The role of unitizing predicates in the construction of logic

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Based on data from a teaching experiment with two undergraduate students, we propose the unitizing of predicates *as a construct to describe how students render various mathematical conditions as predicates such that various theorems have the same logical structure. This may be a challenge when conditions are conjunctions, negative, involve auxiliary objects, or are quantified. We observe that unitizing predicates in theorems and proofs seemed necessary for students in our study to see various theorems as having the same structure. Once they had done so, they reiterated an argument for why contrapositive proofs proved their associated theorems, showing the emergence of logical structure.*

Keywords: Mathematical logic, Student reasoning, Undergraduate mathematics education.

Introduction

Theorem 1. "For every integer x, if x is a multiple of 6, then x is a multiple of 3."

Theorem 2. "For any integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14."

Theorem 3. "For any quadrilateral $\blacksquare ABCD$, if $\blacksquare ABCD$ is a rhombus, then the diagonal $\overline{AC}$ forms two congruent, isosceles triangles ΔABC and ΔCDA."

Theorem 4. "Given any functions f, g that are continuous on the domain $[a, b]$, if $f(a) = g(b)$ and $f(b) = g(a)$, then there exists some c in $[a, b]$ such that $f(c) = g(c)$."

In what ways would we expect students to see these four statements as being "the same"? How would that sameness influence students' reasoning about these statements and related proofs? This is a contextual way of considering the role of logic in students' mathematical reasoning, since the primary analogy between these statements is their logical form: "For all $x \in S$, if $P(x)$, then $Q(x)$." In this study, we investigate how students build such analogies between statements and how it influences their judgments about contrapositive proofs. We propose *the unitizing of predicates* as a construct that describes how students conceptualize the various properties in the statements as entailing properties that each example either has or does not have. This affords a structural analogy between each statement and the corresponding set of objects, which can unify these statements across the mathematical contexts. We provide evidence for how unitizing predicates can be consequential for how students perceive logical structure within each context in the sense of truth conditions (when a statement is true or false) and related types of proof (i.e., direct or contrapositive).

While many studies show that students do not interpret common mathematical statements in a manner compatible with mathematical logic (Epp, 2003; Sellers, Roh, & Parr, 2021; Stylianides, Stylianides, & Phillipou, 2004), our approach to logic learning is to help students construct mathematical logic as they read meaningful mathematical statements. We do not try to use abstract symbols to replace the meaning of statements, which bypasses students' reasoning about the concepts in the statements. Rather, we conducted constructivist teaching experiments (Steffe & Thompson, 2000) to investigate an alternative approach in which students structure their interpretation of each statement (and their proofs) in a manner that affords structural analogies and repeated reasoning. In their survey of research on the teaching and learning of logic, Durand-Guerrier, Boero, Douek, Epp, and Tanguay (2012) argued that "it is important to view logic as dealing with both the syntactic and semantic aspects of the organization of mathematical discourse" (p. 385). To attend to both, logic must capture how Theorems 1-4 refer to mathematical objects, rather than merely how to remove meaning from the statements to render them "the same." The present study contributes to answering the following research question central to our overall agenda of research: How can students abstract logical structure and relationships in a manner that stems from and integrates with their reasoning about various mathematical topics? We propose that *unitizing predicates* provides a construct that characterizes how students can abstract the semantic structure of their mathematical reasoning about particular statements to afford the construction of logical structure that generalizes across contexts.

We focus in this report on a student constructing the principle of *Contrapositive Equivalence*: Contrapositive proofs, which begin "Let not $Q(x)$" and end "Thus, not $P(x)$", prove the given theorem. To understand how students attended to this form of proof and its invariant relationship to the theorems, we designed the theorems they read (which included Theorems 1-4) to vary the relationship between objects and properties in each statement. If we conceptualize a predicate as a truth-function that maps each object to a truth value ("T" if the object has the property or "F" if the object does not), then we can see how the predicates in the statements vary. Both properties in Theorem 1 are somewhat familiar and direct such that the predicate corresponds to a single category of numbers. The antecedent in Theorem 2 is a conjunction, meaning students must somehow combine the properties to yield a single predicate. The consequent in Theorem 3 involves auxiliary objects (triangles), which we may think of as rendering the predicate as a composition of functions (or mental actions). Finally, Theorem 4 is the most complex in that the input consists of pairs of functions, the antecedent is a conjunction, and the consequent is existentially quantified.

