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Revealing Prospective Secondary Teachers' Mathematical Knowledge for Teaching in Teacher-Created Representations of Practice

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To examine differences in dimensions of MKT observable in different creation mediums for teacher-created representations of practice, we analyzed 54 representations of practice (pairs of 27 video and 27 written) created by 27 PSMTs. We found 7 pairs where we observed higher MKT in the video than in the written representation of practice created by the same PSMT. We found 0 cases where we observed higher MKT in the written as compared to the video representation of practice. We examined these 7 pairs for qualities characterizing the pairs as a set. We use findings to problematize issues central to mathematics teacher education: how mathematics is conceptualized, and how representations of practice are used as assessment of dimensions of teachers' MKT.

Keywords: Mathematical Knowledge for Teaching, Prospective Secondary Mathematics Teachers, Geometric Transformations

Enduring Challenges: Revealing Pedagogical Understanding

An enduring challenge of teaching, at any level and for any domain, is revealing learners' understandings (Kennedy, 2016). When the mathematics learners are prospective secondary mathematics teachers (PSMTs), this challenge includes revealing their mathematical knowledge in and for teaching (MKT) (Ball et al., 2008; Rowland et al., 2003) and the way they coordinate mathematical and pedagogical demands. How we, as "mathematician educators" (Tay, 2020, p. 3), provide opportunities for PSMTs to reveal MKT shapes the feedback we give, the growth opportunities we provide, and the next tasks we offer.

We understand our roles as "mathematician educators" because we work with PSMTs in mathematics content courses to develop PSMTs' MKT and "base [our] pedagogy on a strong foundation of mathematics disciplinarity and education theory" (Tay, 2020, p. 3). We differentiate *content courses* from *methods courses* as the former has a primary focus on developing mathematical knowledge and the latter has a primary focus on developing pedagogical knowledge. However, we recognize that in many cases in teacher preparation programs there is a blending between these classifications such that courses for PSMTs often address both mathematical and pedagogical knowledge. Moreover, we posit that PSMTs should be supported in integrating mathematical and pedagogical knowledge during teacher preparation, and that designing supports for PSMTs can be a role that mathematician educators play. The goal of the project within which this study took place is to promote the development of MKT (including its relations with aspects of both content and pedagogical knowledge) in typically content-focused courses.

In our work with PSMTs, we were struck by the potential for video-recorded talk, as opposed to written work, to reveal dimensions of prospective secondary teachers' MKT in the context of a mathematics content course. In both written and video responses for the tasks analyzed here, the foundational mathematical knowledge that determined the characterization of MKT present in PSMTs' response is the idea that definitions are central in identifying, verifying, and constructing transformations. With this mathematical principal as our foundation, we coded 54 (2 x 27) responses from 27 PSMTs from four different institutions to a set of parallel assignments intended to assess MKT in similar domains in analogous ways. We had intended to look for examples where the same teacher showed higher MKT in their written response as compared to their video assignment, and vice versa, and to compare the affordances and limitations of these assignments using these examples. Instead, to our surprise, we found that one of these cases did not exist. Observed MKT differences between the video assignment and written assignment occurred in the responses of 7 teachers, almost one fourth of the teachers. In all 7 cases, without exception, it was the video assignment where we observed higher levels of MKT.

Here, we share these two assignments, which can be used either in secondary content or methods courses. One assignment asked teachers to create a video representation of practice, where they video recorded themselves responding to hypothetical students about sample student work provided in a teaching scenario. The second assignment also provided a teaching scenario with student work, but asked teachers to create a written narrative where they described how they would respond to the hypothetical students in a classroom discussion. In our discussion of the nature of the differences between the video and written responses of the 7 teachers, we focus on the question: What variation is found in dimensions of the MKT observable in video as compared to written teacher-created representations of practice? Then, drawing on our analysis of these responses, we argue that one response medium alone cannot reveal a complete picture of teachers' understandings. In fact, focusing on one response medium may show a distorted image that underestimates dimensions of teachers' knowledge. We conclude by (re)problematizing both the use of prospective teacher-created representations of practice as a method for revealing teachers' MKT and the use of rubrics based at least in part on frameworks of knowledge from a cognitive perspective to evince MKT.

Situating our Work in Literature

In our review of literature, we first describe how we combine Rowland and colleagues' (2013, 2016) notion of observing MKT in episodes of teaching with Simon's (2006) conceptualization of mathematical understanding. This combined conceptualization shaped both our assignments and our analysis of the teacher-created representations of practice in our attempt to reveal PSMT's understandings related to dimensions of MKT, the problem of practice on which we sought to make headway. Following this, we review findings on how the content of PSMTs' representations of practice may be influenced by the medium they used to create the representations (e.g., Amador et al., 2017; Rougée & Herbst, 2018). Then, we describe the context for these assignments and the results of our study. Finally, we discuss the importance of incorporating multiple creation mediums in assignments which seek to reveal prospective teachers' MKT related understandings.

Mathematical Knowledge in and for Teaching (MKT)

The mathematical knowledge teachers utilize while doing the work of teaching (i.e., MKT) is different from the mathematical knowledge one may utilize while only solving mathematics problems. Efforts to conceptualize and understand MKT, along with how it is activated and developed, have intensified over the last three decades. Across the various literature on MKT (e.g., Ball et al., 2008; Davis & Simmt, 2006; Heid et al., 2015; Rowland, 2013; Thompson & Thompson, 1994) and on mathematics learning (e.g., Daro et al., 2011; National Research Council, 2000; Simon, 2006), we have found ideas of Rowland and colleagues' (2013, 2016) and Simon (2006) most generative for our work.

