Decentralized Online Convex Optimization in Networked Systems

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Abstract

We study the problem of networked online convex optimization, where each agent individually decides on an action at every time step and agents cooperatively seek to minimize the total global cost over a finite horizon. The global cost is made up of three types of local costs: convex node costs, temporal interaction costs, and spatial interaction costs. In deciding their individual action at each time, an agent has access to predictions of local cost functions for the next k time steps in an r-hop neighborhood. Our work proposes a novel online algorithm, Localized Predictive Control (LPC), which generalizes predictive control to multi-agent systems. We show that LPC achieves a competitive ratio of $1 + \tilde{O}(\rho_T^k) + \tilde{O}(\rho_S^r)$ in an adversarial setting, where ρ_T and ρ_S are constants in $(0, 1)$ that increase with the relative strength of temporal and spatial interaction costs, respectively. This is the first competitive ratio bound on decentralized predictive control for networked online convex optimization. Further, we show that the dependence on k and r in our results is near optimal by lower bounding the competitive ratio of any decentralized online algorithm.

1. Introduction

A wide variety of multi-agent systems can be modeled as optimization tasks in which individual agents must select actions based on local information with the goal of cooperatively learning to minimize a global objective in an uncertain, time varying environment. This general setting emerges in applications such as formation control (Chen & Wang, 2005; Oh et al., 2015), power systems control (Molzahn et al., 2017; Shi et al., 2021), and multiproduct price optimization (Caro & Gallien, 2012; Candogan et al., 2012). In all these cases, it is key that the algorithms used by agents use only local information due to the computational burden created by the size of the systems, the information constraints in the systems, and the need for fast and/or interpretable decisions.

At this point, there is a mature literature focused on decentralized optimization, e.g. Bertsekas & Tsitsiklis (1989); Boyd et al. (2011); Shi et al. (2015); Nedić et al. (2018), see Xin et al. (2020) for a survey; however, the design of learning policies for uncertain, time-varying environments requires decentralized *online* optimization. The literature studying decentralized online optimization is still nascent (see the related work section for a discussion of recent papers, e.g. Li et al. (2021b); Yuan et al. (2021); Yi et al. (2020)) and many challenging open questions remain.

Three issues of particular importance for real-world applications are the following.

First, temporal coupling in actions is often of first-order importance to applications. For example, startup costs, ramping costs, and switching costs are prominent in settings such as power systems and cloud computing, and lead to penalties for changing actions dramatically over time. The design of online algorithms to address such temporal interaction costs has received significant attention in the single-agent case recently, e.g, smoothed online optimization (Goel et al., 2019; Lin et al., 2020), convex body chasing (Argue et al., 2020b; Sellke, 2020), online optimization with memory (Agarwal et al., 2019; Shi et al., 2020), and dynamic pricing (Besbes & Lobel, 2015; Chen & Farias, 2018).

Second, spatial interaction costs are of broad importance in practical applications. Such costs arise because of the need for actions of nearby agents to be aligned with one another, and are prominent in settings such as economic team theory (Marschak, 1955; Marschak & Radner, 1972), combinatorial optimization over graphs (Hochba, 1997; Gamarnik & Goldberg, 2010), and statistical inference (Wainwright & Jordan, 2008). An example is (dynamic) multiproduct pricing, where the price of a product can impact the demand of other related products (Song & Chintagunta, 2006).

Third, leveraging predictions of future costs has long been recognized as a promising way to improve the performance of online agents (Morari & Lee, 1999; Lin et al., 2012;

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Badiei et al., 2015; Chen et al., 2016; Shi et al., 2019; Li et al., 2020). As learning tools become more prominent, the role of predictions is growing. By collecting data from repeated trials, data-driven learning tools make it possible to provide accurate predictions for near future costs. For example, in multiproduct pricing, good demand forecasts can be constructed up to a certain time horizon and are invaluable in setting prices (Caro & Gallien, 2012).

In addition to the three issues above, we would like to highlight that existing results for decentralized online optimization focus on designing algorithms with low (static) regret (Hosseini et al., 2016; Li et al., 2021b) , i.e., algorithms that (nearly) match the performance of the best static action in hindsight. In a time-varying environment, it is desirable to instead obtain stronger bounds, such as those on the dynamic regret or competitive ratio, which compare to the dynamic optimal actions instead of the best static action in hindsight, e.g., see results in the centralized setting such as Lin et al. (2020); Li et al. (2020); Shi et al. (2020).

This paper aims to address decentralized online optimization with the three features described above. In particular, we are motivated by the open question: *Can a decentralized algorithm make use of predictions to be competitive for networked online convex optimization in an adversarial environment when spatial and temporal costs are considered?*

Contributions. This paper provides the first competitive algorithm for decentralized learning in networked online convex optimization. Agents in a network must each make a decision at each time step, to minimize a global cost which is the sum of convex node costs, spatial interaction costs and temporal interaction costs. We propose a predictive control framework called Localized Predictive Control (LPC, Algorithm 1) and prove that it achieves a competitive ratio of $1 + \tilde{O}(\rho_T^k) + \tilde{O}(\rho_S^r)$, which approaches 1 exponentially fast as the prediction horizon k and communication radius r increase simultaneously. Our results quantify the improvement in competitive ratio from increasing the communication radius r (which also increases the computational requirements) versus increasing the prediction horizon k, and imply that – as a function of problem parameters – one of the two "resources" k and r emerges as the bottleneck to algorithmic performance. Given any target competitive ratio, we find the minimum required prediction horizon k and communication radius r as functions of the temporal interaction strength and the spatial interaction strength, resp.

Further, we show that LPC is order-optimal in terms of k and r by proving a lower bound on the competitive ratio for any online algorithm. We formalize the near optimality of our algorithm by showing that a resource augmentation bound follows from our upper and lower bounds: our algorithm with given k and r performs at least as well as the best possible algorithm that is forced to work with k' and r' which are a constant factor smaller than k and r respectively.

The algorithm we propose, LPC, is inspired by Model Predictive Control (MPC). After fixing the prediction horizon k and the communication radius r, each agent makes an individual decision by solving a k-time-step optimization problem, on a local neighborhood centered at itself and with radius r. In doing so, the algorithm utilizes all available information and makes a "greedy" decision. One benefit of this algorithm is its simplicity and interpretability, which is often important for practical applications. Moreover, since the algorithm is local, the computation needed for each agent is independent of the network size.

Our main results are enabled by a new analysis methodology which obtains two separate decay factors for the propagation of decision errors (a temporal decaying factor ρ_T and spatial decaying factor ρ_S) through a novel perturbation analysis. Specifically, the perturbation analysis seeks to answer the following question: If we perturb the boundary condition of an agent v's r-hop neighborhood at the time step which is τ-th later than the present, how does that affect v's current decision, in terms of spatial distance r and temporal distance τ? With our analysis, we are able to bound the impact on v's current decision by $O(\rho_T^\tau \rho_S^r)$, where the decay factors ρ_T and ρ_S increase with the strength of temporal/spatial interactions among individual decisions. This novel analysis is critical for deriving a competitive ratio that distinguishes the decay rate for temporal and spatial distances.

To illustrate the use of our results in a concrete application, Appendix B provides a detailed discussion of dynamic multiproduct pricing, which is a central problem in revenue management. The resulting revenue maximization problem fits into our theoretical framework, and we deduce from our results that LPC guarantees near optimal revenue, in addition to reducing the computational burden (Schlosser, 2016) and providing interpretable prices (Biggs et al., 2021) for products.

Related Work. This paper contributes to the literature in three related areas, each of which we describe below.

Distributed Online Convex Optimization. Our work relates to a growing literature on distributed online convex optimization with time-varying cost functions over multi-agent networks. Many recent advances have been made including distributed OCO with delayed feedback (Cao & Basar, 2021), coordinating behaviors among agents (Li et al., 2021a; Cao & Başar, 2021), and distributed OCO with a time-varying communication graph (Hosseini et al., 2016; Akbari et al., 2017; Yuan et al., 2021; Li et al., 2021b; Yi et al., 2020). A common theme of the previous literature is the idea that agents can only access partial information of time-varying global loss functions, thus requiring local information exchange between neighboring agents. To the best of our

knowledge, our paper is the first in this literature to provide competitive ratio bounds or consider spatial and temporal costs, e.g., switching costs.

Online Convex Optimization (OCO) with Switching Costs. Online convex optimization with switching costs was first introduced in Lin et al. (2012) to model dynamic power management in data centers. Different forms of cost functions have been studied since then, e.g., Chen et al. (2018); Shi et al. (2020); Lin et al. (2020), in order to fit a variety of applications from video streaming (Joseph & de Veciana, 2012) to energy markets (Kim & Giannakis, 2017). The quadratic form of switching cost was first proposed in Goel & Wierman (2019) and yields connections to optimal control, which were further explored in Lin et al. (2021). The literature has focused entirely on the centralized, single-agent setting. Our paper contributes to this literature by providing the first analysis of switching costs in a networked setting with a decentralized algorithm.

Perturbation Analysis of Online Algorithms. Sensitivity analysis of convex optimization problems studying the properties of the optimal solutions as a function of the problem parameters has a long and rich history (see Fiacco & Ishizuka (1990) for a survey). The works that are most related to ours consider the specific class of problems where the decision variables are located on a horizon axis, or consider a general network and aim to show the impact of a perturbation on a decision variable is exponentially decaying in the graph distance from that variable, e.g., Shin et al. (2021); Shin & Zavala (2021); Lin et al. (2021). The idea of using exponentially decaying perturbation bounds to analyze an online algorithm is first proposed in Lin et al. (2021), where only the temporal dimension is considered. This style of perturbation analysis is key to the proof of our competitive bounds and, to prove our competitive bounds, we provide new perturbation results that separate the impact of spatial and temporal costs in a network for the first time. Additionally, our analysis is enabled by new results on the decay rate of a product of exponential decay matrices, which may be of independent interest.

Notation. A complete notation table can be found in Appendix A. Here we describe the most commonly used notation. In a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we use $d_{\mathcal{G}}(v, u)$ to denote the distance (i.e. the length of the shortest path) between two vertices v and u. N_v^r denotes the r-hop neighborhood of vertex v, i.e., $N_v^r := \{u \in \mathcal{V} \mid d_{\mathcal{G}}(u, v) \leq r\}$. ∂N_v^r denotes the boundary of N_v^r, i.e., $\partial N_v^r = N_v^r \setminus N_v^{r-1}$. We generalize these notations to temporal-spatial graphs as follows. Let $\times$ denote the Cartesian product of sets, and $N_{(t,v)}^{(k,r)} := \{\tau \in \mathbb{Z} \mid t \leq \tau < t+k\} \times N_v^r$, $\partial N_{(t,v)}^{(k,r)} := N_{(t,v)}^{(k,r)} \setminus N_{(t,v)}^{(k-1,r-1)}$. For any subset of vertices S, we use $\mathcal{E}(S)$ to denote the set of all edges that have both endpoints in S, and define $S_+ = \{u \in \mathcal{V} \mid \exists v \in S \text{ s.t. } d_{\mathcal{G}}(u, v) \leq 1\}$ (i.e., S and its 1-hop neighbors). Let Δ denote the maximum degree of any vertex in $\mathcal{G}$; $h(r) := \sup_v |\partial N_v^r|$. We say a function is in C^2 if it is twice continuously differentiable. We use $\|\cdot\|$ to denote the (Euclidean) 2-norm for vectors and the induced 2-norm for matrices.

2. Problem Setting

We consider a set of agents in a networked system where each agent individually decides on an action at each time step and the agents cooperatively seek to minimize a global cost over a finite time horizon H. Specifically, we consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of agents. Each vertex $v \in \mathcal{V}$ denotes an individual agent, and two agents v and u interact with each other if and only if they are connected by an undirected edge $(v, u) \in \mathcal{E}$. At each time step $t = 1, 2, \ldots, H$, each agent v picks an n-dimensional local action $x_t^v \in D_t^v$, where n is a positive integer and $D_t^v \subset \mathbb{R}^n$ is a convex set of feasible actions. The global action at time t is the vector of agent actions $x_t = \{x_t^v\}_{v \in \mathcal{V}}$, and incurs a global state cost, which is the sum of three types of local cost functions:

- **Node costs**: Each agent v incurs a time-varying node cost $f_t^v(x_t^v)$, which characterizes agent v's local preference for its local action x_t^v.
- **Temporal interaction costs**: Each agent v incurs a time-varying temporal interaction cost $c_t^v(x_t^v, x_{t-1}^v)$, that characterizes how agent v's previous local action x_{t-1}^v interacts with its current local action x_t^v.
- **Spatial interaction costs**: Each edge $e = (v, u)$ incurs a time-varying spatial interaction cost[1] $s_t^e(x_t^v, x_t^u)$. This characterizes how agents v and u's current local actions affect each other.

In our model, the node cost is the part of the cost that only depends the agent's current local action. If the other two types of costs are zero functions, each agent will trivially pick the minimizer of its node cost. Temporal interaction costs encourage agents to choose a local action that is "compatible" with their previous local action. For example, a temporal interaction could be a switching cost which penalizes large deviations from the previous action, in order to make the trajectory of local actions "smooth". Such switching costs can be found in work on single-agent online convex optimization, e.g., Chen et al. (2018); Goel et al. (2019); Lin et al. (2020). Spatial interaction costs, on the other hand, can be used to enforce some collective behavior among the agents. For example, spatial interaction can model the probability that one agent's actions affects its neighbor's actions in diffusion processes on social networks (Kempe et al., 2015); or model interactions between comple-

[1] Since e is an undirected edge, the order we write the two inputs (the action of v and the action of u) does not matter. Note that s_t^e can be asymmetric for agents v and u, e.g., $s_t^e(x_t^v, x_t^u) = s_t^e(x_t^u, x_t^v) = \|x_t^v + 2x_t^u\|^2$.

ment/substitute products in multiproduct pricing (Candogan et al., 2012).

Our analysis is based on standard smoothness and convexity assumptions on the local cost functions (see Appendix A for definitions of smoothness and strong convexity):

Assumption 2.1. *For $\mu > 0, \ell_f < \infty, \ell_T < \infty, \ell_S < \infty$, the local cost functions and feasible sets for all t, v, e satisfy:*

- $f_t^v : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is μ-strongly convex, ℓ_f-smooth, and in C^2;*
- $c_t^v : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is convex, ℓ_T-smooth, and in C^2;*
- $s_t^e : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is convex, ℓ_S-smooth, and in C^2;*
- $D_t^v \subseteq \mathbb{R}^n$ *satisfies $int(D_t^v) \neq \emptyset$ and can be written as $D_t^v := \{x_t^v \in \mathbb{R}^n \mid (g_t^v)_i(x_t^v) \leq 0, \forall 1 \leq i \leq m_t^v\}$, where each $(g_t^v)_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function in C^2.*

Note that the assumptions above are common, even in the case of single-agent online convex optimization, e.g., see Li et al. (2020); Shi et al. (2020); Lin et al. (2021).

It is useful to separate the global stage costs into two parts based on whether the cost term depends only on the current global action or whether it also depends on the previous action. Specifically, the part that only depends on the current global action x_t is the sum of all node costs and spatial interaction costs. We refer to this component as the *(global) hitting cost* and denote it as

$$f_t(x_t) := \sum_{v \in \mathcal{V}} f_t^v(x_t^v) + \sum_{(v,u) \in \mathcal{E}} s_t^{(v,u)}(x_t^v, x_t^u).$$

The rest of the global stage cost involves the current global action x_t and the previous global action x_{t-1}. We refer to it as the *(global) switching cost* and denote it as

$$c_t(x_t, x_{t-1}) = \sum_{v \in \mathcal{V}} c_t^v(x_t^v, x_{t-1}^v).$$

Combining the global hitting and switching costs, the networked agents work cooperatively to minimize the total global stage costs in a finite horizon H starting from a given initial global action x_0 at time step 0: $cost(ALG) := \sum_{t=1}^{H}(f_t(x_t) + c_t(x_t, x_{t-1}))$, where ALG denotes the decentralized online algorithm used by the agents. The offline optimal cost is the clairvoyant minimum cost one can incur on the same sequence of cost functions and the initial global action x_0 at time step 0, i.e., $cost(OPT) := \min_{x_{1:H}} \sum_{t=1}^{H}(f_t(x_t) + c_t(x_t, x_{t-1}))$.

We measure the performance of any online algorithm ALG by the competitive ratio (CR), which is a widely-used metric in the literature of online optimization, e.g., Chen et al. (2018); Goel et al. (2019); Argue et al. (2020a).

Definition 2.2. *The competitive ratio of online algorithm ALG is the supremum of $cost(ALG)/cost(OPT)$ over all possible problem instances, i.e., $CR(ALG) := \sup_{\mathcal{G}, H, x_0, \{f_t^v, c_t^v, s_t^e, D_t^v\}} cost(ALG)/cost(OPT)$.*

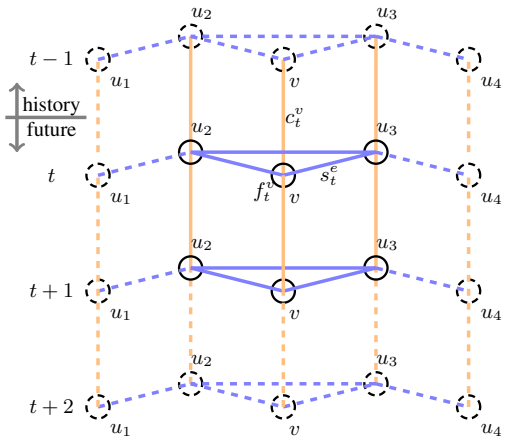

Figure 1: Illustration of available information for agent v at time t in networked online convex optimization with $k = 2$ and $r = 1$, for the network with $\mathcal{V} = \{u_1, u_2, v, u_3, u_4\}$ and $\mathcal{E} = \{(u_1, u_2), (u_2, v), (v, u_3), (u_2, u_3), (u_3, u_4)\}$).

Finally, we define the partial hitting and switching costs over subsets of the agents. In particular, for a subset of agents $S \subseteq \mathcal{V}$, we denote the joint action over S as $x_t^S := \{x_t^v \mid v \in S\}$ and define the partial hitting cost and partial switching cost over S as

$$f_t^S(x_t^{S_+}) := \sum_{v \in S} f_t^v(x_t^v) + \sum_{(v,u) \in \mathcal{E}(S_+)} s_t^{(v,u)}(x_t^v, x_t^u),$$
$$c_t^S(x_t^S, x_{t-1}^S) := \sum_{v \in S} c_t^v(x_t^v, x_{t-1}^v), \quad (1)$$

This notation is useful for presenting decentralized online algorithms where the optimizations are performed over the r-hop neighborhood of each agent.

2.1. Information Availability Model

We assume that each agent has access to local cost functions up to a *prediction horizon* k into the future, for themselves and their neighborhood up to a *communication radius* r. In more detail, recall that N_v^r denotes the r-hop neighborhood of agent v, i.e., $N_v^r := \{u \in \mathcal{V} \mid d_{\mathcal{G}}(u, v) \leq r\}$. Before picking a local action x_t^v at time t, agent v can observe k steps of future node costs, temporal interaction costs, and spatial interaction costs within its r-hop neighborhood, $\{\{(f_\tau^u, c_\tau^u) \mid u \in N_v^r\}, \{s_\tau^e \mid e \in \mathcal{E}(N_v^r)\}\}_{t \leq \tau < t+k}$, and the previous local actions in N_v^r: $\{x_{t-1}^u \mid u \in N_v^r\}$.

We provide an illustration of the local cost functions known to agent v at time t in Figure 1. In the figure, the black circles, blue lines, and orange lines denote the node costs, temporal interaction costs, and spatial interaction costs respectively. The known functions are marked by solid lines. Note that, in addition to the local cost functions, agent v also knows the local actions in N_v^r at time $t-1$, which are

not illustrated in the figure.

To simplify notation, in cases when the prediction horizon exceeds the whole horizon length H, we adopt the convention that $f_t^v(x_t^v) = \frac{\mu}{2}\|x_t^v\|^2$, $c_t^v \equiv s_t^e \equiv 0$ and $D_t^v = \mathbb{R}^n$ for $t > H$. These extended definitions do not affect our original problem with horizon H.

As in many previous works studying the power of predictions in the online optimization literature, e.g., Yu et al. (2020); Lin et al. (2020); Li et al. (2020); Lin et al. (2021), we assume the k-step predictions of cost functions are *exact* and leave the case of *inexact* predictions for future work. This model is reasonable in the case where the predictors can be trained to be very accurate for the near future. Although we focus on exact predictions throughout this paper, we also discuss how to extend this available information model to include inexact predictions in Appendix C.3.

3. Algorithm and Main Results

Algorithm 1 Localized Predictive Control (for agent v)

Parameters: k and r.
for $t = 1$ **to** H **do**
Receive information $\{x_{t-1}^u \mid u \in N_v^r\}$ and

$$\{\{(f_\tau^u, c_\tau^u) \mid u \in N_v^r\}, \{s_\tau^e \mid e \in \mathcal{E}(N_v^r)\}\}_{t \le \tau < t+k}.$$

Choose local action x_t^v to be the (t,v)-th element in

$$\psi_{(t,v)}^{(k,r)}\Big(\{x_{t-1}^u \mid u \in N_v^r\}, \{\theta_\tau^u \mid (\tau,u) \in \partial N_{(t,v)}^{(k,r)}\}\Big)$$

the solution of (2), where $\theta_\tau^u := \arg\min_{y \in D_\tau^u} f_\tau^u(y)$.
end for

In this section we present our main results, which show that our simple and practical LPC algorithm can achieve an order-optimal competitive ratio for the networked online convex optimization problem. We first introduce LPC in Section 3.1. Then, we present the key idea that leads to our competitive ratio bound: a novel perturbation-based analysis (Section 3.2). Next, we use our perturbation analysis to derive bounds on the competitive ratio in Section 3.3. Finally, we show that the competitive ratio of LPC is order-optimal in Section 3.4. An outline that highlights the major novelties in our proofs can be found in Appendix C.

