \frac{1}{|x|}v^4 + \frac{c_2}{8R^3}\int_{|y|\leq 2R}\int \frac{1}{|x-y|}v^4 \\
&\quad + \frac{c_3}{2R}\int \phi\Big(\frac{|x|}{R}\Big)\big[v_t^2 + |\nabla v|^2\big] + \frac{c_3}{4R}\int \chi\Big(\frac{|x|}{R}\Big)v^4 \\
&\quad + 3c_1\int v^2 w\frac{x}{|x|}\cdot\nabla v \\
&\quad + \frac{3c_2}{8R^3}\int_{|y|\leq 2R}\int v^2 w\frac{(x-y)}{|x-y|}\cdot\nabla v + \frac{3c_3}{R}\int v^2 w\chi\Big(\frac{|x|}{R}\Big)x\cdot\nabla v \\
&\lesssim \delta\Big(\int \frac{1}{|x|}v^4\Big) + \delta R E(t)\Big[\frac{1}{R}\|\nabla v\|_{L^2(|x|\leq R)}^2 + \frac{1}{R^3}\|v\|_{L^2(|x|\leq R)}^2\Big] \\
&\quad + \frac{1}{\delta}E(t)\|w\|_{L^6}^2 + \frac{1}{\delta}E(t)\Big(\sum_j 2^{-2j}\|\partial_t w_j\|_{L^6}^2\Big) \\
&\quad + \frac{1}{\delta}E(t)\Big(\sum_j 2^{-2j}\big\||x|^{1/2}\partial_t w_j\big\|_{L^\infty(|x|\geq \frac{R}{2})}^2\Big).
\end{aligned} \tag{4.50}$$

Integrating by parts,

$$3c_1\int v^2 w\frac{x}{|x|}\cdot\nabla v\,dx = -2c_1\int \frac{1}{|x|}v^3 w - c_1\int v^3(\nabla w)\cdot\frac{x}{|x|}. \tag{4.51}$$

Following (4.31),

$$-2c_1\int \frac{1}{|x|}v^3 w\,dx \lesssim \frac{1}{\delta}E(t)\big\|w(t)\big\|_{L_x^6}^2 + \delta\Big(\int \frac{1}{|x|}v^4\,dx\Big). \tag{4.52}$$

The term

$$-c_1\int \big(v^3 - (P_{\leq j-3}v)^3\big)(\nabla w_j)\cdot\frac{x}{|x|}\,dx \tag{4.53}$$

may be estimated using exactly the same arguments as in the estimates for (4.43).

Now, the Fourier support of $(\nabla w_j)(P_{\leq j-3}v)^3$ is $|\xi| \sim 2^j$, so integrating by parts,

$$\begin{aligned} c \int (P_{\leq j-3}v)^3(\nabla w_j) \cdot \frac{x}{|x|}\, dx \\ &= \int \frac{x_l x_k}{|x|^3} \frac{\partial_k}{\Delta}(P_{\leq j-3}v)^3(\partial_l w_j) \\ &\lesssim 2^{-j} \Big\| \frac{1}{|x|^{1/4}} P_{\leq j-3}v \Big\|_{L^4}^2 \Big\| \frac{1}{|x|^{1/2}} P_{\leq j-3}v \Big\|_{L^{10/3}} \|\partial_k w_j\|_{L^5}. \end{aligned} \tag{4.54}$$

Then by the Cauchy–Schwarz inequality,

$$\begin{aligned} \sum_j c \int (P_{\leq j-3}v)^3(\nabla w_j) \cdot \frac{x}{|x|}\, dx \\ &\lesssim \delta\Big(\int \frac{1}{|x|} v^4\Big) + \frac{1}{\delta}\Big(\sum_j 2^{-j} \Big\| \frac{1}{|x|^{1/2}} P_{\leq j-3}v \Big\|_{L^{10/3}} \|\nabla w_j\|_{L^5}\Big)^2, \end{aligned} \tag{4.55}$$

and then by Bernstein's inequality,

$$\lesssim \delta\Big(\int \frac{1}{|x|} v^4\Big) + \frac{1}{\delta} E(t)\Big(\sum_j 2^{j/5} \|w_j\|_{L^5}^2\Big). \tag{4.56}$$

Therefore,

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}(t) + c_1 \pi v(t,0)^2 + \frac{c_2 \pi}{16R^3} \int_{|y|\leq 2R} v(t,y)^2 \\ &\quad + \frac{c_1}{2} \int \frac{1}{|x|} v^4 + \frac{c_2}{8R^3} \int_{|y|\leq 2R} \int \frac{1}{|x-y|} v^4 \\ &\quad + \frac{c_3}{2R} \int \phi\Big(\frac{|x|}{R}\Big)\big[v_t^2 + |\nabla v|^2\big] + \frac{c_3}{4R} \int \chi\Big(\frac{|x|}{R}\Big) v^4 \\ &\quad + \frac{3c_2}{8R^3} \int_{|y|\leq 2R} \int v^2 w \frac{(x-y)}{|x-y|} \cdot \nabla v + \frac{3c_3}{R} \int v^2 w \chi\Big(\frac{|x|}{R}\Big) x \cdot \nabla v \\ &\lesssim \delta\Big(\int \frac{1}{|x|} v^4\Big) + \delta R E(t)\Big[\frac{1}{R}\|\nabla v\|_{L^2(|x|\leq R)}^2 + \frac{1}{R^3}\|v\|_{L^2(|x|\leq R)}^2\Big] \\ &\quad + \frac{1}{\delta} E(t)\|w\|_{L^6}^2 + \frac{1}{\delta} E(t)\Big(\sum_j 2^{j/5}\|w_j\|_{L^5}^2\Big) + \frac{1}{\delta} E(t)\Big(\sum_j 2^{-2j}\|\partial_t w_j\|_{L^6}^2\Big) \\ &\quad + \frac{1}{\delta} E(t)\Big(\sum_j 2^{-2j} \big\| |x|^{1/2} \partial_t w_j \big\|_{L^\infty(|x|\geq \frac{R}{2})}^2\Big) \frac{1}{\delta} E(t)\Big(\sum_j 2^{-2j}\|\nabla w_j\|_{L^6}^2\Big) \\ &\quad + \frac{1}{\delta} E(t)\Big(\sum_j 2^{-2j} \big\| |x|^{1/2} \nabla w_j \big\|_{L^\infty(|x|\geq \frac{R}{2})}^2\Big). \end{aligned} \tag{4.57}$$

Like $\frac{x}{|x|}$, the potentials

$$a(x) = \chi\Big(\frac{2|x|}{R}\Big)\frac{x}{R} \qquad \text{and} \qquad a(x) = \int_{|y| \leq 2R} \frac{(x-y)}{|x-y|}\, dy \tag{4.58}$$

are also bounded radial functions satisfying

$$\nabla \cdot a(x) \lesssim \frac{1}{|x|}, \tag{4.59}$$

and therefore the analysis of

$$+ \frac{3c_2}{8R^3} \int_{|y| \leq 2R} \int v^2 w \frac{(x-y)}{|x-y|} \cdot \nabla v + \frac{3c_3}{R} \int v^2 w \chi\Big(\frac{|x|}{R}\Big) x \cdot \nabla v \tag{4.60}$$

may be carried out in much the same manner as

$$\int v^2 w \frac{x}{|x|} \cdot \nabla v. \tag{4.61}$$

Choosing

$$\frac{1}{R} = \sup_{0 \leq t \leq T} \mathcal{E}(t) \tag{4.62}$$

and absorbing

$$\delta\Big(\int \frac{1}{|x|} v^4\Big) + \delta R E(t) \Big[\frac{1}{R} \|\nabla v\|_{L^2(|x| \leq R)}^2 + \frac{1}{R^3} \|v\|_{L^2(|x| \leq R)}^2\Big] \tag{4.63}$$

into the left-hand side,

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) \lesssim \frac{1}{\delta} E(t) \|w\|_{L^6}^2 &+ \frac{1}{\delta} E(t) \Big(\sum_j 2^{j/5} \|w_j\|_{L^5}^2\Big) \\ &+ \frac{1}{\delta} E(t) \Big(\sum_j 2^{-2j} \|\partial_t w_j\|_{L^6}^2\Big) \\ &+ \frac{1}{\delta} E(t) \Big(\sum_j 2^{-2j} \big\| |x|^{1/2} \partial_t w_j \big\|_{L^\infty(|x| \geq \frac{R}{2})}^2\Big) \\ &+ \frac{1}{\delta} E(t) \Big(\sum_j 2^{-2j} \|\nabla w_j\|_{L^6}^2\Big) \\ &+ \frac{1}{\delta} E(t) \Big(\sum_j 2^{-2j} \big\| |x|^{1/2} \nabla w_j \big\|_{L^\infty(|x| \geq \frac{R}{2})}^2\Big). \end{aligned} \tag{4.64}$$

Since $E(t) \sim \mathcal{E}(t)$, (4.64) implies that

$$\frac{d}{dt}\ln\big(\mathcal{E}(t)\big) \lesssim \frac{1}{\delta}\|w\|_{L^6}^2 + \frac{1}{\delta}\Big(\sum_j 2^{j/5}\|w_j\|_{L^5}^2\Big) + \frac{1}{\delta}\Big(\sum_j 2^{-2j}\|\partial_t w_j\|_{L^6}^2\Big)$$
$$+ \frac{1}{\delta}\Big(\sum_j 2^{-2j}\big\||x|^{1/2}\partial_t w_j\big\|^2_{L^\infty(|x|\geq \frac{1}{2\mathcal{E}(T)})}\Big)\frac{1}{\delta}\Big(\sum_j 2^{-2j}\|\nabla w_j\|_{L^6}^2\Big)$$
$$+ \frac{1}{\delta}\Big(\sum_j 2^{-2j}\big\||x|^{1/2}\nabla w_j\big\|^2_{L^\infty(|x|\geq \frac{1}{2\mathcal{E}(T)})}\Big). \tag{4.65}$$

Now suppose without loss of generality that

$$\mathcal{E}(T) = \sup_{0\leq t\leq T}\mathcal{E}(t). \tag{4.66}$$

Integrating in time and combining (3.38), Corollary 3.3, and (4.5),

$$\ln\big(\mathcal{E}(T)\big) - \ln\big(\mathcal{E}(0)\big) \lesssim \frac{\epsilon^2}{\delta}\ln(T) + \frac{\epsilon^2}{\delta}\ln\big(\mathcal{E}(T)\big) + \epsilon. \tag{4.67}$$

Remark
Corollary 3.3 is only stated for a solution to the linear wave equation. However, since $\|w^3\|_{L_t^1 L_x^{3/2}} \lesssim \epsilon^3$, one may estimate the contribution of the nonlinear term using Duhamel's principle, the principle of superposition, and estimates for the linear wave equation.