Rowland (2013) introduced the Knowledge Quartet framework to analyze teachers' use of knowledge in mathematics teaching, in particular their subject matter knowledge and pedagogical content knowledge. Four dimensions make up the Knowledge Quartet; Foundation (knowledge and understanding of mathematical ideas, the nature of mathematics, as well as principles of mathematical pedagogy), Transformation (presentation of ideas to learners), Connection (unifying material across time for coherence), and Contingency (the ability to respond to unanticipated events ranging from network outages to learners' alternative strategies). In Rowland's studies, he and colleagues argue that researchers might infer Foundation knowledge from actions associated to the other dimensions, because Foundation informs those actions, and moreover it is difficult to observe directly in teaching practice.

In our work, we similarly infer Foundation from actions in the other dimensions, as well as in teacher talk and writing. Foundation knowledge can be thought of as akin to Ball et al.'s (2008) conception of subject matter knowledge, which includes both what school students are learning as well as what teachers may need to know – beyond what students are learning – in order to teach. An analytic difference in how these scholars have approached the study of knowledge is that while Ball and colleagues contrast subject matter knowledge with pedagogical content knowledge as ways to decompose the whole of mathematical knowledge for teaching, Rowland and colleagues view Foundation knowledge as a body that is drawn upon to enact the pedagogical moves needed to present ideas to learners, cohere content over time, and respond to unanticipated events. Because content courses focus on mathematics, and we conceive of pedagogical assignments in content courses as applications of the mathematics, we find Rowland and colleagues' conceptions in greater harmony with our project than other ways to parse mathematical knowledge in and for teaching.

As an analytic framework, Foundation and the other three dimensions of knowledge can apply to teaching across elementary and secondary levels. The dimensions were originally the result of analysis of videos of elementary and middle school lessons, taught by prospective elementary and middle school teachers during student teaching placements (Rowland et al., 2003). In their analysis, Rowland's group focused on actions in teaching that could be inferred to be informed by a prospective teacher's subject matter knowledge or pedagogical content knowledge. Then, they classified these actions, and associated knowledge, into the four dimensions above. In 2016, Rowland, Thwaites, and Jared conducted a follow-up study with prospective secondary teachers in their student placements. They concluded that the

Knowledge Quartet, as distilled from their analysis of elementary and middle school lessons, was a viable analytic tool for novice secondary teaching.

In contrast to Rowland and colleagues (2013, 2016), who analyzed videos of teaching across multiple topics in multiple schools, we examined teacher-created representations of practice responding to a limited set of prompts. Given this smaller scope, we found it useful to delimit and elaborate on the dimension of *Foundation* knowledge as follows. First, we delimit Foundation to knowledge of mathematics because the teacher-created representations of practice were created in mathematics content courses. Second, the dependence of Foundation on mathematical understanding suggested that we be theoretically clear about a conception of mathematical understanding. For this, we used Simon's (2006) characterization: mathematical understanding is the "learned anticipation of the logical necessity of a particular pattern or relationship" (p. 364). For instance, we consider understanding mathematical procedures to include relating that procedure to its underlying definitions or concepts, and to anticipate doing so when explaining procedures or troubleshooting a use of a procedure during the act of teaching. Finally, we note that although we attend to Foundation knowledge, we do so via actions associated with the other dimensions of the Knowledge Quartet.

Teacher-Created Representations of Practice

In all assignments we analyzed, prospective secondary teachers created representations of practice based on a description of a teaching situation provided to them, where the teaching situation included various samples of student work. These assignments offered a potential for "instructive failure" (Grossman et al., 2009, p. 2077): safe spaces to consider possible reactions where no harm is incurred but teachers' learning is possible. By experimenting with responses to student thinking, PSMTs develop their professional practice. At the same time, the teacher-created representations of practice give mathematics instructors a teaching context for PSMTs' Foundation knowledge in teaching (rather than isolated from teaching). These teacher-created representations of practice can thus be thought of as approximations of practice, where novices can engage with challenging teaching practices (Grossman et al., 2009), including framing mathematical ideas in response to student thinking.

As approximations of practice, teacher-created representations of practice can vary in authenticity to teaching practice (Grossman et al., 2009). Less authentic approximations may offer the teacher more chances for revision, whereas more authentic approximations may be closer to "real time", with "no stops and starts" (p. 2079). Less authenticity may also mean that fewer aspects of practice are featured, where more authenticity would mean a more "complete or integrated" representation of practice (p. 2079).

Teacher-created representations of practice are not uncommon in secondary teacher preparation programs. For instance, in many methods and some content courses, mathematics teacher educators ask teachers to consider and respond to sample student thinking (e.g., Grossman et al., 2018; Lischka et al., 2020; Wasserman et al., 2017). The purpose of doing so is twofold: to give opportunities for teachers to engage in practices of teaching, and to provide contexts for mathematics teacher educators to foster prospective teachers' growth as teachers. Whether for efficiency or other reasons, mathematics teacher educators may ask teachers to envision responses to students using different mediums, including writing, video, or even animation (e.g., Amador et al., 2017; Lischka et al., 2020; Rougée & Herbst, 2018).

Written narratives of responses to students may be the least authentic to teaching, as they can be revised multiple times. Animations and video may be more authentic, in different ways. Animations are more authentic in that teachers must consider many aspects of the class (e.g., desk layout, location of teacher), yet also less authentic because they can be edited. Video may be more authentic in that once the recording device begins, there are no stops and starts. Video representation of a response to students is also less authentic in that it typically focuses on only one specific slice of teaching practice.

The Impact of Creation Medium on Teacher-Created Representations of Practice

As Rougée and Herbst (2018) noted, the creation medium must matter because tools mediate human activity and so, if the tools differ, the outcomes of the activity should differ (Engeström, 1999; Vygotsky, 1978). In the literature, we have found three main creation media for teacher-created representations of practice: written (e.g., Amador et al., 2017; Rougée & Herbst, 2018; Zazkis & Marmur, 2018), video (e.g., Lischka et al., 2020), and animation (e.g., Amador et al., 2017; Rougée & Herbst, 2018).