3.1. Localized Predictive Control (LPC)

The design of LPC is inspired by the classical model predictive control (MPC) framework (Garcia et al., 1989), which leverages all available information at the current time step to decide the current local action "greedily". In our context, when an agent v wants to decide its action x_t^v at time t, the available information includes previous local actions in the r-hop neighborhood and k-step predictions of all local node costs, temporal/spatial interaction costs. The boundaries of all available information, which are formed by $\{t-1\} \times N_v^r$ and $\partial N_{(t,v)}^{(k,r)}$, are illustrated in Figure 2.

The pseudocode for LPC is presented in Algorithm 1. For each agent v at time step t, LPC fixes the actions on the boundaries of available information and then solves for the optimal actions inside the boundaries. Specifically, define $\psi_{(t,v)}^{(k,r)}\Big(\{y_{t-1}^u \mid u \in N_v^r\}, \{z_\tau^u \mid (\tau,u) \in \partial N_{(t,v)}^{(k,r)}\}\Big)$ as the optimal solution of the problem

$$\begin{aligned} \min \ & \sum_{\tau=t}^{t+k-1} \Big(f_\tau^{(N_v^{r-1})}\Big(x_\tau^{(N_v^r)}\Big) + c_\tau^{(N_v^r)}\Big(x_\tau^{(N_v^r)}, x_{\tau-1}^{(N_v^r)}\Big)\Big) \\ \text{s.t. } & x_{t-1}^u = y_{t-1}^u, \forall u \in N_v^r, \\ & x_\tau^u = z_\tau^u, \forall (\tau,u) \in \partial N_{(t,v)}^{(k,r)}, \\ & x_\tau^u \in D_\tau^u, \forall (\tau,u) \in N_{(t,v)}^{(k-1,r-1)}, \end{aligned} \tag{2}$$

where the partial hitting cost and partial switching cost f_τ^S and c_τ^S for a subset S of agents were defined in (1). Note that $\psi_{(t,v)}^{(k,r)}\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ is a matrix of actions (in $\mathbb{R}^n$) indexed by $(\tau,u) \in N_{(t,v)}^{(k-1,r-1)}$. (When the context is clear, we use the shorthand $\psi_{(t,v)}^{(k,r)}\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$.) Once the parameters $\{y_{t-1}^u\}$ and $\{z_\tau^u\}$ are fixed, the agent v can leverage its knowledge of the local cost functions to solve (2).

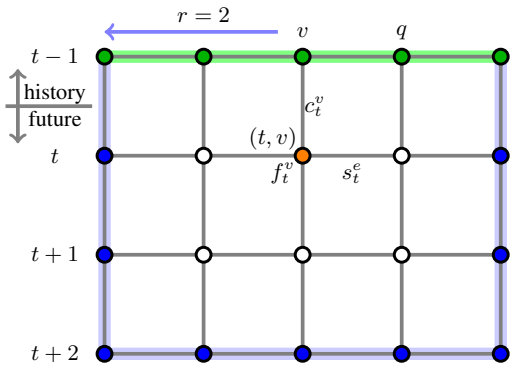

Figure 2: Illustration of LPC with $k = 3, r = 2$ on a line graph (the underlying graph is replicated over the time dimension). The orange node marks the decision variable at (t,v). The green part denotes the decisions in N_v^r at time $(t-1)$. The blue "U" shape denotes the boundary of available information for node v at time t. Edge $e := (v,q)$.

LPC fixes the parameters $\{y_{t-1}^u\}$ to be $\{x_{t-1}^u\}$, which are the previous local actions in N_v^r, and fixes the parameters $\{z_\tau^u\}$ to be the minimizers of local node cost functions at nodes in $\partial N_{(t,v)}^{(k,r)}$. The selection of the parameters at nodes in $\partial N_{(t,v)}^{(k,r)}$ plays a similar role as the terminal cost of classical MPC in centralized settings.

For a single-agent system, MPC-style algorithms are perhaps the most prominent approach for optimization-based control (Garcia et al., 1989) because of their simplicity and excellent performance in practice. LPC extends the ideas of MPC to a multi-agent setting in a networked system by leveraging available information in both the temporal and spatial dimensions, whereas classical MPC focuses only on the temporal dimension. This change leads to significant technical challenges in the analysis.

3.2. Perturbation Analysis

The key idea underlying our analysis of LPC is that the impact of perturbations to the actions at the boundaries of the available information of an agent decay quickly, in fact exponentially fast, in the distance of the boundary from the agent. This quick decay means that small errors cannot build up to hurt algorithm performance.

In this section, we formally study such perturbations by deriving several new results which generalize perturbation bounds for online convex optimization problems on networks. Our bounds capture both the effect of temporal interactions as well as spatial interactions between agent actions, which is a more challenging problem compared to previous literature which considers either temporal interactions (Lin et al., 2021) or spatial interactions (Shin et al., 2021) but not both simultaneously.

More specifically, recall from Section 3.1 that for each agent v at time t, LPC solves an optimization problem $\psi_{(t,v)}^{(k,r)}$ where actions on the boundaries of available information (i.e., $\{t-1\} \times N_v^r$ and $\partial N_{(t,v)}^{(k,r)}$) are fixed. By the principle of optimality, we know that if the actions on the boundaries are selected to be identical with the offline optimal actions, the agent can decide its current action optimally by solving $\psi_{(t,v)}^{(k,r)}$. However, due to the limits on the prediction horizon and communication radius, LPC can only approximate the offline optimal actions on the boundaries (we do this by using the minimizer of node cost functions). The key idea to our analysis of the optimality gap of LPC is by first asking: *If we perturb the parameters of $\psi_{(t,v)}^{(k,r)}$, i.e., the actions on the information boundaries, how large is the resulting change in a local action $x_{t_0}^{v_0}$ for $(t_0, v_0) \in N_{(t,v)}^{(k,r)} \setminus \partial N_{(t,v)}^{(k,r)}$ (in the optimal solution to (2))?*

Ideally, we would like the above impact to decay exponentially fast in the graph distance between node v_0 and the communication boundary for node v (i.e., r minus the graph distance between v_0 and v), as well as in the temporal distance between t_0 and t. We formalize this goal as *exponentially decaying local perturbation bound* in Definition 3.1. We then show in Theorem 3.2 and Theorem 3.3 that such bounds hold under appropriate assumptions.

Definition 3.1. *Define $x_{t_0}^{v_0} := \psi_{(t,v)}^{(k,r)}\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)_{(t_0,v_0)}$, and $(x_{t_0}^{v_0})' := \psi_{(t,v)}^{(k,r)}\big(\{(y_{t-1}^u)'\}, \{(z_\tau^u)'\}\big)_{(t_0,v_0)}$ for arbitrary boundary parameters $\{(y_{t-1}^u)\}, \{(z_\tau^u)\}$ and $\{(y_{t-1}^u)'\}, \{(z_\tau^u)'\}$. We say an **exponentially decaying local perturbation bound** holds if for non-negative constants*

$$C_1 = C_1(\ell_T/\mu, (\Delta\ell_S)/\mu) < \infty,$$
$$C_2 = C_2(\ell_T/\mu, (\Delta\ell_S)/\mu) < \infty,$$
$$\rho_T = \rho_T(\ell_T/\mu) < 1, \rho_S = \rho_S((\Delta\ell_S)/\mu) < 1,$$

for any (t_0, v_0) and arbitrary boundary parameters $\{(y_{t-1}^u)'\}, \{(z_\tau^u)'\}, \{(y_{t-1}^u)\}, \{(z_\tau^u)\}$, we have:

$$\begin{aligned}
&\big\|x_{t_0}^{v_0} - (x_{t_0}^{v_0})'\big\| \\
&\le C_1 \sum_{(u,\tau)\in\partial N_{(t,v)}^{(k,r)}} \rho_T^{|t_0-\tau|}\rho_S^{d_{\mathcal{G}}(v_0,u)}\|z_\tau^u - (z_\tau^u)'\| \\
&\quad + C_2 \sum_{u\in N_v^r} \rho_T^{t_0-(t-1)}\rho_S^{d_{\mathcal{G}}(v_0,u)}\big\|y_{t-1}^u - (y_{t-1}^u)'\big\|.
\end{aligned}$$

Perturbation bounds were recently found to be a promising tool for the analysis of adaptive control and online optimization models(Lin et al., 2021). The exponentially decaying local perturbation bound defined above is similar in spirit to two recent results, i.e., Lin et al. (2021) derives a similar perturbation bound for line graphs and Shin et al. (2021) for general graphs with local perturbations. In fact, one may attempt to derive such a bound by applying these results directly; however, a major weakness of the direct approach is that it will yield $\rho_T = \rho_S$, i.e., it cannot distinguish between spatial and temporal dependencies, and the bound deteriorates as $\max\{\ell_T/\mu, \ell_S/\mu\}$ increases. For instance, even if the temporal interactions are weak (i.e., $\ell_T/\mu \approx 0$), $\rho_T = \rho_S$ can still be close to 1 if ℓ_S/μ is large, leading to a large slack in the perturbation bound for small prediction horizons k.

We overcome this limitation by redefining the action variables. Specifically, to focus on the temporal decay effect, we regroup all local actions in $\{\tau\} \times N_v^r$ as a "large" decision variable for time τ (in Figure 1 we would group each horizontal blue plane in N_v^r to create a new variable). After regrouping, we have $(k+1)$ "large" decision variables located on a line graph, where the strength of the interactions between consecutive variables is upper bounded by ℓ_T. On the other hand, to focus on spatial decay, we regroup all local actions in $\{\tau \mid t-1 \le \tau < t+k\} \times \{v\}$ as a decision variable (in Figure 1 we would group each vertical orange line connecting from $t-1$ to $t+k-1$ to create a new variable). After regrouping, we have $|\mathcal{V}|$ "large" decision variables located on $\mathcal{G}$, where the strength of the interactions between two neighbors is upper bounded by ℓ_S. Averaging over the two perturbation bounds (since we have two valid bounds, their average is also a valid bound) provides the following exponentially decaying local perturbation bound (see (14) in Appendix D.1 for details of the proof).

Theorem 3.2. *Under Assumption 2.1, the exponentially decaying local perturbation bound (Definition 3.1) holds with* $C_1 = C_2 = \frac{2\sqrt{\Delta \ell_S \ell_T}}{\mu}$, *and*

$$\rho_T = \sqrt{1 - 2\Big(\sqrt{1 + (2\ell_T/\mu)} + 1\Big)^{-1}},$$
$$\rho_S = \sqrt{1 - 2\Big(\sqrt{1 + (\Delta\ell_S/\mu)} + 1\Big)^{-1}}.$$

Note that, as ℓ_T/μ (respectively ℓ_S/μ) tends to zero, ρ_T (respectively ρ_S) in Theorem 3.2 also tends to zero with the scaling $\rho_T = \Theta(\sqrt{\ell_T/\mu})$ (resp. $\rho_S = \Theta(\sqrt{\ell_S/\mu})$).

Next, we provide a tighter bound (through a refined analysis) for the regime where μ is much larger than ℓ_T, ℓ_S. Specifically, we establish a bound with the scaling $\rho_T = \Theta(\ell_T/\mu)$ and $\rho_S = \Theta(\ell_S/\mu)$. Again, it is not possible to obtain this result from previous perturbation bounds in the literature.

Theorem 3.3. *Recall* $h(\gamma) := \sup_{v\in\mathcal{V}} |\partial N_v^\gamma|$. *Given any* $b_1, b_2 > 0$, *define* $a = \sum_{\gamma\geq 0}(\frac{1+b_1}{1+b_1+b_2})^\gamma h(\gamma)$, $\tilde{a} = \sum_{\gamma\geq 0}(\frac{1}{1+b_1})^\gamma h(\gamma)$ *and* $\gamma_S = \frac{\sqrt{1+\Delta\ell_S/\mu}-1}{\sqrt{1+\Delta\ell_S/\mu}+1}$. *Suppose Assumption 2.1 holds,* $a, \tilde{a} < \infty$ *and* $\mu \geq \max\{8\tilde{a}\ell_T, \Delta\ell_S(b_1+b_2)/4\}$. *Then the exponentially decaying local perturbation bound (Definition 3.1) holds with* $C_1 = C_2 = \max\{\frac{a^2}{2\tilde{a}(1-4\tilde{a}l_T/\mu)}, \frac{2a^2\Delta\ell_S/\mu}{\gamma_S(1+b_1+b_2)(1-4\tilde{a}l_T/\mu)}\}$

$$\rho_T = \frac{4\tilde{a}\ell_T}{\mu}, \quad \rho_S = (1+b_1+b_2)\gamma_S.$$

Note that $\rho_T, \rho_S < 1$ follow from the condition on μ. Also observe that $\gamma_S = \Theta(\ell_S/\mu)$ as $\ell_S/\mu \to 0$.

The main difference between this result and Theorem 3.2 is, instead of dividing and redefining the action variables, we explicitly write down the perturbations along spatial edges and along temporal edges in the original temporal-spatial graph. We observe that per-time-step spatial interactions are characterized by a banded matrix and that the inverse of the banded matrix exhibits exponential correlation decay, which implies the exponentially decaying local perturbation bounds holds if the perturbed boundary action and the impacted local action we consider are at the same the time step. However, for a multi-time-step problem, to characterize the impact at a local action at some time step due to perturbation at a boundary action at a different time step is a difficult problem. The main technical contribution of this proof is to establish that a product of exponentially decaying matrices still satisfies exponential decay under the conditions in Theorem 3.3. In addition, we obtain a tight bound on the decay rate of the product matrix (see Lemma C.3), which may be of independent interest.

Our condition on $a, \tilde{a} < \infty$ and $\mu > \max\{8\tilde{a}\ell_T, \Delta\ell_S(b_1 + b_2)/4\}$ characterizes a tradeoff between the allowable neighborhood boundary sizes $h(\gamma)$, and how large μ needs to be compared to the interaction cost parameters ℓ_T, ℓ_S. At one extreme, if $h(\gamma) = \Delta^\gamma$, then by setting $b_1 = 2\Delta - 1$ and $b_2 = 4\Delta^2 - 2\Delta$, we obtain $a = \tilde{a} = 2$ but must make a strong requirement on μ, namely, $\mu > \max\{16\ell_T, \Delta^3\ell_S(1 - \frac{1}{4\Delta^2})\}$. At the other extreme, if $h(\gamma) \leq O(poly(\gamma))$ (as is the case if $\mathcal{G}$ is a grid), then $a, \tilde{a} < \infty$ holds for any $b_1, b_2 > 0$, we can impose a weaker requirement on μ: for example, taking $b_1 = b_2 = 1$ yields a requirement $\mu > \max\{8\tilde{a}\ell_T, \Delta\ell_S/2\}$ (where $\tilde{a} = \sum_{\gamma\geq 0}(\frac{1}{2})^\gamma h(\gamma)$); which grows only linearly in Δ, and compares favorably with the $\mu > \Omega(\Delta^3)$ requirement which arose earlier.

Proofs of Theorem 3.2 and Theorem 3.3 are in Appendix D.

3.3. From Perturbations to Competitive Bounds

We now present our main result, which bounds the competitive ratio of LPC using the exponentially decaying local perturbation bounds defined in the previous section.

Before presenting the result, we first provide some intuition as to why the perturbation bounds are useful for deriving the competitive ratio bound. Specifically, to bound the competitive ratio requires bounding the gap between LPC's trajectory and the offline optimal trajectory. This gap comes from the following two sources: (i) the per-time-step error made by LPC due to its limited prediction horizon and communication radius; and (ii) the cumulative impact of all per-time-step errors made in the past. Intuitively, the local perturbation bounds we derive in Section 3.2 allow us to bound the per-step error made jointly by all agents in LPC, and then we use the perturbation bounds from Lin et al. (2021) help us to bound the second type of cumulative errors.

We present our main result in the following theorem and defer a proof outline to Appendix C.2. A formal proof can be found in Appendix E.3.

Theorem 3.4. *Suppose Assumption 2.1 and the exponentially decaying local perturbation bound in Definition 3.1 holds. Define* $\rho_G := 1 - 2 \cdot \Big(\sqrt{1 + (2\ell_T/\mu)} + 1\Big)^{-1}$, *and define* $C_3(r) := \sum_{\gamma=0}^{r} h(\gamma) \cdot \rho_S^\gamma$. *If parameters* r *and* k *of LPC are large enough such that* $O\big(h(r)^2 \cdot \rho_S^{2r} + C_3(r)^2 \cdot \rho_T^{2k} \cdot \rho_G^{2k}\big) \leq \frac{1}{2}$, *then the competitive ratio of LPC is upper bounded by*

$$1 + O\big(h(r)^2 \cdot \rho_S^r\big) + O\big(C_3(r)^2 \cdot \rho_T^k\big).$$

Here the $O(\cdot)$ *notation hides factors that depend polynomially on* $\ell_f/\mu, \ell_T/\mu$, *and* $(\Delta\ell_S)/\mu$; *see Appendix E.3.*

Recall that $h(r)$ denotes the size of the largest r-hop boundary in $\mathcal{G}$. The bound in Theorem 3.4 implies that if $h(r)$ can be upper bounded by $poly(r) \cdot \rho_S^{-\frac{(1-\iota)r}{2}}$ for some constant

$\iota > 0$, the competitive ratio of LPC can be upper bounded by $1 + O(\rho_S^{\iota r}) + O(\rho_T^k)$, because $C_3(r)$ can be upper bounded by some constant that depends on ι in this case. Therefore, the competitive ratio improves exponentially with respect to the prediction horizon k and communication radius r.

Note that the assumption $h(r) \le poly(r) \cdot \rho_S^{-\frac{(1-\iota)r}{2}}$ is not particularly restrictive: For commonly seen graphs like an m-dimensional grid, $h(r)$ is polynomial in r, so $\iota = 1$ works. More generally, for graphs with bounded degree $\Delta < \infty$, there exists $\delta = \delta(\Delta) > 0$ such that, for any graph with node degrees bounded above by Δ and any $\ell_S/\mu \le \delta$, we have ρ_S (from either Theorem 3.2 or Theorem 3.3) will be small enough that, e.g., $h(r) \le \Delta^r = O(\rho_S^{-\frac{r}{4}})$; i.e., $\iota = 1/2$ works. Thus we can eliminate the dependence on $h(r)$ and $C_3(r)$ in the competitive ratio by making additional assumptions on ℓ_S/μ. This result is stated in Corollary 3.5 whose proof is deferred to Appendix E.4. Corollary 3.5 is a corollary of Theorem 3.4 and Theorem 3.3. We use the bound in Theorem 3.3 and not the bound in Theorem 3.2 because Theorem 3.3 is tighter when ℓ_S/μ is small.

Corollary 3.5. *Suppose Assumption 2.1 and inequalities $\ell_S/\mu \le \Delta^{-7}$, and $\ell_T/\mu \le 1/16$ hold. If r and k satisfy that $O\left(\rho_S^r + \rho_T^{2k} \cdot \rho_G^{2k}\right) \le \frac{1}{2}$, then the competitive ratio of LPC is upper bounded by $1 + O\left(\rho_S^{r/2}\right) + O\left(\rho_T^k\right)$, where ρ_S and ρ_T are given by Theorem 3.3 with parameters $b_1 = 2\Delta - 1$ and $b_2 = 4\Delta^2 - 2\Delta$. The $O(\cdot)$ notation hides factors that depend polynomially on $\ell_f/\mu, \ell_T/\mu$, and $(\Delta\ell_S)/\mu$, see Appendix E.4 for the full constants.*

3.4. A Lower Bound

We show that the competitive ratio in Theorem 3.4 is order-optimal by deriving a lower bound on the competitive ratio of any decentralized online algorithm with prediction horizon k and communication radius r. The specific constants and a proof of Theorem 3.6 can be found in Appendix F.

Theorem 3.6. *The competitive ratio of any decentralized online algorithm is lower bounded by $1 + \Omega(\lambda_T^k) + \Omega(\lambda_S^r)$, where $\lambda_T = \left(1 - 2\left(\sqrt{1 + (4\ell_T/\mu)} + 1\right)^{-1}\right)^2$, and $\lambda_S = \left(1 - 2\left(\sqrt{1 + (4\ell_S/\mu)} + 1\right)^{-1}\right)^2$. The $\Omega(\cdot)$ notation hides factors that depend polynomially on $1/\mu, \ell_T$, and ℓ_S.*

While Theorem 3.6 highlights that Theorem 3.4 is order-optimal, the decay factors λ_T, λ_S in the lower bound differ from their counterparts ρ_T, ρ_S in the upper bound for LPC. To understand the magnitude of the difference, we compare the bounds on graphs with bounded degree Δ. The decay factors are a function of the interaction strengths, which are measured by ℓ_S/μ and ℓ_T/μ. Our lower bound on the temporal decay factor λ_T and upper bound ρ_T only differ by a constant factor in the log-scale, and the same holds for the lower/upper bound in terms of the spatial decay factor.

