Path-Connectivity of Fréchet Spaces of Graphs

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Abstract

We examine topological properties of spaces of paths and graphs mapped to $\mathbb{R}^d$ under the Fréchet distance. We show that these spaces are path-connected if the map is either continuous or an immersion. If the map is an embedding, we show that the space of paths is path-connected, while the space of graphs only maintains this property in dimensions four or higher.

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1 Introduction

Motivated by the ubiquitous nature of one-dimensional data in a Euclidean ambient space (road networks in $\mathbb{R}^2$, for example), we investigate spaces of paths and graphs in $\mathbb{R}^d$. In particular, we examine these spaces in relation to the Fréchet distance, which is widely studied in the computational geometry literature [1–3, 5–7]. We work with three classes of paths: the set $\Pi_\mathcal{C}$ of all paths continuously mapped into $\mathbb{R}^d$, the set $\Pi_\mathcal{E}$ of paths embedded in $\mathbb{R}^d$, and the set $\Pi_\mathcal{I}$ of paths immersed in $\mathbb{R}^d$. In addition, we study three analogous spaces of graphs: the set $\mathcal{G}_\mathcal{C}$ of all graphs continuously mapped into $\mathbb{R}^d$, the set $\mathcal{G}_\mathcal{E}$ of graphs embedded in $\mathbb{R}^d$ and the set $\mathcal{G}_\mathcal{I}$ of graphs immersed in $\mathbb{R}^d$. See Figure 1 for examples of

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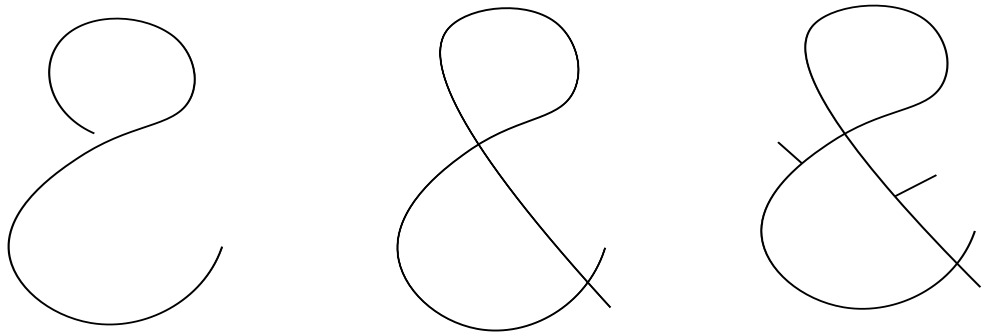


Figure 1 The images of an element in $\Pi_{\mathcal{E}}$, $\Pi_{\mathcal{I}}$, and $\Pi_{\mathcal{C}}$ respectively, mapped in $\mathbb{R}^2$.

paths in $\mathbb{R}^2$. We then topologize these sets using the open ball topology under the Fréchet distance, and study their path-connectedness property.

2 Background

We begin by defining the standard Fréchet distance for paths, adapting the definition from Alt and Godau [1]. Let $\alpha_0, \alpha_1 \in \Pi_{\mathcal{C}}$. The Fréchet distance between α_0 and α_1 is defined as:

$$d_{FP}(\alpha_0, \alpha_1) := \min_{r\colon [0,1]\to[0,1]} \max_{t\in[0,1]} |\alpha_0(t) - \alpha_1(r(t))|$$

Where r ranges over all reparameterizations of the unit interval (that is, homeomorphisms such that $r(0) = 0$ and $r(1) = 1$), and $|\cdot|$ denotes the standard Euclidean norm.

