

Black Holes Hint Towards De Sitter-Matrix Theory

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Abstract

De Sitter black holes and other non-perturbative configurations can be used to probe the holographic degrees of freedom of de Sitter space. For small black holes evidence was first given in seminal work of Banks, Fiol, and Morrise; and followups by Banks and Fischler; showing that dS is described by a form of matrix theory. For large black holes the evidence given here is new: Gravitational calculations and matrix theory calculations of the rates of exponentially rare fluctuations match one another in surprising detail. The occurrence of the Nariai geometry and the “inside-out” transition are especially interesting examples which I explain.

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1 Entanglement in de Sitter Space

In this paper I will assume that there is a holographic description of the static patches of four-dimensional de Sitter space^[1]; but unlike AdS, de Sitter space has no asymptotic boundary where the degrees of freedom are located. Instead, the holographic degrees of freedom are nominally located on the boundary of the static patch (SP) (see for example [1][2][3][4][5][6]); that is to say, the stretched horizon.

Static patches come in opposing pairs. To account for the pair, two sets of degrees of freedom are required. The Penrose diagram of de Sitter space in figure [1] shows such a pair of SPs along with their stretched horizons. The center of the SPs (sometimes thought of as the points where observers are located) will be called the pole and the antipode.

¹The various mechanisms and calculations described in this paper apply to four dimensions. Generalization to other number of dimensions is non-trivial and I will not undertake the task here.

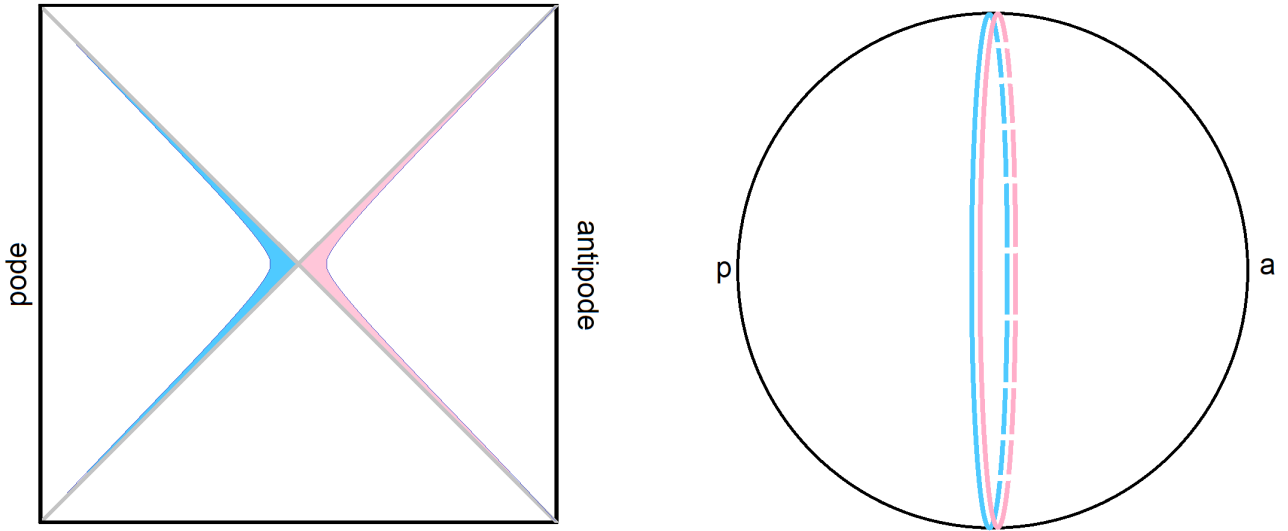


Figure 1: The left panel shows the Penrose diagram for de Sitter space with a particular choice of opposing static patches. The blue and pink regions are the stretched horizons of the two SPs. The right panel shows the spatial geometry of a time-symmetric slice. The blue and pink surfaces represent the stretched horizons.

Although it is clear from the Penrose diagram that the two SPs are entangled in the thermofield-double state, no clear framework similar to the Ryu-Takayanagi formula has been formulated for de Sitter space. This paper is not primarily about such a de Sitter generalization of the RT framework but I will briefly sketch what such a generalization looks like.

We assume that the entanglement entropy of the two sides—pode and antipode—is proportional to the minimum area of a surface homologous to the boundary of one of the two components—let us say the pode side. But what do we mean by the boundary? The full spatial slice at $t = 0$ has no boundary, but the static patch is bounded by the blue stretched horizon. Thus we try the following formulation:

The entanglement entropy of the pde-antipode systems is $1/4G$ times the minimal area of a surface homologous to the stretched horizon (of either side).

This however will not work. Figure 2 shows the spatial slice and the adjacent pair of stretched horizons. The dark blue curve represents a surface homologous to the pde's stretched horizon. It is obvious that that curve can be shrunk to zero, which if the above formula-tion were correct would imply vanishing entanglement between the pde and antipode static patches.

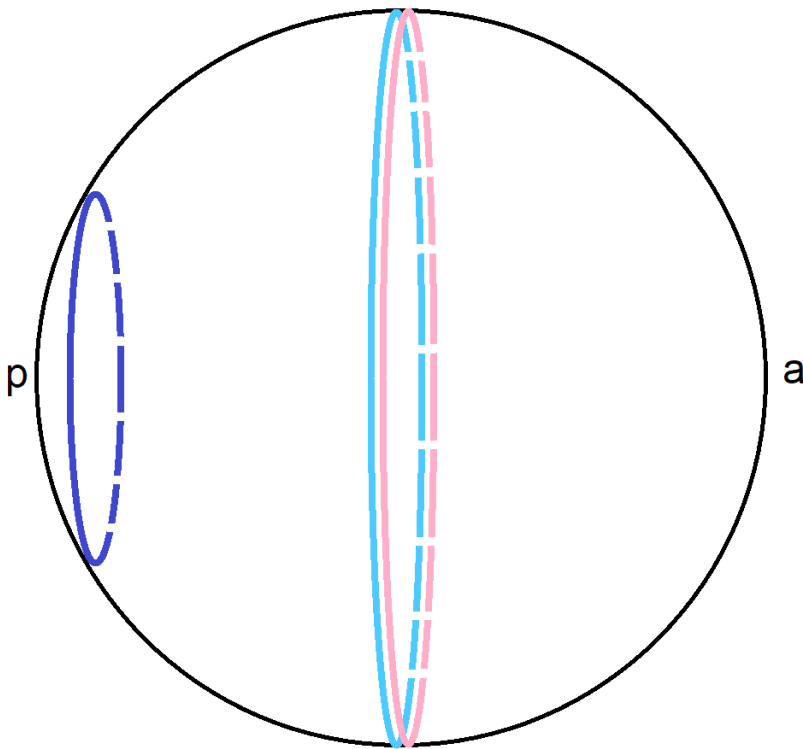


Figure 2: A $t = 0$ slice of dS and the stretched horizons shown as light blue and pink great circles. The dark blue surface is homologous to the light blue horizon. It can be shrunk to a point.

