A framework for analyzing students' reasoning about equivalence across undergraduate mathematics

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Establishing and leveraging equivalence is a central practice in mathematics. Though there have been many studies of students' uses of equivalence, much of the research thus far has been domain-specific, and the literature generally lacks coherence within and across mathematical domains. In this theoretical paper, we propose an initial unifying framework for capturing the different ways that students might establish equivalence. Using constructs born out of the K-12 literature, we discuss how this framework can be applied to student reasoning in undergraduate settings. We do so by presenting the results of conceptual analyses of students' possible uses of equivalence when thinking about vectors, isomorphisms and homeomorphisms, and single-variable limits. We then conclude with a detailed analysis of student data from combinatorics that identifies productive aspects of their uses of equivalence when constructing permutations.

Keywords: Equivalence, Conceptual Analysis, Student Thinking

Equivalence is a pervasive mathematical concept that is fundamental to constructing relationships between mathematical objects at all levels (Carpenter, Franke, & Levi, 2003; Cook, 2018; Lockwood & Reed, 2020; Hamdan, 2006; Kieran & Sfard, 1999; Knuth et al., 2006; Moore, 2013; Ni, 2001; Steffe, 2004; Stylianides et al., 2004). In postsecondary mathematics, equivalence is fundamental to students' thinking about topics such as angle measure (Moore, 2013), logic (Stylianides et al., 2004), combinatorics (Lockwood & Reed, 2020), and abstract algebra (Cook, 2012, 2018; Larsen, 2013). There is, however, evidence that students throughout K-16 mathematics face difficulties in reasoning about equivalence (Chesney et al., 2013; Godfrey & Thomas, 2008; Kieran, 1981; McNeil et al., 2006; Weinberg, 2009). We propose that one reason for these difficulties is that equivalence is often treated in compartmentalized, context-specific ways that emphasize its utility within a context but not its common, overarching structure. This is significant because, as noted by Asghari (2019), "equivalence has had many different faces and […] many different names" (p. 4675).

We note that very little has been done to develop a clear, unifying image of what is involved in productively reasoning with equivalence across domains in undergraduate mathematics. In this theoretical report, we seek to begin to address this need by presenting an initial theoretical framework that characterizes key aspects students' reasoning with equivalence. Specifically, we first present theoretical analyses of the ways that students might operationalize equivalence when reasoning about (1) vectors, (2) isomorphisms and homeomorphisms, and (3) limits in single-variable calculus. Then, we present an analysis of students' mathematical activity in combinatorics that highlights how they conceived of various sets of outcomes as equivalent. In doing so, we demonstrate the utility of the framework for highlighting key aspects of students' productive engagement with equivalence across multiple mathematical domains.

Background Literature

As there is much more literature on equivalence at the K-12 level than at the postsecondary level, we draw on K-12 literature in situating our paper. Because of spatial restrictions, we only discuss the works that largely informed our framework. The K-12 equivalence literature holds two key implications for our theory-building objectives. First, there are a plethora of *explicit* calls for instruction to attend to equivalence (McNeil & Alibali, 2005; McNeil et. al., 2006; Ni, 2001; Smith; 1995; Solares & Kieran, 2013; Stephens, 2006). While this has been somewhat achieved at the K-12 level, we have observed that equivalence in postsecondary domains often remains backgrounded.

Second, the K-12 literature contains descriptions of various in ways in which students might interpret equivalence, specifically in the context of the equals sign. These descriptions provided an initial foundation for our framework. A fundamental distinction in K-12 involves students viewing the equal sign *operationally* (as a indicator to "do something") or *relationally* (as an indicator that the objects in question are in some way the same) (Kieran, 1981; Knuth et al., 2005). But what does a *relational* understanding of the equal sign entail? As an example, we consider the equivalent algebraic expressions $2x + 2y$ and $x + y + x + y$: What does it mean to say that two objects are *in some way* the same? Our framework stems from three possible ways to interpret the equivalence of these two expressions that appear in the K-12 literature (Liebenberg et al., 1999; Saldana & Kieran, 2005; Solares & Kieran, 2013; Zwetzschler & Prediger, 2013):

1. *Numerical*: these two expressions are equivalent because, for any real numbers x and y, the expressions $2x + 2y$ and $x + y + x + y$ have the same numerical value.
2. *Transformational:* these two expressions are equivalent because one can be transformed into the other using algebraic rules (e.g. associativity and commutativity of addition).
3. *Descriptive:* these two expressions are equivalence because they both describe the perimeter of a rectangle with sides x and y.