Amador et al. (2017) and Rougée and Herbst (2018) both compared written and animated teacher-created representations of practice. Amador and colleagues blended written responses and prospective teachers' creation of animations in one task to elicit prospective teachers' mathematical noticing. Rougée and Herbst analyzed separate assignments, where some asked teachers to produce written representations of practice and others asked teachers to produce animated representations of practice. Both sets of authors found differences in teachers' responses by medium. Amador et al. (2017) found that in written responses, teachers focused more on student actions and general descriptions of the content, whereas in the animated responses, teachers focused more on teacher actions and specific descriptions of content. Rougée and Herbst (2018) found that teachers attended more to alternative teaching moves in the written response medium, and that they attended more to students and mathematical representations in the animated response medium.

To our knowledge, there have been no studies comparing video and written responses. This is significant because oral feedback based on student activity, such as could be video-recorded, is central to teaching (National Council of Teachers of Mathematics, 1991), is dependent on Foundation knowledge (Rowland, 2013), and is a learnable, high-leverage teaching practice (Charalambous et al., 2011). Moreover, practicing oral explanation and feedback supports PSMTs' development of mathematical fluency in teaching (Hoover et al., 2016). Our study begins to fill this gap.

Method of Investigation

This investigation was part of a larger National Science Foundation funded project, the MODULE(S^2) Project, whose goals include developing educative curriculum materials (Davis & Krajcik, 2005) to enhance prospective secondary teachers' development of MKT in undergraduate mathematics content courses. In this section, we more fully describe the context in which this project is implemented, and the methods employed in the study.

Context

The MODULE(S^2) project responds to the perceived disconnect between tertiary mathematics instruction and secondary teaching practice (Goulding et al., 2003; Ticknor, 2012; Wasserman et al., 2017; Zazkis & Leikin, 2010). The curricular materials come in four strands: Algebra, Geometry, Modeling, and Statistics. Across all strands, the materials have been piloted at 31 institutions. In this article, we report on an analysis of PSMTs' work from three different instructors who used Geometry materials in the first two years of data collection.

Embedded throughout the materials in all strands are prompts for teachers to create representations of practice. These prompts include a brief description of a secondary mathematics teaching situation, presented in second person ("You are teaching ..."). Each scenario includes samples of student work. Prompts ask prospective teachers to respond to the student work in the context of the teaching situation. Half the prompts ask teachers to write their response; the remaining prompts ask teachers to video-record themselves giving a response. In all cases, the prompts focus on mathematics relevant to course concepts.

The prompts are designed to provide PSMTs with opportunities to use and demonstrate MKT as they respond to learner thinking and plan their ensuing pedagogical moves. Although there is no one specific intended response, instructors can provide feedback to the prospective teachers regarding Foundation knowledge, based on mathematical ideas and language evident in the response, as well as other dimensions of MKT.

Participants

This article reports an analysis of 14 teacher-created representations of practice, consisting of 7 video and 7 written responses, created by 7 prospective secondary teachers in 3 different institutions. These 14 teacher-created representations of practice were drawn from 54 teacher-created representations of practice, consisting of 27 video and 27 written responses, created by 27 teachers enrolled in courses taught by three different instructors piloting the Geometry materials (**Figure 1**). The 14 teacher-created representations of practice were selected because there was a difference in the level of Foundation knowledge observed in the video and written responses, from the same teacher; and in all cases, the video response was scored to have higher Foundation knowledge.

These data were collected in College Geometry, Geometry for Secondary Teachers, and a capstone course for secondary mathematics education majors, depending on the institution. Prospective teachers took these courses at different points in their preparation program, and therefore teachers' direct exposure to learners varied. The instructors were located in institutions in different geographic regions of the US.

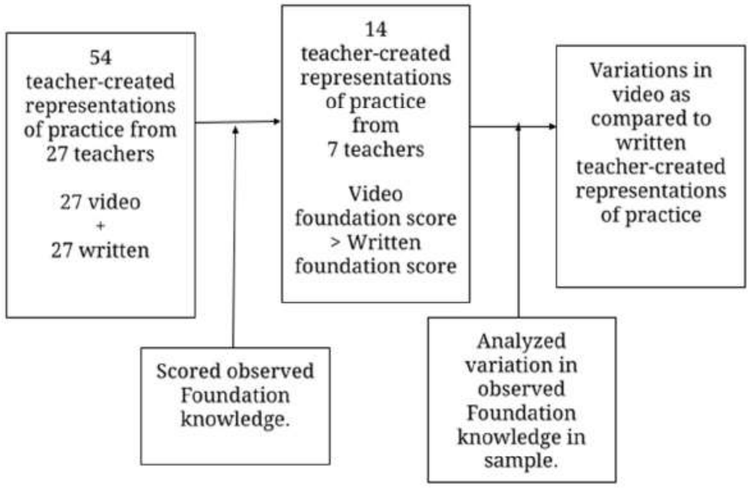

Figure 1. Overview of Method

Mathematical Context

The teacher-created representations of practice used in this study were embedded in a module on congruence from a transformation approach, meaning that congruence is defined in terms of transformations. The module was designed to develop teachers' understanding of transformations as critical to congruence, including reflections, rotations, translations, and glide reflections.

In our materials, the idea that a reflection preserves distance is assumed as an axiom. Prospective teachers begin by exploring a variety of isometric and non-isometric transformations to derive definitions of transformation and isometry. They next collaboratively develop definitions for reflection, rotation, and translation and explore constructions of these isometries based on these definitions while using a variety of tools. After arriving at the community definitions, prospective teachers apply their definitions to construct rotations and reflections or determine whether or not a rotation or reflection exists between given sets of objects. A version of these definitions is given in **Table 1**.

