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Games on signed graphs☆

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ABSTRACT

Decision-makers in foreign policy and national security are constantly confronted with an adversarial, networked environments. In such an environment, the "entities" (i.e., political establishments and military organizations) to whom those decision-makers are answerable are connected through a web of friendly and adversarial relationships. By a "networked political behavior" means that one's behavior can always propagate influences across the world for the world's connectedness. This paper contains a general, mathematical framework of rational agents' strategic interactions in the international security environment, which helps to draw theoretical and practical implications thereof. The theoretical framework called "games on signed graphs" examines how countries strive to survive and succeed in a globally networked environment through security and relation dynamics. Two games will be studied, with one called the power allocation game and the other called the signed network formation game. Theoretical findings, real-world applications, and possible extensions will be presented and discussed.

1. Introduction

A signed graph (Harary et al., 1953) is a graph whose edge(s) between two connected nodes is (are) characterized with either a negative or positive sign, which is commonly used to present friendly or adversarial connections among agents in a network. An application of signed graphs in the social sciences is in the bipartite consensus problem (e.g., Altafini (2012)), where two antagonistic groups of agents converge to a "modulus consensus" – two groups' opinion values equal in modulus but opposite in sign – based on a communication algorithm. This paper uses a signed graph to conveniently depict an international environment, where both cooperative and conflicting relationships coexist. Fig. 1 illustrates the conflicts and cooperation of all countries in the world during the WWI era in 1916.

The games-on-signed-graphs framework consists of two games (Li, 2018). The first game is "the power allocation game (Li & Morse, 2017b)". A power allocation game takes place on a sequence of signed graphs. In the game, countries allocate their power to support their friends and oppose their adversaries. The game would serve as a foundation for learning about how countries compete for survival and success in a networked, antagonistic environment. More broadly, the potential applications of the game include resource allocation scenarios in strategic, tactical, and operational planning, one of which is the power allocation between countries. The second game is "the signed network formation game (Li & Morse, 2017a)". In a signed network formation game, countries strategically change the cooperative and conflictual relationships. The game is a basis for understanding how they may alter the environment to improve the odds of survival and success.

1.1. Contributions

The games on signed graphs, which were inspired by concerns in political science, add a new technological problem to the study of system sciences (Bryen, 2012; Deutsch, 1966). In the network games literature (e.g., Bauso, Tembine, and Başar (2016), Cheng, He, Qi, and Xu (2015), Ding, Li, Lu, and Wang (2021), Myerson (1977), Stier-Moses (1958) and Zhao, Wang,

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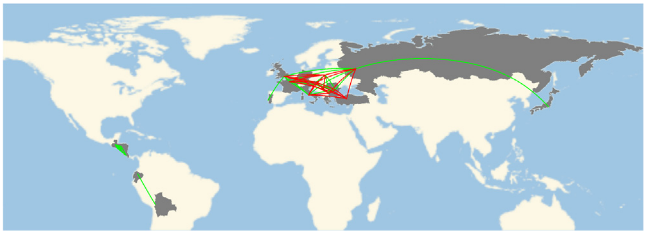


Fig. 1. The relationships in 1916: Red Edge (Conflict) and Green Edge (Cooperation). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and Li (2016)), network resource allocation games have been extensively investigated. This is not surprising, given that resource allocation is a fundamental problem that lends itself well to abstract multidimensional modeling. Many applications in economics, military, and political science have been identified in games in which budget-constrained players strategically distribute resources across numerous, simultaneous fronts (see Roberson (2011)). In our paper, the power allocation game is used as an example of network resource allocation games in international relations. The signed network formation game offers fresh insights into the formation and dissolution of links between agents in the same context. We also provide theoretical results, such as the existence of a pure strategy Nash equilibrium in both games.

In turn, the games contribute a new methodology to the study of international relations. With the exception of Goyal, Vigier, and Dziubinski (2015), Jackson and Nei (2015), and others, much of the game-theoretic literature in international relations focuses on games with two or three players. A game on signed graphs is also distinct from the N-coalition games investigated in papers such as Ye, Hu, Lewis, and Xie (2019). We expect the games in our paper will broaden the theoretical scope of many previous games in international relations to more realistic scenarios, sharpen existing insights derived previously without the use of mathematical modeling, and generate new insights and predictions that may help countries navigate strategic interactions in networked environments. Overall, they propose an alternative "network" approach to international relations, which could supplement the statistical approaches that extract information from international networks data (e.g., Cranmer, Menninga, and Mucha (2015)).