To do better first separate the two stretched horizons a bit. This is a natural thing to do since they will separate after a short time, as is obvious from figure 1. Let us now reformulate a dS-improved version of the RT principle:

The entanglement entropy of the pole-antipode systems is $1/4G$ times the minimal area of a surface homologous to the stretched horizon of the pole, and lying between the two sets of degrees of freedom, i.e., between the two stretched horizons.

This version of the RT principle is illustrated in figure 3

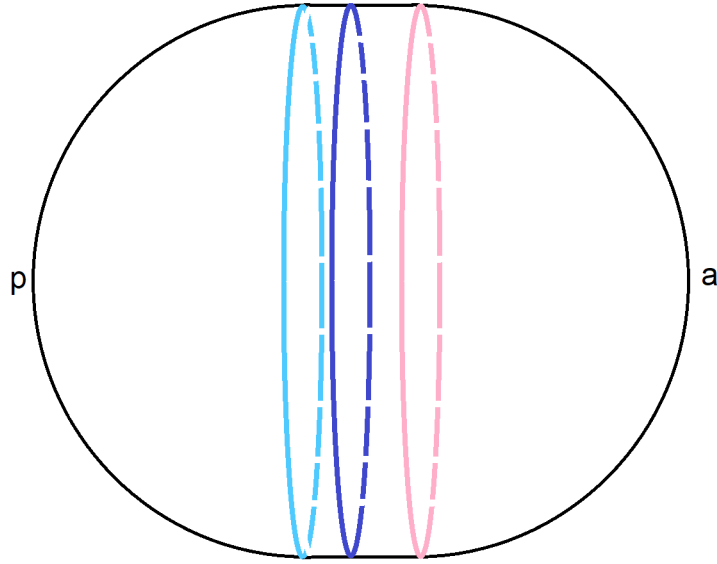


Figure 3: The dark blue curve represents the minimal surface lying between the two stretched horizons shown in light blue and pink.

It is evident from the figure that the area of the dSRT surface is the area of the horizon. This gives the entanglement entropy that

we expect [7], namely

$$\frac{\text{Horizon Area}}{4G}.$$

One thing to note, is that in anti-de Sitter space the phrase “lying between the two sets of degrees of freedom” is redundant. The degrees of freedom lie at the asymptotic boundary and any minimal surface will necessarily lie between them.

This version of the de Sitter RT formula is sufficient for time-independent geometries. A more general “maxmin” formulation goes as follows: Pick a time on the stretched horizons and anchor a three-dimensional surface Σ connecting the two. This is shown in figure 4.

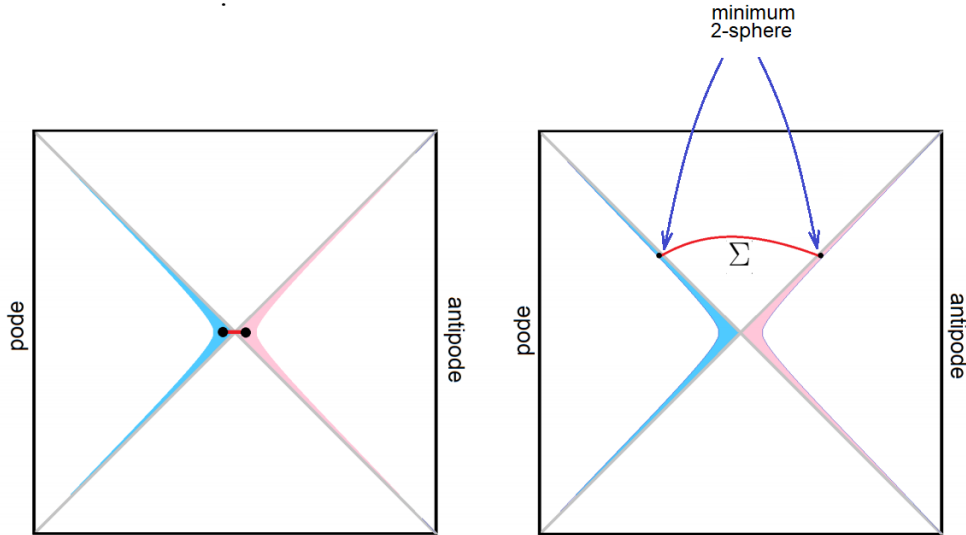


Figure 4: In both panels the black dots represent the anchoring points of a space-like surface Σ connecting the two horizons at a particular time. The minimal two-sphere cutting Σ lies at the anchoring points.

Find the minimum-area two-dimensional sphere that cuts the three-dimensional surface Σ and call its area $A_{min}(\Sigma)$. It is not hard to show that the minimum area sphere hugs one of the two horizons as in figure [4](#). The reason is that in de Sitter space the local 2-sphere grows (exponentially) as one moves behind the horizon.

Now maximize $A_{min}(\Sigma)$ over all space-like Σ . Call the resulting area

$$A_{maxmin}.$$

The entanglement entropy between the pole and antipode static patches is,

$$S_{ent} = \frac{A_{maxmin}}{4G}. \tag{1.1}$$

Because $A_{min}(\Sigma)$ occurs at the anchoring points the maximization of $A_{min}(\Sigma)$ is redundant in the case depicted in figure [1](#).

It should be possible to generalize the dSRT formula to include bulk entanglement term, but I will save this for another time.

Now we turn to the main subject of this paper—dS black holes and their implications for dS holography.

2 From Small Black Holes to Nariai

The properties of black holes in four-dimensional de Sitter space provide hints about the holographic degrees of freedom and their dynamics. These hints will lead us to a remarkably general conclusion: the underlying holographic description of de Sitter space

must be a form a matrix quantum mechanics.

The Schwarzschild de Sitter metric is given by,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2$$

$$f(r) = 1 - \frac{r^2}{R^2} - \frac{2MG}{r} \quad (2.2)$$

where R is the de Sitter radius, M is the black hole mass, and G is Newton's constant.

There are two horizons, the larger cosmic horizon and the smaller black hole horizon. The horizons are defined by $f(r) = 0$. Defining $g(r) = rf(r)$ the horizon condition becomes,

$$g(r) = r - r^3/R^2 - 2MG = 0. \quad (2.3)$$

The function $g(r)$ is shown in figure [5](#).

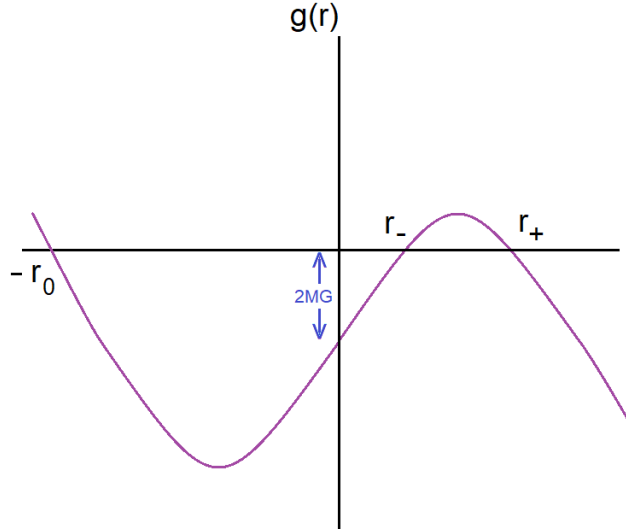


Figure 5: The function $g(r)$ and its zeros, r_0 and $r_{\pm}$.