Table 1.
Definitions for rotation and reflection, given in terms of images and preimages

Transformation	**Definition** Note: These materials teach prospective teachers the convention that P' refers to the *image* of a *preimage P* under the transformation discussed.
Reflection	A reflection across a line L is a transformation of the plane that, for every point P in the plane: • P'=P if P is on L • L is the perpendicular bisector of segment PP' if P is not on L.
Rotation	A rotation with center O and angle α is a transformation of the plane that, for every point P in the plane: • $OP = OP'$ (meaning $P'=O$ if $P = O$), and • $m\angle POP' = \alpha$.

Next, PSMTs are asked to create two representations of practice, one video and one written, where they respond to student thinking (described next). Both prompts for the simulation of teaching practice tasks aligned with the content of the course and approximated teaching practice; the sample student work was drawn from actual classroom practice. As directed in the curriculum materials, creating the video representation of practice was assigned prior to creating the written representation of practice. **Figure 2** shows a snapshot of these activities. The materials continue after this to explore ideas of compositions of transformations, congruence, and proof of triangle congruence theorems.

Prompts for Creating Representations of Teaching Practice

The prompt for creating a *video* representation (**Appendix 1**) provided images of student work on rotations (**Figure 3**) and asked prospective teachers to:

> Record a video of yourself providing a response to both Student 1 and Student 2 where you include:
>
> - A summary of what each student might be thinking and what is worthwhile or reasonable about that students' thinking.
> - A response to each student that does one or more of the following as you deem appropriate: helps the student finish their thought, prompts the student to investigate an error, or helps the student move forward in their thinking.

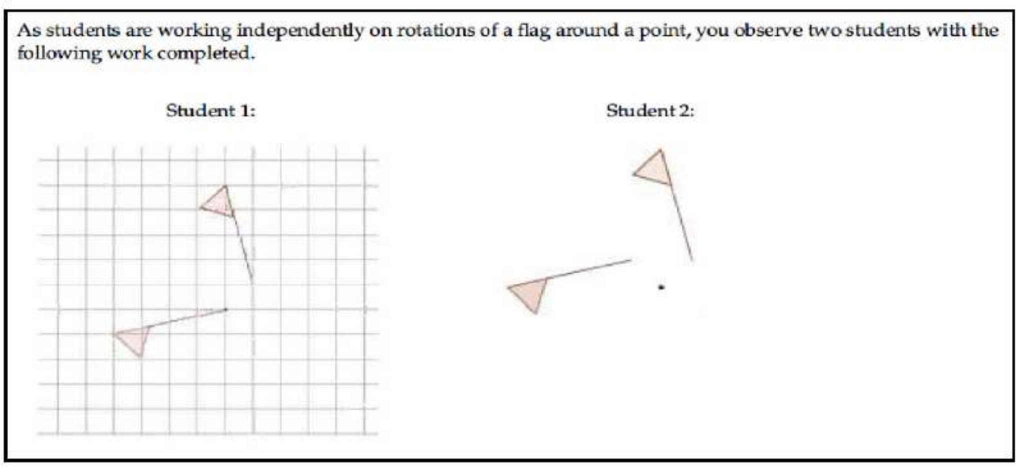

Figure 3. Secondary Student Work provided in Prompt for Teacher-Created Video Representation of Practice

The prompt for creating a written representation of practice (**Appendix 2**) similarly provided samples of two students' work on reflections (**Figure 4**). The prompt asked prospective secondary teachers to:

> Write a paper in which you clearly describe a plan for how you will conduct a whole class discussion which will allow you to elicit student thinking about these reflections, with specific use of the two example students' work, and move the class toward understanding connections between methods of reflection and the definition of reflection. Your plan for the discussion should include discussion questions, descriptions of the ways you anticipate that students might respond to questions, and any appropriate tasks that will move student thinking forward. Your response should

indicate your understandings about the definition of reflection and ways in which this is applied to various methods of construction.

We note that both prompts directed prospective teachers to use the provided student work in their responses. Although the prompts are slightly different, both give prospective secondary teachers an opportunity to use and demonstrate Foundation knowledge of rotations and reflections as they used student thinking to move learning forward.

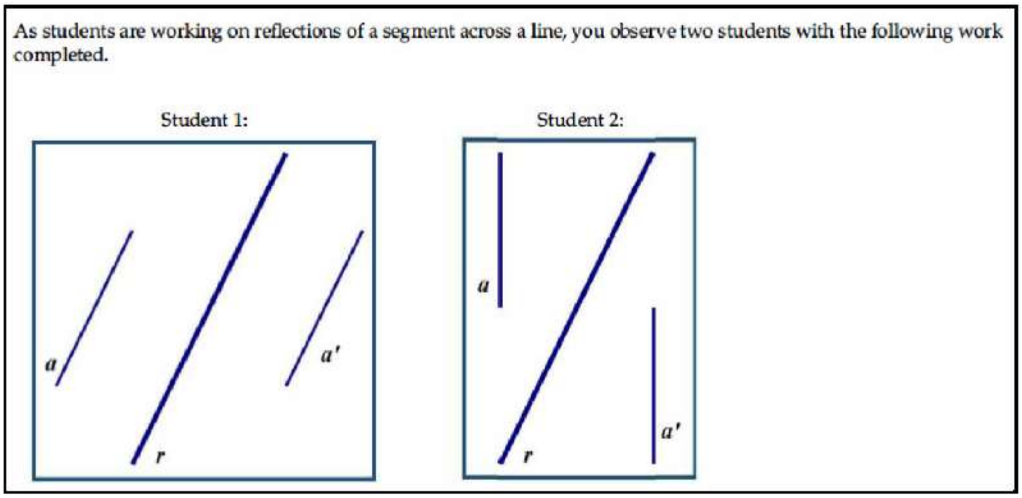

Figure 4. Secondary Student Work provided in Prompt for Teacher-Created Written Representation of Practice

Analysis

To score the Foundation knowledge observed in each of the 54 representations of practice (sets of video and written responses from 27 teachers), we first operationalized Foundation knowledge in the context of constructing images of transformations, the mathematical focus of these representations of practice. The resulting scoring rubric used is shown in **Table 2**. In constructing this rubric, we thought about what it would take to position students to understand that construction methods are not senseless procedures, but rather that one can make sense of them via the definition. For this reason, we determined that teachers' discourse in the representations of practice should specifically use the word "definition", rather than, for example, only using language from the definition but not saying explicitly that the "definition" was being used, or not using the definition at all.