Furthermore, considering Neumann and Morgenstern (1944), a game on signed graphs may pose a challenge to the conventional distinction between "noncooperative" and "cooperative" games. By definition, both the power allocation game and the signed network formation game are noncooperative games. These games, on the other hand, accommodate both agents' "noncooperative" behaviors (with adversaries) and "cooperative" behaviors (with allies). It also calls into question the application of some widely held game theory concepts. The idea of Pareto optimality (Mock, 2011), for example, is frequently used as a criterion for evaluating resource allocative efficiency. Pareto-optimality, defined as a state in which no additional adjustments in allocation may make one person better off without making another worse off, has been widely applied in welfare economics. However, this may not be a desirable goal in resource allocation games on signed graphs, where agents may strive to make themselves better off by making others worse off. Another example is the concept of Coalition-proof Nash equilibrium (Bernheim, Peleg, & Whinston, 1987) in the study of cooperative games. Coalition-proof Nash equilibrium is a state where no subgroup of participants may collectively deviate in a mutually advantageous way. Fundamentally, assuming a group of adversaries will strategize for their mutual benefit is unrealistic, rendering this concept unsuited for a game on signed graphs.

1.2. Paper plan

The paper proceeds as follows. The first part of the theoretical framework is the power allocation game, which takes place on a signed graph. We will study a static power allocation game in both normal form and extensive form, and the applications in countries' survival problem. The second part of the framework is the signed network formation game. A country's network formation strategy is to change the signed graph itself to improve its power allocation outcomes. We will cover a static signed network formation game and the applications in alliance politics, especially great powers' "optimal network design". Lastly, we will discuss future work.

2. The power allocation game

2.1. Static power allocation game in normal form

The normal form of the static power allocation game contains a specification of the players, their strategies, game outcomes, and their preferences for those outcomes. Assume that the players hold complete information of these elements (Fudenberg & Tirole, 1991).

Countries, power, and relationships

Each country in a networked, strategic environment is distinctively characterized by a territory, several ethnic groups, cultures, and languages/dialects. Let n countries in the environment be with labels in $\mathbf{n} = \{1, 2, \ldots, n\}$.

Each country is endowed with a nonnegative quantity of total power p_i, with which it pursues affairs with other countries. Different aspects may have to be weighed in when evaluating a country's total power for different contexts. p_i is a nonnegative integer if it measures the number of i's destroyers, aircraft, and tanks, and a nonnegative real number if it measures i's weaponry in a present-value currency. In this paper, we treat p_i as a nonnegative real number.

Each country has its relationships with other countries in a given environment. A country may have friendly or adversarial relationships with some and no specific relationships with the rest. For each $i \in \mathbf{n}$, the disjoint sets $\mathcal{F}_i$ and $\mathcal{A}_i$ denote the sets of labels of country i's friends and adversaries, respectively. Assume that each country is a friend of itself $i \in \mathcal{F}_i$ and that the relationships are bilateral. The relationships of all countries in an environment make up a *relation configuration*.

An undirected graph $\mathbb{E}_E = \{\mathcal{V}, \mathcal{E}_E\}$ called "an environment graph (Li, Morse, Liu, & Başar, 2017)" represents a networked, strategic environment, which consists of countries, their power, and relationships. The graph's $n \in \mathbb{Z}_{\geq 0}$ vertices represent the countries and $m \in \mathbb{Z}_{\geq 0}$ edges represent relationships between countries. An edge between distinct vertices i and j, denoted by the unordered pair $\{i, j\}$, is labeled with a plus sign if countries i and j are friends and with a minus sign if countries i and j are adversaries. Let the set of all friendly pairs be

$$\mathcal{R}_{\mathcal{F}} = \{\{i, j\} : j \in \mathcal{F}_i, i \in \mathbf{n}\} \tag{1}$$

and the set of all adversarial pairs be

$$\mathcal{R}_{\mathcal{A}} = \{\{i, j\} : j \in \mathcal{A}_i, i \in \mathbf{n}\}. \tag{2}$$