For values of M satisfying

$$0 < \frac{MG}{R} < \frac{1}{3\sqrt{3}}, \quad (2.4)$$

equation (2.3) has three solutions, two with positive values of r , and one with negative r . The two positive solutions, r_- and r_+ define the black hole horizon and the cosmic horizon respectively. The negative solution, $r = -r_0$, is unphysical. Outside the range (2.4) the metric has a naked singularity. Given the values of R (or equivalently the cosmological constant) and G , there is only one parameter in the metric, namely M . Alternatively we may choose the independent parameter to be either r_+ , r_- , or the dimensionless parameter x defined by,

$$x \equiv (r_+ - r_-)/R. \quad (2.5)$$

The variable x runs from $x = -1$ to $x = +1$. Over this range the mass M runs over its allowable values (2.4) twice: once for $x < 0$ and once for $x > 0$. As x increases from -1 to 0 the black hole horizon r_- grows, and the cosmic horizon r_+ shrinks, so that the two become equal at $x = 0$. When x becomes positive the two horizons are exchanged so that $r_- > r_+$. Beyond that r_+ becomes the black hole horizon, r_- the cosmic horizon.

There are two possible ways to think about this. In the first we assume that the range $x > 0$ is redundant and simply describes the same states that were covered for $x < 0$; roughly speaking we think of the choice of the sign of x as a gauge choice. The second possibility is that the two ranges are physically different configurations. We will adopt the latter viewpoint in this paper.

The cubic function $g(r)$ in (2.3) may be written as a product,

$$g(r) = -\frac{(r - r_+)(r - r_-)(r + r_0)}{R^2}. \quad (2.6)$$

Matching (2.3) with (2.6) we find the following relations,

$$r_0 = r_+ + r_- \quad (a)$$

$$R^2 = r_+^2 + r_-^2 + r_+ r_- \quad (b)$$

$$2MGR^2 = r_+ r_- (r_+ + r_-) \quad (c)$$

$$x^2 = \frac{r_+^2 + r_-^2 - 2r_+ r_-}{R^2} \quad (d) \quad (2.7)$$

The last of these equations—(2.7)(d)—is just the square of the defining relation (2.5). By combining (2.7)(d) and (2.7)(b) we find the relation,

$$R^2 - r_+^2 - r_-^2 = \frac{R^2}{3} (1 - x^2) \quad (2.8)$$

The significance of this equation will become clear in subsection 3.2.

3 Entropy Deficit

In thermal equilibrium the entropy is maximized subject to the constraint of a given average energy. In the context of the static patch, the equilibrium entropy is the usual de Sitter entropy which

I will call S_0 ,

$$S_0 = \frac{\text{area}}{4G} = \frac{\pi R^2}{G}. \quad (3.9)$$

Fluctuations may occur in which the entropy is decreased to a smaller value S_1 . The probability for such a fluctuation is given in terms of the entropy deficit,

$$\Delta S \equiv S_0 - S_1 \quad (3.10)$$

by

$$\text{Probability} = e^{-\Delta S} \quad (3.11)$$

3.1 Small Black Holes

By a small black hole I mean one whose mass in Planck units is fixed as R becomes large. It may also be defined by its entropy being parametrically order 1 as the de Sitter entropy is taken to infinity.

Let us consider the probability for a fluctuation in which a black hole of mass M appears at the pole². The thermal equilibrium entropy of de Sitter space is given in (3.9). To compute the entropy S_1 of a state with a black hole at the pole we use equation (2.3) to compute r_+ . To lowest order in M we find,

$$r_+ = R - MG. \quad (3.12)$$

²The center of the causal patch at $r = 0$. The center of the opposing static patch is called the antipode [6].

The area and entropy of the cosmic horizon to lowest order are,

$$\begin{aligned}\text{area} &= 4\pi(R^2 - 2RMG) \\ S &= \frac{\pi(R^2 - 2RMG)}{G},\end{aligned}\tag{3.13}$$

and the entropy deficit is,

$$\Delta S = 2\pi RM.\tag{3.14}$$

The probability for the fluctuation is

$$\text{Prob} = e^{-\Delta S} = e^{-2\pi RM},\tag{3.15}$$

which is also the Boltzmann weight $e^{-\beta M}$.

Another interesting form for the entropy deficit for small black holes can easily be derived and is given by,

$$\Delta S = \sqrt{Ss},\tag{3.16}$$

where s is the black hole entropy $4\pi M^2 G$.

Note that for fluctuations involving small black holes of fixed mass, the entropy deficit goes as $S^{1/2}$. For example the probability of a Planck mass black hole ($s = 1$) is³,

$$\text{Prob} \sim \exp(-\sqrt{S}).\tag{3.17}$$

Equation (3.16) is a strong hint about the holographic degrees of freedom of de Sitter space. It is quite unusual in the way it depends

³In this paper the notation $\exp$ is synonymous with “exponential in.” Thus e^S , $e^{S/3}$, e^{2S} are all $\exp S$.

on the entropies of not only the black hole, but also on the entropy of the cosmic horizon. The question it raises is: without direct reference to gravity, what kind of holographic degrees of freedom can lead to such a relation?

Following [8], we will see in section 5 that the answer is matrix degrees of freedom similar to those of BFSS M(atrix) theory. Moreover 3.17 is suggestive of instantons in large- N matrix quantum mechanics and large- N gauge theories. In subsection 3.2 we will see another more detailed relation for the entropy deficit of large black holes that even more strongly supports the claim for matrix degrees of freedom.

3.2 Large Black Holes

By a large black hole I mean one with Schwarzschild radius parametrically of order the de Sitter radius R . Equivalently, the entropy of a large black hole is a fixed fraction of the de Sitter entropy. Such black holes are characterized by fixed values of the dimensionless variable x defined by (2.5). At $x \approx -1$ the black hole radius is very small compared to the radius of the cosmic horizon. At $x = 0$ the two horizon areas become equal, and of order R^2 . At that point the geometry is called the Nariai geometry. Its properties are reviewed in the appendix.

For $x > 0$ the black hole and cosmic horizon switch roles. As mentioned earlier we will consider $x = -1$ and $x = 1$ to be different states.

The entropy of a horizon of radius r is given by,

$$S = \frac{\text{area}}{4G} = \pi r^2. \quad (3.18)$$

Applying this to the original de Sitter horizon with radius R , and to the horizons $r_{\pm}$ of the Schwarzschild-dS geometry,

$$\begin{aligned} S_0 &= \frac{\pi R^2}{G} \\ S_+ &= \frac{\pi r_+^2}{G} \\ S_- &= \frac{\pi r_-^2}{G}. \end{aligned} \tag{3.19}$$

The total entropy is the sum of the black hole and cosmic entropies,

$$S(x) = S_+(x) + S_-(x). \tag{3.20}$$

Armed with these relations we may write equation (2.8) in the surprisingly simple form,

$$\begin{aligned} \Delta S(x) &= \pi R^2 \left(\frac{1 - x^2}{3G} \right) \\ &= \frac{S_0}{3} (1 - x^2) \end{aligned} \tag{3.21}$$

Equation (3.21) is illustrated in figure 6.