Table 2.
Rubric for Foundation knowledge observed in teacher-created representations of practice

Foundation knowledge observed	**Description** The teacher's talk, examples, use of instructional materials, and/or analysis of students' ideas …	**Examples**

High	Demonstrates a recognition of the logical necessity connecting the definition of a transformation to ways of constructing an image of that transformation	Explaining the method of construction by marking points on a preimage and then "applying the definition to each of the marked points" to obtain the image Reasoning that an attempted image is incorrect by showing that it does not satisfy the definition of the transformation
Medium	Demonstrates the potential for connecting the definition of a transformation to ways of constructing an image of that transformation	Providing a mathematically correct explanation of a construction of an image, and providing a mathematically correct definition, but where talk about construction does not reference the definition.
Low	No evidence of connecting the definition of a transformation to ways of constructing an image of that transformation	Describes the procedure, and never mentions any definition. Mentions a definition, but states definition in a mathematically incorrect way. Language that may be from the definition is used, but the word "definition" is never used.

Comparing the scores given to each prospective secondary teachers' video and written responses across the 27 sets yielded exactly seven cases with differing levels of observed Foundation knowledge. Most notably, in all seven cases, the video responses were scored High and the written responses were scored Low.

To investigate the variation in Foundation knowledge revealed in these seven sets, the first and third authors wrote 14 narrative memos to describe the video and written responses of the PSMTs. These memos described instances of Foundation knowledge in each response. From these narrative memos, we inductively derived themes of similarities and differences that characterized the entire collection of sets (Miles et al., 2014). Because we found these seven sets so distinctive, our attention was not on the range of possible differences between video and written teacher-created representations of practice. Rather, we viewed these seven sets as data for understanding how video teacher-created representations of practice can reveal teachers' knowledge differently than written teacher-created representations of practice, even when the prompts for both are parallel, and the mathematical domain is comparable.

Findings

We report and illustrate three salient variations in video and written teacher-created representations of practice. For each theme, we select the work of one or two prospective teachers to exemplify that theme. The prospective teachers (Sam, Alex, and Sal, all pseudonyms) are each from different institutions, and their work is representative of the findings across the data set. The differences we found involved how PSMTs used the definitions, used embodied sensemaking, and took up the provided student work.

Use of the Definitions of Transformations

Across the seven sets, the prospective teachers used the definition differently in their video and written responses. When responding via video, these prospective teachers clearly stated the definition with correct use of terms such as "center of rotation" and "angle of rotation". In addition, they clearly stated that the standard against which they were measuring the provided student work was the definition of rotation. However, in the written responses, these prospective teachers tended to use language that was likely from the definition but never stated explicitly that they were using the "definition." Further, they emphasized other properties of transformations such as congruence of image and preimage, or reversed or preserved orientation, instead of leveraging components of the definition in their explanations.

For example, prospective secondary teacher Sam's written representation of practice included bulleted lists of steps to construct the image of a reflection. The bulleted list used language that may have been from the definition of reflection, but never explicitly stated so. It featured statements such as: "Draw a perpendicular" and "Measure the distance between the line of reflection and the point." The description, however, is never identified as connected to the definition of reflection. Another prospective teacher, Alex, focused their written response on the orientation change from preimage to image in a reflection, explaining:

> One of the biggest concepts that I would want the students to mention would be how a reflection flips the image over the line. I feel like a lot of students get confused when the image looks different on the other side of the line, and hope that students could understand that concept after this discussion. (Alex, Written Response)

Rather than making sense of the construction method, Alex instead drew attention to visual considerations of the concept of reflection as a whole. Alex never specifically talked about how this or other ideas associated with reflection could be used to understand the student work or how to understand the construction of an image from the definition of reflection. Moreover, the concept of orientation is difficult to see in the case presented in the prompt for the representation of practice, where line segments are being reflected.

In contrast, these prospective teachers' video responses consistently referred to the definition of rotation as the definition and clearly used it as the basis for their response in the video prompt. Alex analyzed the correct and incorrect aspects of the provided student work by referencing specific parts of the definition. Alex indicated that Student 1 seemed to understand the "right angle" but "misunderstood the concept of the center of rotation." After identifying that Student 1 created a right angle using the two flags, Alex stated that "where the student went wrong is that they did not read the definition properly . . . where it says that the point, center of rotation, is equidistant to the original and the final figure."

Sam's video response began by describing the provided student work as related to rotations and then immediately turning to the definition of rotation:

> So if you look at Student A, we obviously can see that the way he was thinking was that he wants to rotate the flag ABCD around point C, which is a point of rotation. And the way he did it was that he started from the center, and then he drew his flag with an

> angle 90 degree with this flag and the new flag. Now *if we go back to the definition of rotation, the definition of rotation says that* a rotation with center O and angle θ is a transformation of the plane, which for every point P, a point P' is found so that the length of OP is equal to the length of OP' and the angle POP' is equal to θ. So obviously here, we can see that the first condition was not met. (Sam, Video Transcript; emphasis ours)

Sam continued to refer back to the definition explicitly throughout their explanation as they described how each pair of corresponding points on the image and preimage could be verified with the definition in order to identify the student error. Sam moved on to describe how a rotation can be constructed according to the definition, concluding, "... Then we would *satisfy the definition of rotation,* where the points and their corresponding points have equal length and the whole thing has similar or same angle of rotation" (Sam, Video Transcript, emphasis ours).