We propose more general, refined versions of these interpretations in the next section and illustrate how they capture key aspects of students' reasoning across different domains.

Theoretical Framework

Our framework takes the form of a *conceptual analysis*, an explicit description of "what students might understand when they know a particular idea in various ways" (Thompson, 2008, p. 43). We find conceptual analyses to be useful for our theory-building objectives in three ways. First, conceptual analyses offer means to identify desirable interpretations of equivalence that can inspire targets of instruction (Thompson, 2008). Conceptual analyses can form unifying threads within and across courses and curricula (O'Bryan, 2018). Finally, conceptual analyses can also enable researchers to create *models of students' thinking* (Clement, 2000; Steffe & Thompson, 2000). These models are useful for both researchers and instructors because they can be employed to explain students' mathematical activity and render it sensible in some way.

In this report, we shall illustrate how results of the cross-domain conceptual analysis of equivalence that we present serves these purposes. The conceptual analysis was informed by our analyses of (1) the K-16 literature on equivalence, and (2) data collected for teaching experiments that had been previously conducted in abstract algebra (Cook, 2018) and combinatorics (Lockwood & Reed, 2020; Reed & Lockwood, 2020). In this framework, we describe three interpretations of equivalence that we hypothesize are useful for reasoning about equivalence across mathematical domains (these are featured in Table 1).

Table 1. A framework for analyzing students' reasoning about equivalence.

Interpretation of equivalence	Description	Example from undergraduate mathematics
Common characteristic	Interpreting or determining equivalence based upon a perceived attribute that the objects in question have in common.	Interpreting that parallel lines are equivalent because "the common property will be the slope" (Hamdan, 2006, p. 143).
Descriptive	Interpreting or determining that objects are equivalent because they describe the same quantity or serve the same purpose with respect to a given situation.	Determining that -3 and 9 are equivalent modulo 12 because they both function as the additive inverse of 3 (Cook, 2012).
Transformational	Interpreting or determining the relationship between equivalent objects in terms of the actions by which one object has been or might be transformed into another.	Interpreting that two matrices are row-equivalent because one can be obtained by applying a sequence of elementary row operations to the other (Berman, Koichu, & Shvartsman, 2013).

In the next two sections, we illustrate the utility of this framework by (a) elaborating theoretical analyses of how these constructs might capture relevant aspects of students' reasoning about equivalence in the context of vectors and magnitudes, isomorphisms and homeomorphisms, and single-variable limits, and (b) using the framework to conduct a detailed analysis of students' reasoning from a teaching experiment in enumerative combinatorics. Together, these will demonstrate ways in which the framework can contribute to a broader, unifying perspective on equivalence that may be applicable across domains.

Using the Framework to Gain Insight into Equivalence Across Domains

We now illustrate how the interpretations detailed above capture productive aspects of reasoning about equivalence in the contexts of vectors, isomorphisms and homeomorphisms, and single-variable limits.

Vectors and Magnitudes

Vector equations provide an example that extends work done at the K-12 level to the undergraduate curriculum. For example, consider the equation $\|5v\| = 5\|v\|$, where $\|\cdot\|$ denotes a vector norm. First, a student might employ *transformational* equivalence to consider that the equality $\|5v\| = 5\|v\|$ follows from allowable operations on vector norms. This transformation might be described as "pulling the 5 out." More formally, the definition of a norm requires that the norm function satisfy the property $\|cv\| = |c| \cdot \|v\|$ for any real constant c and vector v. In a *common characteristic* interpretation of the equation, a student might appeal to the fact that given any vector v, $\|5v\|$ and $5\|v\|$ give the same numerical value[1]. Finally, a descriptive equivalence interpretation could involve reasoning with magnitudes. Following Thompson and colleagues (2014), the magnitude of a quantity A is the size of that quantity measured with respect to a unit[2]. From this perspective, $\|5v\| = 5\|v\|$ could be interpreted *descriptively* as a statement that the measure of the length of $5v$ (when using the length of v as a unit) is 5.