Doing some algebra and choosing $\delta(c_1, c_2, c_3) > 0$ small, and then $\epsilon(\delta) > 0$ sufficiently small,

$$\ln\big(\mathcal{E}(T)\big) \leq \Big(\frac{1}{1-\frac{C\epsilon^2}{\delta}}\Big)\ln\big(\mathcal{E}(0)\big) + \frac{C\epsilon^2}{\delta(1-\frac{C\epsilon^2}{\delta})}\ln(T) + \frac{C\epsilon}{(1-\frac{C\epsilon^2}{\delta})}. \tag{4.68}$$

This proves that for any t, there exists a constant C such that $E(t) \sim \mathcal{E}(t) \lesssim (1+t)^{C\epsilon}$. □

5. Proof of scattering

Having proved that (1.1) is globally well-posed for every radially symmetric $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, the next step is to prove that every such global solution scatters. By Theorem 2.1, proving that a global solution to (1.1) scatters is equivalent to showing that $\|u\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} < \infty$.

By time reversal symmetry, it suffices to show the following.

THEOREM 5.1
For any radial initial data $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, the solution to (1.1) scatters forward in time, with

$$\|u\|_{L^4_{t,x}([0,\infty)\times\mathbf{R}^3)} < \infty. \tag{5.1}$$

By the dominated convergence theorem, there exists $R(u_0, u_1, \epsilon) < \infty$ such that

$$\left\| S(t)(u_0, u_1) \right\|_{L^4_{t,x}(|x| \geq R + |t|)} < \epsilon. \tag{5.2}$$

Then by finite propagation speed, Theorem 2.1, and small data arguments, if u is a global solution to (1.1), then

$$\|u\|_{L^4_{t,x}([0,\infty)\times\{x:|x|\geq R+|t|\})} \lesssim \epsilon. \tag{5.3}$$

Rescaling using (1.2) with $\lambda = 2R$, $(u_0(x), u_1(x)) \mapsto (2Ru_0(2Rx), (2R)^2 \times u_1(2Rx))$, and

$$\|u\|_{L^4_{t,x}(|x|\geq\frac{1}{2}+|t|)} \lesssim \epsilon. \tag{5.4}$$

To prove

$$\|u\|_{L^4_{t,x}([0,\infty)\times\{|x|\leq\frac{1}{2}+t\})} < \infty, \tag{5.5}$$

it is convenient to translate in time so that the space-time integral of (5.5) is over a cone with vertex at the origin. Make a time translation so that

$$u(1,x) = 2Ru_0(2Rx), \qquad u_t(1,x) = (2R)^2 u_1(2Rx). \tag{5.6}$$

After time translation, (5.4) implies that

$$\|u\|_{L^4_{t,x}([1,\infty)\times\{|x|\geq t-\frac{1}{2}\})} \lesssim \epsilon, \tag{5.7}$$

and (5.5) is equivalent to $\|u\|_{L^4_{t,x}([1,\infty)\times\{x:|x|\leq|t|-\frac{1}{2}\})} < \infty$.

Switching to hyperbolic coordinates for the region inside the cone $|x| \leq t$, let

$$\tilde{u}(\tau, s) = \frac{e^\tau \sinh s}{s} u(e^\tau \cosh s, e^\tau \sinh s). \tag{5.8}$$

Making a change of variables,

$$\begin{aligned} &\int_0^\infty \int_0^\infty \tilde{u}(\tau,s)^4 \Big(\frac{s}{\sinh s}\Big)^2 s^2 \, ds \, d\tau \\ &\quad = \int_0^\infty \int_0^\infty u(e^\tau \cosh s, e^\tau \sinh s)^4 e^{2\tau} (\sinh s)^2 e^{2\tau} \, ds \, d\tau \\ &\quad = \int_1^\infty \int_{t^2-r^2\geq 1} u(t,r)^4 r^2 \, dr \, dt \geq \int_2^\infty \int_{t\geq r+\frac{1}{2}} u(t,r)^4 r^2 \, dr \, dt. \end{aligned} \tag{5.9}$$

Therefore,

$$\int_0^\infty \int_0^\infty \tilde{u}(\tau,s)^4 \Big(\frac{s}{\sinh s}\Big)^2 s^2\, ds\, d\tau < \infty \tag{5.10}$$

combined with (5.7) implies that

$$\|u\|_{L^4_{t,x}([2,\infty)\times\mathbf{R}^3)} < \infty. \tag{5.11}$$

The global well-posedness result of Theorem 4.2 implies that

$$\|u\|_{L^4_{t,x}([1,2]\times\mathbf{R}^3)} < \infty, \tag{5.12}$$

which combined with (5.11), after undoing time translation, implies (5.5).

By direct computation,

$$\Big(\partial_{\tau\tau} - \partial_{ss} - \frac{2}{s}\partial_s\Big)\tilde{u}(\tau,s) + \Big(\frac{s}{\sinh s}\Big)^2 \tilde{u}^3 = 0, \tag{5.13}$$

with

$$\tilde{u}|_{\tau=0} = \frac{e^\tau \sinh s}{s} u(e^\tau \cosh s, e^\tau \sinh s)|_{\tau=0} \tag{5.14}$$

and

$$\tilde{u}_\tau|_{\tau=0} = \partial_\tau\Big(\frac{e^\tau \sinh s}{s} u(e^\tau \cosh s, e^\tau \sinh s)\Big)|_{\tau=0}. \tag{5.15}$$

A solution to (5.13) has the conserved energy

$$E(\tau) = \frac{1}{2}\|\tilde{u}_\tau\|_{L^2}^2 + \frac{1}{2}\|\tilde{u}_s\|_{L^2}^2 + \frac{1}{4}\int \tilde{u}(\tau,s)^4\Big(\frac{s}{\sinh s}\Big)^2 s^2\, ds. \tag{5.16}$$

As in the proof of global well-posedness, to use (5.16) we will truncate in frequency. The properties of the initial data (5.14) and (5.15) will be analyzed in more detail later, but for now, assume that (5.14) and (5.15) may be decomposed into an $\dot{H}^1 \times L^2$ piece and an $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ piece.

LEMMA 5.2

There exists a decomposition

$$\tilde{u}|_{\tau=0} = \frac{e^\tau \sinh s}{s} u(e^\tau \cosh s, e^\tau \sinh s)|_{\tau=0} = \tilde{v}_0 + \tilde{w}_0 \tag{5.17}$$

and

$$\tilde{u}_\tau|_{\tau=0} = \partial_\tau\Big(\frac{e^\tau \sinh s}{s} u(e^\tau \cosh s, e^\tau \sinh s)\Big)|_{\tau=0} = \tilde{v}_1 + \tilde{w}_1, \tag{5.18}$$

with

$$\frac{1}{2}\int |\partial_s \tilde{v}_0|^2 s^2 + \frac{1}{2}\int |\tilde{v}_1|^2 s^2 + \frac{1}{4}\int \tilde{v}_0^4 \Big(\frac{s}{\sinh s}\Big)^2 s^2 < \infty \tag{5.19}$$

and

$$\|\tilde{w}_0\|_{\dot{H}^{1/2}} + \|w_1\|_{\dot{H}^{-1/2}} \leq \epsilon. \tag{5.20}$$

Remark
Following (4.13), it is enough to prove $\tilde{u}_0 \in \dot{H}^1 + \dot{H}^{1/2}$ and $\tilde{u}_1 \in L^2 + \dot{H}^{-1/2}$ and then truncate in frequency. The proof of Lemma 5.2 will be postponed.

Proof of Theorem 5.1
Make a Fourier truncation argument. Let $\tilde{v}$ and $\tilde{w}$ solve

$$(\partial_{\tau\tau} - \Delta)\tilde{w} + \Big(\frac{s}{\sinh s}\Big)^2 \tilde{w}^3 = 0, \qquad \tilde{w}(0, y) = \tilde{w}_0, \qquad \tilde{w}_\tau(0, y) = \tilde{w}_1, \tag{5.21}$$

and

$$(\partial_{\tau\tau} - \Delta)\tilde{v} + \Big(\frac{s}{\sinh s}\Big)^2 [\tilde{v}^3 + 3\tilde{v}^2\tilde{w} + 3\tilde{v}\tilde{w}^2] = 0, \qquad \tilde{v}(0, y) = \tilde{v}_0, \qquad \tilde{v}_\tau(0, y) = \tilde{v}_1. \tag{5.22}$$

Define the energy

$$E(\tau) = \frac{1}{2}\int |\partial_s \tilde{v}|^2 s^2 + \frac{1}{2}\int |\partial_\tau \tilde{v}|^2 s^2 + \frac{1}{4}\int \tilde{v}^4 \Big(\frac{s}{\sinh s}\Big)^2 s^2. \tag{5.23}$$

As in the proof of global well-posedness, define the quantity

$$\mathcal{E}(\tau) = E(\tau) + M(\tau) + \int \tilde{v}^3 \tilde{w} \Big(\frac{s}{\sinh s}\Big)^2 s^2\, ds, \tag{5.24}$$

where

$$M(\tau) = c\int \tilde{v}_\tau \tilde{v}_s s^2\, ds + c\int \tilde{v}_\tau \tilde{v} s\, ds, \tag{5.25}$$

and $c > 0$ is a small constant.