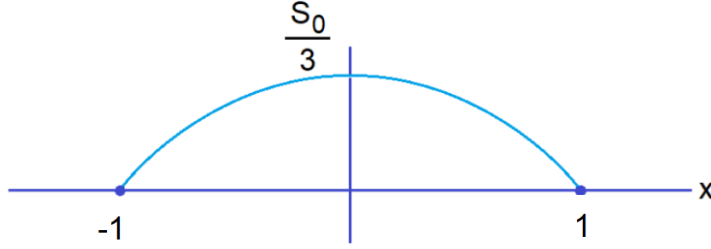


Figure 6: The entropy deficit as a function of x .
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Equation (3.21) may also be written,

$$\Delta S(x) = \Delta S_N (1 - x^2), \quad (3.22)$$

where $\Delta S_N = \frac{1}{3}S_0$ is the entropy deficit of the Nariai geometry. Note that ΔS is symmetric under $x \rightarrow -x$ and perfectly smooth at $x = 0$, the point where the black hole and cosmic horizons cross.

Equation (3.22) gives a detailed relation for how the entropy deficit varies with the parameter x . In principle the ratio $\frac{\Delta S(x)}{\Delta S_N}$ could have been a good deal more complicated, depending in an arbitrary way on x and the entropy S . We will see in section 5 that the relation in (3.22) is characteristic of theories with matrix degrees of freedom.

4 Probabilities and the Entropy Deficit

The importance of the entropy deficit is that it determines probabilities for Boltzmann fluctuations through the formula (3.11)

For example (3.21) implies that the probability for the occurrence

of a freak fluctuation in which a black hole of mass M appears at the pole is,

$$\text{Probability}(x) = e^{-\Delta S(x)} = e^{\pi R^2(x^2-1)/3G}. \quad (4.23)$$

The location of the black hole need not be exactly at the pole. Let us introduce cartesian coordinates X_i centered at the pole. The entropy will then depend not only on x but also X_i . By suitable normalization of the coordinates the entropy deficit in (4.23) can be generalized to [6],

$$\text{Probability}(X) = e^{-\Delta S(X)} = e^{\pi R^2(X^2-1)/3G}. \quad (4.24)$$

where X represents the four component object (x, X_i) .

Now consider the total probability for a black hole to nucleate anywhere in the static patch. It is given by an integral of the form,

$$\text{Probability} = \int \frac{d^4 X}{R^4} e^{\pi R^2(X^2-1)/3G}. \quad (4.25)$$

The range of the integration is from $X = 0$ to $X \sim 1$. The details of the boundary at $X \sim 1$ are not important as long as the components of X are of order 1. At the boundary of the integration the black hole is very small ($x \sim 1$) or its location is close to the horizon.

Defining $u = R^2 X^2$ this may be written,

$$\text{Probability} = e^{-\pi R^2/3} \int_0^{R^2} du \, u e^{\pi u/3G}. \quad (4.26)$$

The integral is straightforward and gives,

$$\text{Probability} = \left(\frac{3}{S_0} - \frac{9}{S_0^2} \right) + \left(\frac{9}{S_0^2} \right) e^{-S_0/3}. \quad (4.27)$$

Let us rewrite (4.27) using $S_0 = \pi R^2/G$,

$$\text{Probability} = \left(\frac{3G}{\pi R^2} - \frac{9G^2}{\pi^2 R^4} \right) + \left(\frac{9G^2}{\pi^2 R^4} \right) e^{-\frac{\pi R^2}{3G}}. \quad (4.28)$$

The first term in (4.28) appears to be perturbative in the Newton constant. It represents contributions from very small black holes which appear close to the horizon and then fall back in. However, one might argue that this is misleading and that we should cut off the integral when the mass of the black hole becomes microscopic. In that case the first term in (4.28) would be replaced by something $\exp(-\sqrt{S})$. This contribution numerically dominates the second term but is non-universal—it depends sensitively on micro-physics.

The second term, although very sub-leading, is what really interests us. It is non-perturbative in G and due to a saddle point in the integrand at $X = 0$. This saddle point represents the contribution of the Nariai geometry to the path integral. It is universal, independent of any micro-physics.

One may wonder whether there is any process for which the non-universal small black hole contribution vanishes and the Nariai geometry dominates. The answer is yes; the Nariai geometry gives the leading contribution to the “inside-out” process (see section 7).

5 dS-Matrix Theory

One can argue on the basis of entropy bounds that the holographic degrees of freedom live at the horizon of the static patch, but that argument does not tell us anything about the nature of those degrees of freedom. I will not try to give a detailed model here, but the properties of de Sitter black holes can tell us more. What we will learn is that the degrees of freedom must be matrices [2][3][4][5], and the Hamiltonian should include a term whose role is to enforce certain constraints.

5.1 Small Blocks

Returning to equation (3.16) for small black holes,

$$\Delta S = \sqrt{S} s,$$

this relation contains a hint about the nature of the holographic degrees of freedom of de Sitter space. Following Banks and collaborators [2] it motivates us to conjecture that the degrees of freedom are matrices (see also [3][4][5]) in the same sense as in M(atric) theory [8]. To see why let's assume that the horizon degrees of freedom are a collection of $N \times N$ Hermitian matrices,

$$A_{m,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,N} \end{pmatrix} \quad (5.29)$$

An implicit index runs over some finite range and may include both bosonic and fermionic matrices. Taking a cue from BFSS M(atrrix) theory [8] we may think of the index m as running over a set of N $D0$ -branes. Very roughly the diagonal elements represent positions of the branes, while the off-diagonal elements represent operators which create and annihilate strings connecting the $D0$ -branes.

For now we will not specify any particular form for the Hamiltonian but we will assume that one exists, as well as a thermal ensemble at the appropriate temperature. We also assume that the total entropy in thermal equilibrium is proportional to the number of degrees of freedom,

$$S_0 = \sigma N^2 \tag{5.30}$$

where σ is the entropy per degree of freedom.

Consider a state with a black hole of entropy

$$s = \sigma m^2 \tag{5.31}$$

located at the pole. Motivated by M(atrrix) Theory we assume that the degrees of freedom split into block-diagonal form with the cosmic horizon degrees of freedom forming an $(N - m) \times (N - m)$ block, and the black hole degrees of freedom forming an $m \times m$ block. The indices labeling the large block and small blocks will be called I and i respectively. The entries in the large and small blocks are a_{IJ} , and a_{ij} . The off-diagonal elements connecting the two blocks are a_{iJ} and a_{Ij} .