Isomorphic and Homeomorphic Spaces

Significant identifications commonly made in advanced mathematics establish spaces as equivalent in the sense of possessing the same essential features. The standard method of determining such an equivalence entails the finding of a (usually bijective) map between the spaces such that the map satisfies certain topological, analytic, or algebraic properties. A homeomorphism, for instance, is a bijective map, f, such that both f and its inverse, f^{-1}, are continuous. A group isomorphism, ϕ, is a bijective map such that ϕ preserves the group operation: $\phi(a * b) = \phi(a) \cdot \phi(b)$, where $*$ and $\cdot$ are the binary operations of the two groups. A student using such mappings to change one space into another would employ *transformational equivalence*, as the maps are the means by which elements of one space are transformed into elements of another. The analytic, algebraic, and topological qualifications of the bijective maps afford other implications, however, that also constitute interpretations of equivalence. Given two isomorphic groups G and H, G is abelian if and only if H is abelian. If M and N are homeomorphic metric spaces, then they share convergent sequences. Put another way, viewing the equivalence between spaces this way focuses on their *common characteristics.* One benefit of this interpretation is that such fundamental results as those we have given above become intuitive (if not obvious). Another is that it can be leveraged to justify that certain spaces are *not* the same: an abelian group cannot be equivalent (isomorphic) to a non-abelian group, and a connected topological space cannot be equivalent (homeomorphic) to one that is disconnected.

Single-Variable Limits

Limits underlie most curricular treatments of fundamental operations in single-variable calculus: derivatives, integrals, and series. One formulation of limits answers the question: *At a given domain value, a, of a function, f, is there a single real number, L, that f approximates to any desired error bound via domain restrictions of f around a?* The mathematical necessity of such a question can be seen by examining $\frac{e^x-1}{x^2}$, which does not admit a readily available output for all domain values. While numerical and graphical methods might allow determination of rather obvious limiting values, L, for certain functions, f, the most efficient way to determine the limits of functions - such as $\frac{e^x-1}{x^2}$ - at points of discontinuity is to find an alternate, continuous function f^* that has the same limit as f at a.

For simpler functions, f^* can be determined algebraically. For instance, $x+1$ can be used to determine the limit of $\frac{x^2-1}{x-1}$ at $x = 1$ by noting that $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$, so that $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2$. We consider the determining f^* in this way to be an example of transformational equivalence, specifically by obtaining $x+1$ from $\frac{x^2-1}{x-1}$ through a series of algebraic

[1] Notice that numerical equivalence from the K-12 literature is subsumed in common characteristic equivalence.

[2] Symbolically, $|A| = m(A) \cdot |u|$ where $|A|$ is the magnitude, $m(A)$ is the measure of A in unit u, and $|u|$ is the magnitude of the unit.

transformations. These operations by themselves, however, do not constitute the utility of interpreting f and f^* as equivalent for the purpose of limit calculations. Rather, f and f^* are also equivalent because of a *common characteristic*: they share the same output values in their common domain (that is, all real numbers except 1). Because of this common characteristic, the output $f^*(1) = 2$ is approximated by values of f for any error bound given a sufficiently small domain interval around $x = 1$, thus constituting the limit of f at $x = 1$. As such, the limit of f is determined because of the *common characteristic* equivalence of f and f^*, yet f^* is likely to be originally determined *transformationally.*

As functions, f, vary in complexity, engaging in algebraic transformations becomes increasingly insufficient, requiring new ways to determine suitable f^*. For instance, while many functions share the same limiting value as $\frac{e^x-1}{x^2}$ at $x = 0, \frac{e^x-1}{x^2}$ admits no readily available algebraic transformations. From this perspective, limit theorems - such as L'Hopital's rule or the squeeze theorem - can be viewed as providing the means of generating useful equivalent functions, f^*. While desired functions f^* have a common characteristic with f that their limits evaluate to the same number, there are many such functions, g, that have this same common characteristic. We consider that students might productively generate more robust understandings of limit theorems as ways to establish equivalence between f^* and f via applications of ideas fundamental to calculus, those of locality and approximation, for instance.