By direct computation, making a slight modification of (4.11) and (3.21),

$$\begin{aligned} \frac{d}{d\tau} M(\tau) = &-\frac{1}{2}\tilde{v}(\tau, 0)^2 - \frac{1}{2}\int \tilde{v}(\tau, s)^4 \Big(\frac{s}{\sinh s}\Big)^2 \Big(\frac{\cosh s}{\sinh s}\Big) s^2\, ds \\ &- 3\int \tilde{v}^2 \tilde{v}_s \tilde{w} \Big(\frac{s}{\sinh s}\Big)^2 s^2\, ds - 3\int \tilde{v}\tilde{v}_s \tilde{w}^2 \Big(\frac{s}{\sinh s}\Big)^2 s^2\, ds \\ &- 3\int \tilde{v}^3 \tilde{w} \Big(\frac{s}{\sinh s}\Big)^2 s\, ds - 3\int \tilde{v}^2 \tilde{w}^2 \Big(\frac{s}{\sinh s}\Big)^2 s\, ds. \end{aligned} \tag{5.26}$$

Therefore,

$$\frac{d}{d\tau}\mathcal{E}(\tau) = -\frac{c}{2}\tilde{v}(\tau,0)^2 - \frac{c}{2}\int \tilde{v}(\tau,s)^4\Big(\frac{s}{\sinh s}\Big)^2\Big(\frac{\cosh s}{\sinh s}\Big)s^2\,ds$$
$$- 3c\int \tilde{v}^2\tilde{v}_s\tilde{w}\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds - 3c\int \tilde{v}\tilde{v}_s\tilde{w}^2\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds$$
$$- 3c\int \tilde{v}^3\tilde{w}\Big(\frac{s}{\sinh s}\Big)^2 s\,ds - 3c\int \tilde{v}^2\tilde{w}^2\Big(\frac{s}{\sinh s}\Big)^2 s\,ds$$
$$- 3\int \Big(\frac{s}{\sinh s}\Big)^2\tilde{v}^3\tilde{w}_\tau s\,ds - 3\int \Big(\frac{s}{\sinh s}\Big)^2\tilde{v}\tilde{v}_\tau\tilde{w}^2 s\,ds. \tag{5.27}$$

By Hardy's inequality and Hölder's inequality,

$$-3c\int \tilde{v}\tilde{v}_s\tilde{w}^2\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds - 3c\int \tilde{v}^2\tilde{w}^2\Big(\frac{s}{\sinh s}\Big)^2 s\,ds$$
$$- 3\int \Big(\frac{s}{\sinh s}\Big)^2\tilde{v}\tilde{v}_\tau\tilde{w}^2 s^2\,ds \lesssim E(\tau)\|\tilde{w}\|_{L^6}^2. \tag{5.28}$$

Also, by Hardy's inequality and the Cauchy–Schwarz inequality,

$$\int \tilde{v}^3\tilde{w}\Big(\frac{s}{\sinh s}\Big)^2 s\,ds \lesssim \delta\Big(\int \tilde{v}^4\Big(\frac{\cosh s}{\sinh s}\Big)\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds\Big) + \frac{1}{\delta}\|\tilde{w}\|_{L^6}^2\Big\|\frac{1}{|x|^{1/2}}\tilde{v}\Big\|_{L^3}^2$$
$$\lesssim \delta\Big(\int \Big(\frac{s}{\sinh s}\Big)^2\Big(\frac{\cosh s}{\sinh s}\Big)\tilde{v}^4 s^2\,ds\Big) + \frac{1}{\delta}\|\tilde{w}\|_{L^6}^2 E(\tau). \tag{5.29}$$

Therefore,

$$\frac{d}{d\tau}\mathcal{E}(\tau) + \frac{c}{2}\tilde{v}(\tau,0)^2 + \frac{c}{2}\int \tilde{v}(\tau,s)^4\Big(\frac{s}{\sinh s}\Big)^2\Big(\frac{\cosh s}{\sinh s}\Big)s^2\,ds$$
$$+ 3c\int \tilde{v}^2\tilde{w}\Big(\frac{s}{\sinh s}\Big)^2\tilde{v}_s s^2\,ds + 3\int \Big(\frac{s}{\sinh s}\Big)^2\tilde{v}^3\tilde{w}_\tau s^2\,ds$$
$$\lesssim \frac{1}{\delta}E(\tau)\|w\|_{L^6}^2 + \delta\Big(\int \tilde{v}^4\Big(\frac{s}{\sinh s}\Big)^2\Big(\frac{\cosh s}{\sinh s}\Big)s^2\,ds\Big). \tag{5.30}$$

Next, integrating by parts,

$$3c\int \tilde{v}^2\tilde{v}_s\tilde{w}\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds = -c\int \tilde{v}^3\tilde{w}_s\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds$$
$$- c\int \tilde{v}^3\tilde{w}\cdot\partial_s\Big(\frac{s^4}{(\sinh s)^2}\Big)\,ds. \tag{5.31}$$

Since

$$\partial_s\Big(\frac{s^4}{(\sinh s)^2}\Big) \lesssim s, \tag{5.32}$$

by (5.29),

$$c\int \tilde{v}^3\tilde{w}\cdot\partial_s\Big(\frac{s^4}{(\sinh s)^2}\Big)\lesssim\delta\Big(\int\tilde{v}^4\Big(\frac{s}{\sinh s}\Big)^2\Big(\frac{\cosh s}{\sinh s}\Big)s^2\,ds\Big)+\frac{1}{\delta}\|\tilde{w}\|_{L^6}^2E(\tau). \tag{5.33}$$

Following (4.42)–(4.44) and using Lemma 3.4,

$$\begin{aligned}-c\sum_j\int_{s\geq\frac{R}{2}}&[\tilde{v}^3-(P_{\leq j-3}\tilde{v})^3](\partial_s\tilde{w}_j)\cdot\Big(\frac{s}{\sinh s}\Big)^2s^2\,ds\\&+3\sum_j\int_{s\geq\frac{R}{2}}[\tilde{v}^3-(P_{\leq j-3}\tilde{v})^3](\partial_\tau\tilde{w}_j)\Big(\frac{s}{\sinh s}\Big)^2s^2\,ds\\&\lesssim\delta\Big(\int\Big(\frac{\cosh s}{\sinh s}\Big)\Big(\frac{s}{\sinh s}\Big)^2\tilde{v}^4s^2\,ds\Big)\\&+\frac{1}{\delta}E(\tau)\Big(\sum_j2^{-2j}\Big\|(\nabla_{\tau,x}\tilde{w}_j)\Big(\frac{\sinh s}{\cosh s}\Big)^{1/2}\Big(\frac{s}{\sinh s}\Big)\Big\|_{L^\infty(|x|\geq\frac{R}{2})}^2\Big).\end{aligned} \tag{5.34}$$

Next, by Hölder's inequality,

$$\begin{aligned}\sum_j\|&(\tilde{v}^3-(P_{\leq j-3}\tilde{v})^3)(\nabla_{\tau,x}\tilde{w}_j)\|_{L^1(|x|\leq\frac{R}{2})}\\&\lesssim\sum_j\|\tilde{v}\|_{L^\infty}\|P_{\geq j-3}\tilde{v}\|_{L^2}\|\nabla_{\tau,x}\tilde{w}_j\|_{L^6}\|\tilde{v}\|_{L^3(|x|\leq\frac{R}{2})}\\&\lesssim E(\tau)\Big(\sum_j2^{-2j}\|\nabla_{\tau,x}\tilde{w}_j\|_{L^6}^2\Big)+RE(\tau)\|v\|_{L^\infty}^2.\end{aligned} \tag{5.35}$$

Following (4.54) and (4.55),

$$\begin{aligned}\int(P_{\leq j-3}\tilde{v})^3(\partial_s\tilde{w}_j)\cdot\Big(\frac{s}{\sinh s}\Big)^2s^2\,ds+\int(P_{\leq j-3}\tilde{v})^3(\partial_\tau\tilde{w}_j)\cdot\Big(\frac{s}{\sinh s}\Big)^2s^2\,ds\\\lesssim\delta\Big(\int\frac{1}{|x|}\tilde{v}^4\Big)+\frac{1}{\delta}E(\tau)\Big(\sum_j2^{-8j/5}\|\nabla_{\tau,x}\tilde{w}_j\|_{L^5}^2\Big).\end{aligned} \tag{5.36}$$

Therefore,

$$\begin{aligned}\frac{d}{d\tau}\mathcal{E}(\tau)+\frac{c}{2}\tilde{v}(\tau,0)^2+\frac{c}{2}\int\tilde{v}(\tau,s)^4\Big(\frac{s}{\sinh s}\Big)^2\Big(\frac{\cosh s}{\sinh s}\Big)s^2\,ds\\\lesssim E(\tau)\Big(\sum_j2^{-2j}\|\nabla_{\tau,x}w_j\|_{L^6}^2\Big)\end{aligned}$$

$$+ \frac{1}{\delta} E(\tau) \Big(\sum_j 2^{-8j/5} \| \nabla_{\tau,x} \tilde{w}_j \|_{L^5}^2 \Big) + R E(\tau) \| v \|_{L^\infty}^2$$

$$+ \frac{1}{\delta} E(\tau) \| w \|_{L^6}^2 + \delta \Big(\int \tilde{v}^4 \Big(\frac{s}{\sinh s} \Big)^2 \Big(\frac{\cosh s}{\sinh s} \Big) s^2 \, ds \Big). \tag{5.37}$$