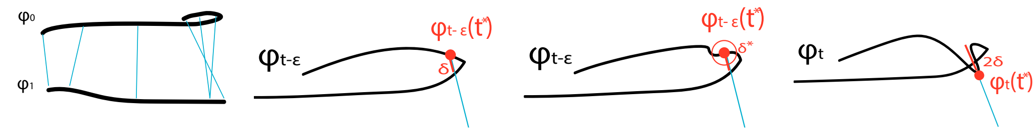
We now define the Fréchet distance for graphs, inspired by the Fréchet distance among paths. Let G be a one-dimensional simplicial complex, and let $\phi, \psi\colon G \to \mathbb{R}^d$ be continuous, rectifiable maps. Given any homeomorphism $h\colon G \to G$, we say that the *induced L_∞ distance* between the maps ϕ and $\psi \circ h$ is $||\phi - \psi \circ h||_\infty = \max_{x\in G} |\phi(x) - \psi(h(x))|$. With this distance in hand, we define the Fréchet distance between (G, ϕ) and (G, ψ) by minimizing over all homeomorphisms:[1]

$$d_{FG}\left((G, \phi), (G, \psi)\right) := \min_h ||\phi - \psi \circ h||_\infty$$

We now define and provide context for the underlying spaces that are studied in this work. Recall from above that $\Pi_{\mathcal{C}}$ denotes the set of all continuous mappings $\alpha : [0, 1] \to \mathbb{R}^d$. The set $\Pi_{\mathcal{E}}$ of embedded paths in $\mathbb{R}^d$ results from further specifying that α is injective, and the set $\Pi_{\mathcal{I}}$ of immersed paths in $\mathbb{R}^d$ results from requiring only local injectivity of α. Note that $\Pi_{\mathcal{E}} \subsetneq \Pi_{\mathcal{I}} \subsetneq \Pi_{\mathcal{C}}$ and elements of $\Pi_{\mathcal{C}}, \Pi_{\mathcal{E}}$, and $\Pi_{\mathcal{I}}$ are deemed equivalent if the image of their underlying map α is equivalent, giving a path-Fréchet distance (denoted d_{FP}) of zero.

We define the analogous spaces of graphs, letting G be a one-dimensional simplicial complex and $\mathcal{G}_{\mathcal{C}}(G)$ denote the set of all continous mappings $\phi : G \to \mathbb{R}^d$. Similarly, we define the set of embeddings $\mathcal{G}_{\mathcal{E}}(G)$ with the added requirement that ϕ be injective, and the set of

[1] Other generalizations of the Fréchet distance minimize over all "orientation-preserving" homeomorphisms, which can be defined in several ways for stratified spaces. We drop this requirement in our definition.



(a) Interpolate **(b)** δ from crossing **(c)** Inflate δ^*-nbhd **(d)** Self-cross by 2δ

Figure 2 The sequence of moves to continuously conduct self crossings in $\Pi_{\mathcal{I}}$.

immersions $\mathcal{G}_{\mathcal{I}}$ with the requirement that ϕ need be only locally injective. Note that elements of $\mathcal{G}_{\mathcal{C}}, \mathcal{G}_{\mathcal{I}}$, and $\mathcal{G}_{\mathcal{E}}$ are equivalent (with graph Fréchet distance zero) if their underlying graphs belong to the same homeomorphism class, and if the image of their accompanying map ϕ is equivalent.

3 Results

▶ **Theorem 1** (Continuous Mappings). *The topological spaces of continuous mappings of paths* $(\Pi_{\mathcal{C}}, d_{FP})$ *and continuous mappings of graphs* $(\mathcal{G}_{\mathcal{C}}(G), d_{FG})$ *in* $\mathbb{R}^d$ *are path-connected.*

Proof Sketch. Let $\phi_0, \phi_1 \in \Pi_{\mathcal{C}}$. Naively, a path may be constructed from ϕ_0 to ϕ_1 by interpolating ϕ_0 to ϕ_1 along the pointwise matchings (so-called leashes) defining $d_{FP}(\phi_0, \phi_1)$. The same technique may be extended to demonstrate the path-connectivity of $\mathcal{G}_{\mathcal{C}}(G)$. ◀

▶ **Theorem 2** (Immersions). *The topological spaces of immersions of paths* $(\Pi_{\mathcal{I}}, d_{FP})$ *and immersions of graphs* $(\mathcal{G}_{\mathcal{I}}(G), d_{FG})$ *in* $\mathbb{R}^d$ *are path-connected.*