Literature on student understanding of conditionals and proof thereof

The mathematical use of conditional statements differs from the everyday meaning. One of the best-supported models for everyday interpretation is the suppositional account that posits people affirm a conditional based on the conditional probability of Q given P (Evans & Over, 2004). This means that people affirm some conditionals in the presence of counterexamples and explicitly think such statements are only about cases where P is true. Inglis (2006) found that mathematicians used a similar criterion, though they sought to know whether Q was certain given P, meaning no counterexamples were allowed. Two important corollaries of this are that the meaning of a conditional is strongly rooted in the meaning of the statement and the semantic links between P and Q. As a result, semantic links are hard to generalize across the statements above since they are context-specific, so to construct logic, students must attend to something more about the conditionals.

There are few studies on how students understand contrapositive equivalence. Stylianides et al. (2004) found that students directly taught the rule often gave answers in conflict with it, especially in mathematical contexts. Yopp (2017) provides one of the only accounts for how students should understand why contrapositives are equivalent. He argues that students should interpret conditionals by *eliminating counterexamples* (cases where P is true and Q is false), similar to how Inglis noted mathematicians reasoning. This supports both an explanation for why direct proofs prove and why contrapositive proofs prove, since a counterexample of the contrapositive statement is equivalent to a counterexample to the original. Hub and Dawkins (2018) observed a student providing a different justification rooted in the subset meaning, which states that a conditional is true whenever the truth set of P is a subset of the truth set of Q. Their student argued for contrapositive equivalence by noting that non-Q cases have no overlap with P cases, which they called the *empty intersection meaning*.

Theoretical framing

Consistent with our interest in how students construct logic as a shared structure across various mathematical statements, we conducted our study using the Radical Constructivist (von Glasersfeld, 1995) notions of assimilation and accommodation. Assimilation occurs when someone interprets a new experience as an instance of something known. Assimilation occurs in the context of goal-oriented activity, and assimilating to a scheme induces some action with an expectation to meet the reasoner's goal. When the goal is not met or something unexpected occurs, the reasoner may experience perturbation and thus engage in accommodation to assimilate the experience to a new scheme or modify the scheme to fit the new experience. This is often the opportunity for learning. Accommodation is often prompted by interlocutors (i.e., teachers) who introduce goals and prompt reflection on previous activities. In particular, we frequently asked the two students in our teaching experiment to compare their reasoning across various theorems and proofs and invited them to repeat their reasoning if possible. Such questions invited them to accommodate so as to construct a shared structure between the theorems and proofs, so as to promote reflexive abstraction by which they could project their images of their own activity onto a higher level where they could re-present the shared structure across the statements and proofs, which would for us constitute logical understandings.

Our notion of *unitizing predicates* is inspired by the work of Steffe (1983) and his colleagues in modeling students' construction of number concepts. They use *unitizing* to describe how children operate on quantities (counting or partitioning) to create new quantities (10 or 1/5) that are simultaneously a new unit and in fixed relation to the original unit (1). We can think of this as closure of numeric units under certain transformations. We use unitizing predicates to describe how students interpret the conjunction of predicates (Theorem 2), the composition of predicates (Theorem 3), and quantified predicates (Theorem 4) as new predicates with the same structure as simple predicates (Theorem 1). We call this unitizing predicates, since it is a form of closure under mental operations.

Methods

The teaching experiment reported here is part of an ongoing sequence of experiments focusing on how students can construct logic by reflecting on their mathematical activity. The participants, whom we call April and Moria, were recruited from a Calculus 3 (multi-variable and vector calculus) course at a public university in the United States. We recruited such students since we anticipated their mathematical understandings would be sufficiently strong, they would not have learned logic before, and they might go on from this course into a proof-based course (depending on their degree program). To further affirm these criteria, we asked them whether they had learned logic and had them complete a logic assessment (Roh & Lee, 2018). April and Moria were computer science majors at the time of participation. We met with them once per week outside of class time for 1-1.5 hours and paid them modestly for their time. We met with April and Moria for six sessions during which they read eight theorems, each paired with 2–4 proofs. Their task in each case was to determine whether each proof proved the statement and, if not, what statement it proved.