Absorbing

$$\delta \Big(\int \tilde{v}^4 \Big(\frac{s}{\sinh s} \Big)^2 \Big(\frac{\cosh s}{\sinh s} \Big) s^2 \, ds \Big) \tag{5.38}$$

into the left-hand side,

$$\frac{d}{d\tau} \mathcal{E}(\tau) + \frac{c}{4} \int \tilde{v}(\tau,s)^4 \Big(\frac{s}{\sinh s} \Big)^2 \Big(\frac{\cosh s}{\sinh s} \Big) s^2 \, ds$$

$$\lesssim E(\tau) \Big(\sum_j 2^{-2j} \| \nabla_{\tau,x} w_j \|_{L^6}^2 \Big) + \frac{1}{\delta} E(\tau) \Big(\sum_j 2^{-8j/5} \| \nabla_{\tau,x} \tilde{w}_j \|_{L^5}^2 \Big)$$

$$+ R E(\tau) \| v \|_{L^\infty}^2 + \frac{1}{\delta} E(\tau) \| w \|_{L^6}^2. \tag{5.39}$$

Since $E(\tau) \sim \mathcal{E}(\tau)$,

$$\frac{d}{d\tau} \ln\big(\mathcal{E}(\tau)\big) + \frac{c}{4\mathcal{E}(\tau)} \int \tilde{v}(\tau,s)^4 \Big(\frac{s}{\sinh s} \Big)^2 \Big(\frac{\cosh s}{\sinh s} \Big) s^2 \, ds$$

$$\lesssim \Big(\sum_j 2^{-2j} \| \nabla_{\tau,x} w_j \|_{L^6}^2 \Big) + \frac{1}{\delta} \Big(\sum_j 2^{-8j/5} \| \nabla_{\tau,x} \tilde{w}_j \|_{L^5}^2 \Big)$$

$$+ R \| v \|_{L^\infty}^2 + \frac{1}{\delta} \| w \|_{L^6}^2$$

$$+ \frac{1}{\delta} \Big(\sum_j 2^{-2j} \Big\| (\nabla_{\tau,x} \tilde{w}_j) \Big(\frac{\sinh s}{\cosh s} \Big)^{1/2} \Big(\frac{s}{\sinh s} \Big) \Big\|_{L^\infty(s \geq \frac{R}{2})}^2 \Big). \tag{5.40}$$

Suppose that T is such that $\mathcal{E}(T) = \sup_{0<\tau<T} \mathcal{E}(\tau)$. Integrating in τ,

$$\ln\big(\mathcal{E}(T)\big) - \ln\big(\mathcal{E}(0)\big) + \frac{c}{4} \int_0^T \frac{1}{\mathcal{E}(\tau)} \int \tilde{v}(\tau,s)^4 \Big(\frac{s}{\sinh s} \Big)^2 \Big(\frac{\cosh s}{\sinh s} \Big) s^2 \, ds \, d\tau$$

$$\lesssim \frac{\epsilon^2}{\delta} \big(1 - \ln(R) \big) + \epsilon^2 + \int_0^T R \| \tilde{v} \|_{L^\infty}^2 \, d\tau. \tag{5.41}$$

Now by direct computation,

$$\Big\| \Big(\frac{s}{\sinh s} \Big)^{1/2} \tilde{u} \Big\|_{L^4} \lesssim \Big\| \Big(\frac{s}{\sinh s} \Big)^{1/2} \Big(\frac{\cosh s}{\sinh s} \Big)^{1/4} \tilde{v} \Big\|_{L^4} + \| \tilde{w} \|_{L^4}. \tag{5.42}$$

If I is an interval on which $\| (\frac{s}{\sinh s})^{1/2} \tilde{u} \|_{L^4_{\tau,x}(I)} \ll 1$, then by (2.11) and (5.22),

$$\|\tilde{v}\|_{L^2_\tau L^\infty_x(I\times\mathbf{R}^3)} \lesssim \|\nabla\tilde{v}\|_{L^\infty_\tau L^2_x} + \|\tilde{v}_\tau\|_{L^\infty_\tau L^2_x} + \|\tilde{v}\|_{L^2_\tau L^\infty_x}\Big(\int_I\int \tilde{u}^4\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds\,d\tau\Big)^{1/2}, \tag{5.43}$$

which implies that

$$\|\tilde{v}\|_{L^2_\tau L^\infty_x(I\times\mathbf{R}^3)} \lesssim \|\nabla\tilde{v}\|_{L^\infty_\tau L^2_x} + \|\tilde{v}_\tau\|_{L^\infty_\tau L^2_x}, \tag{5.44}$$

and therefore,

$$\int_0^T R\|\tilde{v}\|^2_{L^\infty}\,d\tau \lesssim R\mathcal{E}(T)\Big(\int_0^T\int \tilde{v}^4\Big(\frac{s}{\sinh s}\Big)^2 s^2\,ds\,d\tau\Big). \tag{5.45}$$

Choosing $R = \delta\frac{1}{\mathcal{E}(T)^2}$, (5.45) can be absorbed into the left-hand side of (5.41), proving

$$\ln\big(\mathcal{E}(T)\big) - \ln\big(\mathcal{E}(0)\big) \lesssim \frac{\epsilon^2}{\delta}\Big(\ln\Big(\frac{1}{\delta}\Big) + \ln\big(\mathcal{E}(T)\big)\Big) + \epsilon^2. \tag{5.46}$$

This implies a uniform bound on $\mathcal{E}(T)$. Plugging the uniform bound on $\mathcal{E}(\tau)$ for all τ into (5.40) implies a uniform bound on

$$\int_0^T\int\Big(\frac{s}{\sinh s}\Big)^2\Big(\frac{\cosh s}{\sinh s}\Big)\tilde{v}(\tau,s)^4 s^2\,ds\,d\tau < \infty. \tag{5.47}$$

This proves scattering, assuming Lemma 5.2 is true. □

Proof of Lemma 5.2

By Duhamel's formula, for $t > 1$,

$$u(t) = S(t)(u_0,u_1) + \int_0^t S(t-t')(0,u^3)\,dt' = u_l + u_{nl}. \tag{5.48}$$

The contributions of u_l and u_{nl} to (5.17) and (5.18) will be analyzed separately.

First take the term u_{nl}. When f and g are radial and $r > t$,

$$rS(t)(f,g) = \frac{1}{2}f(r-t) + \frac{1}{2}f(r+t) + \frac{1}{2}\int_{r-t}^{r+t} sg(s)\,ds. \tag{5.49}$$

Because the curve $t^2 - r^2 = 1$ has slope $\frac{dr}{dt} > 1$ everywhere, (5.48) implies that

$$s\tilde{u}_{nl}(\tau,s)|_{\tau=0} = \int_1^{e^\tau\cosh s}\int_{e^\tau\sinh s - e^\tau\cosh s + t}^{e^\tau\sinh s + e^\tau\cosh s - t} ru^3(t,r)\,dr\,dt|_{\tau=0}. \tag{5.50}$$

By direct computation,

$$\begin{aligned}
&\int_0^\infty \big(\partial_\tau(s\tilde{u}_{nl})|_{\tau=0}\big)^2\,ds \\
&\quad\lesssim \int_0^\infty e^{2s}\Big(\int_1^{\cosh s}(e^s-t)u^3(t,e^s-t)\,dt\Big)^2\,ds \\
&\qquad + \int_0^\infty e^{-2s}\Big(\int_1^{\cosh s}(t-e^{-s})u^3(t,t-e^{-s})\,dt\Big)^2\,ds. \qquad (5.51)
\end{aligned}$$

By Hölder's inequality and a change of variables,

$$\begin{aligned}
&\int_0^\infty e^{2s}\Big(\int_1^{\cosh s}(e^s-t)u^3(t,e^s-t)\,dt\Big)^2\,ds \\
&\quad\lesssim \int_0^\infty\int_1^{\cosh s} e^{3s}(e^s-t)^2u^6(t,e^s-t)\,dt\,ds \\
&\quad\lesssim \int_0^\infty\int_{t^2-r^2\le 1} u^6(t,r)r^4\,dt\,dr<\infty. \qquad (5.52)
\end{aligned}$$

The last inequality follows from global well-posedness of u, which implies $\|u\|_{L^4_{t,x}([1,3]\times\mathbf{R}^3)}<\infty$, (5.7), Strichartz estimates, and the radial Sobolev embedding theorem, which implies that

$$\big\||x|^{1/3}u\big\|_{L^6_{t,x}(\mathbf{R}\times\mathbf{R}^3)}\lesssim\big\||\nabla|^{1/6}u\big\|_{L^6_tL^3_x(\mathbf{R}\times\mathbf{R}^3)}. \qquad (5.53)$$

Also, by a change of variables and Hölder's inequality, since $(t-e^{-s})\gtrsim 1$ for $s\ge 1$ and $t\ge 1$,

$$\begin{aligned}
&\int_1^\infty e^{-2s}\Big(\int_1^{\cosh s}(t-e^{-s})u^3(t,t-e^{-s})\,dt\Big)^2\,ds \\
&\quad\lesssim \int_1^\infty\int_1^{\cosh s} e^{-s}(t-e^{-s})^2u^6(t,e^s-t)\,dt\,ds \\
&\quad\lesssim \int_0^\infty\int_{t^2-r^2\le 1} u^6(t,r)r^4\,dt\,dr<\infty. \qquad (5.54)
\end{aligned}$$

Finally, by the radial Sobolev embedding theorem, Young's inequality, and a change of variables,

$$\begin{aligned}
&\int_0^1 e^{-2s}\Big(\int_1^{\cosh s}(t-e^{-s})u^3(t,t-e^{-s})\,dt\Big)^2\,ds \\
&\quad\lesssim \int_1^3\Big(\int_{t^2-r^2\le 1} u(t,r)^6r^2\,dr\Big)^{1/2}\,dt \\
&\quad\lesssim \int_1^3\frac{1}{(t-1)^{3/4}}\,dt<\infty. \qquad (5.55)
\end{aligned}$$

Therefore,

$$\int \big(\partial_\tau (s\tilde{u}_{nl})|_{\tau=0}\big)^2 \, ds = \int \tilde{u}_\tau^2 s^2 \, ds|_{\tau=0} < \infty. \tag{5.56}$$

Integrating by parts,

$$\int_{s_0}^\infty u_s^2 s^2 \, ds = \int_{s_0}^\infty \big(\partial_s (su)\big)^2 \, ds + s_0 u(s_0)^2, \tag{5.57}$$

so taking $s_0 = 0$,

$$\int \tilde{u}_s^2 s^2 \, ds < \infty. \tag{5.58}$$

This shows that the contribution of the nonlinear term to (5.17) and (5.18) lies in $\dot{H}^1 \times L^2$.