Throughout the set of prospective teachers' representations of practice, the video representations showcased highly articulated use of the definition in connection to the construction, where the written representations did not mention the definition, mentioned concepts that were not used to make sense of the specific student work of the situation, or where connections to the definition might be inferred by someone already familiar with the ideas, but perhaps not a learner.

Embodied Sense-Making with Transformations

Another contrast found among these video and written responses involved their use of embodied ways of making sense of the transformations, such as paper-folding activities or kinesthetic activities involving learners' bodies and/or mirrors to demonstrate reflections.

In the written responses, these prospective teachers discussed reflections using descriptions of moving one's right hand to match the left or of drawing a preimage on patty paper and then folding the paper and tracing to create the image. However, in the video responses, their descriptions relied primarily on the definition of rotation. Although constructing a rotation with patty paper is slightly more complex than constructing a reflection with patty paper, the work of verifying a rotation with patty paper can be done by tracing the preimage and then turning the patty paper while holding the center of rotation fixed to see if the preimage matches the image. Surprisingly, PSMTs did not focus on such embodied descriptions when responding with video, even though the video medium would appear to be ideal for such a description.

As an example, we consider prospective teacher Alex. In their written response, Alex described methods for constructing reflections with the following:

> One method that the students might use to perform the reflection of segment *a* over the line *r* would be to fold the paper on the reflection line . . . [and] make a copy on the other side of the line of reflection. This method is great for students who are new learners of reflections because it is a very hands-on way to make the reflection. (Alex, Written Response).

Alex continued on to explain a second method of construction:

> The second method that students could use to perform the reflection would be to make perpendicular bisectors at the two endpoints of the line and the line of reflection. These perpendicular bisectors will make a 90-degree angle and will create equal lengths between line a and line a'. This method is more for students who are comfortable with

> reflections because sometimes it can be difficult to visualize the transformation. (Alex, Written Response)

Here, Alex may have been alluding to the definition, but then discounted the use of this method as one only accessible to advanced students. Alex continued in the response to describe a whole-class discussion that would begin with eliciting thinking on students' real-world experiences with reflections, such as mirrors and the symmetry of butterflies, and concluded with discussion of reflections as "flipping" over a line. Although connecting ideas to real-world and embodied experiences can be powerful, Alex also drew no links from these ideas to the specific problem at hand.

Alex's video response on rotations, in contrast, seemed to privilege the sort of method they discounted in the written response. Alex indicated an appropriate response to Student 1 would direct the student to compare their work to the definition of rotation, specifically considering each pair of corresponding points individually. Alex provided similar feedback for Student 2, although indicating that Student 2's work was correct and concluded by explaining the need for attention to "all of the key points of the definition" (Alex, Video Transcript) as students examine rotations.

Although constructions of both reflections and rotations can be conceptualized in both embodied and more abstract ways, Alex chose to focus on embodied approaches in the written representation of practice and an abstract approach in the video representation of practice.

Making Use of Student Thinking

A final theme we found among our data related to the different ways in which prospective teachers attended to the provided student work in responding to written and video tasks. Although both tasks directed prospective teachers to use the student work, these prospective teachers largely disregarded the student samples in their written responses but specifically referred to sample student work as useful in learning to construct rotations in their video responses.

For example, in the written representation of practice, prospective teacher Sal indicated that students would be given an activity sheet on which they would practice constructing reflections using the definition. In describing this activity, Sal stated: "Considering the two examples of students' work, I can anticipate that students will identify the incorrect line of reflection, have incongruent reflected figures, and/or have non-equidistant points" (Sal, Written Response). Rather than taking up the student work specifically in response to the learners, Sal considered the student work as anticipatory only and does not bring examination of it into a class discussion.

In contrast, Sal's video response clearly refers to the provided student rotation samples in order to move learning forward. Early in the video, Sal says:

> I would ask the student if this precise definition applies to the figure he or she drew. And if not, then what is a reason that these two pairs of points are not equal distance? If we consider all the points on the original figure, we should make sure that the definition applies to every two, all corresponding points, of the rotated figure to the center of rotation. (Sal, Video Transcript)

This response demonstrates that Sal's efforts to move learning forward begin with the student work in specific connection to the underlying mathematics.

Implications for the Knowledge and Practice of Mathematics Teacher Educators

With regard to providing opportunities for prospective secondary teachers to reveal their MKT related understandings in teacher-created representations of practice, our work informs several questions. Namely: How does the creation medium matter? How does a mathematician educator's conception of knowledge for teaching matter? Is the portrait of a teacher's emergent practice shown through one medium consistent with the image through another? When the medium is more authentic to practice, for instance a live performance compared to a written narrative, how might this shape teachers' responses?

In this study, we considered how different creation mediums may impact the mathematical knowledge evinced by analyzing teacher-created representation of practices. We found, by analyzing representations of practice through the lens of Foundation knowledge as conceptualized by combining Rowland's (2013) and Simon's (2006) formulations, that video representations of practice may reveal higher knowledge levels for a specific dimension of MKT than written representations of practice. We also found that there were zero cases, among our 27 sets of written and video pairs of representations of practice, where the written representations of practice revealed higher Foundation knowledge than video representations of practice did. Based on these results, we problematize two issues of concern to mathematician educators: conceptions of mathematics, and the use of representations of practice as an assessment, whether formative or not, of a teacher's knowledge.