An Analysis of Students' Reasoning in Combinatorics

We now demonstrate the utility of the framework for capturing key aspects of students' reasoning with equivalence in combinatorics. Lockwood and Reed (2020) characterized an *equivalence way of thinking* to describe a general approach that students in teaching experiments (Steffe & Thompson, 2000) used to solve enumerative combinatorics problems successfully. Broadly, their *equivalence way of thinking* involved identifying outcomes of counting processes as the same and then using division to account for such 'duplicate' outcomes. Our analysis here furthers this work by explicating *how* the students employed equivalence in multiple ways. Specifically, we discuss the counting activity exhibited by novice counters (pseudonyms Carson, Anne-Marie, and Aaron) when solving the Horse Race Problem, which states: "There are 10 horses in a race. In how many different ways can the horses finish in first, second, and third place?"

The students first answered $10 \cdot 9 \cdot 8$, enumerating the sequence of events in which 10 horses finish the race, but only 9 horses remain after the first horse finishes, followed by 8 horses that compete for a third-place spot. The interviewer then introduced the notation $\frac{10!}{7!}$ as another way to express the solution and asked the students to justify why $\frac{10!}{7!}$ was also a solution. The students first argued that $\frac{10!}{7!}$ gave another way of writing $10 \cdot 9 \cdot 8$ as $\frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!}$, yielding cancellation of 7!. This first response employed *transformational equivalence*, as the students enacted algebraic transformations in which $\frac{10!}{7!}$ transformed into $10 \cdot 9 \cdot 8$. Wanting to give the students opportunities to make other – combinatorially based - connections, the interviewer asked, "can you explain why this answer might make sense aside from the fact that its numerically equivalent to 10 times 9 times 8?" The following conversation ensued:

Carson: So, the way I'm thinking about it, is that we know kind of the method to get the number of ways that 10 horses can finish a race, and that's 10!. … So, there's 10! total outcomes, and then we know <u>for any given first 3 there's gonna be 7!</u>, because that's saying <u>we know the first 3 horses have finished. How can the last 7 horses finish?</u> So that's gonna be 7!. <u>But all we care about is how many given first 3s there are.</u> So, if we divide the total number of outcomes by the number of potential of outcomes for the last 7 horses - that will give us the potential number of outcomes for the first 3. If that makes sense?

Interviewer: It makes sense to me. Are you guys following what he's saying?

Anne-Marie: I see why, like 10! would be looking at all 10 positions for each 10 horses. I just feel like it'd be more intuitive to subtract the 7! than it would be divide but I see why dividing works better.

We note two complementary interpretations of equivalence that Carson engaged in for the Horse Race Problem. First, Carson employed descriptive equivalence to establish similitude of the expressions $10 \cdot 9 \cdot 8$ and $\frac{10!}{7!}$. By establishing that $10 \cdot 9 \cdot 8$ and $\frac{10!}{7!}$ counted the same total collection of the first three race finishers, Carson argued that the expressions described the same outcome set. This use of descriptive equivalence is commonly employed in combinatorial proof. Second, in the underlined portions, Carson argued that there were 7! orderings of 10 horses that represented each single desired ordering of the first three horses. This representation of the single outcome in 7! ways was the first time that the students identified what they would later call "duplicate" outcomes and set the foundation for what Reed & Lockwood (2020) called an *equivalence way of thinking*. For Carson, the assumption that there were 7! representations of the same desired outcome provided the impetus for the division of 10! by 7!, and constitutes another use of descriptive equivalence, as the 7! duplicates represent the same desired quantity.

This discussion of the utility in dividing versus subtracting, initiated above by Anne-Marie, became a prevalent distinction for these students. While the students could articulate that there were 7! arrangements of the 10 horses for any specific arrangement of gold, silver and bronze medalists, at this point in the experiment only Carson could articulate why division meaningfully accounted for those 7! extraneous arrangements to produce a single desired outcome.