Now consider the contribution u_l. First suppose that $u_1 = 0$ and

$$u_l = S(t-1)(u_0, 0). \tag{5.59}$$

Then by (5.49),

$$\begin{aligned} s\tilde{u}_l(\tau, s) &= e^\tau \sinh s \cdot u_l(e^\tau \cosh s, e^\tau \sinh s) \\ &= \frac{1}{2}\big[u_0(e^{\tau+s}-1)(e^{\tau+s}-1) + u_0(1-e^{\tau-s})(1-e^{\tau-s})\big]. \end{aligned} \tag{5.60}$$

Let $\chi \in C_0^\infty(\mathbf{R})$ be a partition of unity function satisfying

$$1 = \sum_{k \geq 0} \chi(s-k), \tag{5.61}$$

for any $s \in [0, \infty)$, and where $\chi(s-k)$ is supported on $(k-1) \cdot \ln(2) \leq s \leq (k+1) \cdot \ln(2)$. Split

$$\begin{aligned} \tilde{u}_l(\tau, s) = \tilde{u}_l^{(1)}(\tau, s) + \tilde{u}_l^{(2)}(\tau, s) + \tilde{u}_l^{(3)}(\tau, s) + \tilde{u}_l^{(4)}(\tau, s) \\ + \tilde{u}_l^{(5)}(\tau, s) + \tilde{u}_l^{(6)}(\tau, s), \end{aligned} \tag{5.62}$$

where

$$s\tilde{u}_l^{(1)}(\tau, s) = \sum_{k \geq 0} \chi(s-k)(P_{\leq -k} u_0)(e^{\tau+s}-1) \cdot (e^{\tau+s}-1),$$

$$s\tilde{u}_l^{(2)}(\tau, s) = P_{\geq 0} \sum_{k \geq 0} \chi(s-k)(P_{>-k} u_0)(e^{\tau+s}-1) \cdot (e^{\tau+s}-1),$$

$$
\begin{aligned}
s\tilde{u}_l^{(3)}(\tau,s) &= P_{\leq 0}\sum_{k\geq 0}\chi(s-k)(P_{>-k}u_0)(e^{\tau+s}-1)\cdot(e^{\tau+s}-1),\\
s\tilde{u}_l^{(4)}(\tau,s) &= \sum_{k\geq 0}\chi(s-k)(P_{\leq k}u_0)(1-e^{\tau-s})\cdot(1-e^{\tau-s}),\\
s\tilde{u}_l^{(5)}(\tau,s) &= P_{\leq 0}\sum_{k\geq 0}\chi(s-k)(P_{>k}u_0)(1-e^{\tau-s})\cdot(1-e^{\tau-s}),\\
s\tilde{u}_l^{(6)}(\tau,s) &= P_{\geq 0}\sum_{k\geq 0}\chi(s-k)(P_{>k}u_0)(1-e^{\tau-s})\cdot(1-e^{\tau-s}).
\end{aligned}
\tag{5.63}
$$

Remark

If g is a radial function in $\mathbf{R}^3$, then

$$
g\in \dot{H}^1(\mathbf{R}^3)\Leftrightarrow \int_0^\infty (g_r r)^2\,dr<\infty \tag{5.64}
$$

and

$$
g\in L^2(\mathbf{R}^3)\Leftrightarrow \int_0^\infty \big(g(r)r\big)^2\,dr<\infty. \tag{5.65}
$$

Taking the derivative,

$$
\begin{aligned}
s\partial_\tau(\tilde{u}_l^{(1)})|_{\tau=0} &= \partial_\tau(s\tilde{u}_l^{(1)})(\tau,s)|_{\tau=0}\\
&= \sum_{k\geq 0}\chi(s-k)(P_{\leq -k}u_0')(e^s-1)\cdot(e^s-1)e^s\\
&\quad+\sum_{k\geq 0}\chi(s-k)(P_{\leq -k}u_0)(e^s-1)\cdot e^s.
\end{aligned}
\tag{5.66}
$$

Then by a change of variables, Hardy's inequality, and Young's inequality,

$$
\begin{aligned}
\|(5.66)\|_{L^2[0,\infty)} &\lesssim \Big(\sum_{k\geq 0}2^k\Big(\sum_{j\leq -k}\big\|\chi(s-k)(P_j\nabla u_0)(e^s-1)\big\|_{L^2}\Big)^2\Big)^{1/2}\\
&\quad+\Big(\sum_{k\geq 0}2^k\Big(\Big\|\chi(s-k)\frac{1}{|x|}(P_ju_0)(e^s-1)\Big\|_{L^2}\Big)^2\Big)^{1/2}\\
&\lesssim \|u_0\|_{\dot{H}^{1/2}}.
\end{aligned}
\tag{5.67}
$$

The computation of $\partial_s(s\tilde{u}_l^{(1)}(\tau,s))|_{\tau=0}$ is similar, except that, in addition, it is necessary to compute

$$
\sum_k\big\|\chi'(s-k)(P_{\leq -k}u_0)(e^s-1)\cdot(e^s-1)\big\|_{L^2}^2. \tag{5.68}
$$

By a change of variables and Hardy's inequality,

$$(5.68) \lesssim \sum_{k\geq 0} 2^k \Big(\sum_{j\leq -k} \Big\| \chi'(s-k)\frac{1}{|x|}(P_j u_0)(e^s-1) \Big\|_{L^2} \Big)^2 \lesssim \|u_0\|_{\dot H^{1/2}}^2. \tag{5.69}$$

By the product rule,

$$s\partial_s \tilde u_l(\tau,s) = \partial_s\big(s\tilde u_l(\tau,s)\big) - \tilde u_l(\tau,s). \tag{5.70}$$

By the support properties of $\chi(s-k)$ and the Sobolev embedding theorem,

$$\Big\| \sum_{k\geq 0} \chi(s-k)(P_{\leq -k}u_0)(e^s-1)\cdot(e^s-1) \Big\|_{L^\infty} \lesssim \|u_0\|_{\dot H^{1/2}}, \tag{5.71}$$

and therefore,

$$\begin{aligned} \big\|\tilde u_l(\tau,s)|_{\tau=0}\big\|_{L^2([0,\infty)} &= \Big\| \frac{1}{s}\sum_{k\geq 2}\chi(s-k)(P_{\leq -k}u_0)(e^s-1)\cdot(e^s-1)\Big\|_{L^2([0,\infty)} \\ &\lesssim \Big(\int_1^\infty \frac{1}{s^2}\,ds\Big)^{1/2}\|u_0\|_{\dot H^{1/2}} \lesssim \|u_0\|_{\dot H^{1/2}}. \end{aligned} \tag{5.72}$$

Also, by the support properties of $\chi(s-k)$ and (5.71),

$$\Big\| \sum_{k=0,1}\chi(s-k)P_{\leq -k}u_0(e^s-1)\cdot\frac{(e^s-1)}{s}\Big\|_{L^2([0,\infty)} \lesssim \|u_0\|_{\dot H^{1/2}}. \tag{5.73}$$

Therefore, $\tilde u_l^{(1)}(\tau,s)|_{\tau=0}$ has finite energy.

Next, for any $k\geq 0$, $j>-k$, by the product rule and change of variables,

$$\begin{aligned} &\big\|\partial_\tau\big(\chi(s-k)(P_ju_0)(e^{s+\tau}-1)\cdot(e^{s+\tau}-1)\big)|_{\tau=0}\big\|_{L^2([0,\infty)} \\ &\lesssim \big\|\chi(s-k)(P_j\nabla u_0)(e^s-1)\cdot(e^s-1)e^s\big\|_{L^2([0,\infty)} \\ &\quad + \big\|\chi(s-k)(P_ju_0)(e^s-1)\cdot e^s\big\|_{L^2([0,\infty)} \\ &\lesssim 2^{k/2}\|P_j\nabla u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})} \\ &\quad + 2^{k/2}\Big\|\frac{1}{|x|}P_ju_0\Big\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}. \end{aligned} \tag{5.74}$$

Therefore, if $f\in \dot H^{1/2}(\mathbf{R}^3)$ is a radial function, then by Bernstein's inequality,

$$\begin{aligned} &\int_0^\infty \big(P_lf(s)\big)s\cdot\partial_\tau\big(\chi(s-k)(P_ju_0)(e^{s+\tau}-1)\cdot(e^{s+\tau}-1)\big)|_{\tau=0}\,ds \\ &\lesssim \|P_lf\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\Big[2^{k/2}\|P_j\nabla u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})} \\ &\quad + 2^{k/2}\Big\|\frac{1}{|x|}P_ju_0\Big\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\Big]. \end{aligned} \tag{5.75}$$

Summing up, by Young's inequality and Bernstein's inequality,

$$\sum_{l\geq j+k>0} \|P_l f\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\Big[2^{k/2}\|P_j\nabla u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})} + 2^{k/2}\Big\|\frac{1}{|x|}P_j u_0\Big\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\Big] \lesssim \|f\|_{\dot{H}^{1/2}}\|u_0\|_{\dot{H}^{1/2}}. \tag{5.76}$$

Next, by a change of variables,

$$\big\|\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\big\|_{L^2([0,\infty)} \lesssim 2^{-k/2}\|P_j u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}. \tag{5.77}$$