Power allocation strategy

An allocation of this power p_i, called a *power allocation strategy*, is a nonnegative $1 \times n$ row vector u_i whose jth component u_{ij} is that part of p_i which country i allocates under the strategy in the support of country j if $j \in \mathcal{F}_i$ or to the demise of country j

if $j \in \mathcal{A}_i$; accordingly $u_{ij} = 0$ if $j \notin \mathcal{F}_i \cup \mathcal{A}_i$ and the allocations are subject to the total power constraint

$$u_{i1} + u_{i2} + \cdots + u_{in} = p_i. \tag{3}$$

Each set of country strategies $\{u_i : i \in \mathbf{n}\}$ determines an $n \times n$ matrix U whose ith row is u_i. Thus

$$U = \left[u_{ij}\right]_{n\times n} \tag{4}$$

is a nonnegative matrix such that, for each $i \in \mathbf{n}$, $u_{i1} + u_{i2} + \cdots + u_{in} = p_i$. Any such matrix is called a strategy matrix and $\mathcal{U}$ is the set of all $n \times n$ strategy matrices.

A weighted directed graph $\mathbb{E}_A = \{\mathcal{V}, \mathcal{E}_A\}$ called "an allocation graph" represents the power allocations in a networked, strategic environment. Two edges with opposite directions are denoted as ordered pairs (i, j) and (j, i). The edges are labeled with a plus sign if countries i and j are friends and with a minus sign if countries i and j are adversaries. In addition, u_{ii} is the vertex weight of i, and u_{ij} is the edge weight of (i, j), $i \in \mathbf{n}$.

Accordingly, each strategy matrix U determines for each $i \in \mathbf{n}$, the total support $\sigma_i(U)$ of country i and the total threat $\tau_i(U)$ against country i. Here $\sigma_i : \mathcal{U} \to \mathbb{R}$ and $\tau_i : \mathcal{U} \to \mathbb{R}$ are nonnegative valued maps defined by

$$\sigma_i(U) = \sum_{j\in\mathcal{F}_i} u_{ji} + \sum_{j\in\mathcal{A}_i} u_{ij} \tag{5}$$

and

$$\tau_i(U) = \sum_{j\in\mathcal{A}_i} u_{ji} \tag{6}$$

respectively. Thus country i's total support is the sum of the amounts of power each of country i's friends allocate to its support plus the sum of the amounts of power country i allocates to the destruction of all its adversaries. Country i's total threat, on the other hand, is the sum of the amounts of power country i's adversaries allocate to its destruction. These allocations, in turn, determine country i's *state* $x_i(U)$ which may be safe, precarious, or unsafe depending on the relative values of $\sigma_i(U)$ and $\tau_i(U)$. $x_i : \mathcal{U} \to \{\text{safe, precarious, unsafe}\}$ is the map defined such that

$$\begin{cases} x_i(U) = \text{safe} & \sigma_i(U) > \tau_i(U), \\ x_i(U) = \text{precarious} & \sigma_i(U) = \tau_i(U), \\ x_i(U) = \text{unsafe} & \sigma_i(U) < \tau_i(U). \end{cases} \tag{7}$$

where country i is said to *survive* if $x_i(U) =$ safe or precarious (Li et al., 2017). A row vector $x(U) = [x_i(U)]_{1\times n}$ is the state vector, which is an element of the state space,