Again, motivated by M(atrrix) theory, we will assume that in a state composed of two well-separated components—in this case

the small black hole and the large cosmic horizon—the off-diagonal degrees of freedom a_{iJ} and a_{Ij} are constrained⁴ to be in their ground states, and therefore carry no entropy [2][3][4][5]. In other words the state is constrained by $2m(N - m)$ constraints which express the condition that there are no strings connecting the $D0$ -branes in the two blocks. Classically these constraints take to form,

$$a_{iJ} = a_{Ij} = 0. \quad (5.32)$$

Subject to these constraints the entropy of this state is

$$S_1 = \sigma(N - m)^2 + \sigma m^2 \quad (5.33)$$

Assuming $m \ll N$ and working to lowest order in m the entropy deficit is,

$$\Delta S = 2\sigma N m. \quad (5.34)$$

Using (5.30) and (5.31) gives

$$\Delta S = 2\sqrt{S_s} \quad (5.35)$$

which reproduces (3.16) within a factor of 2. I don't know of any other mechanism that will accomplish this.

The factor of two discrepancy between (3.16) and (5.35) may be another significant hint whose meaning we will discuss in section 6.

⁴Banks and Fischler have proposed that the connection between localized objects and constrained states of holographic variables is the basis for understanding locality on scales smaller than that set by the cosmological constant [4][5].

5.2 Remark on Higher Dimensions

In higher dimensions things are more complicated. I will quote the d dimensional generalization of (3.16), valid for $s \ll S_0$:

$$\Delta S = \left(\frac{d-2}{2} \right) S_0^{\frac{1}{d-2}} s^{\frac{d-3}{d-2}}. \quad (5.36)$$

This formula, derived from the d -dimensional Schwarzschild solution, can be reproduced with matrix degrees of freedom but at a cost. It is necessary to allow the entropy per degree of freedom to depend on N according to⁵,

$$\sigma(N) \sim \frac{1}{N^{\left(\frac{d-4}{d-3}\right)}}. \quad (5.37)$$

I will leave any further discussion of the higher dimensional generalization to future work.

5.3 Large Blocks

Returning to the case $d = 4$ let us now consider fixed values of s/S_0 . The entropy deficit is proportional to the number of constraints,

$$\Delta S = 2\sigma m(N - m). \quad (5.38)$$

⁵A different view of the holography of higher-dimensional dS based on multidimensional matrices was given in [4][5].

Following (2.5) and using the fact that $r_{\pm}$ are proportional to the square roots of the entropies of the horizons, we define,

$$x = \frac{\sqrt{s} - \sqrt{S}}{\sqrt{S_0}}. \quad (5.39)$$

Using,

$$\begin{aligned} S &= \sigma(N - m)^2 \\ s &= \sigma m^2 \end{aligned} \quad (5.40)$$

we find,

$$x = \frac{2m}{N} - 1, \quad (5.41)$$

which leads to a relation identical to (3.22),

$$\begin{aligned} \Delta S(x) &= \Delta S_N(1 - x^2), \\ \Delta S_N &= \frac{1}{2}S_0. \end{aligned} \quad (5.42)$$

As in the earlier small-block case (5.35) the only difference between the gravitational result and the dS-matrix result is the numerical factor. At the Nariai point $x = 0$ the entropy deficit in the matrix theory is $S_0/2$ instead of $S_0/3$. In section 6 we will see that there is room in the matrix theory to decrease the numerical constant in (5.42) and bring it closer to its gravitational value of $1/3$.

There is nothing inevitable about the relation (5.42). It does not follow from any general statistical or thermodynamic principles. It

is a consequence of the matrix degrees of freedom and the particular assumptions concerning the way the system is decomposed into subsystems. The close correspondence between the matrix theory and general relativity calculations of $\Delta S(x)$ seems remarkable to me. I don't know any other holographic mechanism that can lead to it. However it is important to explain the discrepancy between the numerical factors. In the next section we will see that there is plenty of room in the matrix theory to modify the constants and bring them into alignment with their gravitational counterparts.

6 Dynamics of the Constraints

The degrees of freedom and Hamiltonian of the static patch are highly constrained by the symmetries of de Sitter space [6]. Implementing those symmetries is a very hard problem which I will not try to solve in this paper. My purpose is more modest; namely to illustrate a dynamical mechanism for how the constraints (5.32) can be enforced by energy considerations.

Let us add to the matrix degrees of freedom (5.29) one more $N \times N$ matrix denoted by $\mathcal{R}$,

$$\mathcal{R}_{m,n} = \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,N} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \cdots & r_{m,N} \end{pmatrix} \quad (6.43)$$

The notation is chosen to indicate that the eigenvalues of $\mathcal{R}$ represent radial position in the static patch.

To enforce the constraints we will assume the Matrix-theory La-

grangian contains the term,

$$L = \sum \text{Tr} (c^2 R^4 \dot{A}^2 - [\mathcal{R}, A][A, \mathcal{R}]) \quad (6.44)$$

where c is a numerical constant and the sum in (6.44) is over all the other matrices—bosonic and fermionic⁶—that comprise the degrees of freedom of the matrix theory.

Now consider a configuration representing an object well separated from the cosmic horizon. For simplicity the object could be at the pole at $r = 0$. The cosmic horizon is at $r = R$. To represent this we assume the matrix $\mathcal{R}$ has approximately block-diagonal form,

$$\begin{aligned} r_{IJ} &= R\delta_{IJ} + \epsilon_{IJ} \\ r_{ij} &= \epsilon_{ij} \\ r_{iJ} &= \epsilon_{iJ} \\ r_{Ij} &= \epsilon_{Ij}. \end{aligned} \quad (6.45)$$

where ϵ is a numerically small matrix representing quantum fluctuations.

Ignoring ϵ , the commutator term in (6.44) gives

$$\text{Tr} \sum [\mathcal{R}, A][A, \mathcal{R}] = R^2 \sum |a_{iJ}|^2. \quad (6.46)$$

Combining this with the kinetic term in (6.44) gives,

$$L = \sum (c^2 R^4 \dot{a}_{iJ} \dot{a}_{Ji} - R^2 a_{iJ} a_{Ji}) \quad (6.47)$$

⁶For fermionic matrices the quadratic kinetic term in (6.44) should be replaced by the usual Dirac term linear in time derivatives.

The effective Hamiltonian for the off-diagonal elements is a sum of harmonic oscillator Hamiltonians with frequency,

$$\omega = \frac{1}{cR}. \quad (6.48)$$

If ω is much larger than the other energy scales the oscillators will be forced to their ground states and the off-diagonal degrees of freedom will carry no entropy. In that case the analysis leading up to equations (5.35) and (5.42) applies unmodified.

The energy scale with which ω is to be compared is the temperature $T = 1/(2\pi R)$. If the numerical constant c is much smaller than 2π then the constraints will be tightly enforced, but the more interesting situation is when $c \sim 2\pi$. In that case the constraints will not be tightly enforced; the off diagonal elements will carry some entropy, but only a fraction of σ (the thermal entropy per degree of freedom in (5.30)). If we carry out the analysis leading up to equations (5.35) and (5.42) we will find that the only effect of relaxing the constraints is to change the numerical coefficients in these equations. For example it should be possible to choose c so as to change the constant in (5.35) from 2 to the gravitational value 1. At the same time that will decrease the value of factor 1/2 in (5.42) but to bring it to exactly 1/3 would require subtle and possibly fine-tuned properties of H . One might speculate that if the Hamiltonian satisfies the symmetry requirements of de Sitter space, this would be automatic.