By the product rule,

$$\begin{aligned}\partial_\tau\big(\chi(s-k)(P_j u_0)(e^{s+\tau}-1)\cdot(e^{s+\tau}-1)\big)|_{\tau=0} &= \partial_s\big(\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\big)\\ &\quad -\chi'(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1).\end{aligned} \tag{5.78}$$

Integrating by parts,

$$\begin{aligned}&\int_0^\infty \big(P_l f(s)\big)s\cdot\partial_s\big(\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\big)\,ds\\ &\quad= -\int_0^\infty\big[\big(P_l\nabla f(s)\big)s+\big(P_l f(s)\big)\big]\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\,ds\\ &\quad\lesssim 2^{-k/2}\Big[\|P_l\nabla f\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\\ &\qquad+\Big\|\frac{1}{|x|}P_l f\Big\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\Big]\|P_j u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}.\end{aligned} \tag{5.79}$$

Summing up, by Bernstein's inequality,

$$\begin{aligned}&\sum_{0\leq l<j+k} 2^{-k/2}\Big[\|P_l\nabla f\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\\ &\qquad+\Big\|\frac{1}{|x|}P_l f\Big\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\Big]\|P_j u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\\ &\quad\lesssim \|f\|_{\dot{H}^{1/2}}\|u_0\|_{\dot{H}^{1/2}}.\end{aligned} \tag{5.80}$$

Also,

$$\begin{aligned}&\int_0^\infty \big(P_l f(s)\big)s\cdot\chi'(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\,ds\\ &\quad\lesssim \|P_l f\|_{L^2(2^{k-1}-1\leq s\leq 2^{k+1})}2^{-k/2}\|P_j u_0\|_{L^2(2^{k-1}-1\leq s\leq 2^{k+1})}.\end{aligned} \tag{5.81}$$

Then by Bernstein's inequality,

$$\sum_{0\leq l<j+k} 2^{-k/2}\|P_l f\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\|P_j u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})} \lesssim \|f\|_{\dot H^{1/2}}\|u_0\|_{\dot H^{1/2}}. \tag{5.82}$$

Therefore,

$$\big\|\partial_\tau\big(\tilde u_l^{(2)}(\tau,s)\big)|_{\tau=0}\big\|_{\dot H^{-1/2}(\mathbf{R}^3)} \lesssim \|u_0\|_{\dot H^{1/2}}. \tag{5.83}$$

The proof that $\|\tilde u_l^{(2)}(\tau,s)|_{\tau=0}\|_{\dot H^{1/2}}$ is bounded is quite similar. By the product rule and change of variables, compute

$$\begin{aligned}&\big\|\partial_s\big(\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\big)\big\|_{L^2}\\ &\quad\lesssim 2^{-k/2}\|P_j u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\\ &\qquad+2^{k/2}\|(P_j\nabla u_0)\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\\ &\qquad+2^{k/2}\Big\|\frac{1}{|x|}(P_j u_0)\Big\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}\end{aligned} \tag{5.84}$$

and

$$\begin{aligned}&\Big\|\frac{1}{s}\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\Big\|_{L^2}\\ &\qquad\lesssim 2^{-k/2}\|P_j u_0\|_{L^2(2^{k-1}-1\leq r\leq 2^{k+1})}.\end{aligned} \tag{5.85}$$

By Bernstein's inequality, Young's inequality, and the support properties of $\chi(s-k)$,

$$\begin{aligned}&\sum_l 2^l\Big\|P_l\Big(\sum_{l\leq k+j,k+j>0}\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\Big)\Big\|_{L^2}^2\\ &\quad\lesssim\sum_l 2^l\sum_k\Big(\sum_{l\leq k+j,k+j>0}\|\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\|_{L^2}\Big)^2\\ &\quad\lesssim\|u_0\|_{\dot H^{1/2}}^2.\end{aligned} \tag{5.86}$$

Also, by Bernstein's inequality and (5.84),

$$\begin{aligned}&\sum_l 2^l\Big\|P_l\Big(\sum_{0<k+j<l}\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\Big)\Big\|_{L^2}^2\\ &\quad\lesssim\sum_l 2^l\sum_k\Big(\sum_{0<k+j<l}\|\chi(s-k)(P_j u_0)(e^s-1)\cdot(e^s-1)\|_{L^2}\Big)^2\\ &\quad\lesssim\|u_0\|_{\dot H^{1/2}}^2.\end{aligned} \tag{5.87}$$

Therefore, we have proved that

$$\big\|\partial_\tau\big(\tilde{u}_l^{(2)}(\tau,s)\big)|_{\tau=0}\big\|_{\dot{H}^{-1/2}} + \big\|\tilde{u}_l^{(2)}(\tau,s)|_{\tau=0}\big\|_{\dot{H}^{1/2}} \lesssim \|u_0\|_{\dot{H}^{1/2}}. \tag{5.88}$$

Next, following (5.80)–(5.83) with P_l, $l \geq 0$ replaced by $P_{\leq 0}$ and $f \in L^2(\mathbf{R}^3)$,

$$\big\|\partial_\tau\big(\tilde{u}_l^{(3)}(\tau,s)\big)|_{\tau=0}\big\|_{L^2} + \big\|\tilde{u}_l^{(3)}(\tau,s)|_{\tau=0}\big\|_{\dot{H}^1} \lesssim \|u_0\|_{\dot{H}^{1/2}}. \tag{5.89}$$

Next consider $\tilde{u}_l^{(4)}(\tau,s)$. By the product rule,

$$\begin{aligned}\partial_\tau\big(s\tilde{u}_l^{(4)}(\tau,s)\big)|_{\tau=0} = &-\sum_{k\geq 0}\chi(s-k)(P_{\leq k}\nabla u_0)(1-e^{-s})\cdot(1-e^{-s})e^{-s}\\ &-\sum_{k\leq 0}\chi(s-k)(P_{\leq k}u_0)(1-e^{-s})e^{-s}.\end{aligned} \tag{5.90}$$

Then, by Young's inequality,

$$\begin{aligned}&\big\|\partial_\tau\big(s\tilde{u}_l^{(4)}(\tau,s)\big)|_{\tau=0}\big\|_{L^2([0,\infty))}\\ &\lesssim \sum_{k\geq 0}2^{-k}\Big(\sum_{j\leq k}\|\nabla P_j u_0\|_{L^2(1-2^{-k-1}\leq r\leq 1-2^{-k+1})}\Big)^2\\ &\quad+\sum_{k\geq 0}2^{-k}\Big(\sum_{j\leq k}\Big\|\frac{1}{|x|}P_j u_0\Big\|_{L^2(1-2^{-k-1}\leq r\leq 1-2^{-k+1})}\Big)^2\\ &\lesssim \|u_0\|^2_{\dot{H}^{1/2}}.\end{aligned} \tag{5.91}$$

Also, by the product rule,

$$\begin{aligned}\partial_s\big(s\tilde{u}_l^{(4)}(\tau,s)\big) = &-\partial_\tau\big(s\tilde{u}_l^{(4)}(\tau,s)\big)\\ &+\sum_{k\geq 0}\chi'(s-k)(P_k u_0)(1-e^{-s})\cdot(1-e^{-s}).\end{aligned} \tag{5.92}$$

Then by the finite overlapping property of $\chi(s-k)$ and the radial Sobolev embedding theorem,

$$\begin{aligned}&\Big\|\sum_{k\geq 0}\chi'(s-k)(P_k u_0)(1-e^{-s})\cdot(1-e^{-s})\Big\|^2_{L^2([0,\infty)}\\ &\qquad\lesssim \sum_{k\geq 0}\|P_k u_0\|^2_{\dot{H}^{1/2}} \lesssim \|u_0\|^2_{\dot{H}^{1/2}}.\end{aligned} \tag{5.93}$$

Therefore,

$$\left\| \partial_s \big(s\tilde{u}_l^{(4)}(\tau, s) \big)|_{\tau=0} \right\|_{L^2([0,\infty)} + \left\| \partial_\tau \big(s\tilde{u}_l^{(4)}(\tau, s) \big)|_{\tau=0} \right\|_{L^2([0,\infty)} \lesssim \|u_0\|_{\dot{H}^{1/2}}. \tag{5.94}$$

Next, by a change of variables,

$$\left\| \chi(s-k)(P_j u_0)(1-e^{-s}) \cdot (1-e^{-s}) \right\|_{L^2} \lesssim 2^{k/2} \|P_j u_0\|_{L^2(1-2^{-k-1} \leq r \leq 1-2^{-k+1})}. \tag{5.95}$$

Therefore, by Young's inequality,

$$\left\| s\tilde{u}_l^5(\tau, s)|_{\tau=0} \right\|^2_{L^2([0,\infty)} \lesssim \sum_{k \geq 0} 2^k \Big(\sum_{j>k} \|P_j u_0\|_{L^2(1-2^{-k-1} \leq r \leq 1-2^{-k+1})} \Big)^2 \lesssim \|u_0\|^2_{\dot{H}^{1/2}}. \tag{5.96}$$

Therefore, by the Fourier support of $\tilde{u}_l^{(5)}$,

$$\left\| \tilde{u}_l^{(5)}(\tau, s)|_{\tau=0} \right\|_{\dot{H}^1(\mathbf{R}^3)} \lesssim \|u_0\|_{\dot{H}^{1/2}(\mathbf{R}^3)}. \tag{5.97}$$

Also, if $f \in L^2$ and f is supported on $|\xi| \leq 1$, then

$$\begin{aligned} &\int_0^\infty f(s)s \cdot \partial_\tau \big(s\tilde{u}_l^{(5)}(\tau, s) \big)|_{\tau=0}\, ds \\ &\quad = -\int_0^\infty f(s)s \cdot \partial_s \big(s\tilde{u}_l(\tau, s) \big)|_{\tau=0}\, ds \\ &\quad\quad - \int_0^\infty f(s)s \cdot \sum_{k \geq 0} \chi'(s-k)(P_{\geq k} u_0)(1-e^{-s}) \cdot (1-e^{-s})\, ds. \end{aligned} \tag{5.98}$$