To formalize this comparison, we derive a resource augmentation bound that bounds the additional "resources" that LPC needs to outperform the optimal decentralized online algorithm.[2] Here the prediction horizon k and the communication radius r can be viewed as the "resources" available to a decentralized online algorithm in our setting. We ask *how large do k and r given to LPC need to be, to ensure that it beats the optimal decentralized online algorithm given a communication radius r^* and prediction horizon k^*?*

We formally state our result in the following corollary and provide a proof in Appendix G.

Corollary 3.7. *Under Assumption 2.1, suppose the optimal decentralized online algorithm achieves a competitive ratio of $c(k^*, r^*)$ with prediction horizon k^* and communication radius r^*. Additionally assume that $h(\gamma) = \tilde{O}\left(\rho_S^{-\gamma/4}\right)$ and $\Delta \ge 2$, where the $\tilde{O}$ notation hides a factor that depends polynomially on γ. As $k^*, r^* \to \infty$, LPC achieves a competitive ratio at least as good as that of the optimal decentralized online algorithm when LPC uses a prediction horizon of $k = (4 + o(1))k^*$ and a communication radius of $r = (16\Delta \log \Delta + o(1))r^*$.*

Finally, note that we establish Corollary 3.7 based on the local perturbation bound in Theorem 3.2 rather than Theorem 3.3 for simplicity, because it does not make assumptions on the relationship among μ, ℓ_T, and ℓ_S. We expect that Theorem 3.3 can give better resource augmentation bounds under stronger assumptions on μ, ℓ_T, and ℓ_S.

4. Concluding Remarks

In this work, we introduce and study a novel form of decentralized online convex optimization in a networked system, where the local actions of each agent are coupled by temporal interactions and spatial interactions. We propose a decentralized online algorithm, LPC, which leverages all available information within a prediction horizon of length k and a communication radius of r to achieve a competitive ratio of $1 + \tilde{O}(\rho_T^k) + \tilde{O}(\rho_S^r)$. Our lower bound result shows that this competitive ratio is order optimal. Our results imply that the two types of resources, the prediction horizon and the communication radius, must be improved *simultaneously* in order to obtain a competitive ratio that converges to 1. That is, it is not enough to either have a large communication radius or a long prediction horizon, the combination of both is necessary to approach the hindsight optimal performance.

[2] See, e.g., Roughgarden (2020), for an introduction to this flavor of bounds for expressing the near-optimality of an algorithm.

A limitation of this work is that we have considered only exact predictions in the available information model, the LPC algorithm, and its analysis. Considering inexact predictions is an important future goal and we are optimistic that our work can be generalized in that direction using the roadmap in Appendix C.3.

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A. Notation Summary and Definitions

The notation we use throughout the paper is summarized in the following two tables.

Table 1: Notation related to the graph/network structures.

Notation	Meaning
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$	The network of agents connected by undirected edges;
$d_{\mathcal{G}}(u, v)$	The graph distance (i.e. the length of the shortest path) between two vertices u and v in $\mathcal{G}$;
N_v^r	The r-hop neighborhood of vertex/agent v in $\mathcal{G}$, i.e., $N_v^r := \{u \in \mathcal{V} \mid d_{\mathcal{G}}(u, v) \leq r\}$;
∂N_v^r	The boundary of the r-hop neighborhood of vertex/agent v, i.e., $\partial N_v^r = N_v^r \setminus N_v^{r-1}$;
$h(r)$	The size of the largest r-hop boundary in $\mathcal{G}$, i.e., $h(r) := \sup_v \lvert\partial N_v^r\rvert$;
Δ	The maximum degree of any vertex v in $\mathcal{G}$;
$\mathcal{E}(S)$	The set of all edges that have both endpoints in S, where $S \subseteq \mathcal{V}$;
S_+	The extension of S by 1-hop, i.e., $S_+ = S \cup \{u \mid \exists v \in S \text{ s.t. } (u, v) \in \mathcal{E}\}$;
$N_{(t,v)}^{(k,r)}$	$\{\tau \in \mathbb{Z} \mid t \leq \tau < t + k\} \times N_v^r$, which is a set of (time, vertex) pairs;
$\partial N_{(t,v)}^{(k,r)}$	$N_{(t,v)}^{(k,r)} \setminus N_{(t,v)}^{(k-1,r-1)}$, which is a set of (time, vertex) pairs;

Table 2: Notation related to the optimization problems.

Notation	Meaning
$\lVert\cdot\rVert$	The (Euclidean) 2-norm for vectors and the induced 2-norm for matrices;
H	The whole horizon length of Networked OCO problem;
$[H]$	The sequence of integers $1, 2, \ldots, H$;
$\mathbb{S}^m$	For any positive integer m, $\mathbb{S}^m$ denotes the set of all symmetric real $m \times m$ matrices;
$y_{t_1:t_2}$	The sequence $y_{t_1}, y_{t_1+1}, \ldots, y_{t_2}$, for $t_2 \geq t_1$;
x_t^v	The individual action of agent v at time step t. It is a vector in $\mathbb{R}^n$;
x_t^S	The joint action of all agents in set $S \subseteq \mathcal{V}$ at time t, i.e., $x_t^S = \{x_t^v\}_{v \in S}$;
x_t	The joint action of all agents in $\mathcal{V}$ at time t, i.e., $x_t = \{x_t^v\}_{v \in \mathcal{V}}$. It is a shorthand of $x_t^{\mathcal{V}}$;
x_t^*	The offline optimal joint action of all agents at time t;
$x_{\tau\mid t}^*$	The clairvoyant joint decision of all agents at time τ given that the joint decision is x_t at time t;
$f_t^v(x_t^v)$	The node cost function for agent $v \in \mathcal{V}$ at time step t;
$c_t^v(x_t^v, x_{t-1}^v)$	The temporal interaction cost function for agent $v \in \mathcal{V}$ at time step t;
$s_t^e(x_t^v, x_t^u)$	The spatial interaction cost for edge $e = (v, u) \in \mathcal{E}$ at time step t;
μ	The strong convexity constant of node costs f_t^v;
ℓ_f, ℓ_T, ℓ_S	The smoothness constant of node costs, temporal interaction costs, and spatial interaction costs;
D_t^v	The feasible set of x_t^v for agent v at time t. It is a convex subset of $\mathbb{R}^n$;
θ_t^v	The minimizer of node cost function for v at time t subject to D_t^v, i.e., $\theta_t^v = \arg\min_{y \in D_t^v} f_t^v(y)$;
$f_t^S(x_t^{S_+})$	The total node costs and spatial interaction costs over a subset $S \subseteq \mathcal{V}$ at time t, i.e., $f_t^S(x_t^S) := \sum_{v \in S} f_t^v(x_t^v) + \sum_{(v,u) \in \mathcal{E}(S)} s_t^{(v,u)}(x_t^v, x_t^u)$;
$c_t^S(x_t^S)$	The total temporal interaction costs over a subset $S \subseteq \mathcal{V}$ at time t, i.e., $c_t^S(x_t^S) := \sum_{v \in S} c_t^v(x_t^v, x_{t-1}^v)$;
$f_t(x_t)$	The total node costs and spatial interaction costs over a $\mathcal{V}$ at time t. A shorthand of $f_t^{\mathcal{V}}(x_t^{\mathcal{V}})$;
$c_t(x_t)$	The total temporal interaction costs over $\mathcal{V}$ at time t. A shorthand of $c_t^{\mathcal{V}}(x_t^{\mathcal{V}})$;
$\psi_{(t,v)}^{(k,r)}(\cdot, \cdot)$	The optimal individual decisions in $N_{(t,v)}^{(k,r)}$ when the decision boundaries formed by $\{t-1\} \times N_v^r$ and $\partial N_{(t,v)}^{(k,r)}$ are fixed as parameters;
$\tilde{\psi}_t^p(\cdot, \cdot)$	The optimal global trajectory $x_t, x_{t+1}, \ldots, x_{t+p-2}$ when x_{t-1} and x_{t+p-1} are fixed as parameters;
$\tilde{\psi}_t(\cdot)$	The optimal global trajectory $x_t, x_{t+1}, \ldots, x_T$ when x_{t-1} is fixed as the parameter;

In addition to the notation in the tables above, we make use of the concepts of strong convexity and smoothness throughout this paper.

Definition A.1. *For a fixed dimension $m \in \mathbb{Z}_+$, let $D \subset \mathbb{R}^m$ be a convex set, and suppose function $\hat{h} : D \to \mathbb{R}$ is a differentiable function. Then, $\hat{h}$ is called ℓ-smooth for some constant $\ell \in \mathbb{R}_{\geq 0}$ if*

$$\hat{h}(y) \leq \hat{h}(x) + \langle \nabla \hat{h}(x), y - x \rangle + \frac{\ell}{2}\|y - x\|_2^2, \forall x, y \in \mathbb{R}^m,$$

and is called μ-strongly convex for some constant $\mu \in \mathbb{R}_{\geq 0}$ if

$$\hat{h}(y) \geq \hat{h}(x) + \langle \nabla \hat{h}(x), y - x \rangle + \frac{\mu}{2}\|y - x\|_2^2, \forall x, y \in \mathbb{R}^m.$$

Here $\langle \cdot, \cdot \rangle$ denotes the dot product of vectors.

B. Example: Multiproduct Pricing

The networked online convex optimization problem captures many applications where individual agents must make decisions online in a networked system. In this section, we give a concrete motivating example from multiproduct pricing problems. Many recent books of revenue management (Talluri & Ryzin, 2006; Gallego & Topaloglu, 2019) and papers (Song & Chintagunta, 2006; Caro & Gallien, 2012; Candogan et al., 2012; Chen & Chen, 2015) describe numerous instances of this problem. In our example below, we follow a similar pricing model as in Candogan et al. (2012).

Consider a setting where a large company sells n different products and wants to maximize its revenue by adjusting prices adaptively in a time-varying market. Each vertex/agent $v \in \mathcal{V}$ corresponds to a product and x_t^v denotes its price at time t. Two products v and u are connected by an edge if they interact, e.g., because the products are complements or substitutes.

We assume a classical linear demand model (Talluri & Ryzin, 2006; Gallego & Topaloglu, 2019), where the demand of v at time t, denoted as d_t^v, is given by

$$d_t^v = \underbrace{a_t^v - k_t^v x_t^v}_{\text{Part 1}} \overbrace{- \sum_{u \in N_v^1 \setminus \{v\}} \eta_t^{(u \to v)} x_t^u}^{\text{Part 2}} \underbrace{+ b_t^v x_{t-1}^v}_{\text{Part 3}},$$

with parameters $a_t^v, k_t^v, b_t^v > 0$ and $\eta_t^{(u \to v)} \in \mathbb{R}$. Here, Part 1 corresponds to the nominal demand at price x_t^v; Part 2 adds the network externalities from v's complements/substitutes, and Part 3 reflects the pent up demand of product v due to high price at time $t-1$. Note that the coefficient $\eta_t^{(u \to v)}$ can be different with $\eta_t^{(v \to u)}$. To simplify the notations, for each undirected edge $e = (u, v)$, we define an aggregate coefficient $\gamma_t^e := \frac{1}{2}\left(\eta_t^{(u \to v)} + \eta_t^{(v \to u)}\right)$.

The full revenue maximization problem can be written as

$$\begin{aligned} \max \quad & \sum_{t=1}^{H} \sum_{v \in \mathcal{V}} x_t^v d_t^v = \sum_{t=1}^{H} \sum_{v \in \mathcal{V}} x_t^v (a_t^v - k_t^v x_t^v - \sum_{u \in N_v^1 \setminus \{v\}} \eta_t^{(u \to v)} x_t^u + b_t^v x_{t-1}^v) \\ \text{s.t.} \quad & 0 \leq x_t^v \leq \overline{p}_t^v, \end{aligned} \tag{3}$$

which is equivalent to the following:

$$\begin{aligned} \min \quad & -\sum_{t=1}^{H} \sum_{v \in \mathcal{V}} x_t^v d_t^v = \sum_{t=1}^{H} \sum_{v \in \mathcal{V}} \left[x_t^v (-a_t^v + k_t^v x_t^v + \sum_{u \in N_v^1 \setminus \{v\}} \eta_t^{(u \to v)} x_t^u - b_t^v x_{t-1}^v) \right] \\ \text{s.t.} \quad & 0 \leq x_t^v \leq \overline{p}_t^v \end{aligned} \tag{4}$$

We assume that the product's own price elasticity coefficient k_t^v is uniformly larger than the sum of magnitudes of cross-elasticity coefficients, i.e., exist $\mu > 0$, s.t.

$$\xi_t^v := k_t^v - \sum_{u \in N_v^1 \setminus v} |\gamma_t^{(u,v)}| - \frac{b_t^v + b_{t+1}^v}{2} \geq \mu/2 > 0$$

holds for any (t, v). Further, we assume $\sup_{v\in\mathcal{V},t\in H} k_t^v \le \ell_f/2$, $\sup_{v\in\mathcal{V},t\in H} b_t^v \le b$, and $\sup_{(u,v)\in\mathcal{E},t\in H} |\eta_t^{(u\to v)}| \le \gamma$.

We now observe that this problem fits within our framework with node, spatial, and temporal costs which are quadratic and defined as follows.

$$f_t^v(x_t^v) \coloneqq \xi_t^v\left(x_t^v - \frac{a_t^v}{2\xi_t^v}\right)^2,$$
$$s_t^{(u,v)}(x_t^u, x_t^v) \coloneqq |\gamma_t^{(u,v)}|\left(x_t^u + sgn\left(\gamma_t^{(u,v)}\right)\cdot x_t^v\right)^2,$$
$$c_t^v(x_t^v, x_{t-1}^v) \coloneqq \frac{b_t^v}{2}\left(x_t^v - x_{t-1}^v\right)^2.$$

Note that $f_t^v(x_t^v)$ is $\xi_t^v(x_t^v)^2 - a_t^v x_t^v$ plus a constant $(a_t^v)^2/(4\xi_t^v)$, and the interaction functions can be rewritten as

$$s_t^{(u,v)}(x_t^u, x_t^v) = |\gamma_t^{(u,v)}|\left((x_t^u)^2 + (x_t^v)^2\right) + 2\gamma_t^{(u,v)} x_t^v x_t^u, \quad c_t^v(x_t^v, x_{t-1}^v) = \frac{b_t^v}{2}\left((x_t^v)^2 + (x_{t-1}^v)^2\right) - b_t x_{t-1}^v x_t^v.$$

Summing, we see that

$$\sum_{t=1}^{H}\sum_{v\in\mathcal{V}} f_t^v(x_t^v) + c_t^v(x_t^v, x_{t-1}^v) + \sum_{t=1}^{H}\sum_{e\in\mathcal{E}} s_t^e(x_t^u, x_t^v) = \text{(Objective in (4))} + \sum_{t=1}^{H}\sum_{v\in\mathcal{V}} (a_t^v)^2/(4\xi_t^v)\,. \tag{5}$$

Hence the optimal solution of (4) is the same as the following problem:

$$\begin{aligned} \min \quad & \sum_{t=1}^{H}\sum_{v\in\mathcal{V}} f_t^v(x_t^v) + c_t^v(x_t^v, x_{t-1}^v) + \sum_{t=1}^{H}\sum_{e\in\mathcal{E}} s_t^e(x_t^u, x_t^v) \\ \text{s.t.} \quad & 0 \le x_t^v \le \overline{p}_t^v \end{aligned} \tag{6}$$

where the node cost function $f_t^v(x_t^v)$ is nonnegative, μ-strongly convex, and ℓ_f-smooth; the spatial interaction function is nonnegative, convex and (4γ)-smooth; the temporal interaction function is nonegative, convex and $(2b)$-smooth.

The decentralized nature of our policy is important in this setting. Interpretable pricing algorithms (Biggs et al., 2021) are attractive in practice. Our local pricing algorithm is indeed interpretable, since the current price of a given product is transparently determined by reliable predictions of demand in the near future as well as interactions with directly related products.

In addition, exactly solving the global multiproduct pricing problem can be computationally challenging in practice, especially when the network is large. For example, large online e-commerce companies maintain millions of products, which makes the entire network difficult to store, let alone do computation over. Moreover, due to the ease of changing prices, e-commerce companies often use dynamic pricing and change prices on a daily (or quicker) basis, which magnifies the computational burden.

B.1. Competitive Bound

We end our discussion of multiproduct pricing by showing how the competitive bound in Theorem 3.4 can be applied to the revenue maximization problem of product networks through the use of the following lemma.

Lemma B.1. *Suppose the competitive ratio of our general cost minimization problem is $CR(k, r)$, which a function of prediction horizon k and communication radius r. Suppose $\sup_{(u,v)\in\mathcal{E},t\in[H]} a_t^u/a_t^v \le \tilde{b}$, $\sup_{v\in\mathcal{V},t\in[H]} \frac{a_t^v}{\overline{p}_t^v} \le \tilde{c}$, then the competitive ratio for the corresponding revenue maximization problem, defined as $rev(ALG)/rev(OPT)$, is at least $1 - \frac{\eta}{2}(CR(k, r) - 1)$, where Δ denotes the degree of the product network and $\eta \coloneqq \max\{2(\ell_f + \Delta\tilde{b}\gamma)/\mu, \tilde{c}/\mu\}$.*

Proof. We define $C \coloneqq \sum_{t,v}(a_t^v)^2/(4\xi_t^v)$. Suppose

$$CR(k, r)\cdot cost(OPT) \ge cost(ALG),$$

then

$$CR(k,r)\cdot(-rev(OPT)+C)\geq(-rev(ALG)+C).$$

Rearranging the terms yields

$$(CR(k,r)-1)\cdot C\geq CR(k,r)rev(OPT)-rev(ALG). \tag{7}$$

To find a lower bound on $rev(OPT)$, we choose a pricing strategy such that $x_t^v=\frac{a_t^v}{\eta\mu}\geq 0$ where $\eta=\max\{2(\ell_f+\Delta\tilde{b}\gamma)/\mu,\tilde{c}/\mu\}$. We first check that the demand is always nonnegative under this strategy:

$$\begin{aligned}
a_t^v-k_t^v\frac{a_t^v}{\eta\mu}-\sum_{u\in N_v^1\setminus\{v\}}\eta_t^{(u\to v)}\frac{a_t^u}{\eta\mu}+b_t^v\frac{a_{t-1}^v}{\eta\mu}&\geq a_t^v-k_t^v\frac{a_t^v}{\eta\mu}-\sum_{u\in N_v^1\setminus\{v\}}\eta_t^{(u\to v)}\frac{a_t^u}{\eta\mu}\\
&\geq a_t^v-k_t^v\frac{a_t^v}{\eta\mu}-\sum_{u\in N_v^1\setminus\{v\}}\tilde{b}\gamma\frac{a_t^v}{\eta\mu}\\
&\geq a_t^v(1-\frac{\ell_f+\Delta\tilde{b}\gamma}{\eta\mu})\\
&\geq\frac{a_t^v}{2}.
\end{aligned}$$

Moreover,

$$x_t^v\leq a_t^v/\tilde{c}\leq\overline{p}_t^v.$$

Hence this is a feasible price strategy.

We lower bound the optimal revenue:

$$\begin{aligned}
rev(OPT)&\geq\sum_{t=1}^{H}\sum_{v\in\mathcal{V}}\frac{a_t^v}{\eta\mu}(a_t^v-k_t^v\frac{a_t^v}{\eta\mu}-\sum_{u\in N_v^1\setminus\{v\}}\eta_t^{(u\to v)}\frac{a_t^u}{\eta\mu}+b_t^v\frac{a_{t-1}^v}{\eta\mu})\\
&\geq\sum_{t=1}^{H}\sum_{v\in\mathcal{V}}\frac{a_t^v}{\eta\mu}\frac{a_t^v}{2}\\
&\geq\frac{2}{\eta}C.
\end{aligned}$$

We further divide Equation (7) by $rev(OPT)$ to obtain

$$(CR(k,r)-1)\frac{C}{rev(OPT)}\geq CR(k,r)-rev(ALG)/rev(OPT).$$

Since $CR(k,r)\geq 1$ for the cost minimization problem,

$$rev(ALG)/rev(OPT)\geq 1-(CR(k,r)-1)\frac{C}{rev(OPT)}.$$

This allows us to complete the proof as follows

$$rev(ALG)/rev(OPT)\geq 1-\frac{\eta}{2}(CR(k,r)-1).$$

□

C. Proof Outline

In this section, we outline the major novelties in our proofs for the tighter exponentially decaying local perturbation bound in Theorem 3.3 and the main competitive ratio bound for LPC in Theorem 3.4. The full details of the proofs of these and other results are in the appendices following this one.

C.1. Refined Analysis of Perturbation Bounds

We begin by outlining the four-step structure we use to prove Theorem 3.3. Our goal is to highlight the main ideas, while deferring a detailed proof to Appendix D.2.

STEP 1. ESTABLISH FIRST ORDER EQUATIONS

We define h as the objective function in (2), where actions on the boundary are fixed as $\{z_\tau^u | (\tau, u) \in \partial N_{(t,v)}^{(k,r)}\}$ and the actions at time $t-1$ are fixed as $\{x_{t-1}^u | u \in N_v^r\}$. We denote those fixed actions as system parameter

$$\zeta := (x_{t-1}^{(N_v^r)}, \{z_\tau^u | (\tau, u) \in \partial N_{(t,v)}^{(k,r)}\}).$$

To avoid writing the time index t repeatedly, we use $\hat{x}_i$ to denote actions at time $t-1+i$ for $0 \le i \le k$. The main lemma in for this step is the following.