7 The Inside-Out Process

Classically the Nariai geometry is stable, but not quantum-mechanically [9]. Initially the two horizons are at the same temperature (see appendix C) Now suppose a statistical fluctuation occurs and the left horizon emits a bit of energy which is absorbed by the right horizon. The effect is to increase T_L and decrease T_R . This creates a tendency (heat flows from hot to cold) for more energy to flow from left to right. The statistical tendency is for the left horizon to shrink down to a small black hole, while the right horizon grows to the full size of the de Sitter horizon. Eventually the small black hole will disappear, transferring all its energy to the cosmic horizon on the right side. Of course it could have happened the other way—the right horizon shrinking and the left growing.

How long does the entire process of evaporation take? The answer is roughly the Page time $t_p \sim S_0 R$. Note that this process does not violate the second law—the entropy increases from $\frac{2}{3}S_0$ to S_0 .

But now instead of running the system forward in time with e^{-iHt} we run it backwards with e^{iHt} . What will happen is the time reverse in which the system back-evolves to some micro-state of de Sitter space with either the left or right horizon growing. This implies that there are fluctuations in the thermal state which begin with dS, pass through Nariai space and eventually decay back to dS. The entire history from dS to N to dS is a massive Boltzmann fluctuation in which the de Sitter horizon emits a small black hole which then grows to the Nariai size, and then one of the two Nariai horizons shrinks back to nothing, while the other grows back to the

dS size.

In particular, the process can proceed so that the two horizons are exchanged. One may think of it, in terms of the diagram in figure 6, as a process in which the system migrates from $x = -1$ to $x = +1$, passing through the Nariai state at $x = 0$. This process of exchange of the horizons is the “inside-out” process. An observer (figure 7) watching this take place would literally see the dS turn itself inside out—the tiny black hole growing and becoming the surrounding cosmic horizon while the cosmic horizon shrinks to a tiny black hole (or no black hole at all).

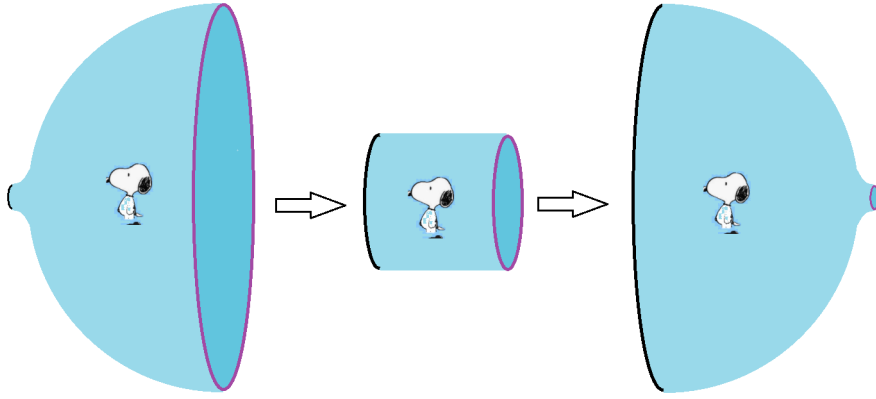


Figure 7: The inside-out transition as seen by an observer in the static patch.

For the inside-out process to take place the system must pass through the Nariai state at $x = 0$. Since the probability for this is $e^{-S_0/3}$ it is obviously not allowed perturbatively. Passing through the Nariai state gives the leading contribution to the transition $(x = -1) \rightarrow (x = 1)$.

It is tempting to think of the inside-out process as a quantum tunneling event mediated by some kind of conventional instanton,

i.e., a solution of the classical Euclidean equations of motion interpolating from $x = -1$ to $x = +1$. This is not correct—there is no such solution. What does exist is the classical Nariai solution eternally sitting at the point $x = 0$. This is similar to a process in which a system gradually thermally up-tunnels over a broad potential barrier, mediated by an so-called Hawking Moss instanton [10]. In the Hawking-Moss framework the exponential of the Euclidean action (in this case the action of the Euclidean Nariai geometry) gives the probability to find the system at the top of the potential [11]; in other words at the Nariai point. The probability is given by $e^{-S_0/3}$.

The HM instanton does control the rate at which such inside-out processes occur. There are two time scales of interest. The first which I'll call δt is how long does the process take from beginning ($x = -1$) to end ($x = 1$)? The answer is that it takes a time of order the Page time, $\delta t \sim t_{page} \sim S_0 R$. The other time scale, Δt , is the average time between inside-out events. That time is very much longer: δt is essentially instantaneous on the longer time scale Δt . This is shown in figure 8.

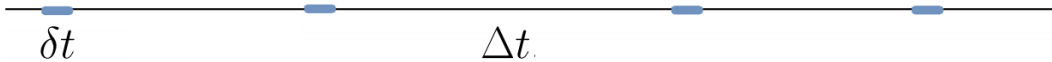


Figure 8: Time scales for the inside-out process. The short blue intervals δt represent the duration of process which takes place over a time of order the Page time. The long intervals Δt between them represent the times between inside-out transitions.

Under this circumstance the probability to find the system close

to the Nariai state would be the ratio,

$$\text{Probability} = \frac{\delta t}{\Delta t} \quad (7.49)$$

The probability to find the Nariai state is of order $e^{-S_0/3}$ from which we conclude,

$$\Delta t \sim RS_0 e^{S_0/3}. \quad (7.50)$$

The prefactor in (7.50) is not very reliable but it does show that the rate of inside-out events is determined by the exponential $e^{-S_0/3}$. This can be compared with the longest possible decay time for Coleman DeLuccia tunneling to a terminal vacuum, if in fact such decays are allowed. That time scale can in principle be as long as e^{S_0} although it can be much shorter. If we suppose the decay rate to terminal vacua is as long as possible then there is plenty of time for the inside-out process to occur many times before the de Sitter vacuum decays.

The inside-out process is especially interesting because its rate is controlled by the Nariai saddle at $x = 0$, with no contribution from small black holes. In section 4 the Nariai saddle was a tiny subleading effect in the probability for a black hole fluctuation, but the inside-out transition can only occur if the system passes through the Nariai point. Therefore the rate is determined by the universal saddle at $x = 0$.

It is obvious what the inside-out transition means in the dS-matrix theory. The matrix representation of the unconstrained thermal equilibrium state has all N^2 degrees of freedom fluctuating

in thermal equilibrium. The state with a small black hole is a constrained state [2][3][4][5] represented by block-diagonal matrices; one small block for the black hole, and one large block for the cosmic horizon. In the inside-out process the small block grows while the large block shrinks until they become equal, and then continues until the blocks are exchanged. In the process the system must pass through the configuration with two equal blocks which is the matrix version of the Nariai geometry.