Consistent with teaching experiment methodology (Steffe & Thompson, 2000), the first author served as the teacher/researcher and the second author as an outside observer. All sessions were video recorded for iterative and retrospective analysis. During each session, we sought to build second-order models of their reasoning and test those models through questioning. This occurred at multiple levels such as meaning for concepts, construal of proofs, attention to logical structure, etc. Once Moria introduced Euler diagrams to represent the sets of objects referred to in each theorem/proof, we consistently invited the students to produce such diagrams. We hoped such activity would support relating the structure of different statements and re-presenting the logic of conditionals.

For this paper, our analysis focused on the students' construction of analogies between the statements and proofs. We attended to cases when they compared or contrasted theorems/proofs and for the opportunities to assimilate different theorem/proofs that shared the same logical structure (according to mathematical logic). This allowed us to attend carefully to their assimilation and accommodation activity and the particular means by which they connected theorems and proofs across contexts.

Results

We focus in this paper on April and Moria's reasoning about three proofs by contraposition, which proved Theorems 1, 3, and 4 stated above. April and Moria read these proofs during the first, fourth, and sixth sessions, respectively. For brevity, we provide only two of the proofs below in Figure 1 and omit the definitions students saw. "Proof 1.3" denoted the third proof associated with Theorem 1.

Proof 1.3

Regarding Proof 1.3, April produced her first argument for why a contrapositive proof did prove, to which we hoped she might assimilate later such proofs. Based on April's reading of Proofs 1.1 and 1.2, it was clear that she understood that all multiples of 6 are multiples of 3, but that some multiples of 3 were not multiples of 6. Upon reading Proof 1.3, April responded, "I think this is proving the theorem because it's saying if it's not a multiple of 3, then it can't be a multiple of 6." She elaborated in the context that "Because I already know that you have to have 3 times 2 times something to be a multiple of 6… Without being a multiple of 3, it's never going to equal a multiple of 6." She elaborated this argument relative to the proof saying, "Everything that you throw into it is going to give a remainder, how it's set up… you've already eliminated the fact that there's ever going to be a 3 in this, so it just doesn't formulate. It's like making a cake without the flour or the sugar."

We see a few important aspects of April's reasoning that are worth describing in some detail. First, her meaning for "multiple of d" was to factor out a d in an algebraic expression. Hence, she recognized that being a multiple of 6 entailed being "3 times 2 times something." Her meaning for multiple of 6 entailed her meaning for multiple of 3, inducing a sense of necessity between them.

Second, she gave a meaning to the negation of "multiple of 3" in terms of "give a remainder" which she could connect to the various equations in Proof 1.3. We have found that students often want to substitute a positive description for a negative predicate, which greatly facilitates their reasoning. April unitized the negation by giving it a positive meaning.

Third, based on the suppositional account of everyday conditionals (Evans & Over, 2004), we may expect students to think Proof 1.3 is irrelevant to Theorem 1 because it is not about multiples of 6. April explained how not being a multiple of 3 justifies the theorem using the analogy to ingredients. Since the factored out 6 is composed of a 2 and 3, then to try to make a 6 without a 3 is "like making a cake without the flour or the sugar." This argument implicitly draws upon the *empty intersection meaning* in that it shows no number can both be a non-multiple of 3 and be a multiple of 6.

Theorem to be proven 1: For every integer x, if x is a multiple of 6, then x is a multiple of 3.

Proof 1.3: Let x be any integer that is not a multiple of 3.
That means when we divide x by 3, we get a remainder of 1 or 2.
Then there exists some integer k such that $x = k * 3 + 1$ or $x = k * 3 + 2$.
If k is even, then there exists some integer s such that $k = s * 2$.
Substituting into the equations for x, we see:

$$x = (s * 2) * 3 + 1$$
$$= s * 6 + 1$$
or
$$x = (s * 2) * 3 + 2$$
$$= s * 6 + 2.$$

This means x is not a multiple of 6, because x is 1 or 2 greater than a multiple of 6.
If k is odd, then there exists some integer t such that $k = t * 2 + 1$.
Substituting into the equations for x, we see

$$x = (t * 2 + 1) * 3 + 1$$
$$= t * 6 + 4$$
or
$$x = (t * 2 + 1) * 3 + 2$$
$$= t * 6 + 5$$

This means x is not a multiple of 6, because it is 4 or 5 greater than a multiple of 6.