Lemma C.1. *Given $\theta \in \mathbb{R}$, system parameter ζ and perturbation vector e, we have*

$$\frac{d}{d\theta}\psi(\zeta + \theta e) = M^{-1}\left(R^{(1)}e_0 + R^{(k-1)}e_k + \sum_{\tau=1}^{k-1} K^{(\tau)} e_\tau\right)$$

where

$$\begin{aligned} M &= \nabla^2_{\hat{x}_{1:k-1}} h(\psi(\zeta+\theta e), \zeta + \theta e), \\ R^{(1)} &:= -\nabla_{\hat{x}_0}\nabla_{\hat{x}_{1:k-1}} h(\psi(\zeta+\theta e), \zeta + \theta e), \\ R^{(k-1)} &:= -\nabla_{\hat{x}_k}\nabla_{\hat{x}_{1:k-1}} h(\psi(\zeta+\theta e), \zeta + \theta e), \\ K^{(\tau)} &:= -\nabla_{\hat{x}_\tau^{(\partial N_v^r)}}\nabla_{\hat{x}_{1:k-1}} h(\psi(\zeta+\theta e), \zeta + \theta e). \end{aligned}$$

The proof for Lemma C.1 using first order conditions at the global optimal solution for convex function $h(\cdot, \zeta + \theta e)$ and then takes derivatives with respect to to θ. See Appendix D.2 for a proof.

STEP 2: EXPLOIT THE STRUCTURE OF MATRIX M

M is a hierarchical block matrix with the first level of dimension $(k-1)\times(k-1)$. When fixing the first level indices (i.e. time indices) in M, the lower level matrices are non-zero only if the difference in the time indices is ≤ 1. Hence we decompose M to a block diagonal matrix D and tri-diagonal block matrix A with zero matrix on the diagonal. Each diagonal block in D is a graph-induced banded matrix, which captures the Hessian of h in a single time step. Denote each diagonal block as $D_{i,i}$ for $1 \le i \le k-1$. Further, for $1 \le i \le k-1$, $A_{i,i-1}$ (similarly $A_{i,i+1}$) captures the temporal correlation of individual's action between consecutive time steps. Given this decomposition,

$$M^{-1} = (D + A)^{-1} = D^{-1}(I + AD^{-1})^{-1}.$$

For the ease of notation, we denote $I + AD^{-1}$ by P. Note that P is not necessarily a symmetric matrix. Nevertheless, under technical conditions on P's eigenvalues, we have the following power series expansion (Shin et al., 2020). The details are presented in the Lemma C.2 in Appendix D.2.

Lemma C.2. *Under the condition $\mu > 2\ell_T$, we have*

$$P^{-1} = \sum_{\ell \ge 0} (I - P)^\ell. \tag{8}$$

To understand the the power series in (8), consider the special case where each block $A_{i,j} = \ell_T \cdot I$, and $D_{i,i} = Q$. Denote $P - I$ as J, which is equivalent to AD^{-1}. Then, we have $J_{i,i} = 0$, $J_{i,i-1} = J_{i,i+1} = \ell_T Q^{-1}$, $J_{i,j} = 0$ when $|i-j| > 1$. Intuitively, J captures the "correlation over actions" after one time step. More generally, for $\ell \ge 0$ and any two time indices τ', τ,

$$J^\ell_{\tau',\tau} = \ell_T^\ell Q^{-\ell} b(\ell, \tau, \tau'),$$

where $b(\ell, \tau, \tau')$ is a constant depending on ℓ, τ, τ' and equal to zero if $\ell < |\tau - \tau'|$.

Given that Q is a graph-induced banded matrix, Q^{-1} satisfies exponential-decay properties, which makes it plausible that $Q^{-\ell}$ is an exponential decay matrix with a slower rate.

For the general case, we need to bound terms similar to $\left\|(D^{-1}_{i_1,i_1} D^{-1}_{i_2,i_2} \cdots D^{-1}_{i_\ell,i_\ell})_{u,v}\right\|$. This is the goal of Step 3.

STEP 3: PROPERTIES FOR GENERAL EXPONENTIAL-DECAY MATRICES

The goal of this step is to establish that a product of exponential decay matrices still exhibits exponential decay property under technical conditions about the underlying graph.

Lemma C.3. *Given any graph $\mathcal{M} = (\mathcal{V}', \mathcal{E}')$ and integers $d, \ell \geq 1$, suppose block matrices $A_i \in \mathbb{R}^{|\mathcal{V}'|d \times |\mathcal{V}'|d}$ all satisfy exponential decay properties, i.e. exists $C_i \geq 0$, and $0 \leq \lambda < 1$, s.t.,*

$$\|(A_i)_{u,q}\| \leq C_i \lambda^{d_{\mathcal{M}}(u,q)} \quad \text{for any node } u, q \in \mathcal{M}.$$

Select some $\delta > 0$ s.t. $\lambda' = \lambda + \delta < 1$. If $\tilde{a} \coloneqq \sum_{k=0}^{\infty} (\frac{\lambda}{\lambda'})^k (\sup_{u \in \mathcal{V}'} |\partial N_u^k|) < \infty$, then $\prod_{i=1}^{\ell} A_i$ satisfies exponential decay properties with decay rate λ', i.e.,

$$\left\| (\prod_{i=1}^{\ell} A_i)_{u,v} \right\| \leq C' (\lambda')^{d_M(u,v)}.$$

where $C' = (\tilde{a})^{\ell} \prod_{i=1}^{\ell} C_i$.

A proof of Lemma C.3 can be found in the Appendix D.2.

STEP 4: ESTABLISH CORRELATION DECAY PROPERTIES OF MATRIX M

The last step of the proof is to study the properties of M. To accomplish this, we first show that, for time indices $i, j \geq 1$, J^{ℓ} has the following properties:

- $(J^{\ell})_{i,j} = 0$ if $\ell < |i - j|$ or $\ell - |i - j|$ is odd.
- $(J^{\ell})_{i,j}$ is a summation of terms $\prod_{k=1}^{\ell} A_{j_k, i_k} D_{i_k, i_k}^{-1}$ and the number of such terms is bounded by $\binom{\ell}{(\ell - |i-j|)/2}$.

We formally state and prove the above observation in Lemma D.3. We can further use Theorem C.3 on block matrices $A_{j_k, i_k} D_{i_k}^{-1}$, which gives the following lemma.

Lemma C.4. *Recall $\gamma_S \coloneqq \frac{\sqrt{1+(\Delta \ell_S/\mu)}-1}{\sqrt{1+(\Delta \ell_S/\mu)}+1}$. Select $\delta > 0$ s.t. $\gamma_S' = \gamma_S + \delta < 1$ and $b \coloneqq \sum_{\gamma \geq 0} (\frac{\gamma_S}{\gamma_S'})^{\gamma} h(\gamma)$. Given $\ell, i, j \geq 1$ and $u, q \in \mathcal{V}$, we have*

$$\left\| ((J^{\ell})_{i,j})_{u,q} \right\| \leq \binom{\ell}{(\ell - |i-j|)/2} (b \frac{2\ell_T}{\mu})^{\ell} (\gamma_S')^{d_{\mathcal{G}}(u,q)}.$$

Intuitively speaking, Lemma C.4 bounds the correlations over actions for node u at time step $t - 1 + i$ and action for node q at time step $t - 1 + j$. We present its proof in the Appendix D.2.

Recall that, for $1 \leq i, j \leq k - 1$,

$$M_{i,j}^{-1} = D_{i,i}^{-1} \sum_{\ell \geq 0} (-J)_{i,j}^{\ell}.$$

With the exponential decaying bounds on matrix J^{ℓ}, we can thus conclude Theorem 3.3 by following a similar procedure as in the proof of Theorem 3.1 of (Lin et al., 2021). We present the details in Appendix D.4.

C.2. From Perturbation to Competitive Ratio

We now show how to use the result proven in the previous section to prove our competitive ratio bounds in Theorem 3.4. Our starting point is the assumption that the exponentially decaying local perturbation bound in Definition 3.1 holds for some $C_1, C_2 > 0$ and $\rho_S, \rho_T \in [0, 1)$, which is established using the proof approach outlined in the Section 3.3.

As we discussed in Section 3.3, our proof contains two key parts: (i) we bound the per-time-step error of LPC (Lemma C.7); and (ii) show that the per-time-step error does not accumulate to be unbounded (Theorem C.8).

A key observation that enables the above analysis approach is that the aggregation of local per-time-step error made by each agent at x_t^v can be viewed as a global per-time-step error in the joint global action x_t. Following this observation, we

first introduce a global perturbation bound that focuses on the global action x_t rather than the local actions x_t^v. Recall that f_t denotes the global hitting cost (see Section 2). Define the optimization problem that solves the optimal trajectory from global state x_{t-1} to x_{t+p-1}

$$\tilde{\psi}_t^p(y,z) = \operatorname*{arg\,min}_{x_{t:t+p-1}} \sum_{\tau=t}^{t+p-1} \left(f_\tau(x_\tau) + c_\tau(x_\tau, x_{\tau-1})\right) \\ \text{s.t. } x_{t-1} = y, x_{t+p-1} = z, \tag{9}$$

and another one that solves the optimal trajectory from global state x_{t-1} to the end of the game

$$\tilde{\psi}_t(y) = \operatorname*{arg\,min}_{x_{t:T}} \sum_{\tau=t}^{T} (f_\tau(x_\tau) + c_\tau(x_\tau, x_{\tau-1})) \\ \text{s.t. } x_{t-1} = y. \tag{10}$$

The following global perturbation bound can be derived from Theorem 3.1 in Lin et al. (2021):

Theorem C.5 (Global Perturbation Bound)**.** *Under Assumption 2.1, the following perturbation bounds hold for optimization problems (9) and (10):*

$$\left\|\tilde{\psi}_t^p(y,z)_{t_0} - \tilde{\psi}_t^p(y',z')_{t_0}\right\| \le C_G \rho_G^{t_0-t+1}\|y-y'\| + C_G\rho_G^{t+p-1-t_0}\|z-z'\|,$$
$$\left\|\tilde{\psi}_t(y)_{t_0} - \tilde{\psi}_t(y')_{t_0}\right\| \le C_G \rho_G^{t_0-t+1}\|y-y'\|,$$

where $\rho_G = 1 - 2\cdot\left(\sqrt{1+\frac{2\ell_T}{\mu}}+1\right)^{-1}$ *and* $C_G = \frac{2\ell_T}{\mu}$.

To make the concept of *per-time-step error* rigorous, we formally define it as the distance between the actual next action picked by LPC and the clairvoyant optimal next action from previous action x_{t-1} to the end of the game:

Definition C.6 (Per-step error magnitude)**.** *At time step t, given the previous state x_{t-1}, the (decentralized) online algorithm ALG picks $x_t \in D_t$. We define error e_t as*

$$e_t := \left\|x_t - \tilde{\psi}_t(x_{t-1})_t\right\|.$$

Using the local perturbation bound in Definition 3.1, we show the per-time-step error of LPC decays exponentially with respect to prediction length k and communication range r. This result is stated formally in Lemma C.7, and the proof can be found in Appendix E.2.

Lemma C.7. *For LPC with parameters r and k, e_t satisfies*

$$e_t^2 = O\big(h(r)^2\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2k}\rho_G^{2k}\big)\cdot\left\|x_{t-1}-x_{t-1}^*\right\|^2 \\ + O\big(h(r)^2\cdot\rho_S^{2r}\big)\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t} f_\tau(x_\tau^*) \\ + O\big(C_3(r)^2\cdot\rho_T^{2k}\big) f_{t+k-1}(x_{t+k-1}^*),$$

where $C_3(r) := \sum_{\gamma=0}^{r} h(\gamma)\cdot\rho_S^\gamma$.

Using the global perturbation bound in Theorem C.5, we show $\sum_{t=1}^T\|x_t - x_t^*\|^2$ can be upper bounded by the sum of per-time-step errors of LPC in Theorem C.8. The proof can be found in Appendix E.1.

Theorem C.8. *Let $x_0, x_1^*, x_2^*, \ldots, x_H^*$ denote the offline optimal global trajectory and $x_0, x_1, x_2, \ldots, x_H$ denote the trajectory of ALG. The trajectory of ALG satisfies that*

$$\sum_{t=1}^{H}\|x_t - x_t^*\|^2 \le \frac{C_0^2}{(1-\rho_G)^2}\sum_{t=1}^{H} e_t^2,$$

where $C_0 := \max\{1, C_G\}$ and C_G is defined in Theorem C.5.

To understand the bound in Theorem C.8, we can set all per-time-step error e_t to be zero except a single time step τ. We see the impact of e_τ on the total squared distance $\sum_{t=1}^{T}\|x_t - x_t^*\|^2$ is up to some constant factor of e_τ. This is because the impact of e_τ on $\|x_t - x_t^*\|$ decays exponentially as t increases from τ to T.

By substituting the per-time-step error bound in Lemma C.7 into Theorem C.8, one can bound $\sum_{t=1}^{T}\|x_t - x_t^*\|^2$ by the offline optimal cost, which can be converted to the competitive ratio bound in Theorem 3.4.

C.3. Roadmap to Generalize the Proof to Inexact Predictions

In this section, we present a roadmap to generalize our proof to the case where predictions of future cost functions are inexact. In the information availability model in Section 2.1, one can study inexact predictions by introducing additional disturbance parameters $\{\delta_t^v, w_t^v, w_t^e\}$ to the 3 types of cost functions and generalize them to $f_t(x_t^v, \delta_t^v), c_t^v(x_t^v, x_{t-1}^v, w_t^v), s_t^e(x_t^v, x_t^u, w_t^e)$ for all $t \in [H], v \in \mathcal{V}, e = (v,u) \in \mathcal{E}$. $\{\delta_t^v, w_t^v, w_t^e\}$ represents the disturbances in the cost functions that are hard to predict exactly. Before the decentralized online algorithm decides each local action, it receives the generalized cost functions $f_t^v(\cdot,\cdot), c_t^v(\cdot,\cdot,\cdot), s_t^e(\cdot,\cdot,\cdot)$ and noisy predictions of the true disturbance parameters $\{\delta_t^v, w_t^v, w_t^e\}$ within k time steps and an r-hop neighborhood. In the LPC algorithm (Algorithm 1), the optimization problem $\psi_{(t,v)}^{(k,r)}$ can then be solved with the noisy predictions of disturbance parameters. To analyze the performance of LPC in the presence of prediction errors, one can first generalize the exponentially decaying perturbation bounds in Theorem 3.2 and Theorem 3.3 to include the perturbations on disturbance parameters similar to what we already did in Theorem D.2. The prediction error on disturbance parameters will result in an additional additive term in the per-step error bound in Lemma C.7. If one is willing to assume that the total sum of prediction errors is $O(cost(OPT))$, as in Antoniadis et al. (2020), one can derive a competitive ratio for LPC by substituting the per-step error bound into Theorem C.8. It is worth noting that the resulting competitive ratio will inevitably depend on the quality of predictions, and will converge to a limit larger than 1 (under imperfect predictions) as the prediction horizon k and the communication radius r increase.

D. Perturbation Bounds

This section provides the full proofs of the perturbation bounds stated in Section 3.2.

D.1. Proof of Theorem 3.2

We begin with a technical lemma. Recall that for any positive integer m, $\mathbb{S}^m$ denotes the set of all symmetric $m \times m$ real matrices.

Lemma D.1. *For a graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, suppose A is a positive definite matrix in $\mathbb{S}^{\sum_{i\in\mathcal{V}'} p_i}$ formed by $|\mathcal{V}'| \times |\mathcal{V}'|$ blocks, where the (i,j)-th block has dimension $p_i \times p_j$, i.e., $A_{i,j} \in \mathbb{R}^{p_i \times p_j}$. Assume that A is q-banded for an even positive integer q; i.e.,*

$$A_{i,j} = 0, \forall d_{\mathcal{G}'}(i,j) > q/2.$$

Let a_0 denote the smallest eigenvalue value of A, and b_0 denote the largest eigenvalue value of A. Assume that $b_0 \geq a_0 > 0$. Suppose $D = diag(D_1, \ldots, D_{|\mathcal{V}'|})$, where $D_i \in \mathbb{S}^{p_i}$ is positive semi-definite. Let $M = \left((A+D)^{-1}\right)_{S_R, S_C}$, where $S_R, S_C \subseteq \{1, \ldots, |\mathcal{V}'|\}$. Then we have $\|M\| \leq C\gamma^{\hat{d}}$, where

$$C = \frac{2}{a_0}, \gamma = \left(\frac{\sqrt{cond(A)} - 1}{\sqrt{cond(A)} + 1}\right)^{2/q}, \hat{d} = \min_{i\in S_R, j \in S_c} d_{\mathcal{G}'}(i,j).$$

Here $cond(A) = b_0/a_0$ denotes the condition number of matrix A.

We can show Lemma D.1 using the same method as Lemma B.1 in Lin et al. (2021). We only need to note that even when the size of blocks are not identical, the m th power of a q-banded matrix is a qm-banded matrix for any positive integer m.

With the help of Lemma D.1, we can proceed to show a local perturbation bound on a general $\mathcal{G}'$ in Theorem D.2, where $\mathcal{G}'$ can be different from the network $\mathcal{G}$ of agents in Section 2. Compared with Theorem 3.1 in Lin et al. (2021), Theorem D.2 is more general because it considers a general network of decision variables while Theorem 3.1 in Lin et al. (2021) only consider the special case of a line graph. Although Theorem D.2 does not consider the temporal dimension which features in the local perturbation bound defined in Definition 3.1, we will use it to show Theorem 3.2 later by redefining the variables from two perspectives.

Theorem D.2. *For a network $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ with undirected edges, suppose that each node $v \in \mathcal{V}'$ is associated with a decision vector[3] $\hat{x}_v \in \mathbb{R}^{p_v}$ and a cost function $\hat{f}_v : \mathbb{R}^{p_v} \to \mathbb{R}_{\geq 0}$, and each edge $e = (u, v) \in \mathcal{E}'$ is associated with an edge cost $\hat{c}_e : \mathbb{R}^{p_v} \times \mathbb{R}^{p_u} \times \mathbb{R}^q \to \mathbb{R}_{\geq 0}$. Assume that $\hat{f}_v$ is μ-strongly convex for all $v \in \mathcal{V}'$ and $\hat{c}_e$ is ℓ-smooth for all $e \in \mathcal{E}'$. For some subset $S \subset \mathcal{V}'$, define*

$$E_0 := \{(u, v) \in \mathcal{E}' \mid u, v \in \mathcal{V}' \setminus S\},$$
$$E_1 := \{(u, v) \in \mathcal{E}' \mid u \in \mathcal{V}' \setminus S, v \in S\}.$$

For the disturbance vectors[4] $w \in \mathbb{R}^{(|E_0|+|E_1|)\times q}$ indexed by $e \in E_0 \cup E_1$ and $y \in \mathbb{R}^{\sum_{v\in S} p_v}$ indexed by $v \in S$, let $\psi(w, y)$ denote the optimal solution of the optimization problem

$$\psi(w, y) := \operatorname*{arg\,min}_{x \in \mathbb{R}^{|\mathcal{V}'\setminus S|\times d}} \sum_{v\in\mathcal{V}'\setminus S} \hat{f}_v(\hat{x}_v) + \sum_{(u,v)\in E_0} \hat{c}_{(u,v)}(\hat{x}_u, \hat{x}_v, w_{(u,v)}) + \sum_{(u,v)\in E_1} \hat{c}_{(u,v)}(\hat{x}_u, y_v, w_{(u,v)}).$$

Then, we have that for any vertex $u_0 \in \mathcal{V}' \setminus S$, the following inequality holds:

$$\|\psi(w, y)_{u_0} - \psi(w', y')_{u_0}\| \leq C\left(\sum_{e\in E_0\cup E_1} \lambda^{d_{\mathcal{G}'}(h,e)-1}\|w_e - w'_e\| + \sum_{v\in S} \lambda^{d_{\mathcal{G}'}(h,v)-1}\|y_v - y'_v\|\right), \forall w, y, w', y',$$

where $C := (2\ell)/\mu$ and $\lambda := 1 - 2 \cdot \left(\sqrt{1 + (\Delta'\ell/\mu)} + 1\right)^{-1}$. Here, Δ' denote the maximum degree of any vertex $v \in \mathcal{V}'$ in graph $\mathcal{G}'$. For $e = (u, v) \in \mathcal{E}'$, we define $d_{\mathcal{G}'}(u_0, e) := \min\{d_{\mathcal{G}'}(u_0, u), d_{\mathcal{G}'}(u_0, v)\}$.