8 Instantons and Giant Instantons

The processes of small black hole formation, and the inside-out transition, exhibit some interesting parallels with instanton-mediated processes in large- N gauge and matrix theories.

8.1 Some Probabilities

This subsection summarizes the results of some probability calculations in gravity and dS-matrix theory so that we can compare them with instanton amplitudes.

The thermal formation of the smallest black hole—one with entropy of order unity—has a matrix theory description in which the small block is a single matrix element and the number of constrained is $\sim 2N$. The entropy deficit is

$$\Delta S \sim N \tag{8.51}$$

and the corresponding probability is

$$P = \exp(-N). \tag{8.52}$$

Now consider the probability in de Sitter gravity for a minimal size black hole with $s = 1$,

$$P = \exp(-\sqrt{S}). \quad (8.53)$$

Using $S = \sigma N^2$ we see that (8.52) and (8.53) are essentially the same.

Next: a bigger fluctuation in the matrix theory, namely a fluctuation all the ways to the matrix version of the Nariai state $m = N/2$, in which the two blocks are equal. The entropy deficit is,

$$\Delta S = \frac{\sigma N^2}{2} \quad (8.54)$$

Compare that with the gravity result for the same process,

$$\Delta S = S_0/3. \quad (8.55)$$

Using $S = \sigma N^2$ shows that (8.54) and (8.55) scale in the same way.

8.2 Instantons

Now let us turn to instantons, first in matrix quantum mechanics and then in gauge theories. The simplest example is single-matrix quantum mechanics with Lagrangian,

$$L = \frac{1}{2g^2} \left(\text{Tr} \dot{A}^\dagger \dot{A} - \text{Tr} V(A) \right), \quad (8.56)$$

and V being a double well potential like the one in figure 9.

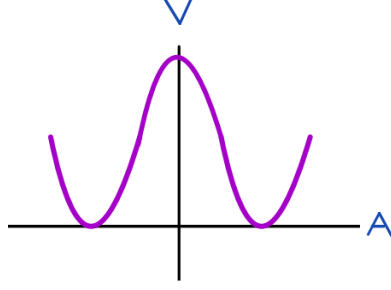


Figure 9: A double-well potential for a matrix model.

By standard arguments this can be reduced to the quantum mechanics of a one dimensional system of N fermions which represent the eigenvalues of A .

An individual eigenvalue can tunnel from the left well to the right well with probability given by an instanton. The probability for a single eigenvalue tunneling is

$$P_1 \sim \exp(-1/g^2). \quad (8.57)$$

In the 't Hooft large- N limit,

$$g^2 N = \lambda \quad (\lambda \sim 1). \quad (8.58)$$

We find,

$$P_1 \sim \exp(-N). \quad (8.59)$$

This simple instanton process scales with N the same as in (8.52), suggesting that the formation of a Planck-mass black hole is an instanton-mediated process in the dS-matrix theory.

8.3 Giant Instantons

We may also consider a process in which all the eigenvalues tunnel from one side to the other. I'll call it a “giant instanton.” The action for a giant instanton is N times larger than the simple instanton and the probability for the “giant transition” is,

$$P_N \sim \exp(-N^2). \quad (8.60)$$

The probability for the giant transition scales the same way as the inside-out transition, namely $\exp -S_0$. We note that this transition, much like the inside-out transition takes the system between states related by a symmetry.

Instantons and giant instanton transitions also exist in Yang Mills theory. Recall that an instanton in an $SU(N)$ theory lives in an $SU(2)$ subgroup and describes a tunneling transition of the $SU(2)$ Chern-Simons invariant by one unit. The rate also scales like $\exp(-N)$.

One can also consider a transition in which all $N/2$ commuting $SU(2)$ -subgroups tunnel. The rate for such giant instantons is $\exp(-N^2)$. Thus we see a common pattern governing non-perturbative transition rates in large- N gauge theories, and matrix theories, and also Boltzmann fluctuations in de Sitter space.

9 Remarks about the Holographic Principle in dS

Semiclassically, the static patch of de Sitter space is a holographic quantum system with the degrees of freedom localized at the stretched horizon. This is reasonable in semiclassical-gravity but things are

more complicated in the full non-perturbative theory. Large Boltzmann fluctuations can lead to higher topologies such as the Nariai geometry, and the horizon can break up into multiple horizons. Where, under those circumstances, do the holographic degrees of freedom reside? On the outermost or largest horizon? On the union of all of the horizons? Or is the hologram more abstract and not localized at all?

1. The outermost horizon? Consider a state very near the Nariai limit but with one horizon being slightly bigger than other. In this case the largest horizon has entropy slightly greater than $S_0/3$. That is clearly not enough degrees of freedom to describe both horizons which in sum have entropy $2S/3$. Moreover the sizes of the horizons can change with time; the largest can become the smallest and vice versa.
2. The union of horizons? This also does not seem right. The corresponding matrix description would be that the hologram is the union of blocks, but the Hilbert space does not factor into Hilbert spaces for the blocks. This is obvious from the fact that there are off-diagonal components a_{Ij} and a_{Ji} . In an approximation these elements may be unexcited if the constraints are tight, but in order to match the numerical coefficients the constraints cannot be infinitely tight.
3. The dS-matrix theory shows that the hologram is a single system, with a number of degrees of freedom as large as the largest area of the cosmic horizon when it forms a single connected whole. It is large enough to describe any state of the system but not much larger. In that sense it may be identified with the

horizon of the dominant saddle point in the path integral. In individual branches of the wave function no single component of the horizon may be large enough to describe the whole, but the hologram itself is.

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A Nariai Geometry

As the mass M of the black hole increases the two solutions r_- and r_+ come together. The limiting geometry is the Nariai solution. It occurs at the point where $dg/dr = 0$. From (2.3),

$$r_N = \frac{R}{\sqrt{3}}. \tag{A.61}$$

The entropy of each horizon is given by $\frac{\text{area}}{4G} = \pi r_N^2 = S_0/3$. The combined entropy of the two horizons is the total Nariai entropy S_N ,

$$\begin{aligned} S_N &= \frac{2}{3}S_0 \\ &= \frac{2\pi}{3}R^2. \end{aligned} \tag{A.62}$$

The entropy deficit of a state ρ is defined as the difference of

the de Sitter entropy and the entropy of ρ . For the Nariai state the entropy deficit is,

$$\begin{aligned}\Delta S_N &= S_0 - S_N \\ &= \frac{S_0}{3}.\end{aligned}\tag{A.63}$$

To understand the Nariai geometry we begin with the near-Nariai geometry in which the two roots $r_{\pm}$ in figure [5](#) are very close as can be seen from figure [5](#). The function $f(r)$ in the small interval between the roots may be expanded to quadratic order. Define,

$$r - \frac{R}{\sqrt{3}} = u.\tag{A.64}$$

Then f is given by,

$$f = \frac{3}{R^2}(\epsilon^2 - u^2)\tag{A.65}$$

and,

$$ds^2 = -\frac{3}{R^2}(\epsilon^2 - u^2) + \frac{R^2}{3(\epsilon^2 - u^2)}du^2 + \frac{R^2}{3}d\Omega^2.\tag{A.66}$$

Now define,

$$\begin{aligned}v &= \frac{u}{\epsilon} \\ \tau &= \frac{3\epsilon}{R^2}t\end{aligned}\tag{A.67}$$

and the Nariai metric becomes,

$$\frac{R^2}{3} \left(-(1 - v^2) d\tau^2 + \frac{1}{1 - v^2} dv^2 + d\Omega^2 \right). \quad (\text{A.68})$$

The Nariai geometry is $dS_2 \times S_2$. The Euclidean continuation is simply $S_2 \times S_2$ with both S_2 -factors having radius $R/\sqrt{3}$.