Theorem to be proven 3: For any quadrilateral $■ABCD$, if $■ABCD$ is a rhombus, then the diagonal $\overline{AC}$ forms two congruent, isosceles triangles ΔABC and ΔCDA.

Proof 3.3: Let $■ABCD$ be a quadrilateral such that when we form the diagonal $\overline{AC}$, the triangles ΔABC and ΔCDA are not both isosceles and congruent.
This means either the triangles are not isosceles, not congruent, or both non-isosceles and non-congruent.
If the triangles are not isosceles, that means that none of their sides are congruent.
This means $AB \neq BC$, which means $■ABCD$ is not a rhombus.
If the triangles are not congruent, that means at least one pair of corresponding sides are not congruent.
Clearly, $AC = CA$, so it must be the case that $AB \neq CD$ or $AB \neq DA$.
In both cases, $■ABCD$ is not a rhombus.
Therefore, $■ABCD$ is not a rhombus.

Figure 1: Theorems 1 and 3 and their contrapositive proofs.

Proof 3.3

Proof 3.3 was the second proof by contraposition that April and Moria read in a new mathematical context and with a more complex predicate in the conclusion. It provided our first opportunity to see whether April would assimilate this proof to her prior argument affirming proof by contraposition. She did not do so initially, but later accommodated once she had unitized predicates relevant to the theorem and developed a strong sense that being a rhombus entailed the theorem's conclusion.

By the time April and Moria read Theorem 3 and its proofs, they had introduced Euler diagrams and the interviewer asked them to produce such diagrams for each proof to portray the relationships between the sets of objects discussed. April and Moria recognized there were two relationships in the early theorems: one set was "nested" in another (proper subset) and one was "nested and exhaustive" in the other (equal sets). Both of these are compatible with the subset meaning for the truth of a conditional, which we intended for them to develop.

Theorem 3 was the first task we introduced with a highly unfamiliar property. After reading Proof 3.1 (Direct Proof), April summarized it saying, "It proves why a rhombus is an isosceles." April thus used the term "isosceles" to stand for the antecedent condition in the theorem. In discussing Proof 3.2 (Disproof of Converse), they used the phrases "quadrilateral" and "rest of the sentence" to refer to that condition. They represented this in an initial Euler diagram with three nested regions: the innermost "rhombus," the outermost "quads," and the middle unlabeled.

At this point, April was searching for an effective way to refer to the consequent condition. She asked if there was a name for this category. The interviewer invited Moria and April to name this middle set, for which April chose the name "fancy." Moria wondered how this category related to parallelograms, and concluded that they are "more conditional than what we are going for in this."

When they then read Proof 3.3 (Proof by Contrapositive), April used the diagram to interpret the proof. She explained, "We're in quadrilaterals [pointing to outer region] and we can't go into fancy quadrilaterals, and therefore we cannot go into rhombuses." However, neither student at this point judged that Proof 3.3 proved Theorem 3. When the interviewer asked them to explain what Proof 3.3 proved, neither one explicitly connected the first line of that proof to "not fancy," despite April's association of the hypothesis with the outer ring of their diagram. Indeed, they seemed to struggle to articulate the hypotheses for Proof 3.3. Moria explained how, if one of the two triangles was scalene, the unequal sides or the triangles having unequal area meant they did not have a rhombus. Thus, while they found multiple ways to conceptualize how Proof 3.3's hypothesis entailed not being a rhombus, they did not judge this as relevant to Theorem 3. They failed to assimilate this proof to the line of reasoning April exhibited for Proof 1.3, though we see their arguments as having the same structure.

Toward the end of the session, the interviewer wrote abbreviated forms of Theorem 3, its converse, its inverse, and its contrapositive on the board (using the phrase "not fancy" in the last two) with a reproduction of their diagram with three nested regions. He asked Moria and April to interpret all four statements using the diagram. They were very comfortable explaining that Theorem 3 corresponded to the subset relation between rhombi and fancy quadrilaterals and that the converse was false because the relationship was "not exhaustive" and "you are going to have some fancy quadrilaterals that are not rhombuses."