Figure 1: Arrangements of A-E

ABCDE	ACBDE	BACDE	BCADE	CABDE	CBADE	DABCE	DBACE	EABCD	EBACD
ABCED	ACBED	BACED	BCAED	CABED	CBADE	DABED	DBACE	EABDC	EBACD
ABDCE	ACDBE	BADCE	BCDAE	CADBE	CBDAE	DACBE	DBCAE	EACBD	EBCAD
ABDEC	ACDEB	BADEC	BCDEA	CADEB	CBDEA	DACEB	DBCEA	EACDB	EBCDA
ABECD	ACEBD	BAECD	BCEAD	CAEBD	CBEAD	DAEBC	DBEAC	EADBC	EBDAC
ABEDC	ACEDB	BAEDC	BCEDA	CAEDB	CBEDA	DAECB	DBECA	EADCB	EBDCA
ADBCE	AEBCD	BDACE	BEACD	CDABE	CEABD	DCABE	DEABC	ECABD	EDABC
ADBEC	AEBDC	BDAEC	BEADC	CDAEB	CEBAD	DCAEB	DEACB	ECADB	EDABC
ADCBE	AECBD	BDCAE	BECAD	CDBAE	CEBAD	DCBAE	DEBAC	ECBAD	EDBAC
ADEBC	AEDBC	BDEAC	BEDAC	CDEAB	CEDAB	DCEAB	DECAB	ECDAB	EDCAB
ADECB	AEDCB	BDECA	BEDCA	CDEBA	CEDBA	DCEBA	DECBA	ECDBA	EDCBA

To elicit further reflection, the interviewer provided a printed list of the 5! arrangements of the letters A through E (Figure 1) and asked the students to find the 20 groups of letters that could represent first and second place finishers. Notice that the entries in the list were spaced apart according to the fixed first two letters. The students noticed this arrangement, and subsequently circled the 20 groups that reflected this spacing. Seeing that there were six elements to each grouping, Aaron asked why division by 3! made more sense than "getting rid of the other 5". The following exchange occurred after the interviewer pointed out that 3! was 6:

Aaron: Well, since there are 6 options for each AB, then dividing by 6 would just mean you would get 1, because that's all you're looking for. But then 5! would give you the number of groups (i.e. arrangements of A-E) and 3! would give you the number of combinations in each group (i.e. arrangements of the 3^{rd}-5^{th} letters).

Carson: Well, 3! gives you the number of ways you can arrange the last 3 letters given the first 2 letters.

Anne-Marie similarly explained that she understood why division by 6 created the single desired outcome, and that the 6 was achieved by 3!. As with Carson in the Horse Race Problem, the students' generation of a desired outcome from a collection of representative outcomes constitutes employment of descriptive equivalence. Accordingly, the students' motivations for division were rooted in considering each of the 6 outcomes as a version of the desired singular outcome from which generation of the 1 desired from the 6 duplicates could follow.

Following this activity, the students expressed solutions to permutation problems through division, and explained their process as "getting rid of unwanted" outcomes. In general, the students throughout the rest of the teaching experiment *explicitly* attended to whether certain outcomes generated by a counting process could be seen as duplicates of other outcomes under the constraints of the problem, thus continuing to employ descriptive equivalence. This was a notable component of students' determination of when multiplication was appropriate and when addition was appropriate. As determining the operations appropriate for the constraints of a particular counting problem is an area of difficulty for students (e.g., Batanero et al., 1997), the students' use of descriptive equivalence was productive for their overall counting.

Conclusion

In this report, we have presented and discussed an initial framework for analyzing students' reasoning about equivalence across undergraduate mathematics. We exemplified the utility of this framework by demonstrating its constructs through a discussion of three different mathematical concepts, and by presenting student data from a combinatorial context. We are motivated by the fact that despite the fundamental nature of equivalence in K-16 mathematics, few frameworks offer constructs and language that span domains and levels of mathematics.

As exemplified in our analyses, students might interpret established equivalences between objects and spaces in myriad ways, each of which might have implications for the ways students carry out goal-oriented activity with the objects. In addition to providing unifying accounts of the associations that students make between various mathematical objects, this framework also offers tools for identifying productive aspects of students' engagements with equivalence, such as the productivity of the combinatorics students' uses of *descriptive* equivalence to determine whether subtraction or division was appropriate in a permutation calculation.

Our hope is that we and other researchers can refine this framework by applying it to empirical data in a variety of domains and topics. Moreover, conceptual analyses such as those in this report can serve as a foundation for design research that targets these concepts. We offer these theoretical analyses as inspiration for future conceptual analyses and empirical studies in which equivalence is considered to serve a key role in students' reasoning.

Acknowledgments

This project was funded by **[Blinded funding source]**

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