Proof of Theorem D.2. Let $e = [\pi^\top, \epsilon^\top]^\top$ be a vector where $\epsilon = \{\epsilon_v\}_{v\in S}$ for $\epsilon_v \in \mathbb{R}^{p_v}$ and $\pi = \{\pi_e\}_{e\in E_0\cup E_1}$, for $\pi_e \in \mathbb{R}^q$. Let θ be an arbitrary real number. Define function $\hat{h} : \mathbb{R}^{\sum_{v\in\mathcal{V}'\setminus S} p_v} \times \mathbb{R}^{(|E_0|+|E_1|)\times q} \times \mathbb{R}^{\sum_{v\in S} p_v} \to \mathbb{R}_{\geq 0}$ as

$$\hat{h}(\hat{x}, w, y) = \sum_{v\in\mathcal{V}'\setminus S} \hat{f}_v(\hat{x}_v) + \sum_{(u,v)\in E_0} \hat{c}_{(u,v)}(\hat{x}_u, \hat{x}_v, w_{(u,v)}) + \sum_{(u,v)\in E_1} \hat{c}_{(u,v)}(\hat{x}_u, y_v, w_{(u,v)}).$$

To simplify the notation, we use ζ to denote the tuple of system parameters, i.e.,

$$\zeta := (w, y).$$

From our construction, we know that $\hat{h}$ is μ-strongly convex in x, so we use the decomposition $\hat{h} = \hat{h}_a + \hat{h}_b$, where

$$\hat{h}_a(\hat{x}, \zeta) = \sum_{v\in\mathcal{V}'\setminus S} \frac{\mu}{2}\|\hat{x}_v\|^2 + \sum_{(u,v)\in E_0} \hat{c}_{(u,v)}(\hat{x}_u, \hat{x}_v, w_{(u,v)}) + \sum_{(u,v)\in E_1} \hat{c}_{(u,v)}(\hat{x}_u, y_v, w_{(u,v)}),$$
$$\hat{h}_b(\hat{x}, \zeta) = \sum_{v\in\mathcal{V}'\setminus S} \left(\hat{f}_v(\hat{x}_v) - \frac{\mu}{2}\|\hat{x}_v\|^2\right).$$

Since $\psi(\zeta + \theta e)$ is the minimizer of convex function $\hat{h}(\cdot, \zeta + \theta e)$, we see that

$$\nabla_{\hat{x}}\hat{h}(\psi(\zeta + \theta e), \zeta + \theta e) = 0.$$

Taking the derivative with respect to θ gives that

$$\nabla^2_{\hat{x}}\hat{h}(\psi(\zeta + \theta e), \zeta + \theta e)\frac{d}{d\theta}\psi(\zeta + \theta e) = -\sum_{v\in S} \nabla_{y_v}\nabla_{\hat{x}}\hat{h}(\psi(\zeta + \theta e), \zeta + \theta e)\epsilon_v - \sum_{e\in E_1\cup E_2} \nabla_{w_e}\nabla_{\hat{x}}\hat{h}(\psi(\zeta + \theta e), \zeta + \theta e)\pi_e.$$

[3] We add a hat over the decision vector $\hat{x}_v$ to distinguish it with the local action x_t^v and global action x_t defined in Section 2. We assume $\hat{x}_v$ is a p_v dimensional real vector.

[4] We do not consider the disturbance vectors in the exponentially decaying local perturbation bounds defined in Definition 3.1, but adding w into the edge costs makes Theorem D.2 more general. For each edge e, w_e is a q-dimensional real vector.

To simplify the notation, we define

$$\begin{aligned}
M &:= \nabla^2_{\hat{x}} \hat{h}(\psi(\zeta+\theta e), \zeta+\theta e), \text{which is a } |\mathcal{V}' \setminus S| \times |\mathcal{V}' \setminus S| \text{ block matrix,} \\
R^{(v)} &:= -\nabla_{y_v} \nabla_{\hat{x}} \hat{h}(\psi(\zeta+\theta e), \zeta+\theta e), \forall v \in S, \text{which are } |\mathcal{V}' \setminus S| \times 1 \text{ block matrix,} \\
K^{(e)} &:= -\nabla_{w_e} \nabla_{\hat{x}} \hat{h}(\psi(\zeta+\theta e), \zeta+\theta e), \forall e \in E_0 \cup E_1, \text{which are } |\mathcal{V}' \setminus S| \times 1 \text{ block matrices,}
\end{aligned}$$

where in M, the block size is $p_u \times p_v, \forall (u,v) \in (\mathcal{V}' \setminus S)^2$; in $R^{(v)}$, the block size is $p_u \times p_v, \forall u \in \mathcal{V}' \setminus S$; in $K^{(e)}$, the block size is $p_u \times q, \forall u \in \mathcal{V}' \setminus S$. Hence we can write

$$\frac{d}{d\theta}\psi(\zeta+\theta e) = M^{-1}\left(\sum_{v \in S} R^{(v)} \epsilon_v + \sum_{e \in E_1 \cup E_2} K^{(e)} \pi_e\right).$$

Recall that $\{R^{(v)}\}_{v \in S}$ are $|\mathcal{V}' \setminus S| \times 1$ block matrices with block size $p_u \times p_v, \forall u \in \mathcal{V}' \setminus S$; $\{K^{(e)}\}_{e \in E_0 \cup E_1}$ are $|\mathcal{V}' \setminus S| \times 1$ block matrices with block size $p_u \times q, \forall u \in \mathcal{V}' \setminus S$. Let $N(v)$ denote the set of neighbors of vertex v on $\mathcal{G}'$. For $R^{(v)}, v \in S$, the $(u,1)$-th block can be non-zero only if $u \in (\mathcal{V}' \setminus S) \cap N(v)$. For $K^{(e)}, e \in E_0 \cup E_1$, the $(u,1)$-th block can be non-zero only if $u \in e$ and $u \in \mathcal{V}' \setminus S$. Hence we see that

$$\frac{d}{d\theta}\psi(\zeta+\theta e)_{u_0} = \sum_{v \in S} (M^{-1})_{u_0, (\mathcal{V}' \setminus S) \cap N(v)} R^{(v)}_{(\mathcal{V}' \setminus S) \cap N(v), 1} \epsilon_v + \sum_{e \in E_0 \cup E_1} (M^{-1})_{u_0, \{u \in e | u \in \mathcal{V}' \setminus S\}} K^{(\tau)}_{\{u \in e | u \in \mathcal{V}' \setminus S\}, 1} \pi_e.$$

Since the switching costs $c_\tau(\cdot,\cdot,\cdot), \tau = 1, \ldots, p$ are ℓ-strongly smooth, we know that the norms of

$$R^{(v)}_{(\mathcal{V}' \setminus S) \cap N(v), 1}, \text{ and } K^{(\tau)}_{\{u \in e | u \in \mathcal{V}' \setminus S\}, 1}$$

are all upper bounded by ℓ. Taking norms on both sides gives that

$$\left\| \frac{d}{d\theta}\psi(\zeta+\theta e)_{u_0} \right\| \leq \sum_{v \in S} \ell \left\| (M^{-1})_{u_0, (\mathcal{V}' \setminus S) \cap N(v)} \right\| \|\epsilon_v\| + \sum_{e \in E_0 \cup E_1} \ell \left\| (M^{-1})_{u_0, \{u \in e | u \in \mathcal{V}' \setminus S\}} \right\| \|\pi_e\|. \tag{11}$$

Note that M can be decomposed as $M = M_a + M_b$, where

$$\begin{aligned}
M_a &:= \nabla^2_{\hat{x}} \hat{h}_a(\psi(\zeta+\theta e), \zeta+\theta e), \\
M_b &:= \nabla^2_{\hat{x}} \hat{h}_b(\psi(\zeta+\theta e), \zeta+\theta e).
\end{aligned}$$

Since M_a is block tri-diagonal and satisfies $(\mu + \Delta'\ell) I \succeq M_a \succeq \mu I$, and M_b is block diagonal and satisfies $M_b \succeq 0$, we obtain the following using Lemma D.1:

$$\left\| (M^{-1})_{u_0, (\mathcal{V}' \setminus S) \cap N(v)} \right\| \leq \frac{2}{\mu} \lambda^{d_{\mathcal{G}'}(u_0, v) - 1}, \text{ and } \left\| (M^{-1})_{u_0, \{u \in e | u \in \mathcal{V}' \setminus S\}} \right\| \leq \frac{2}{\mu} \lambda^{d_{\mathcal{G}'}(u_0, e) - 1},$$

where $\lambda := (\sqrt{cond(M_a)} - 1)/(\sqrt{cond(M_a)} + 1) = 1 - 2 \cdot \left(\sqrt{1 + (2\ell/\mu)} + 1\right)^{-1}$.

Substituting this into (11), we see that

$$\left\| \frac{d}{d\theta}\psi(\zeta+\theta e)_{u_0} \right\| \leq C\left(\sum_{v \in S} \lambda^{d_{\mathcal{G}'}(u_0, v) - 1} \|\epsilon_v\| + \sum_{e \in E_0 \cup E_1} \lambda^{d_{\mathcal{G}'}(u_0, e) - 1} \|\pi_e\|\right),$$

where $C = (2\ell)/\mu$.

Finally, by integration we can complete the proof

$$\begin{aligned}
\|\psi(\zeta)_{u_0} - \psi(\zeta+e)_{u_0}\| &= \left\| \int_0^1 \frac{d}{d\theta}\psi(\zeta+\theta e)_{u_0} d\theta \right\| \\
&\leq \int_0^1 \left\| \frac{d}{d\theta}\psi(\zeta+\theta e)_{u_0} \right\| d\theta \\
&\leq C\left(\sum_{v \in S} \lambda^{d_{\mathcal{G}'}(u_0, v) - 1} \|\epsilon_v\| + \sum_{e \in E_0 \cup E_1} \lambda^{d_{\mathcal{G}'}(u_0, e) - 1} \|\pi_e\|\right).
\end{aligned}$$

□

Now we return to the proof of Theorem 3.2. For simplicity, we temporarily assume the individual decision points are unconstrained, i.e., $D_t^v = \mathbb{R}^n$. We discuss how to relax this assumption in Appendix D.3.

We first consider the case when $\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ and $\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ only differ at one entry y_{t-1}^u or z_τ^u. If the difference is at z_τ^u, by viewing each subset $\{\tau\} \times N_v^r$ for $\tau \in \{t-1, t, \dots, t+k\}$ in the original problem as a vertex in the new graph $\mathcal{G}'$ and applying Theorem D.2, we obtain that

$$\big\|x_{t_0}^{v_0} - (x_{t_0}^{v_0})'\big\| \le C_1^0 \cdot (\rho_T^0)^{|t_0-\tau|} \|z_\tau^u - (z_\tau^u)'\|, \tag{12}$$

where $C_1^0 = (2\ell_T)/\mu$ and $\rho_T^0 = 1 - 2 \cdot \Big(\sqrt{1 + (2\ell_T/\mu)} + 1\Big)^{-1}$. On the other hand, by viewing each subset $\{\tau \mid t-1 \le \tau < t+k\} \times \{u\}$ for $u \in N_v^r$ in the original problem as a vertex in the new graph $\mathcal{G}'$ and applying Theorem D.2, we obtain that

$$\big\|x_{t_0}^{v_0} - (x_{t_0}^{v_0})'\big\| \le C_1^1 \cdot (\rho_S^0)^{d_\mathcal{G}(u,v_0)} \|z_\tau^u - (z_\tau^u)'\|, \tag{13}$$

where $C_1^1 = (2\Delta\ell_S)/\mu$ and $\rho_S^0 = 1 - 2 \cdot \Big(\sqrt{1 + (2\Delta\ell_S/\mu)}\Big)^{-1}$. Combining (12) and (13) gives that

$$\begin{aligned}
\big\|x_{t_0}^{v_0} - (x_{t_0}^{v_0})'\big\| &\le \min\{C_1^0 \cdot (\rho_T^0)^{|t_0-\tau|}, C_1^1 \cdot (\rho_S^0)^{d_\mathcal{G}(u,v_0)}\} \cdot \|z_\tau^u - (z_\tau^u)'\| \\
&\le \sqrt{C_1^0 \cdot C_1^1} \cdot (\rho_T^0)^{|t_0-\tau|/2} \cdot (\rho_S^0)^{d_\mathcal{G}(u,v_0)/2} \cdot \|z_\tau^u - (z_\tau^u)'\| \\
&\le C_1 \cdot \rho_T^{|t_0-\tau|} \rho_S^{d_\mathcal{G}(v_0,u)} \|z_\tau^u - (z_\tau^u)'\|
\end{aligned} \tag{14}$$

when $\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ and $\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ only differ at one entry z_τ^u for $(\tau, u) \in \partial N_{(t,v)}^{(k,r)}$.

We can use the same method to show that when $\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ and $\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ only differ at one entry y_{t-1}^u for $u \in N_v^r$, we have

$$\big\|x_{t_0}^{v_0} - (x_{t_0}^{v_0})'\big\| \le C_2 \rho_T^{t_0-(t-1)} \rho_S^{d_\mathcal{G}(v_0,u)} \big\|y_{t-1}^u - (y_{t-1}^u)'\big\|. \tag{15}$$

In the general case where $\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ and $\big(\{y_{t-1}^u\}, \{z_\tau^u\}\big)$ differ not only at one entry, we can perturb the entries of parameters one at a time and apply the triangle inequality. Then, the conclusion of Theorem 3.2 follows from (14) and (15).

D.2. Proof of Theorem 3.3

The proof follows a four step structure outlined in Appendix C.1.

STEP 1. ESTABLISH FIRST ORDER EQUATIONS

Given any system parameter $\zeta = (x_{t-1}^{(N_v^r)}, \{z_\tau^u | (\tau, u) \in \partial N_{(t,v)}^{(k,r)}\})$, we can define function $\hat{h}$ as follows:

$$\hat{h}(\hat{x}_{[k-1]}, \zeta) = \sum_{i=1}^{k-1} \sum_{u \in N_v^{r-1}} f_{t-1+i}^u(\hat{x}_i^u) + \sum_{i=1}^{k-1} \sum_{(u,u') \in \mathcal{E}(N_v^r)} s_{t-1+i}^{(u,u')}(\hat{x}_i^u, \hat{x}_i^{u'}) + \sum_{i=1}^{k} \sum_{u \in N_v^r} c_{t-1+i}^u(\hat{x}_i^u, \hat{x}_{i-1}^u).$$

$\hat{x}_0$ coincides with x_{t-1} on every node in N_v^r. $\hat{x}_k$ coincides with z_{t-1+k} on every node in N_v^r. For $1 \le i \le k-1$, $\hat{x}_i^u$ coincides with z_{t-1+i}^u on the boundary, i.e., $u \in \partial N_v^r$.

Let perturbation vector $e = [e_0^T, e_1^T, \cdots, e_{k-1}^T, e_k^T]^T$ where $e_0, e_k \in \mathbb{R}^{|N_v^r| \times n}$ and $e_i = \mathbb{R}^{|\partial N_v^r| \times n}$ for $1 \le i \le k-1$.

Given $\theta \in \mathbb{R}$, $\psi(\zeta + \theta e)$ is the global minimizer of convex function $\hat{h}(\cdot, \zeta + \theta e)$, and hence we have

$$\nabla_{\hat{x}_{1:k-1}^{(N_v^{r-1})}} \hat{h}(\psi(\zeta + \theta e), \zeta + \theta e) = 0.$$

Taking the derivative with respect to θ, we establish the following set of equations:

$$\begin{aligned}\nabla^2_{\hat{x}^{(N_v^{r-1})}_{1:k-1}} \hat{h}(\psi(\zeta+\theta e),\zeta+\theta e)\frac{d}{d\theta}\psi(\zeta+\theta e) = &-\nabla_{\hat{x}_0^{(N_v^r)}}\nabla_{\hat{x}^{(N_v^{r-1})}_{1:k-1}}\hat{h}(\psi(\zeta+\theta e),\zeta+\theta e)e_0 \\ &-\nabla_{\hat{x}_k^{(N_v^r)}}\nabla_{\hat{x}^{(N_v^{r-1})}_{1:k-1}}\hat{h}(\psi(\zeta+\theta e),\zeta+\theta e)e_k \\ &-\sum_{\tau=1}^{k-1}\nabla_{\hat{x}_\tau^{(\partial N_v^r)}}\nabla_{\hat{x}^{(N_v^{r-1})}_{1:k-1}}\hat{h}(\psi(\zeta+\theta e),\zeta+\theta e)e_\tau.\end{aligned} \tag{16}$$

We adopt the following short-hand notation:

- $M \coloneqq \nabla^2_{\hat{x}^{(N_v^{r-1})}_{1:k-1}} \hat{h}(\psi(\zeta+\theta e),\zeta+\theta e)$, which is a hierarchical block matrix with the first level of dimension $(k-1)\times(k-1)$, the second level of dimension $|N_v^{r-1}|\times|N_v^{r-1}|$ and the third level of dimension $n\times n$.

- $R^{(1)} \coloneqq -\nabla_{\hat{x}_0^{(N_v^r)}}\nabla_{\hat{x}^{(N_v^{r-1})}_{1:k-1}}\hat{h}(\psi(\zeta+\theta e),\zeta+\theta e)$, which is also a hierarchical block matrix with the first level of dimension $(k-1)\times 1$, the second level of dimension $|N_v^{r-1}|\times|N_v^r|$ and the third level of dimension $n\times n$.

- $R^{(k-1)} \coloneqq -\nabla_{\hat{x}_k^{(N_v^r)}}\nabla_{\hat{x}^{(N_v^{r-1})}_{1:k-1}}\hat{h}(\psi(\zeta+\theta e),\zeta+\theta e)$, which is also a hierarchical block matrix with the first level of dimension $(k-1)\times 1$, the second level of dimension $|N_v^{r-1}|\times|N_v^r|$ and the third level of dimension $n\times n$.

- $K^{(\tau)} \coloneqq -\nabla_{\hat{x}_\tau^{(\partial N_v^r)}}\nabla_{\hat{x}^{(N_v^{r-1})}_{1:k-1}}\hat{h}(\psi(\zeta+\theta e),\zeta+\theta e)$, which is also a hierarchical block matrix with the first level of dimension $(k-1)\times 1$. the second level of dimension $|N_v^{r-1}|\times|\partial N_v^r|$ and the third level of dimension $n\times n$.

Using the above, we can rewrite (16) as follows:

$$\frac{d}{d\theta}\psi(\zeta+\theta e) = M^{-1}\left(R^{(1)}e_0 + R^{(k-1)}e_k + \sum_{\tau=1}^{k-1}K^{(\tau)}y_\tau\right).$$

Due to the structure of temporal interaction cost functions, for $R^{(1)}$(resp. $R^{(k-1)}$), only when the first level index is 1 (resp. $k-1$), the lower level block matrix is non-zero; due to the structure of spatial interaction cost functions, for $K^{(\tau)}$, only when the first level index is τ, the lower level block matrix is non-zero. Hence, for $1\le\tau'\le k-1$, we have

$$(\frac{d}{d\theta}\psi(\zeta+\theta e))_{\tau'} = M^{-1}_{\tau',1}R^{(1)}_1 e_0 + M^{-1}_{\tau',k-1}R^{(k-1)}_{k-1}e_k + \sum_{\tau=1}^{k-1}M^{-1}_{\tau',\tau}K^{(\tau)}_\tau y_\tau, \tag{17}$$

where the subscripts on the right hand side denote the first level index of hierarchical block matrices M, $R^{(1)}$, $R^{(k-1)}$ and $K^{(\tau)}$.

STEP 2. EXPLOIT THE STRUCTURE OF MATRIX M

We decompose M to block diagonal matrix D and tri-diagonal block matrix A such that $M = D + A$. We denote each diagonal block in D as $D_{i,i}$ for $1 \le i \le k-1$. Other blocks in D are zero matrices.

$$D := \begin{bmatrix}
* & 0 & \cdots & * & & & & & & & & & & & & \\
0 & * & & 0 & & & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & & & & & \\
* & 0 & \cdots & * & & & & & & & & & & & & \\
 & & & & * & 0 & \cdots & * & & & & & & & & \\
 & & & & 0 & * & & 0 & & & & & & & & \\
 & & & & \vdots & & \ddots & & & & & & & & & \\
 & & & & * & 0 & \cdots & * & & & & & & & & \\
 & & & & & & & & \ddots & & & & & & & \\
 & & & & & & & & & * & 0 & \cdots & * & & & \\
 & & & & & & & & & 0 & * & & 0 & & & \\
 & & & & & & & & & \vdots & & \ddots & & & & \\
 & & & & & & & & & * & 0 & \cdots & * & & &
\end{bmatrix}$$

Each non-zero block in A is a diagonal block matrix, which captures the Hessian of temporal interaction cost between consecutive time steps. Denote each block as $A_{i,j}$ for $1 \le i, j \le k-1$.

$$A := \begin{bmatrix}
 & & & & * & 0 & \cdots & 0 & & & & & & & & \\
 & & & & 0 & * & & 0 & & & & & & & & \\
 & & & & \vdots & & \ddots & & & & & & & & & \\
 & & & & 0 & 0 & \cdots & * & & & & & & & & \\
* & 0 & \cdots & 0 & & & & & * & 0 & \cdots & 0 & & & & \\
0 & * & & 0 & & & & & 0 & * & & 0 & & & & \\
\vdots & & \ddots & & & & & & \vdots & & \ddots & & & & & \\
0 & 0 & \cdots & * & & & & & 0 & 0 & \cdots & * & & & & \\
 & & & & & & & & & & & & * & 0 & \cdots & 0 \\
 & & & & & & & & & & & & 0 & * & & 0 \\
 & & & & & & & & & & \ddots & & \vdots & & \ddots & \\
 & & & & & & & & & & & & 0 & 0 & \cdots & * \\
 & & & & & & & & * & 0 & \cdots & 0 & & & & \\
 & & & & & & & & 0 & * & & 0 & & & & \\
 & & & & & & & & \vdots & & \ddots & & & & & \\
 & & & & & & & & 0 & 0 & \cdots & * & & & &
\end{bmatrix}$$

We rewrite the inverse of M as follows:

$$M^{-1} = (D + A)^{-1} = D^{-1}(I + AD^{-1})^{-1} = D^{-1}P^{-1}.$$

Next we show the proof for Lemma C.2.

Proof of Lemma C.2. We claim the eigenvalues of P are in $\{\lambda \in \mathbb{C} | |\lambda - z| \le R\}$ for some $R \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \setminus \{0\}$ such that $R < |z|$. We first establish Lemma C.2 based on the claim and then prove the claim.