Both students agreed that the contrapositive statement (abbreviated as "If not fancy, then not a rhombus") must be true. April explained that non-fancy quadrilaterals were in the outer region (covering it with both hands) and you cannot find a rhombus anywhere except the inner circle (the *empty intersection meaning*). The interviewer then returned to the question of whether Proof 3.3 proved Theorem 3. April now shifted to claiming that it can prove the theorem, explaining "You need to have it so that it forms the isosceles and you need to have it that the sides are congruent. You need to have those properties to have a rhombus." Once again, she was able to use her sense that rhombus entailed fancy to argue why failing to have fancy is sufficient not to be a rhombus.

What allowed April to shift from denying that Proof 3.3 proved Theorem 3 to affirming that it did prove? First, she desired a way to conceptualize the “fancy” condition in Theorem 3 as a predicate of quadrilaterals. By naming the category “fancy” and exploring its relationship to more familiar properties such as parallelograms, she constituted it as a predicate of quadrilaterals.

Regarding Proof 3.3, despite other relevant connections April made, she did not assimilate the hypothesis condition as equivalent to “not fancy,” which we consider a failure to unitize the negation. Once the interviewer wrote down the contrapositive using the phrase “not fancy,” April and Moria were able to assimilate Proof 3.3 to that statement and relate it to their understanding that “rhombus” entailed “fancy.” April’s empty intersection argument (that there are no rhombi among the non-fancy quadrilaterals) closely matched her argument for Proof 1.3. In the same way, having a factor of 2 and 3 was necessary to have a factor of 6, forming isosceles, congruent triangles was necessary to be a rhombus. Thus, April’s abstraction of her argument to the new mathematical context depended upon her ability to unitize the predicate “fancy,” to unitize its negation in Proof 3.3, and to coordinate the entailments among the properties.

Proof 4.3

Due to space limitations, we cannot present the text of Proof 4.3, which proves Theorem 4 from the introduction by contraposition. We will simply note that April initially saw the proof as irrelevant to the statement. Still, as with Proof 3.3 was later able to affirm the proof once she had conceptualized and named the conditions in the theorem. She later paraphrased the theorem as saying that if the endpoints of the functions “swapped” then the functions intersected. She then paraphrased the proof by contraposition saying, “If there is no intersection, there is definitely no endpoint swapping.”

Discussion

We make three claims about April’s ability to assimilate the theorems and proofs to her argument by contraposition. First, she could not assimilate the more complex theorems to the subset meaning (“nested” conditions) until she had unitized each predicate. Regarding Theorem 3, this entailed both seeing the condition on the triangles as a property of the quadrilateral and naming it. Regarding Theorem 4, this entailed conceptualizing the complex conditions and giving them names. Second, she needed to construct a relationship of entailment between the antecedent and the consequent in each theorem. She understood how being a multiple of 3 required being a multiple of 6. She had to construct similar senses of necessity regarding rhombus /fancy and endpoints swapping/intersection. Third, in the latter two cases, we see how unitizing the predicates and constructing a sense of entailment induced a shift from denying that the contrapositive proof proved to affirming it. This suggests that reconstructing each theorem as a relationship between two predicates was necessary to assimilate each new proof by contraposition into her original empty intersection meaning argument.

Based on these claims, we argue that unitizing predicates stands as an essential cognitive precondition for being able to render all of these theorems as having the “same logical structure.” Students must be able to treat more complex conditions as entailing a truth-functional predicate on the relevant set of objects. Furthermore, we note that unitizing predicates is non-trivial. Moria consistently had trouble in the experiment treating conjunctions as single predicates. For her, claims like Theorem 4 were relationships between three sets of objects instead of two, which prevented her from assimilating these theorems to the subset structure of Theorem 1.

We see two primary points of significance to unitizing predicates. First, we hope future work will continue to explore the necessary cognitive work involved in constructing various statements and proofs as having the same logical structure. Second, we hope logic instruction will begin to attend to this critical learning process rather than beginning with logical syntax. We fear that this does not help students like April learn how to unitize predicates and construct relationships of entailment, which we see as central to meaningful learning of mathematical logic.

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