Integrating by parts, by (5.96),

$$-\int_0^\infty f(s)s \cdot \partial_s \big(s\tilde{u}_l^{(5)}(\tau, s) \big)|_{\tau=0}\, ds = \int_0^\infty \partial_s \big(f(s)s \big) \cdot s\tilde{u}_l^{(5)}(\tau, s)|_{\tau=0}\, ds \lesssim \|f\|_{L^2} \|u_0\|_{\dot{H}^{1/2}}. \tag{5.99}$$

Also, by (5.96),

$$\int_0^\infty f(s)s \cdot \sum_{k \geq 0} \chi'(s-k)(P_{\geq k} u_0)(1-e^{-s}) \cdot (1-e^{-s})\, ds \lesssim \|f\|_{L^2} \|u_0\|_{\dot{H}^{1/2}}. \tag{5.100}$$

Therefore,

$$\|\partial_\tau(s\tilde{u}_l^{(5)}(\tau,s))|_{\tau=0}\|_{L^2([0,\infty)} + \|\partial_s(s\tilde{u}_l^{(5)}(\tau,s))|_{\tau=0}\|_{L^2([0,\infty)} \lesssim \|u_0\|_{\dot{H}^{1/2}}. \tag{5.101}$$

Finally, take $\tilde{u}_l^{(6)}(\tau,s)$. Take $f \in \dot{H}^{1/2}$ supported in Fourier space on $|\xi| \geq 1$. Then by the product rule and (5.96),

$$\begin{aligned}&\|\partial_s(\chi(s-k)(P_j u_0)(1-e^{-s})\cdot(1-e^{-s}))\|_{L^2([0,\infty)}\\ &\quad\lesssim 2^{k/2}\|P_j u_0\|_{L^2(1-2^{-k-1}\leq r\leq 1-2^{-k+1})}\\ &\qquad + 2^{-k/2}\|P_j\nabla u_0\|_{L^2(1-2^{-k-1}\leq r\leq 1-2^{-k+1})}\\ &\qquad + 2^{-k/2}\left\|\frac{1}{|x|}P_j u_0\right\|_{L^2(1-2^{-k-1}\leq r\leq 1-2^{-k+1})}.\end{aligned} \tag{5.102}$$

Also, by (5.96) and (5.95),

$$\left\|\frac{1}{s}\chi(s-k)(P_j u_0)(1-e^{-s})\cdot(1-e^{-s})\right\|_{L^2([0,\infty)} \lesssim 2^{k/2}\|P_j u_0\|_{L^2(1-2^{-k-1}\leq r\leq 1-2^{-k+1})}. \tag{5.103}$$

Therefore, by Young's inequality,

$$\sum_{l<j+k} 2^l \sum_k \Big(\sum_{j>k}\|\chi(s-k)(P_j u_0)(1-e^{-s})\cdot(1-e^{-s})\|_{L^2}\Big)^2 \lesssim \|u_0\|_{\dot{H}^{1/2}}^2. \tag{5.104}$$

Also, by Bernstein's inequality,

$$\sum_{l\geq j+k} 2^{-l} \sum_k \Big(\sum_{j>k}\|\partial_s(\chi(s-k)(P_j u_0)(1-e^{-s})\cdot(1-e^{-s}))\|_{L^2}\Big)^2 \lesssim \|u_0\|_{\dot{H}^{1/2}}^2. \tag{5.105}$$

Therefore, we have finally proved that if $u_1 = 0$, then

$$\tilde{u}_l(\tau,s)|_{\tau=0} \in \dot{H}^{1/2}(\mathbf{R}^3) + \dot{H}^1(\mathbf{R}^3) \tag{5.106}$$

and

$$\partial_\tau(\tilde{u}_l(\tau,s))|_{\tau=0} \in \dot{H}^{-1/2}(\mathbf{R}^3) + L^2(\mathbf{R}^3). \tag{5.107}$$

To compute the contribution to $\tilde{u}_l(\tau,s)$ of

$$S(t)(0,u_1), \tag{5.108}$$

observe that

$$\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f = \partial_t \Big(\frac{\cos(t\sqrt{-\Delta})}{\Delta} f \Big). \tag{5.109}$$

Plugging the formula for a solution into to the wave equation when $r > t$, let $w(t, r) = \cos(t\sqrt{-\Delta}) f$. Then,

$$\begin{aligned} \partial_t \big(w(t, r) \big) &= \frac{1}{2r} \partial_t \big(f(t + r)(t + r) + f(r - t)(r - t) \big) \\ &= \frac{1}{2r} \big[f(t + r) + f'(t + r)(t + r) \\ &\quad - f(r - t) - f'(r - t)(r - t) \big]. \end{aligned} \tag{5.110}$$

Then decompose $\tilde{u}_l(\tau, s) = \tilde{u}_l^{(1)}(\tau, s) + \tilde{u}_l^{(2)}(\tau, s) + \tilde{u}_l^{(3)}(\tau, s)$, where

$$\begin{aligned} s\tilde{u}_l^{(1)}(\tau, s) &= \frac{1}{2} \big[f'(e^{\tau+s} - 1) \cdot (e^{\tau+s} - 1) - f'(1 - e^{\tau-s}) \cdot (1 - e^{\tau-s}) \big], \\ s\tilde{u}_l^{(2)}(\tau, s) &= \frac{1}{2} \big(1 - \chi(s) \big) \big[f(e^{\tau+s} - 1) - f(1 - e^{\tau-s}) \big], \\ s\tilde{u}_l^{(3)}(\tau, s) &= \frac{1}{2} \chi(s) \big[f(e^{\tau+s} - 1) - f(1 - e^{\tau-s}) \big]. \end{aligned} \tag{5.111}$$

Since

$$f = \frac{u_1}{\Delta} \in \dot{H}^{3/2}(\mathbf{R}^3), \tag{5.112}$$

the contribution of

$$f'(e^{\tau+s} - 1) \cdot (e^{\tau+s} - 1) - f'(1 - e^{\tau-s}) \cdot (1 - e^{\tau-s}) \tag{5.113}$$

to

$$\big(\tilde{u}_l(\tau, s)|_{\tau=0}, \partial_\tau \tilde{u}_l(\tau, s)|_{\tau=0} \big) \tag{5.114}$$

may be analyzed in exactly the same manner as the contribution of $S(t)(u_1, 0)$. Therefore,

$$\tilde{u}_l^{(1)}(\tau, s)|_{\tau=0} \in \dot{H}^{1/2} + \dot{H}^1 \tag{5.115}$$

and

$$\partial_\tau \big(\tilde{u}_l^{(1)}(\tau, s) \big)|_{\tau=0} \in \dot{H}^{-1/2} + L^2. \tag{5.116}$$

Next take $\tilde{u}_l^{(2)}(\tau, s)$. By a change of variables,

$$\int_1^\infty \big(\partial_s f(e^s-1)\big)^2\,ds = \int_1^\infty \big(f'(e^s-1)\cdot e^s\big)^2\,ds \lesssim \int |f'(r)|^2 r\,dr \lesssim \|f\|_{\dot H^{3/2}(\mathbf{R}^3)}^2 \tag{5.117}$$

and

$$\int_1^\infty \big(\partial_s f(1-e^{-s})\big)^2\,ds = \int_1^\infty \big(f'(1-e^{-s})\cdot e^{-s}\big)^2\,ds \lesssim |f'(r)|^2 r\,dr \lesssim \|f\|_{\dot H^{3/2}(\mathbf{R}^3)}^2. \tag{5.118}$$

By an identical calculation,

$$\int_1^\infty \big(\partial_\tau f(e^{s+\tau}-1)|_{\tau=0}\big)^2\,ds = \int_1^\infty \big(f'(e^s-1)\cdot e^s\big)^2\,ds \lesssim \int |f'(r)|^2 r\,dr \lesssim \|f\|_{\dot H^{3/2}(\mathbf{R}^3)}^2 \tag{5.119}$$

and

$$\int_1^\infty \big(\partial_s f(1-e^{\tau-s})|_{\tau=0}\big)^2\,ds = \int_1^\infty \big(f'(1-e^{-s})\cdot e^{-s}\big)^2\,ds \lesssim \int |f'(r)|^2 r\,dr \lesssim \|f\|_{\dot H^{3/2}(\mathbf{R}^3)}^2. \tag{5.120}$$

Next, by the fundamental theorem of calculus, for $s_0 \sim 1$,

$$s_0\big[f(e^{s_0}-1)-f(1-e^{-s_0})\big]^2 = s_0\Big[\int_{1-e^{-s_0}}^{e^{s_0}-1} f'(r)\,dr\Big]^2 \lesssim \int |f'(r)|^2 r\,dr \lesssim \|f\|_{\dot H^{3/2}}^2. \tag{5.121}$$

Therefore, by (5.118) and (5.119),

$$\big\|\partial_\tau\big(\tilde u_l^{(2)}(\tau,s)\big)|_{\tau=0}\big\|_{L^2} \lesssim \|f\|_{\dot H^{3/2}} \tag{5.122}$$

and

$$\big\|\tilde u_l^{(2)}(0,s)\big\|_{\dot H^1} \lesssim \|f\|_{\dot H^{3/2}}. \tag{5.123}$$

Finally, consider

$$f(e^{\tau+s}-1)-f(1-e^{\tau-s}) \tag{5.124}$$

when $s<1$. By direct computation,

$$\partial_\tau\big[f(e^{\tau+s}-1)-f(1-e^{\tau-s})\big]|_{\tau=0}$$
$$= f'(e^s-1)\cdot e^s + f'(1-e^{-s})\cdot e^{-s}. \tag{5.125}$$

Then for $g \in \dot{H}^{1/2}$, by Hardy's inequality,

$$\int f'(e^s-1)\cdot e^s\cdot g(s)s\,ds + \int f'(1-e^{-s})\cdot e^{-s}\cdot g(s)s\,ds$$
$$\lesssim \|f\|_{\dot{H}^{3/2}}\|g\|_{\dot{H}^{1/2}}. \tag{5.126}$$