We follow the argument as in the proof of Thm 4 in Shin et al. (2020). Since any eigenvalue λ of P satisfies $|\lambda - z| \le R$, $|\lambda/z - 1| \le R/|z| < 1$. Thus, the eigenvalues of $I - (1/z)P$ lie on $\{\tilde{\lambda} \in \mathbb{C} : |\tilde{\lambda}| \le R/|z|\}$, which guarantees

$\rho(I-(1/z)P)<1$. Therefore,

$$P^{-1}=\frac{1}{z}\left(I-(I-\frac{1}{z}P)\right)^{-1}=\frac{1}{z}\sum_{q\geq 0}(I-\frac{1}{z}P)^q.$$

We let $z=1$ and $R=\frac{2\ell_T}{\mu}$ and prove the above claim by utilizing Gershgorin circle theorem for block matrices.

By Theorem 1.13.1 and Remark 1.13.2 of Tretter (2008), the following holds: Consider $\mathcal{A}=(A_{ij})\in\mathbb{R}^{dn\times dn}$ $(d,n\geq 1)$ where $A_{ij}\in\mathbb{R}^{d\times d}$ and A_{ii} is symmetric. Suppose $\sigma(\cdot)$ is the spectrum of a matrix. Define set

$$G_i\coloneqq\sigma(A_{ii})\cup\left\{\cup_{k=1}^{d}B\left(\lambda_k(A_{ii}),\sum_{j\neq i}\|A_{ij}\|\right)\right\}$$

where $B(\cdot,\cdot)$ denotes a disk $B(c,r)=\{\lambda:\|\lambda-c\|\leq r\}$ and λ_k is the k-th smallest eigenvalues of A_{ii}. Then,

$$\sigma(\mathcal{A})\in\cup_{i=1}^{n}G_i.$$

Next, we use the above fact to find a superset of $\sigma(P)$. Every diagonal block of P is I. Moreover, $P_{i,j}=0$ for $|i-j|>1$, $P_{i,i-1}=A_{i,i-1}D_{i-1,i-1}^{-1}$, $P_{i,i+1}=A_{i,i+1}D_{i+1,i+1}^{-1}$. Hence we have

$$\begin{aligned}\sum_{j\neq i}\|P_{i,j}\|&\leq\|A_{i,i-1}\|\left\|D_{i-1,i-1}^{-1}\right\|+\|A_{i,i+1}\|\left\|D_{i+1,i+1}^{-1}\right\|\\&\leq\frac{2\ell_T}{\mu}.\end{aligned}$$

The last inequality is by Assumptions 2.1. Therefore, $G_i=B(1,\frac{2\ell_T}{\mu})$. This implies all eigenvalues of P are in $B(1,\frac{2\ell_T}{\mu})$. □

To further simplify the notation in the power series expansion, we define $J\coloneqq AD^{-1}=P-I$. Given any time indices τ' and τ, we have

$$\begin{aligned}(M^{-1})_{\tau',\tau}&=(D^{-1})_{\tau',\tau'}(P^{-1})_{\tau',\tau}\\&=(D^{-1})_{\tau',\tau'}\times\sum_{\ell\geq 0}(-J)^{\ell}_{\tau',\tau},\end{aligned}\tag{18}$$

where the first equality is since D^{-1} is a diagonal block matrix, the second equality is due to Lemma C.2.

STEP 3: PROPERTY FOR GENERAL EXPONENTIAL-DECAY MATRICES

This step simply requires proving Lemma C.3.

Proof of Lemma C.3. Under the assumptions, we see that

$$\begin{aligned}\sum_q(\frac{1}{\lambda'})^{d_{\mathcal{M}}(u,q)}\|(A_1A_2\cdots A_\ell)_{u,q}\|&=\sum_q(\frac{1}{\lambda'})^{d_{\mathcal{M}}(u,q)}\left\|\sum_{s_1,\cdots,s_{\ell-1}}(A_1)_{u,s_1}(A_2)_{s_1,s_2}\cdots(A_\ell)_{s_{\ell-1},q}\right\|\\&\leq\sum_q(\frac{1}{\lambda'})^{d_{\mathcal{M}}(u,q)}\sum_{s_1,\cdots,s_{\ell-1}}(C_1\lambda^{d_{\mathcal{M}}(u,s_1)})(C_2\lambda^{d_{\mathcal{M}}(s_1,s_2)})\cdots(C_\ell\lambda^{d_{\mathcal{M}}(s_{\ell-1},q)})\\&\leq\sum_q\sum_{s_1,\cdots,s_{\ell-1}}\prod_{i=1}^{\ell}C_i(\frac{\lambda}{\lambda'})^{d_{\mathcal{M}}(u,s_1)+d_{\mathcal{M}}(s_1,s_2)+\cdots+d_{\mathcal{M}}(s_{\ell-1},q)}\\&\leq(\tilde{a})^{\ell}\prod_{i=1}^{\ell}C_i.\end{aligned}\tag{19}$$

Hence, we obtain that

$$\left\|(\prod_{i=1}^{\ell} A_i)_{u,q}\right\| \leq C'(\lambda')^{d_M(u,q)}.$$

□

STEP 4: ESTABLISH CORRELATION DECAY PROPERTIES OF MATRIX M

In this step, we use the property developed for general exponential-decay matrices on M and derive the perturbation bound in the Theorem 3.3.

Lemma D.3. *For $\ell \geq 1$, time index $i, j \geq 1$, J^ℓ has the following properties:*

- $(J^\ell)_{i,j} = 0$ *if* $\ell < |i-j|$ *or* $\ell - |i-j|$ *is odd.*
- $(J^\ell)_{i,j}$ *is a summation of terms* $\prod_{k=1}^{\ell} A_{j_k,i_k} D^{-1}_{i_k,i_k}$ *and the number of such terms is bounded by* $\binom{\ell}{(\ell-|i-j|)/2}$.

Note for integers $m, k \geq 1$, we define $\binom{m}{k/2} = 0$ if k is odd.

Proof. Since J is a tri-diagonal banded matrix, $J^\ell_{i,j} = 0$ for $\ell < |i-j|$. We prove the rest of properties of J by induction on ℓ. When $\ell = 1$,

$$J_{i,i} = 0, \quad J_{i,i-1} = A_{i,i-1} D^{-1}_{i-1,i-1}, \quad J_{i,i+1} = A_{i,i+1} D^{-1}_{i+1,i+1}.$$

Lemma D.3 holds for the base case. Suppose Lemma D.3 holds for J^q for $q \leq \ell - 1$. Let $q = \ell$, then

$$J^\ell_{i,j} = \sum_k J^{\ell-1}_{i,k} J_{k,j} = J^{\ell-1}_{i,j-1} A_{j-1,j} D^{-1}_{j,j} + J^{\ell-1}_{i,j+1} A_{j+1,j} D^{-1}_{j,j}.$$

By induction hypothesis, $J^{\ell-1}_{i,j}$ is a summation of terms $\prod_{k=1}^{\ell-1} A_{j_k,i_k} D^{-1}_{i_k,i_k}$. Moreover, the number of such terms is bounded by $\binom{\ell-1}{(\ell-1-|i-j-1|)/2} + \binom{\ell-1}{(\ell-1-|i-j+1|)/2}$. Next we will show $\binom{\ell-1}{(\ell-1-|i-j-1|)/2} + \binom{\ell-1}{(\ell-1-|i-j+1|)/2} = \binom{\ell}{(\ell-|i-j|)/2}$ case by case.

Case 1: $\ell - |i-j|$ is odd.

If $\ell - |i-j|$ is odd, then $\ell - 1 - |i-j-1|$ and $\ell - 1 - |i-j+1|$ are both odd. Under this case,

$$\binom{\ell-1}{(\ell-1-|i-j-1|)/2} + \binom{\ell-1}{(\ell-1-|i-j+1|)/2} = 0,$$

which is equal to $\binom{\ell}{(\ell-|i-j|)/2}$.

Case 2: $\ell - |i-j|$ is even and $i = j$. Under this case, we have

$$\binom{\ell-1}{(\ell-1-|i-j-1|)/2} + \binom{\ell-1}{(\ell-1-|i-j+1|)/2} = \binom{\ell-1}{\ell/2-1} + \binom{\ell-1}{\ell/2-1}.$$

Since ℓ is even, $\binom{\ell-1}{\ell/2-1} + \binom{\ell-1}{\ell/2-1} = \binom{\ell}{\ell/2} = \binom{\ell}{(\ell-|i-j|)/2}$.

Case 3: $\ell - |i-j|$ is even and $i \neq j$.

If $\ell - |i-j|$ is even, then $\ell - 1 - |i-j-1|$ and $\ell - 1 - |i-j+1|$ are both even. We denote $(\ell - |i-j|)/2$ as k_0. By triangle inequality, $(\ell - 1 - |i-j-1|)/2$ and $(\ell - 1 - |i-j+1|)/2$ are in $\{k_0 - 1, k_0\}$. Since $i \neq j$,

$$\binom{\ell-1}{(\ell-1-|i-j-1|)/2} + \binom{\ell-1}{(\ell-1-|i-j+1|)/2} = \binom{\ell-1}{k_0-1} + \binom{\ell-1}{k_0},$$

which sums to $\binom{\ell}{k_0}$ by Pascal's triangle.

□

Next we present the proof of Lemma C.4.

Proof of Lemma C.4. By Lemma D.3, $(J^\ell)_{i,j}$ is a summation of terms $\prod_{k=1}^{\ell} A_{j_k,i_k} D^{-1}_{i_k,i_k}$ and the number of such terms is bounded by $\binom{\ell}{(\ell-|i-j|)/2}$.

Define $B_k \coloneqq A_{j_k,i_k} D^{-1}_{i_k,i_k}$. Recall A_{j_k,i_k} is a diagonal matrix and D_{i_k,i_k} is a graph-induced banded matrix.

$$\|(B_k)_{u,q}\| = \big\|(A_{j_k,i_k} D^{-1}_{i_k,i_k})_{u,q}\big\| = \big\|(A_{j_k,i_k})_{u,u}(D^{-1}_{i_k,i_k})_{u,q}\big\| \le \ell_T \big\|(D^{-1}_{i_k,i_k})_{u,v}\big\| \le \frac{2\ell_T}{\mu}\gamma_S^{d_{\mathcal{G}}(u,v)}.$$

where the last inequality is by using Lemma D.1 on D_{i_k,i_k}.

Under the condition $b < \infty$, we can use Lemma C.3 to obtain the following bound,

$$\left\|\Big(\prod_{k=1}^{\ell} A_{j_k,i_k} D^{-1}_{i_k}\Big)_{u,v}\right\| \le \Big(b\frac{2\ell_T}{\mu}\Big)^{\ell}(\gamma_S')^{d_{\mathcal{G}}(u,v)}.$$

Since the number of such terms is bounded by $\binom{\ell}{(\ell-|i-j|)/2}$, we have

$$\big\|((J^\ell)_{i,j})_{u,q}\big\| \le \binom{\ell}{(\ell-|i-j|)/2}\Big(b\frac{2\ell_T}{\mu}\Big)^{\ell}(\gamma_S')^{d_{\mathcal{G}}(u,v)}.$$

□

Lemma D.4. *Given $1 \le \tau', \tau \le k-1$, $y \in \mathbb{R}^{|\partial N_v^r| \times n}$ and $v_0 \in N_v^{r-1}$, we have*

$$\left\|\left((M)^{-1}_{\tau',\tau} K_\tau^{(\tau)} y\right)_{v_0}\right\| \le C_1 \rho_T^{|\tau'-\tau|} \sum_{u \in \partial N_v^r} \rho_S^{d_{\mathcal{G}}(v_0,u)-1}\|y_u\|,$$

and for $i \in \{1, k-1\}$, $e \in \mathbb{R}^{|N_v^r| \times n}$,

$$\left\|\left((M^{-1})_{\tau',i} R_i^{(i)} e\right)_{v_0}\right\| \le C_2 \rho_T^{|\tau'-i|+1} \sum_{u \in N_v^r} \rho_S^{d_{\mathcal{G}}(v_0,u)}\|e_u\|,$$

where $\rho_T = \frac{4\tilde{a}\ell_T}{\mu}$ and $\rho_S = (1+b_1+b_2)\gamma_S$. We let $C_1 = C_2 = \max\{\frac{a^2}{2\tilde{a}(1-4\tilde{a}\ell_T/\mu)}, \frac{2a^2\Delta\ell_S/\mu}{\gamma_S(1+b_1+b_2)(1-4\tilde{a}\ell_T/\mu)}\}$.

Proof. Given $1 \le \tau, \tau' \le k-1$ and $v_0 \in N_v^{r-1}$, since $M^{-1} = D^{-1}\sum_{\ell \ge 0}(-J)^\ell$, we have

$$\left\|\left((M)^{-1}_{\tau',\tau} K_\tau^{(\tau)} y\right)_{v_0}\right\| = \left\|\left(D^{-1}_{\tau',\tau'} \sum_{\ell \ge 0}(-J)^\ell_{\tau',\tau} K_\tau^{(\tau)} y\right)_{v_0}\right\|. \tag{20}$$

With slight abuse of notation, we use K to denote $K_\tau^{(\tau)}$, and Q^{-1} to denote $D^{-1}_{\tau',\tau'}$ in this proof from now. We can rewrite the right hand side of (20) using the new notation as follows:

$$\begin{aligned}
\left\|\left(Q^{-1}\sum_{\ell \ge 0}(-J)^\ell_{\tau',\tau} K y\right)_{v_0}\right\| &\le \sum_{\ell \ge 0}\left\|\left(Q^{-1}(-J)^\ell_{\tau',\tau} K y\right)_{v_0}\right\| \\
&= \sum_{\ell \ge 0}\left\|\sum_{q \in N_v^{r-1}}\left(Q^{-1}(-J)^\ell_{\tau',\tau}\right)_{v_0,q}(Ky)_q\right\| \\
&\le \sum_{\ell \ge 0}\sum_{q \in N_v^{r-1}}\left\|\left(Q^{-1}(-J)^\ell_{\tau',\tau}\right)_{v_0,q}\right\|\|(Ky)_q\|.
\end{aligned} \tag{21}$$

For a given $q \in N_v^{r-1}$ and $y \in \mathbb{R}^{|\partial N_v^r|d}$,

$$\|(Ky)_q\| = \left\| \sum_{u \in \partial N_v^r} K_{q,u} y_u \right\| = \left\| \sum_{u \in \partial N_v^r \cap N_q^1} K_{q,u} y_u \right\|.$$

where the last equality is since spacial interaction costs are only among neighboring nodes.

For a given $u \in \partial N_v^r$, since the spacial interaction cost for each edge is ℓ_S smooth,

$$\|K_{q,u} y_u\| \le \|K_{q,u}\| \|y_u\| \le \ell_S \|y_u\|,$$

which gives

$$\|(Ky)_q\| \le \sum_{u \in \partial N_v^r \cap N_q^1} \ell_S \|y_u\|.$$

Therefore,

$$\left\| \left(Q^{-1} \sum_{\ell \ge 0} (-J)^\ell_{\tau',\tau} K y \right)_{v_0} \right\| \le \ell_S \sum_{\ell \ge 0} \sum_{q \in N_v^{r-1}} \left\| \left(Q^{-1} (-J)^\ell_{\tau',\tau} \right)_{v_0,q} \right\| \sum_{u \in \partial N_v^r \cap N_q^1} \|y_u\|. \tag{22}$$

By Lemma C.4, $(-J)^\ell_{\tau',\tau}$ satisfies the following exponential decay properties: for any $u, q \in N_v^{r-1}$,

$$\left\| ((J^\ell)_{\tau',\tau})_{u,q} \right\| \le \binom{\ell}{(\ell - |\tau' - \tau|)/2} (\tilde{a} \frac{2\ell_T}{\mu})^\ell (\gamma_S')^{d_{\mathcal{G}}(u,q)},$$

where we choose $\delta = b_1 \cdot \gamma_S$, $\gamma_S' = (1 + b_1)\gamma_S$ and $\tilde{a} = \sum_{\gamma \ge 0} (\frac{1}{1+b_1})^\gamma h(\gamma)$.

Moreover, Q^{-1} (which denotes $D^{-1}_{\tau',\tau'}$) is the inverse of a graph-induced banded matrix. Q^{-1} satisfies: for any $u, q \in N_v^{r-1}$,

$$\left\| (Q^{-1})_{u,q} \right\| \le \frac{2}{\mu} \gamma_S^{d_{\mathcal{G}}(u,q)} < \frac{2}{\mu} (\gamma_S')^{d_{\mathcal{G}}(u,q)},$$

where the first inequality is again by using Lemma D.1 on $D_{\tau',\tau'}$.

Applying Lemma C.3 on Q^{-1} and $\left\| ((J^\ell)_{\tau',\tau}) \right\|$, we have for any $u, q \in N_v^{r-1}$, and $\ell \ge 1$,

$$\left\| \left(Q^{-1} (-J)^\ell_{\tau',\tau} \right)_{u,q} \right\| \le a^2 \frac{2}{\mu} \binom{\ell}{(\ell - |\tau' - \tau|)/2} (\tilde{a} \frac{2\ell_T}{\mu})^\ell (\lambda')^{d_{\mathcal{G}}(u,q)},$$

where $\lambda' \coloneqq \gamma_S' + b_2 \cdot \gamma_S < 1$ and $a \coloneqq \sum_{\gamma \ge 0} (\frac{1+b_1}{1+b_1+b_2})^\gamma h(\gamma)$. Note that $J^0 \coloneqq I$, it is straightforward to verify that the above inequality holds when $\ell = 0$.

With the exponential decay properties of $Q^{-1}(-J)^\ell_{\tau',\tau}$, we have

$$\begin{aligned}
\left\| \left(Q^{-1} \sum_{\ell \ge 0} (-J)^\ell_{\tau',\tau} K y \right)_{v_0} \right\| &\le \ell_S a^2 \frac{2}{\mu} \sum_{\ell \ge 0} \binom{\ell}{(\ell - |\tau' - \tau|)/2} (\tilde{a} \frac{2\ell_T}{\mu})^\ell \sum_{q \in N_v^{r-1}} (\lambda')^{d_{\mathcal{G}}(v_0,q)} \sum_{u \in \partial N_v^r \cup N_q^1} \|y_u\| \\
&\le \ell_S a^2 \frac{2}{\mu} \sum_{\ell \ge |\tau' - \tau|} \binom{\ell}{(\ell - |\tau' - \tau|)/2} (\tilde{a} \frac{2\ell_T}{\mu})^\ell \sum_{u \in \partial N_v^r} \Delta (\lambda')^{d_{\mathcal{G}}(v_0,u)-1} \|y_u\| \\
&\le \Delta \ell_S a^2 \frac{2}{\mu} \sum_{\ell \ge |\tau' - \tau|} (\frac{4\tilde{a}\ell_T}{\mu})^\ell \sum_{u \in \partial N_v^r} (\lambda')^{d_{\mathcal{G}}(v_0,u)-1} \|y_u\| \\
&\le \frac{2\Delta \ell_S a^2}{\mu - 4\tilde{a}\ell_T} (\frac{4\tilde{a}\ell_T}{\mu})^{|\tau' - \tau|} \sum_{u \in \partial N_v^r} (\lambda')^{d_{\mathcal{G}}(v_0,u)-1} \|y_u\| \\
&= \frac{2\Delta \ell_S a^2}{\lambda'(\mu - 4\tilde{a}\ell_T)} (\frac{4\tilde{a}\ell_T}{\mu})^{|\tau' - \tau|} \sum_{u \in \partial N_v^r} (\lambda')^{d_{\mathcal{G}}(v_0,u)} \|y_u\|.
\end{aligned} \tag{23}$$

The third inequality uses $\binom{\ell}{(\ell-|\tau'-\tau|)/2} \leq 2^\ell$, which can be proved using the following version of Stirling's approximation: For all $n \geq 1$, e denotes the natural number,

$$\sqrt{2\pi n}(n/e)^n e^{1/(12n+1)} < n! < \sqrt{2\pi n}(n/e)^n e^{1/(12n)}.$$

Similarly, consider $\left\|((M^{-1})_{\tau',i})R_i^{(i)}e)_{v_0}\right\|$ for $i \in \{1, k-1\}$. With slight abuse of notation, in this proof, we use R to denote $R_i^{(i)}$ and use the notation Q^{-1} to denote $D^{-1}_{\tau',\tau'}$. Following the same steps as before, we have

$$\left\|\left((M^{-1})_{\tau',i})R_i^{(i)}e\right)_{v_0}\right\| \leq \sum_{\ell\geq 0}\sum_{q\in N_v^r}\left\|\left(Q^{-1}(-J)^\ell_{\tau',i}\right)_{v_0,q}\right\|\|(Re)_q\|. \tag{24}$$

Since temporal interactions occurs for the same node under consecutive time steps, R is a diagonal block matrix. Hence,

$$\|(Re)_q\| = \|R_{q,q}e_q\| \leq \ell_T\|e_q\|.$$

Moreover, using the exponential decay properties of $Q^{-1}(-J)^\ell_{\tau',i}$, we have for $u, q \in N_v^{r-1}$,

$$\left\|\left(Q^{-1}(-J)^\ell_{\tau',i}\right)_{u,q}\right\| \leq a^2\frac{2}{\mu}\binom{\ell}{(\ell-|\tau'-i|)/2}(\tilde{a}\frac{2\ell_T}{\mu})^\ell(\lambda')^{d_{\mathcal{G}}(u,q)}.$$