Proof Sketch. Let $\phi_0, \phi_1 \in \Pi_{\mathcal{I}}$, and construct a path $\Gamma : [0,1] \to \Pi_{\mathcal{I}}$ as in Theorem 1 by interpolating ϕ_0 to ϕ_1 along the pointwise matchings defining $d_{FP}(\phi_0, \phi_1)$. We next show that this is well defined. Suppose not, then, at some $t \in [0,1]$, $\phi_t = \Gamma(t)$ could create an intersection not present in ϕ_0. This may collapse an entire region of the image of ϕ_t, rendering ϕ_t no longer an immersion. Then, there exists $\epsilon > 0$ such that $\Gamma(t-\epsilon) = \phi_{t-\epsilon}$ has $t^* \in [0,1]$ where $\phi_{t-\epsilon}(t^*)$ is $\delta > 0$ away from a new self-intersection, and t^* comes sufficiently close to minimizing δ. At this time $t-\epsilon$, suspend interpolation along all leashes, and continuously inflate a small δ^*-neighborhood $\phi_{t-\epsilon}|_{(t^*-\delta^*, t^*+\delta^*)}$ about the point $\phi_{t-\epsilon}(t^*)$ in the image of $\phi_{t-\epsilon}$ so that the leash lengths for every point in the δ^*-neighborhood equal the leash length defined at $\phi_{t-\epsilon}(t^*)$. Then directly perturb $\phi_{t-\epsilon}(t^*)$ by 2δ along its unique leash such that the crossing at $\phi_{t-\epsilon}(t^*)$ occurs, and the crossing point defined by t^* again lies δ away from a self intersection, and 2δ away from its original position in the final image of ϕ_t. See Figure 2. Repeat the process for any subsequent crossings in the interpolation. An analogous path can be constructed for graphs. ◀

▶ **Theorem 3** (Path Embeddings). *The space* $(\Pi_{\mathcal{E}}, d_{FP})$ *is path-connected.*

Proof Sketch. Let $\phi_0, \phi_1 \in \Pi_{\mathcal{E}}$. There exists a canonical path from ϕ_0 to ϕ_1 by condensing each map toward its center until the images are "nearly straight", continuously mapping each image to a straight segment, and then interpolating as in Theorem 1. ◀

▶ **Theorem 4** (Graph Embeddings). *The topological space of graphs* $(\mathcal{G}_{\mathcal{E}}(G), d_{FG})$ *embedded in* $\mathbb{R}^d$ *is path-connected if* $d \geq 4$.

Proof Sketch. Examining the path-connectivity of $\mathcal{G}_\mathcal{E}$ under the Fréchet distance reduces to a knot theory problem for $d \leq 3$. For $d \geq 4$, there exists a sequence of Reidemeister moves from any tame knot to another. Hence, if $\phi_0, \phi_1 \in \mathcal{G}_\mathcal{E}$, we construct a path by interpolating along the pointwise matchings between ϕ_0 and ϕ_1 as in Theorem 1. If a self intersection would be created, we suspend interpolation elsewhere and conduct the corresponding Reidemeister move. Repeat the process for all intersections thereafter, until attaining the image of ϕ_1. ◀

▶ **Corollary 5** (Path-Connectivity of Metric Balls). *Metric balls in the space $\Pi_\mathcal{C}, \mathcal{G}_\mathcal{C}(G), \Pi_\mathcal{I}$, and $\mathcal{G}_\mathcal{I}(G)$ are path-connected.*

Proof Sketch. Note that the techniques used in Theorem 1 and Theorem 2 never strictly increase the Frechet distance among two images of corresponding maps, so metric balls in each space are path-connected. For Theorem 2 this relies on the inflation step in Figure 2c, which assures that the Fréchet distance is fixed during a crossing event. The paths constructed in Theorem 3 and Theorem 4 do not necessarily maintain this property. ◀

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