Also, by the fundamental theorem of calculus,

$$f(e^s-1)-f(1-e^{-s})$$
$$= \int_{s-\frac{s^2}{2}+\frac{s^3}{3!}-\cdots}^{s+\frac{s^2}{2}+\frac{s^3}{3!}+\cdots} f'(r)\,dr$$
$$= \int_0^1 f'\Big(s+\theta\Big(\frac{s^2}{2}+\frac{s^3}{3!}+\cdots\Big)\Big)\cdot\Big(\frac{s^2}{2}+\frac{s^3}{3!}+\cdots\Big)\,d\theta$$
$$+\int_{-1}^0 f'(s+\theta\Big(\frac{s^2}{2}-\frac{s^3}{3!}+\cdots\Big)\cdot\Big(\frac{s^2}{2}+\frac{s^3}{3!}+\cdots\Big)\,d\theta. \tag{5.127}$$

Therefore, since $\chi(s)$ is supported on $s \le 1$,

$$\|f(e^s-1)-f(1-e^{-s})\|_{\dot{H}^{1/2}} \lesssim \|f\|_{\dot{H}^{3/2}}. \tag{5.128}$$

This proves that

$$\big\|\tilde{u}_l^{(3)}(\tau,s)|_{\tau=0}\big\|_{\dot{H}^{1/2}} + \big\|\partial_\tau\tilde{u}_l^{(3)}(\tau,s)|_{\tau=0}\big\|_{\dot{H}^{-1/2}} \lesssim \|f\|_{\dot{H}^{3/2}}. \tag{5.129}$$

This finally completes the proof of Lemma 5.2. □

6. Proof of Theorem 1.4

To complete the proof of Theorem 1.4, it remains to prove a bound on the scattering size of a solution to (1.1) that depends only on the size of the initial data. Previous work has only shown that for any $(u_0,u_1)\in \dot{H}^{1/2}\times\dot{H}^{-1/2}$, (1.1) has a global solution that scatters both forward and backward in time. However, this fact does not preclude the existence of some $A<\infty$ for which $\|u\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)}$ may be arbitrarily large for $\|u_0\|_{\dot{H}^{1/2}}+\|u_1\|_{\dot{H}^{-1/2}}\le A$.

To preclude this possibility and prove (1.12), it suffices to prove that if (u_n^0,u_n^1) is a sequence of initial data

$$\|u_0^n\|_{\dot{H}^{1/2}}+\|u_1^n\|_{\dot{H}^{-1/2}}\le A<\infty, \tag{6.1}$$

then

$$\|u^n\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \tag{6.2}$$

is uniformly bounded, where u^n is the solution to (1.1) with initial data (u_0^n, u_1^n).

Remark
Observe that this gives no quantitative bound on (1.12).

To prove this, make a profile decomposition.

THEOREM 6.1 (Profile decomposition)
Suppose that there is a uniformly bounded, radially symmetric sequence

$$\|u_0^n\|_{\dot{H}^{1/2}(\mathbf{R}^3)} + \|u_1^n\|_{\dot{H}^{-1/2}(\mathbf{R}^3)} \leq A < \infty. \tag{6.3}$$

Then there exists a subsequence, also denoted $(u_0^n, u_1^n) \subset \dot{H}^{1/2} \times \dot{H}^{-1/2}$*, such that for any* $N < \infty$*,*

$$S(t)(u_0^n, u_1^n) = \sum_{j=1}^{N} \Gamma_n^j S(t)(\phi_0^j, \phi_1^j) + S(t)(R_{0,n}^N, R_{1,n}^N), \tag{6.4}$$

with

$$\lim_{N\to\infty} \limsup_{n\to\infty} \big\| S(t)(R_{0,n}^N, R_{1,n}^N) \big\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} = 0. \tag{6.5}$$

$\Gamma_n^j = (\lambda_n^j, t_n^j)$ *belongs to the group* $(0,\infty)\times\mathbf{R}$*, which acts by*

$$\Gamma_n^j F(t,x) = \lambda_n^j F\big(\lambda_n^j (t - t_n^j), \lambda_n^j x\big). \tag{6.6}$$

The Γ_n^j*'s are pairwise orthogonal; that is, for every* $j \neq k$*,*

$$\lim_{n\to\infty} \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + (\lambda_n^j)^{1/2}(\lambda_n^k)^{1/2}|t_n^j - t_n^k| = \infty. \tag{6.7}$$

Furthermore, for every $N \geq 1$*,*

$$\begin{aligned} \big\|(u_{0,n}, u_{1,n})\big\|^2_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} = {} & \sum_{j=1}^{N} \big\|(\phi_0^j, \phi_0^k)\big\|^2_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \\ & + \big\|(R_{0,n}^N, R_{1,n}^N)\big\|^2_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} + o_n(1). \end{aligned} \tag{6.8}$$

In the course of proving Theorem 6.1, [18] proved that

$$S(\lambda_n^j t_n^j)\Big(\frac{1}{\lambda_n^j}u_0^n\Big(\frac{x}{\lambda_n^j}\Big), \frac{1}{(\lambda_n^j)^2}u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big) \rightharpoonup \phi_0^j(x) \tag{6.9}$$

weakly in $\dot{H}^{1/2}(\mathbf{R}^3)$ and that

$$\partial_t S(t+\lambda_n^j t_n^j)\Big(\frac{1}{\lambda_n^j}u_0^n\Big(\frac{x}{\lambda_n^j}\Big), \frac{1}{(\lambda_n^j)^2}u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\Big|_{t=0} \rightharpoonup \phi_1^j(x) \tag{6.10}$$

weakly in $\dot{H}^{-1/2}(\mathbf{R}^3)$.

First suppose that $\lambda_n^j t_n^j$ is uniformly bounded. Then after passing to a subsequence, $\lambda_n^j t_n^j$ converges to some t^j. Changing (ϕ_0^j, ϕ_1^j) to $S(-t^j)(\phi_0^j, \phi_1^j)$ and absorbing the error into $(R_{0,n}^N, R_{1,n}^N)$,

$$\Big(\frac{1}{\lambda_n^j}u_0^n\Big(\frac{x}{\lambda_n^j}\Big), \frac{1}{(\lambda_n^j)^2}u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big) \rightharpoonup \phi_0^j(x) \tag{6.11}$$

and

$$\partial_t S(t)\Big(\frac{1}{\lambda_n^j}u_0^n\Big(\frac{x}{\lambda_n^j}\Big), \frac{1}{(\lambda_n^j)^2}u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\Big|_{t=0} \rightharpoonup \phi_1^j(x). \tag{6.12}$$

Then, if u^j is the solution to (1.1) with initial data (ϕ_0^j, ϕ_1^j),

$$\|u^j\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \leq M_j. \tag{6.13}$$

Next, suppose that after passing to a subsequence, $\lambda_n^j t_n^j \nearrow +\infty$. In this case, Theorem 5.1 also implies that for any $(\phi_0, \phi_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, there exists a solution u to (1.1) that is globally well-posed and scattering, and furthermore, that u scatters to $S(t)(\phi_0, \phi_1)$ as $t \searrow -\infty$:

$$\lim_{t\to-\infty} \big\| u - S(t)(\phi_0, \phi_1) \big\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} = 0. \tag{6.14}$$

Indeed, by Strichartz estimates, the dominated convergence theorem, and small data arguments, for some $T < \infty$ sufficiently large, (1.1) has a solution u on $(-\infty, -T]$ such that

$$\|u\|_{L^4_{t,x}((-\infty,-T]\times\mathbf{R}^3)} \lesssim \epsilon, \qquad \big(u(-T,x), u_t(-T,x)\big) = S(-T)(\phi_0, \phi_1). \tag{6.15}$$

and by Strichartz estimates,

$$\lim_{t\to+\infty} \big\| S(t)\big(u(-t), u_t(-t)\big) - (\phi_0, \phi_1) \big\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \lesssim \epsilon^3. \tag{6.16}$$

Then by the inverse function theorem, there exists some $(u_0(-T), u_1(-T))$ such that (1.1) has a solution that scatters backward in time to $S(t)(\phi_0, \phi_1)$, and by Theorem 5.1, this solution must also scatter forward in time. Therefore,

$$S(-t_n^j)\big(\lambda_n^j \phi_0^j(\lambda_n^j x), (\lambda_n^j)^2 \phi_1^j(\lambda_n^j x)\big) \tag{6.17}$$

converges strongly to

$$\big(\lambda_n^j u^j(-\lambda_n^j t_n^j, \lambda_n^j x), (\lambda_n^j)^2 u_t^j(-\lambda_n^j t_n^j, \lambda_n^j x)\big) \tag{6.18}$$

in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, where u^j is the solution to (1.1) that scatters backward in time to $S(t)(\phi_0^j, \phi_1^j)$, and the remainder may be absorbed into $(R_{0,n}^N, R_{1,n}^N)$. In this case as well, for some M_j,

$$\|u^j\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \leq M_j < \infty. \tag{6.19}$$

The proof for $\lambda_n^j t_n^j \searrow -\infty$ is similar.

By (6.8), there are only finitely many j's, say, J, such that $\|\phi_0^j\|_{\dot{H}^{1/2}} + \|\phi_1^j\|_{\dot{H}^{-1/2}} > \epsilon$. For all other j's, small data arguments imply that

$$\|u^j\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \lesssim \|\phi_0^j\|_{\dot{H}^{1/2}} + \|\phi_1^j\|_{\dot{H}^{-1/2}}. \tag{6.20}$$

Then by the decoupling property (6.7), (6.8), (6.13), (6.20), and Lemma 2.2,

$$\limsup_{n\nearrow\infty} \|u^n\|^2_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \lesssim \sum_j \|u^j\|^2_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \lesssim \sum_{j=1}^J M_j^2 + A^2 < \infty. \tag{6.21}$$

This completes the proof of Theorem 1.4. □

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