Our first finding, that there is a different use of definitions across the media, prompts us to problematize conceptions of mathematics. This finding broaches the question: What results would we have found had we conceptualized knowledge differently, both in our analytic process, and more generally, as mathematician educators? Gutierrez (2017) argued that the dominant mathematics culture privileges algebraic representations over spatial representations, abstraction over context, and mind over body. Further, humanizing mathematics must involve broadening the reasoning that is valued. Although we stand by the importance of connecting definition to construction, we also recognize, in retrospect, the narrowness of our coding and the adherence of this coding to dominant mathematics culture. Had we conceptualized knowledge differently, in our materials, coding, and scoring rubrics, we may have found different results because the prospective teachers might have absorbed different messages about mathematics in the course.

Our results also prompt us to problematize the use of representations of practice as an assessment of teachers' mathematical knowledge, regardless of how knowledge is conceptualized. The differences noted among the selected group of prospective teachers' representations of practice indicate that there is value in providing opportunities for prospective teachers to reveal MKT in multiple ways. Although our data showed these differences in only seven of 27 completed sets of responses, this is enough to cause us to question the validity of assessing prospective teachers' MKT in only one way. For example, if we had only used the written response task to gather information on these prospective teachers' MKT, we would have grossly underestimated the knowledge these prospective teachers held. In each of these cases, the video responses indicated a higher level of MKT than the written responses.

An interesting post-analysis observation, that calls for further research, involves the existence of language learners in this sample. Although we do not have specific demographic data on each of these students, it is clear in some of the videos (in this set and in other samples of our data), that some of these students are language learners. As we as mathematician educators work to practice equitable teaching in our preparation of mathematics teachers, it is possible that we underestimate the knowledge with which PSMTs enter our courses, particularly the knowledge of language learners. In this data set, PSMTs demonstrated higher levels for a dimension of MKT in video (oral) form than in written form. As many language learners speak English before they learn to write it, this finding is not surprising. But it does speak to the need for mathematician educators to attend to the ways in which we provide opportunities for PSMTs to reveal their knowledge to us, specifically, our response medium choices.

A limitation of this work is that the video and written responses relate to slightly different geometry concepts. Although reflections and rotations are closely related by similar fundamental notions, there are some differences in them. However, the order in the curriculum materials from which these tasks are drawn called for the video task on rotations to precede the written task on reflections. With that order, we might expect the written task to show higher foundation MKT levels and it does not. Although we do not have specific data on the instructional decisions made by the prospective teachers' instructors in terms of order of presentation, the structure of the materials leads us to consider the format of the response as a possible factor in the revealing of different levels of MKT.

These findings leave us with implications for future research. Our data was bound by the MODULE(S^2) curriculum materials from which the sample student work was collected and thus limited our choice of medium to explore PMSTs' MKT related to transformation concepts. We may learn more about the impact of the medium by gathering video responses related to reflections and written responses related to rotations, thus reversing the medium used to gather PSMTs' MKT on these concepts. In addition, we wonder if these findings are specific to the geometric topic of transformations. MODULE(S^2) materials include teacher-created representation of practice activities across four content areas: Geometry, Algebra, Statistics, and Mathematical Modeling. Exploration of the differences in observed MKT produced across video and written responses in these other content areas is an important area of future research.

It matters how we choose to provide opportunities for PSMTs to reveal their understandings. When the knowledge we are attempting to observe is MKT, teacher-created representations of practice are an appropriate way to reveal PSMTs' understandings. However, more accurate assessments of the MKT PSMTs hold can be found by providing opportunities to reveal this knowledge in more than one medium. Here, we reported on the mediums of written and video responses, but perhaps even more aspects of this knowledge might be revealed if other mediums were offered. Working with PSMTs to develop MKT begins with building understanding of the MKT they hold.

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Appendix 1
Video Response to Student Thinking on Rotations Assignment

Students in your 8th grade math class have defined the isometric transformation rotation with the definition shown below and are working on performing transformations in preparation for exploration of the properties of the transformations (CCSS-M 8.G.A.1).

A **ROTATION** with center *O* and angle is a transformation of the plane which, for every point *P*, a point *P′* is found so that $OP \cong OP'$ and $\angle POP' = \alpha$.

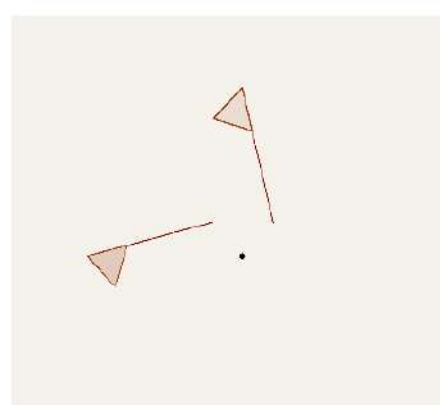
You have given students the following task:

Using the method of your choice, rotate the flag 90 degrees counterclockwise about the given point.

As students are working independently on rotations of a flag around a point, you observe two students with the following work completed.

Student 1:

Student 2:



Record a video of yourself providing a response to both Student 1 and Student 2 where you include:

- A summary of what each student might be thinking and what is worthwhile or reasonable about that student's thinking.
- A response to each student that does one or more of the following as you deem appropriate: helps the student finish their thought, prompts the student to investigate an error, or helps the student move forward in their thinking.
- Your response should indicate both your understanding of the student thinking and your understandings about the definition of rotation and ways in which this is applied to various methods of construction.