Therefore,

$$\begin{aligned}
\left\|((M^{-1})_{\tau',i})R_i^{(i)}e)_{v_0}\right\| &\leq \sum_{\ell\geq 0}\sum_{q\in N_v^r} a^2\frac{2}{\mu}\binom{\ell}{(\ell-|\tau'-i|)/2}(\tilde{a}\frac{2\ell_T}{\mu})^\ell(\lambda')^{d_{\mathcal{G}}(v_0,q)}\ell_T\|e_q\| \\
&\leq \sum_{\ell\geq|\tau'-i|}\sum_{q\in N_v^r} a^2\frac{2}{\mu}\binom{\ell}{(\ell-|\tau'-i|)/2}(\tilde{a}\frac{2\ell_T}{\mu})^\ell(\lambda')^{d_{\mathcal{G}}(v_0,q)}\ell_T\|e_q\| \\
&\leq \frac{2\ell_T a^2}{\mu}\sum_{\ell\geq|\tau'-i|}(\frac{4\tilde{a}\ell_T}{\mu})^\ell\sum_{q\in N_v^r}(\lambda')^{d_{\mathcal{G}}(v_0,q)}\|e_q\| \\
&\leq \frac{2\ell_T a^2}{\mu-4\tilde{a}\ell_T}(\frac{4\tilde{a}\ell_T}{\mu})^{|\tau'-i|}\sum_{q\in N_v^r}(\lambda')^{d_{\mathcal{G}}(v_0,q)}\|e_q\| \\
&= \frac{a^2\mu}{2\tilde{a}(\mu-4\tilde{a}\ell_T)}(\frac{4\tilde{a}\ell_T}{\mu})^{|\tau'-i|+1}\sum_{q\in N_v^r}(\lambda')^{d_{\mathcal{G}}(v_0,q)}\|e_q\|.
\end{aligned} \tag{25}$$

□

Given time index $1 \leq \tau' \leq k-1$, node $v_0 \in N_v^{r-1}$, and perturbation vector $e = (e_0, e_1, \cdots, e_k)$,

$$\begin{aligned}
\left\|(\frac{d}{d\theta}\psi(\zeta+\theta e))_{\tau',v_0}\right\| &\leq \left\|\left(M^{-1}_{\tau',1}R_1^{(1)}e_0\right)_{v_0}\right\| + \left\|\left(M^{-1}_{\tau',k-1}R_{k-1}^{(k-1)}e_k\right)_{v_0}\right\| + \sum_{\tau=1}^{k-1}\left\|\left(M^{-1}_{\tau',\tau}K_\tau^{(\tau)}e_\tau\right)_{v_0}\right\| \\
&\leq \frac{a^2\mu}{2\tilde{a}(\mu-4\tilde{a}\ell_T)}\left[\rho_T^{\tau'}\sum_{q\in N_v^r}\rho_S^{d_{\mathcal{G}}(v_0,q)}\|(e_0)_q\| + \rho_T^{k-\tau'}\sum_{q\in N_v^r}\rho_S^{d_{\mathcal{G}}(v_0,q)}\|(e_k)_q\|\right] \\
&+ \sum_{\tau=1}^{k-1}\frac{2\Delta\ell_S a^2}{\lambda'(\mu-4\tilde{a}\ell_T)}\rho_T^{|\tau'-\tau|}\sum_{u\in\partial N_v^r}(\rho_S)^{d_{\mathcal{G}}(v_0,u)}\|(e_\tau)_u\|
\end{aligned}$$

where $\rho_T = \frac{4\tilde{a}\ell_T}{\mu}$ and $\rho_S = \lambda' = (1+b_1+b_2)\gamma_S$. We let $C = \max\{\frac{a^2}{2\tilde{a}(1-4\tilde{a}\ell_T/\mu)}, \frac{2a^2\Delta\ell_S/\mu}{\gamma_S(1+b_1+b_2)(1-4\tilde{a}\ell_T/\mu)}\}$. Under the condition $\mu \geq \max\{8\tilde{a}\ell_T, \Delta\ell_S(b_1+b_2)/4\}$, $\rho_T < 1$ and $\rho_S < 1$.

Then,

$$\begin{aligned}&\left\|(\frac{d}{d\theta}\psi(\zeta+\theta e))_{\tau',v_0}\right\|\\ &\leq C\left[\rho_T^{\tau'}\sum_{q\in N_v^r}\rho_S^{d_{\mathcal{G}}(v_0,q)}\|(e_0)_q\|+\rho_T^{k-\tau'}\sum_{q\in N_v^r}\rho_S^{d_{\mathcal{G}}(v_0,q)}\|(e_k)_q\|+\sum_{\tau=1}^{k-1}\rho_T^{|\tau'-\tau|}\sum_{u\in\partial N_v^r}(\rho_S)^{d_{\mathcal{G}}(v_0,u)}\|(e_\tau)_u\|\right].\end{aligned}$$

Finally, let $\zeta=\{y_{t-1}^u, z_\tau^u|(\tau,u)\in\partial N_{(v,t)}^{(k,r)}\}$ and $e=\{(y_{t-1}^u)'-y_{t-1}^u,(z_\tau^u)'-z_\tau^u\}$. By integration,

$$\left\|\psi_{(t,v)}^{(k,r)}\left(\{y_{t-1}^u\},\{z_\tau^u\}\right)_{(t_0,v_0)}-(\psi_{(t,v)}^{(k,r)}\left(\{(y_{t-1}^u)'\},\{(z_\tau^u)'\}\right)_{(t_0,v_0)}\right\|\leq\int_0^1\left\|(\frac{d}{d\theta}\psi(\zeta+\theta e))_{t_0,v_0}\right\|d\theta,$$

which is bounded by

$$C\sum_{u\in N_v^r}\rho_T^{t_0-(t-1)}\rho_S^{d_{\mathcal{G}}(v_0,u)}\left\|y_{t-1}^u-(y_{t-1}^u)'\right\|+C\sum_{(u,\tau)\in\partial N_{(t,v)}^{(k,r)}}\rho_T^{|t_0-\tau|}\rho_S^{d_{\mathcal{G}}(v_0,u)}\|z_\tau^u-(z_\tau^u)'\|.$$

D.3. Adding Constraints to Perturbation Bounds

Recall that in Appendix D.1 and D.2, we showed Theorem 3.2 and Theorem 3.3 under the assumption that the individual decisions are unconstrained to simplify the analysis. In this section, we present a general way to relax this assumption by incorporating logarithm barrier functions, which also applies for Theorem C.5.

Recall that in Assumption 2.1, we assume that D_t^v is convex with a non-empty interior, and can be expressed as

$$D_t^v:=\{x_t^v\in\mathbb{R}^n\mid(g_t^v)_i(x_t^v)\leq 0,\forall 1\leq i\leq m_t^v\},$$

where the i th constraint $(g_t^v)_i:\mathbb{R}^n\to\mathbb{R}$ is a convex function in C^2. For any time-vertex pair (τ,v), we can approximate the individual constraints

$$(g_\tau^v)_i(x_\tau^v)\leq 0,\forall 1\leq i\leq m_\tau^v,$$

by adding the *logarithmic barrier function* $-\mu\sum_{i=1}^{m_\tau^v}\ln\left(-(g_\tau^v)_i(x_\tau^v)\right)$ to the original node cost function f_τ^v. Here, parameter μ is a positive real number that controls how "good" the barrier function approximates the indicator function

$$\mathbf{I}_{D_\tau^v}(x_\tau^v)=\begin{cases}0 & \text{if } (g_\tau^v)_i(x_\tau^v)\leq 0,\forall 1\leq i\leq m_\tau^v,\\ +\infty & \text{otherwise.}\end{cases}$$

The approximation improves as parameter μ becomes closer to 0. Thus, the new node cost function will be

$$B_\tau^v(x_\tau^v;\mu):=f_\tau^v(x_\tau^v)-\mu\sum_{i=1}^{m_\tau^v}\ln\left(-(g_\tau^v)_i(x_\tau^v)\right).$$

As an extension of the original notation, we use $\psi_{(t,v)}^{(k,r)}(\{y_{t-1}^u\},\{z_\tau^u\};\mu)$ denote the optimal solution of the following optimization problem

$$\begin{aligned}\underset{\{x_\tau^u|(\tau,v)\in N_{(t,v)}^{(k-1,r-1)}\}}{\arg\min}\ &\sum_{\tau=t}^{t+k-1}\left(\sum_{u\in N_v^r}B_\tau^u(x_\tau^u;\mu)+\sum_{u\in N_v^r}c_\tau^u(x_\tau^u,x_{\tau-1}^u)+\sum_{(u,q)\in\mathcal{E}(N_v^r)}g_t^{(u,q)}(x_t^u,x_t^q)\right)\\ \text{s.t. }\ &x_{t-1}^u=y_{t-1}^u,\forall u\in N_v^r,\\ &x_\tau^u=z_\tau^u,\forall(\tau,u)\in\partial N_{(t,v)}^{(k,r)}.\end{aligned}$$

Compared with $\psi_{(t,v)}^{(k,r)}(\{y_{t-1}^u\},\{z_\tau^u\})$ defined in Section 3.1, the constraints $x_\tau^u\in D_\tau^u$ are removed and the node costs $f_\tau^u(x_\tau^u)$ are replaced with $B_\tau^u(x_\tau;\mu)$.

A key observation we need to point out is that the perturbation bounds we have shown in Appendix D.1 and D.2 do not depend on the smoothness constant ℓ_f of node cost functions. That means the perturbation bound

$$\begin{aligned}&\left\|\psi_{(t,v)}^{(k,r)}\left(\{y_{t-1}^u\},\{z_\tau^u\};\mu\right)_{(t_0,v_0)}-\psi_{(t,v)}^{(k,r)}\left(\{(y_{t-1}^u)'\},\{(z_\tau^u)'\};\mu\right)_{(t_0,v_0)}\right\|\\ \le\ & C_1\sum_{(u,\tau)\in\partial N_{(t,v)}^{(k,r)}}\rho_T^{|t_0-\tau|}\rho_S^{d_{\mathcal{G}}(v_0,u)}\|z_\tau^u-(z_\tau^u)'\|+C_2\sum_{u\in N_v^r}\rho_T^{t_0-(t-1)}\rho_S^{d_{\mathcal{G}}(v_0,u)}\left\|y_{t-1}^u-(y_{t-1}^u)'\right\|\end{aligned}$$

holds for arbitrary μ, where C_1, C_2, ρ_S, ρ_T are specified in Theorem 3.2 or Theorem 3.3 and are independent of μ. Theorem 3.10 in Forsgren et al. (2002) guarantees that $\psi_{(t,v)}^{(k,r)}(\{y_{t-1}^u\},\{z_\tau^u\};\mu_k)$ converge to $\psi_{(t,v)}^{(k,r)}(\{y_{t-1}^u\},\{z_\tau^u\})$ for any positive sequence $\{\mu_k\}_{k=1}^\infty$ that tends to zero. Thus the above perturbation bound also holds for $\psi_{(t,v)}^{(k,r)}(\{y_{t-1}^u\},\{z_\tau^u\})$ which includes the constraints on individual decisions.

Note that the argument we present in this section also works for Theorem C.5.

E. Competitive Bounds

This appendix includes the proofs of the competitive bounds presented in Section 3.3.

E.1. Proof of Theorem C.8

We first derive an upper bound on the distance between x_t and x_t^*.

Note that for any time step t, we have

$$\left\|x_t-\tilde{\psi}_t(x_{t-1})_t\right\|\le e_t. \tag{26}$$

Thus we see that

$$\begin{aligned}\|x_t-x_t^*\|&=\left\|x_t-\tilde{\psi}_1(x_0)_t\right\|\\ &\le\left\|x_t-\tilde{\psi}_t(x_{t-1})_t\right\|+\sum_{i=1}^{t-1}\left\|\tilde{\psi}_{t-i+1}(x_{t-i})_t-\tilde{\psi}_{t-i}(x_{t-i-1})_t\right\|\\ &\le\left\|x_t-\tilde{\psi}_t(x_{t-1})_t\right\|+\sum_{i=1}^{t-1}C_G\rho_G^i\left\|x_{t-i}-\tilde{\psi}_{t-i}(x_{t-i-1})_{t-i}\right\| &\text{(27a)}\\ &\le\sum_{i=0}^{t-1}C_0\rho_G^i\left\|x_{t-i}-\tilde{\psi}_{t-i}(x_{t-i-1})_{t-i}\right\| &\text{(27b)}\\ &\le\sum_{i=1}^{t}C_0\rho_G^{t-i}e_i, &\text{(27c)}\end{aligned}$$

where in (27a), we used Theorem C.5 and the fact that $\tilde{\psi}_{t-i}(x_{t-i-1})_t$ can be written as

$$\tilde{\psi}_{t-i}(x_{t-i-1})_t=\tilde{\psi}_{t-i+1}\left(\tilde{\psi}_{t-i}(x_{t-i-1})_{t-i}\right)_t.$$

We also used $C_0=\max\{1,C_G\}$ in (27b) and (26) in (27c).

By (27) and the Cauchy-Schwarz Inequality, we see that

$$\|x_t-x_t^*\|^2\le C_0^2\left(\sum_{i=1}^t\rho_G^{t-i}e_i\right)^2\le C_0^2\left(\sum_{i=1}^t\rho_G^{t-i}\right)\cdot\left(\sum_{i=1}^t\rho_G^{t-i}e_i^2\right)\le\frac{C_0^2}{1-\rho_G}\cdot\left(\sum_{i=1}^t\rho_G^{t-i}e_i^2\right).$$

Summing up over t gives that

$$\sum_{t=1}^H\|x_t-x_t^*\|^2\le\frac{C_0^2}{(1-\rho_G)^2}\cdot\sum_{t=1}^H e_t^2.$$

E.2. Proof of Lemma C.7

In this section, we show Lemma C.7 holds with following specific constants:

$$\begin{aligned} e_t^2 &:= \left\|x_t - x^*_{t|t-1}\right\|^2 \\ &\le 4C_1^2C_0^2\left(\frac{h(r)^2\rho_G^2}{(1-\rho_T)(1-\rho_G^2\rho_T)}\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2(k-1)}\cdot\rho_G^{2k}\right)\left\|x_{t-1}-x^*_{t-1}\right\|^2 \\ &\quad + \frac{8C_1^2}{\mu}\left(\frac{h(r)^2}{1-\rho_T}\cdot\rho_S^{2r}\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}f_\tau(x^*_\tau) + C_3(r)^2\cdot\rho_T^{2(k-1)}f_{t+k-1}(x^*_{t+k-1})\right) \end{aligned} \tag{28}$$

Note that, by the principle of optimality, we have

$$\begin{aligned} x_t^v &= \psi^{(k,r)}_{(t,v)}\left(\{x^u_{t-1}\},\{\theta^u_\tau\}\right)_{(t,v)}, \\ (x^v_{t|t-1})^* &= \psi^{(k,r)}_{(t,v)}\left(\{x^u_{t-1}\},\{(x^u_{\tau|t-1})^*\}\right)_{(t,v)}. \end{aligned}$$

Recall that we define the quantity $C_3(r) := \sum_{\gamma=0}^{r} h(\gamma)\cdot\rho_S^\gamma$ to simplify the notation.

Since the exponentially decaying local perturbation bound holds in Definition 3.1, we see that

$$\begin{aligned} \left\|x_t^v - (x^v_{t|t-1})^*\right\| \le{}& C_1\rho_S^r\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}\sum_{u\in\partial N_v^r}\left\|(x^u_{\tau|t-1})^* - \theta^u_\tau\right\| \\ &+ C_1\rho_T^{k-1}\sum_{u\in N_v^r}\rho_S^{d_{\mathcal{G}}(u,v)}\left\|(x^u_{t+k-1|t-1})^* - \theta^u_{t+k-1}\right\|, \end{aligned} \tag{29}$$

which implies that

$$\begin{aligned} \left\|x_t^v - (x^v_{t|t-1})^*\right\|^2 \le{}& 2C_1^2\rho_S^{2r}\left(\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}\sum_{u\in\partial N_v^r}\left\|(x^u_{\tau|t-1})^* - \theta^u_\tau\right\|\right)^2 \\ &+ 2C_1^2\rho_T^{2(k-1)}\left(\sum_{u\in N_v^r}\rho_S^{d_{\mathcal{G}}(u,v)}\left\|(x^u_{t+k-1|t-1})^* - \theta^u_{t+k-1}\right\|\right)^2 && \text{(30a)} \\ \le{}& 2C_1^2\rho_S^{2r}\left(\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}\sum_{u\in\partial N_v^r}1\right)\left(\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}\sum_{u\in\partial N_v^r}\left\|(x^u_{\tau|t-1})^* - \theta^u_\tau\right\|^2\right) \\ &+ 2C_1^2\rho_T^{2(k-1)}\left(\sum_{u\in N_v^r}\rho_S^{d_{\mathcal{G}}(u,v)}\right)\left(\sum_{u\in N_v^r}\rho_S^{d_{\mathcal{G}}(u,v)}\left\|(x^u_{t+k-1|t-1})^* - \theta^u_{t+k-1}\right\|^2\right) && \text{(30b)} \\ \le{}& \frac{2C_1^2h(r)}{1-\rho_T}\cdot\rho_S^{2r}\left(\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}\sum_{u\in\partial N_v^r}\left\|(x^u_{\tau|t-1})^* - \theta^u_\tau\right\|^2\right) \\ &+ 2C_1^2C_3(r)\cdot\rho_T^{2(k-1)}\left(\sum_{u\in N_v^r}\rho_S^{d_{\mathcal{G}}(u,v)}\left\|(x^u_{t+k-1|t-1})^* - \theta^u_{t+k-1}\right\|^2\right), && \text{(30c)} \end{aligned}$$

where we used the AM-GM Inequality in (30a); we used the Cauchy-Schwarz Inequality in (30b); we used the definitions of functions $h(r)$ and $C_3(r)$ in (30c).

Summing up (30) over all $v \in \mathcal{V}$ and reorganizing terms gives

$$\begin{aligned}
&\sum_{v\in\mathcal{V}}\left\|x_t^v - (x_{t|t-1}^v)^*\right\|^2 \\
&\le \frac{2C_1^2 h(r)}{1-\rho_T}\cdot\rho_S^{2r}\sum_{v\in\mathcal{V}}\left(\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}\sum_{u\in\partial N_v^r}\left\|(x_{\tau|t-1}^u)^* - \theta_\tau^u\right\|^2\right) \\
&\quad + 2C_1^2 C_3(r)\cdot\rho_T^{2(k-1)}\sum_{v\in\mathcal{V}}\left(\sum_{u\in N_v^r}\rho_S^{d_\mathcal{G}(u,v)}\left\|(x_{t+k-1|t-1}^u)^* - \theta_{t+k-1}^u\right\|^2\right) \\
&\le \frac{2C_1^2 h(r)^2}{1-\rho_T}\cdot\rho_S^{2r}\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}\left\|x_{\tau|t-1}^* - \theta_\tau\right\|^2 + 2C_1^2 C_3(r)^2\cdot\rho_T^{2(k-1)}\left\|x_{t+k-1|t-1}^* - \theta_{t+k-1}\right\|^2,
\end{aligned} \tag{31}$$

where we used the facts that

$$\sum_{v\in\mathcal{V}}\sum_{u\in\partial N_v^r}\left\|(x_{\tau|t-1}^u)^* - \theta_\tau^u\right\|^2 \le h(r)\sum_{v\in\mathcal{V}}\left\|(x_{\tau|t-1}^v)^* - \theta_\tau^v\right\|^2 = h(r)\cdot\left\|x_{\tau|t-1}^* - \theta_\tau\right\|^2,$$

and

$$\begin{aligned}
\sum_{v\in\mathcal{V}}\sum_{u\in\partial N_v^r}\rho_S^{d_\mathcal{G}(u,v)}\left\|(x_{t+k-1|t-1}^u)^* - \theta_{t+k-1}^u\right\|^2 &\le C_3(r)\sum_{v\in\mathcal{V}}\left\|(x_{t+k-1|t-1}^v)^* - \theta_{t+k-1}^v\right\|^2 \\
&= C_3(r)\cdot\left\|x_{t+k-1|t-1}^* - \theta_{t+k-1}\right\|^2.
\end{aligned}$$

We also note that by the principle of optimality, the following equations hold for all $\tau \ge t$:

$$\begin{aligned}
x_{\tau|t-1}^* &= \tilde{\psi}_t(x_{t-1})_\tau, \\
x_\tau^* &= \tilde{\psi}_t(x_{t-1}^*)_\tau.
\end{aligned}$$

Recall that $C_0 := \max\{1, C_G\}$. By Theorem C.5, we see that

$$\left\|x_{\tau|t-1}^* - x_\tau^*\right\| \le C_0\rho_G^{\tau-t+1}\|x_{t-1} - x_{t-1}^*\|, \tag{32}$$

which implies

$$\begin{aligned}
\left\|x_{\tau|t-1}^* - \theta_\tau\right\|^2 &\le 2\left\|x_{\tau|t-1}^* - x_\tau^*\right\|^2 + 2\|x_\tau^* - \theta_\tau\|^2 && \text{(33a)} \\
&\le 2C_0^2\rho_G^{2(\tau-t+1)}\left\|x_{t-1} - x_{t-1}^*\right\|^2 + 2\|x_\tau^* - \theta_\tau\|^2, && \text{(33b)}
\end{aligned}$$

where we used the triangle inequality and the AM-GM inequality in (33a); we used (32) in (33b).