This casts a new light on the second term of (4.28). It is evidently the saddle point contribution to the path integral coming from the classical Nariai geometry—a geometry with a different topology. The contribution to the path integral can be worked out by calculating the action I of the classical $S_2 \times S_2$ geometry. One finds the unsurprising result

$$I_{\text{Nariai}} = S_N. \quad (\text{A.69})$$

Consider the contributions to the Euclidean path integral from the original de Sitter space S_4 , and the Nariai space $S_2 \times S_2$. Schematically (ignoring prefactors), the path integral is given by,

$$e^{S_0} + e^{2S_0/3} = e^{S_0} \left(1 + e^{-S_0/3} \right). \quad (\text{A.70})$$

The geometry of de Sitter space is the non-compact group $O(4, 1)$ which is a continuation of the compact group $O(5)$. The compact group describes the symmetry of the Euclidean continuation of dS, namely the 4-sphere. $O(4, 1)$ and $O(5)$ are the symmetries of the semiclassical theory. The geometry of Nariai space is $dS(2) \times S(2)$, or in the Euclidean signature, $dS(2) \times S(2)$. The symmetry of Nariai space is $O(2, 1) \times O(3)$ or $O(3) \times O(3)$. The dS symmetry is

larger than the Nariai symmetry, but it does not contain the Nariai symmetry as a subgroup.

In the semiclassical limit in which the entropy is infinite the probability of transitions between the two geometries is zero. There is no obstacle to the symmetries being realized. But as we have seen, in the full quantum theory there are transitions, and it does not seem possible for either $O(4, 1)$ or $O(2, 1) \times O(3)$ to be exact. This clash was discussed in [6].

B The Equilibrium Shell

A non-relativistic particle at rest in static coordinates will be in equilibrium at a point where $\frac{df(r)}{dr} = 0$. For an ordinary Schwarzschild-black hole in flat space no such point exists; $f(r)$ is monotonic in that case. But the Schwarzschild-de Sitter black hole does have an equilibrium point. From (2.2) we see that the (unstable) equilibrium shell is given by,

$$r_{eq}^3 = MGR^2 \tag{B.71}$$

Consider the Penrose diagram for a Schwarzschild-de Sitter black hole in figure 10.

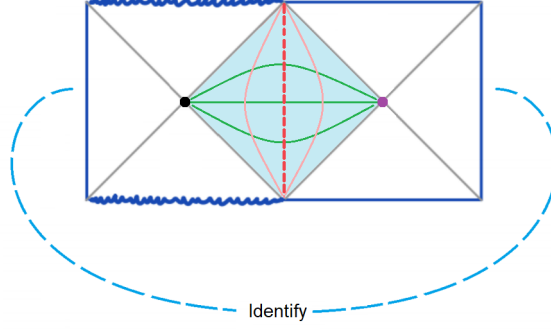


Figure 10: Penrose diagram for Schwarzschild-de Sitter black hole and a static patch along with the equilibrium surface shown as a red dashed line.

The static patch shown in light blue surrounds the black hole so that the black hole remains static at the center of the static patch. The dotted red line is the equilibrium shell that also surrounds the black hole at the equilibrium position.

The equilibrium shell is a natural place to introduce observers. We can think of it as a substitute for the pole. If the observer looks in one direction he sees the black hole horizon and in the other direction he sees the cosmic horizon surrounding him.

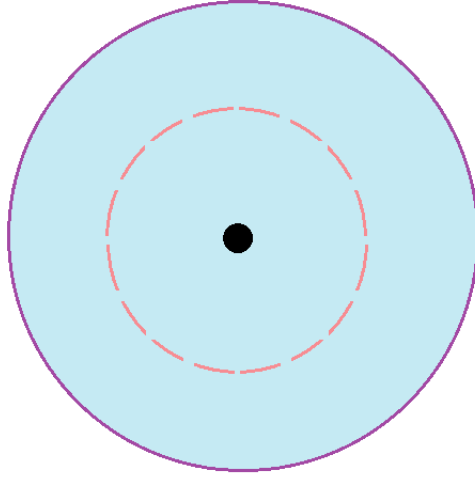


Figure 11: A small black hole in the static patch. The red dashed curve is the equilibrium surface.

Figure 11 shows the static patch surrounded by the cosmic horizon and the black hole at the center of the patch. The dotted red circle (really a sphere) represents the equilibrium shell.

In the Nariai limit the metric takes the form (A.68) and one sees that the equilibrium shell is at the symmetry point $v = 0$, midway between the horizons. One should note that although the value of r is the same at the two horizons, the distance between them is not zero. It is given by,

$$\begin{aligned}
 \text{distance} &= \frac{R}{\sqrt{3}} \int_{-1}^1 \frac{1}{\sqrt{1-v^2}} dv \\
 &= \frac{\pi R}{\sqrt{3}}.
 \end{aligned} \tag{B.72}$$

C The Temperature of the Nariai Geometry

Now let us consider the Minkowski-signature Nariai geometry (A.68). In this limit the two horizons become equal and the geometry is symmetric with respect to a reflection about $v = 0$. The equilibrium shell is the two-sphere at $v = 0$.

Let us consider the temperature of the Nariai geometry. The temperature of a black hole in flat or AdS space is usually defined as the temperature registered by a thermometer located at spatial infinity. In de Sitter space there is no asymptotic spatial infinity so we must choose another rule for defining the temperature. One possibility is to locate the thermometer at the equilibrium shell. To compute the temperature we may consider the Euclidean continuation and compute the circumference of the time-circle and identify it with inverse temperature. Alternatively we may use the fact that the Minkowski geometry is $dS_2 \times S_2$ with a radius $R/\sqrt{3}$. It follows from both arguments that the temperature at the equilibrium shell is,

$$T_N = \frac{\sqrt{3}}{2\pi R} \tag{C.73}$$

(which is larger by a factor $\sqrt{3}$ than the temperature of the original de Sitter space.) The observer at the equilibrium position is bathed in radiation at temperature T_N .

Being at the same temperature, the two horizons are in thermal equilibrium with each other, but the equilibrium is unstable.

We should note that the pole of the two dimensional de Sitter space is exactly the point $v = 0$, so in that sense the temperature

at the equilibrium shell is the temperature at the pole of dS_2 . The temperature of the original dS_4 is the proper temperature at the 4-D pole, and is given by,

$$T = \frac{1}{2\pi R}. \quad (\text{C.74})$$

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