Upload your video to the following link:

Feedback Chart

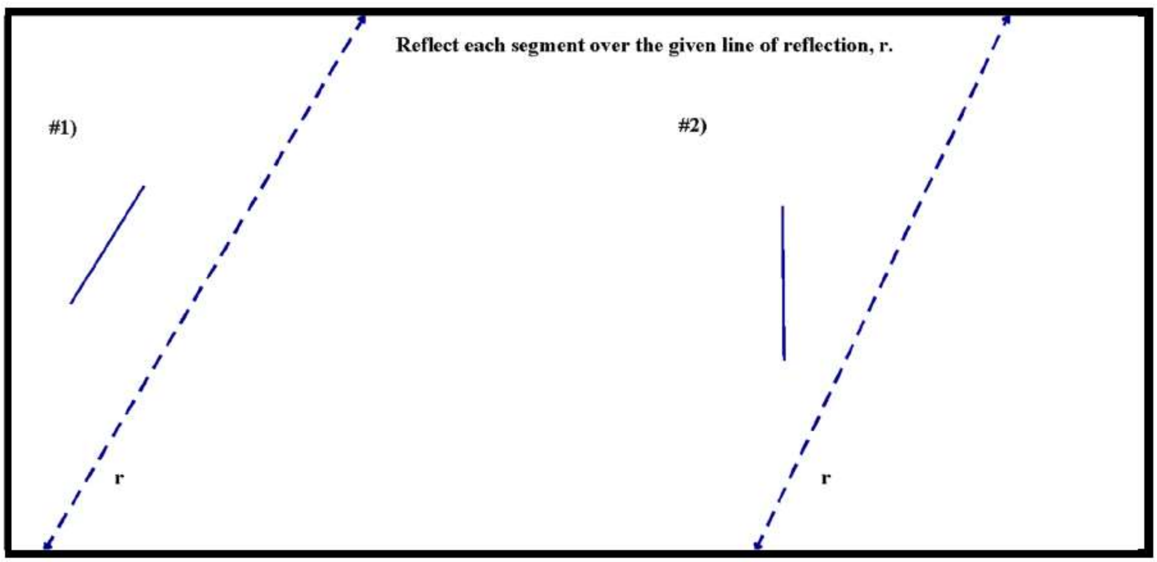
Descriptor	Meets Expectations	Does Not Meet Expectations
Is the summary of student thinking reasonable?	Summary points to reasonable explanation of student responses.	Summary does not attend to what students might have been thinking.
Is the response to student 1 reasonable?	Response to student 1 appropriately helps the student complete their thinking, prompts the student to investigate an error, or helps the student move forward in their thinking.	Response to student 1 does not accurately assess student understanding and move the student in a reasonable direction.
Is the response to student 2 reasonable?	Response to student 2 appropriately helps the student complete their thinking, prompts the student to investigate an error, or helps the student move forward in their thinking.	Response to student 2 does not accurately assess student understanding and move the student in a reasonable direction.
Is the mathematical language used appropriate?	Oral description of mathematical ideas uses accurate mathematical language.	Oral description of mathematical ideas does not use accurate mathematical language.

Appendix 2
Written Response to Student Work on Reflections Assignment

Students in your 8th grade math class have defined the isometric transformation reflection with the definition shown below and are working on performing transformations in preparation for exploration of the properties of the transformations (CCSS-M 8.G.A.1).

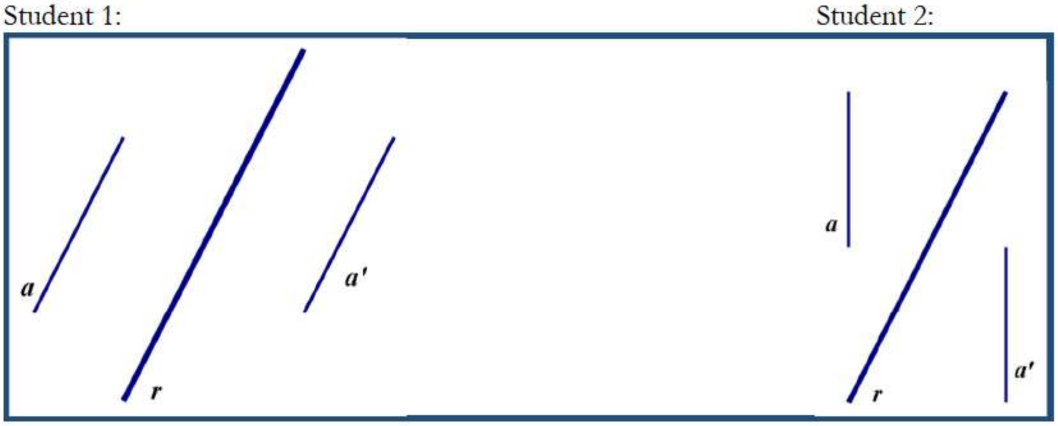
> A **REFLECTION** in line l is a transformation of the plane which, for every point P in the plane:
> $P' = P$ (if P is on l) and,
> l is the perpendicular bisector of PP' (if P is not on l).

You have provided students with the following task:



As students are working on the task, you observe two students with the following work completed.

Student 1: Student 2:



<u>Write a paper</u> in which you clearly describe a plan for how you will conduct a whole class discussion which will allow you to elicit student thinking about these reflections, with specific use of the two example students' work, and move the class toward understanding connections between methods of reflection and the definition of reflection. Your plan for the discussion should include discussion questions, descriptions of the ways you anticipate that students might respond to questions, and any appropriate tasks that will move student thinking forward. Your response should indicate your understandings about the definition of reflection and ways in which this is applied to various methods of construction.

Feedback Chart

Descriptor	Meets Expectations	Does Not Meet Expectations
Are anticipated solutions realistic?	Both correct and incorrect solutions are generated using a variety of methods.	Solutions are not reasonable and/or do not include a variety of method and/or no correct solution is provided.
Is the whole class discussion plan reasonable?	Discussion questions are included that follow a logical path and will move student thinking forward.	Discussion questions are not logically sequenced and/or not appropriate to move thinking forward.
Are anticipated student responses for the whole class discussion reasonable?	Anticipated student responses are reasonable.	Anticipated student responses are unreasonable or not included.
Will the task posed move thinking toward properties of reflections?	Task provided will move students toward properties of reflections.	Task provided does not logically follow from previous and/or will not move toward properties of reflections.