Substituting (33) into (31) gives

$$\begin{aligned}
&\sum_{v\in\mathcal{V}}\left\|x_t^v - (x_{t|t-1}^v)^*\right\|^2 \\
&\le 4C_1^2C_0^2\left(\frac{h(r)^2\rho_G^2}{(1-\rho_T)(1-\rho_G^2\rho_T)}\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2(k-1)}\cdot\rho_G^{2k}\right)\left\|x_{t-1} - x_{t-1}^*\right\|^2 \\
&\quad + 4C_1^2\left(\frac{h(r)^2}{1-\rho_T}\cdot\rho_S^{2r}\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}\|x_\tau^* - \theta_\tau\|^2 + C_3(r)^2\cdot\rho_T^{2(k-1)}\left\|x_{t+k-1}^* - \theta_{t+k-1}\right\|^2\right) \\
&\le 4C_1^2C_0^2\left(\frac{h(r)^2\rho_G^2}{(1-\rho_T)(1-\rho_G^2\rho_T)}\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2(k-1)}\cdot\rho_G^{2k}\right)\left\|x_{t-1} - x_{t-1}^*\right\|^2 \\
&\quad + \frac{8C_1^2}{\mu}\left(\frac{h(r)^2}{1-\rho_T}\cdot\rho_S^{2r}\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}f_\tau(x_\tau^*) + C_3(r)^2\cdot\rho_T^{2(k-1)}f_{t+k-1}(x_{t+k-1}^*)\right),
\end{aligned} \tag{34}$$

where we used the fact that the node cost function f_τ^v is non-negative and μ-strongly convex for all τ, v, thus

$$f_\tau(x_\tau^*) \geq \sum_{v\in\mathcal{V}} f_\tau^v((x_\tau^v)^*) \geq \frac{\mu}{2}\sum_{v\in\mathcal{V}}\|(x_\tau^v)^* - \theta_\tau^v\|^2 = \frac{\mu}{2}\|x_\tau^* - \theta_\tau\|^2.$$

Note that $\sum_{v\in\mathcal{V}}\left\|x_t^v - (x_{t|t-1}^v)^*\right\|^2 = \left\|x_t - x_{t|t-1}^*\right\|^2 = e_t^2$. Thus we have finished the proof of (28).

E.3. Proof of Theorem 3.4

In this section, we show Theorem 3.4 holds with the following specific constants:

$$1 + \left(1 + \frac{32C_0^2C_1^2(\ell_f + \Delta\ell_S + 2\ell_T)\cdot h(r)^2}{\mu(1-\rho_G)^2(1-\rho_T)^2}\right)\cdot\rho_S^r + \left(1 + \frac{32C_0^2C_1^2(\ell_f + \Delta\ell_S + 2\ell_T)C_3(r)^2}{\mu(1-\rho_G)^2}\right)\rho_T^{k-1}. \tag{35}$$

under the assumption that

$$\frac{4C_1^2C_0^4}{(1-\rho_G)^2}\left(\frac{h(r)^2\rho_G^2}{(1-\rho_T)(1-\rho_G^2\rho_T)}\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2(k-1)}\cdot\rho_G^{2k}\right) \leq \frac{1}{2}. \tag{36}$$

Recall that C_0 is defined in Theorem C.8. Note that Theorem 3.2 and Theorem C.5 hold under Assumption 2.1. One can check that $C_0, C_1, (1-\rho_G)^{-1}$, and $(1-\rho_T)^{-1}$ are bounded by polynomials of $\ell_f/\mu, \ell_T/\mu$, and $(\Delta\ell_S)/\mu$.

In the proof, we need to use Lemma F.2 in Lin et al. (2021) to bound LPC's total cost by a weighted sum of the offline optimal cost and the sum of squared distances between their trajectories. For completeness, we present Lemma F.2 in Lin et al. (2021) below:

Lemma E.1. *For a fixed dimension $m \in \mathbb{Z}_+$, assume a function $h : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is convex, ℓ-smooth and continuously differentiable. For all $x, y \in \mathbb{R}^m$, for all $\eta > 0$, we have*

$$h(x) \leq (1+\eta)h(y) + \frac{\ell}{2}\left(1 + \frac{1}{\eta}\right)\|x - y\|^2.$$

Now we come back to the proof of Theorem 3.4. We first bound the sum of squared distances between LPC's trajectory and the offline optimal trajectory:

$$\sum_{t=1}^H\|x_t - x_t^*\|^2 \leq \frac{C_0^2}{(1-\rho_G)^2}\sum_{t=1}^H e_t^2 \tag{37a}$$

$$\leq \frac{4C_1^2C_0^4}{(1-\rho_G)^2}\left(\frac{h(r)^2\rho_G^2}{(1-\rho_T)(1-\rho_G^2\rho_T)}\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2(k-1)}\cdot\rho_G^{2k}\right)\sum_{t=1}^H\left\|x_{t-1} - x_{t-1}^*\right\|^2$$

$$+ \frac{8C_0^2C_1^2}{\mu(1-\rho_G)^2}\sum_{t=1}^H\left(\frac{h(r)^2}{1-\rho_T}\cdot\rho_S^{2r}\sum_{\tau=t}^{t+k-1}\rho_T^{\tau-t}f_\tau(x_\tau^*) + C_3(r)^2\cdot\rho_T^{2(k-1)}f_{t+k-1}(x_{t+k-1}^*)\right), \tag{37b}$$

where we used Theorem C.8 in (37a); we used Lemma C.7 with the specific constants given in Appendix E.2 in (37b).

Recall that in (36), we assume r and k are sufficient large so that the coefficient of the first term in (37) satisfies

$$\frac{4C_1^2C_0^4}{(1-\rho_G)^2}\left(\frac{h(r)^2\rho_G^2}{(1-\rho_T)(1-\rho_G^2\rho_T)}\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2(k-1)}\cdot\rho_G^{2k}\right) \leq \frac{1}{2}.$$

Substituting this bound into (37) gives that

$$\sum_{t=1}^H\|x_t - x_t^*\|^2 \leq \frac{16C_0^2C_1^2}{\mu(1-\rho_G)^2}\left(\frac{h(r)^2}{(1-\rho_T)^2}\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2(k-1)}\right)\cdot\sum_{t=1}^H f_t(x_t^*). \tag{38}$$

By Lemma E.1, since f_t is $(\ell_f + \Delta\ell_S)$-smooth, convex, and non-negative on $\mathbb{R}^n$, and c_t is ℓ_T-smooth, convex, and non-negative on $\mathbb{R}^n \times \mathbb{R}^n$, we know that

$$\begin{aligned} f_t(x_t) &\le (1+\eta) f_t(x_t^*) + \frac{\ell_f + \Delta\ell_S}{2}\left(1+\frac{1}{\eta}\right)\|x_t - x_t^*\|^2 \\ c_t(x_t, x_{t-1}) &\le (1+\eta) c_t(x_t^*, x_{t-1}^*) + \frac{\ell_T}{2}\left(1+\frac{1}{\eta}\right)\left(\|x_t - x_t^*\|^2 + \left\|x_{t-1} - x_{t-1}^*\right\|^2\right) \end{aligned} \tag{39}$$

holds for any $\eta > 0$. Summing the above inequality over t gives

$$\begin{aligned} &\sum_{t=1}^{H}(f_t(x_t) + c_t(x_t, x_{t-1})) \\ &\le (1+\eta)\sum_{t=1}^{H}\left(f_t(x_t^*) + c_t(x_t^*, x_{t-1}^*)\right) + \frac{(\ell_f + \Delta\ell_S + 2\ell_T)}{2}\left(1+\frac{1}{\eta}\right)\sum_{t=1}^{H}\|x_t - x_t^*\|^2 \\ &\le (1+\eta) cost(OPT) + \left(1+\frac{1}{\eta}\right)\frac{16C_0^2C_1^2(\ell_f + \Delta\ell_S + 2\ell_T)}{\mu(1-\rho_G)^2}\left(\frac{h(r)^2}{(1-\rho_T)^2}\cdot\rho_S^{2r} + C_3(r)^2\cdot\rho_T^{2(k-1)}\right)\cdot cost(OPT), \end{aligned} \tag{40}$$

where we used (38) and $\sum_{t=1}^{H} f_t(x_t^*) \le cost(OPT)$ in the last inequality. Setting $\eta = \rho_S^r + \rho_T^{k-1}$ in (40) finishes the proof of (35).

As a remark, we require the local cost function (f_t^v, c_t^v, s_t^e) to be non-negative, convex, and smooth in the whole Euclidean spaces $(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n)$ in Assumption 2.1 because we want to apply Lemma E.1 in (39).

E.4. Proof of Corollary 3.5

We first show $\Delta^2\rho_S \le \sqrt{\rho_S}$ holds under Assumption 2.1 and the assumptions that $\frac{\ell_S}{\mu} \le \frac{1}{\Delta^7}$, $\frac{\ell_T}{\mu} \le \frac{1}{16}$. To see this, note that as we discussed in Section 3.2, by setting $b_1 = 2\Delta - 1$ and $b_2 = 4\Delta^2 - 2\Delta$, Theorem 3.4 holds with

$$\rho_S = \frac{4\Delta^2(\sqrt{1+\Delta\ell_S/\mu} - 1)}{\sqrt{1+\Delta\ell_S/\mu} + 1}.$$

Hence we see that

$$\Delta^2\sqrt{\rho_S} = 2\Delta^3\left(\frac{\sqrt{1+(\Delta\ell_S/\mu)} - 1}{\sqrt{1+(\Delta\ell_S/\mu)} + 1}\right)^{\frac{1}{2}} \le 2\Delta^3\left(\frac{\sqrt{1+\Delta^{-6}} - 1}{2}\right)^{\frac{1}{2}} \le 1,$$

which implies that

$$\Delta^2\rho_S \le \sqrt{\rho_S}. \tag{41}$$

Recall that function $C_3(r) := \sum_{\gamma=0}^{r} h(\gamma)\cdot\rho_S^\gamma$. Hence we see that

$$C_3(r) \le \sum_{\gamma=0}^{r}\Delta^\gamma\cdot\rho_S^\gamma \le \sum_{\gamma=0}^{r}\left(\frac{\sqrt{\rho_S}}{\Delta}\right)^\gamma \le \frac{\Delta}{\Delta - \sqrt{\rho_S}}. \tag{42}$$

Substituting (41) and (42) into the competitive ratio bound in (35) shows that the competitive ratio of LPC is upper bound by

$$1 + \left(1 + \frac{32C_0^2C_1^2(\ell_f + \Delta\ell_S + 2\ell_c)}{\mu(1-\rho_G)^2(1-\rho_T)^2}\right)\cdot\rho_S^{\frac{r}{2}} + \left(1 + \frac{32C_0^2C_1^2(\ell_f + \Delta\ell_S + 2\ell_c)\Delta^2}{\mu(1-\rho_G)^2(\Delta - \sqrt{\rho_S})^2}\right)\rho_T^{k-1}.$$

F. Proof of Theorem 3.6

In this appendix we prove a lower bound on the competitive ratio of any online algorithm. Our proof focuses on temporal and spatial lower bounds separately first, and then combines them.

STEP 1: TEMPORAL LOWER BOUNDS

We first show that the competitive ratio of any online algorithm with k steps of future predictions is lower bounded by $1+\Omega(\lambda_T^k)$. To show this, we consider the special case when there are no spatial interaction costs (i.e., $s_t^e \equiv 0$ for all t and e). In this case, since all agents are independent with each other, it suffices to assume there is only one agent in the network $\mathcal{G}$. Thus we will drop the agent index in the following analysis. To further simplify the problem, we assume dimension $n=1$, $c_t(x_t, x_{t-1}) = \frac{\ell_T}{2}(x_t - x_{t-1})^2$, and the feasible set is $D_t \equiv D = [0,1]$ for all t. Let R denote the diameter of D, i.e., $R = \sup_{x,y\in D}|x-y| = 1$.

By Theorem 2 in Li et al. (2020) and Case 1 in its proof, we know that for any online algorithm ALG with k steps of future predictions and $L_T \in (2R, RH)$, there exists a problem instance with quadratic functions $f_1, f_2, \ldots, f_H$ that have the form $f_t(x_t) = \frac{\mu}{2}(x_t - \theta_t)^2, \theta_t \in D$ such that

$$cost(ALG) - cost(OPT) \geq \frac{\mu^3(1-\sqrt{\lambda_T})^2}{96(\mu+1)^2} \cdot \lambda_T^k \cdot R \cdot L_H, \tag{43}$$

where $L_H \geq \sum_{t=1}^H |\theta_t - \theta_{t-1}|$. Note that

$$\begin{aligned} R \cdot L_T &\geq \sum_{t=1}^H |v_t - v_{t-1}|^2 \\ &= \frac{2}{\ell_T} \cdot \sum_{t=1}^H (f_t(v_t) + c_t(v_t, v_{t-1})) \\ &\geq \frac{2}{\ell_T} \cdot cost(OPT). \end{aligned}$$

Substituting this into (43) gives

$$cost(ALG) \geq \left(1 + \frac{\mu^3(1-\sqrt{\lambda_T})^2}{48(\mu+1)^2\ell_T} \cdot \lambda_T^k\right) \cdot cost(OPT). \tag{44}$$

Note that (43) implies $cost(ALG) > 0$, hence the competitive ratio can be unbounded if $cost(OPT) = 0$.

STEP 2: SPATIAL LOWER BOUNDS

We next show that the competitive ratio of any online algorithm that can communicate within r-hop neighborhood according to the scheme defined in Section 2.1 is lower bounded by $1+\Omega(\lambda_S^r)$. To show this, consider the special case when there are no temporal interaction costs (i.e., $c_t^v \equiv 0$ for all t and v). In this case, since x_t and x_{t-1} are independent with each other, it suffices to assume the horizon length $H=1$. That means the game only has one round, and each agent v decides x_1^v after observing f_1^v for $v \in N_v^r$ and s_1^e for $e \in \mathcal{E}(N_v^r)$. To further simplify the problem, we assume the graph $\mathcal{G}$ of agents is a line, and all agents are labeled by $1, 2, \ldots, |\mathcal{V}|$ from left to right.

Suppose that there exists a one-step algorithm ALG that achieves a competitive ratio of c in the setting we mentioned above. We can also use ALG as an online algorithm to solve the setting defined in Section F. Specifically, we assume $H = |\mathcal{V}|$ and view each time step in the finite horizon H as an agent. When $k = r+1$ and $\ell_T = \ell_S$, the prediction model defined in Section F allows ALG to observe $f_{t-r}, f_{t-r+1}, \ldots, f_{t+r}$ (which are the cost functions in r-hop neighborhood of "agent" t) before picking decision point x_t. The temporal interaction costs in Section F play the role of the spatial interaction costs for ALG, and they are common knowledge according to the prediction model. Therefore, by the lower bound results we have shown in Section F, we know that

$$c \geq 1 + \frac{\mu^3(1-\sqrt{\lambda_S})^2}{48(\mu+1)^2\ell_S} \cdot \lambda_S^{r+1}.$$

STEP 3: COMBINE TEMPORAL AND SPATIAL LOWER BOUNDS

Combining the results of Steps 1 and 2 together, we know that the competitive ratio of any decentralized online algorithm is lower bounded by

$$\max\{1 + \frac{\mu^3(1-\sqrt{\lambda_T})^2}{48(\mu+1)^2\ell_T} \cdot \lambda_T^k, 1 + \frac{\mu^3(1-\sqrt{\lambda_S})^2}{48(\mu+1)^2\ell_S} \cdot \lambda_S^{r+1}\} = 1 + \Omega(\lambda^k) + \Omega(\lambda_S^r).$$

G. Proof of Corollary 3.7

In this appendix we prove a resource augmentation bound for LPC. To simplify the notation, we define the shorthand $a_T := \ell_T/\mu$ and $a_S := \ell_S/\mu$. a_T and a_S are positive real numbers. We first show two lemmas about the relationships between the decay factors ρ_T and λ_T, and ρ_S and λ_S.

Lemma G.1. *Under the assumptions of Theorem 3.2, we have $\rho_T^4 \le \lambda_T \le \rho_T^2$.*

Proof of Lemma G.1. Recall that ρ_T is given by

$$\rho_T = \sqrt{1 - \frac{2}{\sqrt{1+2a_T}+1}}$$

in Theorem 3.2. Thus we see that

$$\rho_T^4 = \left(1 - \frac{2}{\sqrt{1+2a_T}+1}\right)^2 \le \left(1 - \frac{2}{\sqrt{1+4a_T}+1}\right)^2 = \lambda_T.$$

On the other hand, we have that

$$\lambda_T - \rho_T^2 = \left(1 - \frac{2}{\sqrt{1+4a_T}+1}\right)^2 - 1 + \frac{2}{\sqrt{1+2a_T}+1} = \frac{4\sqrt{(1+2a_T)}\left(\sqrt{1+2a_T} - \sqrt{1+4a_T}\right)}{\left(\sqrt{1+2a_T}+1\right)\left(\sqrt{1+4a_T}+1\right)^2} \le 0.$$

□

Lemma G.2. *Under the assumptions of Theorem 3.2, we have $\rho_S^{8\Delta \log \Delta} \le \lambda_S$.*

Proof of Lemma G.2. Recall that ρ_T is given by

$$\rho_S = \sqrt{1 - \frac{2}{\sqrt{1+\Delta a_S}+1}}$$

in Theorem 3.2. We consider the following 3 cases separately.

Case 1: $a_S \ge 16\Delta - 8$.

We first show that the following inequality holds for any $x \in [0, 1/(2\Delta)]$:

$$(1-x)^{2\Delta} \le 1 - \Delta x. \tag{45}$$

To see this, define function $g(x) = (1-x)^{2\Delta} + \Delta x - 1$. Note that g is a convex function with $g(0) = 0$ and

$$g\left(\frac{1}{2\Delta}\right) = \left(1 - \frac{1}{2\Delta}\right)^{2\Delta} - \frac{1}{2} \le e^{-1} - \frac{1}{2} < 0.$$

Thus, we see that $g(x) \le 0$ holds for all $x \in [0, 1/(2\Delta)]$. Hence (45) holds.

Note that under the assumption $a_S \ge 16\Delta - 8$, we have

$$0 \le \frac{2}{\sqrt{1+\Delta a_S}+1} \le \frac{1}{2\Delta}.$$

Thus substituting $x = \frac{2}{\sqrt{1+\Delta a_S}+1} \le \frac{1}{2\Delta}$ gives

$$\rho_S^{4\Delta} = \left(1 - \frac{2}{\sqrt{1+\Delta a_S}+1}\right)^{2\Delta} \le 1 - \frac{2\Delta}{\sqrt{1+\Delta a_S}+1} \le 1 - \frac{2}{\sqrt{1+4a_S}+1} = \lambda_S.$$

Case 2: $a_S \le \frac{\Delta^2}{(\Delta^2-1)^2}$. Recall that $\Delta \ge 2$. Thus, in this case, we have $a_S < 1$ and

$$\sqrt{\lambda_S} = 1 - \frac{2}{\sqrt{1+4a_S}+1} \le \frac{1}{\Delta^2}.$$

Note that

$$\rho_S^2 = \frac{(\sqrt{1+\Delta a_S}-1)^2}{\Delta a_S} \le \frac{\Delta a_S}{4} = \Delta \cdot \frac{(\sqrt{1+2a_S+a_S^2}-1)^2}{4a_S} \le \Delta \cdot \frac{(\sqrt{1+4a_S}-1)^2}{4a_S} = \Delta\sqrt{\lambda_S}.$$

Thus we see that

$$\rho_S^4 \le (\Delta^2 \cdot \lambda_S) \cdot \lambda_S \le \lambda_S.$$

Case 3: $\frac{\Delta^2}{(\Delta^2-1)^2} < a_S < 16\Delta - 8$.

In this case, we have

$$\lambda_S = \left(1 - \frac{2}{\sqrt{1+4a_S}+1}\right)^2 \ge \frac{1}{\Delta^4},$$
$$\rho_S = \left(1 - \frac{2}{\sqrt{1+\Delta a_S}+1}\right)^{\frac{1}{2}} \le \sqrt{1-\frac{1}{2\Delta}}.$$

Since $(1-1/(2\Delta))^{2\Delta} < e^{-1}$, we see that $\rho_S^{8\Delta\log(\Delta)} \le \lambda_S$. □

Now we come back to the proof of Corollary 3.7. By Theorem 3.4 and Theorem 3.6, we know that the optimal competitive ratio is lower bounded by

$$c(k^*, r^*) \ge 1 + C_\lambda\left(\lambda_T^{k^*} + \lambda_S^{r^*}\right)$$

and LPC's competitive ratio is upper bounded by

$$c_{LPC}(k, r) := 1 + C_\rho\big(C_3(r)^2 \cdot \rho_T^k + h(r)^2 \cdot \rho_S^r\big),$$

where C_λ and C_ρ are some positive constants. To achieve $c_{LPC}(k, r) \le c(k^*, r^*)$, it suffices to guarantee that

$$C_\rho \cdot C_3(r)^2 \cdot \rho_T^k \le C_\lambda \lambda_T^{k^*} \text{ and } C_\rho \cdot h(r)^2 \cdot \rho_S^r \le C_\lambda \lambda_S^{r^*}.$$

Note that $C_3(r)$ can be upper bounded by some constant and $h(r)^2 \le poly(r) \cdot \rho_S^{-\frac{r}{2}}$ under our assumptions. Applying Lemma G.1 and Lemma G.2 finishes the proof.