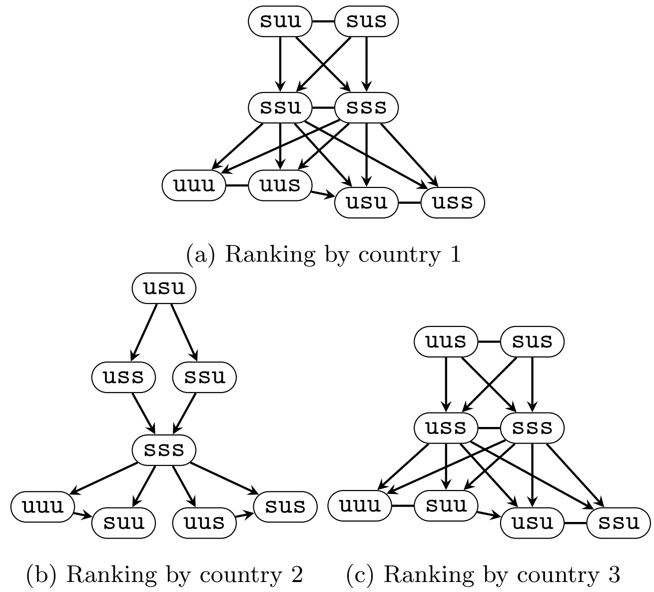
$$\mathcal{X} = \{\text{safe, precarious, unsafe}\}^n. \tag{8}$$

Preference axioms

The following axioms are *sufficient conditions* for three basic preference relations – weak preference, strong preference, and the indifference relation – regarding power allocation matrices and a basis on which countries optimally determine their power allocation strategies.

(1) $U \preceq_i V$, which means country i weakly prefers strategy matrix V over U, if
   (a) $\forall j \in \mathcal{F}_i$, $(x_j(V) \in \{\text{safe, precarious}\}) \vee (x_j(U) = \text{unsafe})$ and
   (b) $\forall j \in \mathcal{A}_i$, $(x_j(V) \in \{\text{unsafe, precarious}\}) \vee (x_j(U) = \text{safe})$.

(2) $U \sim_i V$, which means country i is indifferent between strategy matrices V and U, if
   (a) $\forall j \in \mathcal{A}_i \cup \mathcal{F}_i, x_j(U) = x_j(V)$.



(a) Ranking by country 1

(b) Ranking by country 2

(c) Ranking by country 3

Fig. 2. A partial order of the state vectors.

(3) $U \prec_i V$, which means country i strongly prefers strategy matrix V over U, if
   (a) $(x_i(V) \in \{\text{safe, precarious}\}) \wedge (x_i(U) = \text{unsafe})$.

By the first and the second axioms, country i cares positively about its friends' survival, negatively about its adversaries' survival, and indifferently about the rest's survival. By the third axiom, amongst all, country i is to achieve self-survival as its first priority, which is consistent with Waltz (1979). By the preference axioms, country i can partially order the state vectors in $\mathcal{X}$. By the following Theorem 1 whose proof is in Li and Morse (2018a), a utility function for i that satisfies the axioms exists, thus extending the partial order into a total order.

Theorem 1. *A utility function that satisfies the preference axioms exists.*

Example 1. Consider a power allocation game where

(1) Countries: $\mathbf{n} = \{1, 2, 3\}$.
(2) Relationships: $\mathcal{F}_1 = \{1\}$ and $\mathcal{A}_1 = \{2\}$; $\mathcal{F}_2 = \{2\}$ and $\mathcal{A}_2 = \{1, 3\}$; $\mathcal{F}_3 = \{3\}$ and $\mathcal{A}_3 = \{2\}$.
(3) Preferences: assume the preference axioms.

There are a total of twenty-seven state vectors in $\mathcal{X}$. However, by the preference axioms, country i will be indifferent as to whether its friends are safe or precarious or whether its adversaries are unsafe or precarious. Then, to have a total order of these twenty-seven vectors, country i only needs to totally order the following eight state vectors, abbreviated as [sss], [ssu], [sus], [suu], [uss], [usu], [uus], and [uuu], where "s" means the state being safe or precarious and "u" means the state being unsafe or precarious.

Fig. 2 illustrates the three countries' preference orders using three chain graphs. Each node in a graph denotes one of the above eight vectors. An edge directed from node A to node B means that the country weakly prefers A to B, and an undirected edge between A and B means the country is indifferent between them. Transitivity holds for the three preference orders, by which any two nodes on the same chain are comparable.

Fig. 3 shows that country 1 and country 3's preference orders are total orders, and that country 2's preference order is a partial order for its inability to compare [ssu] to [uss], and [suu] to [uus].

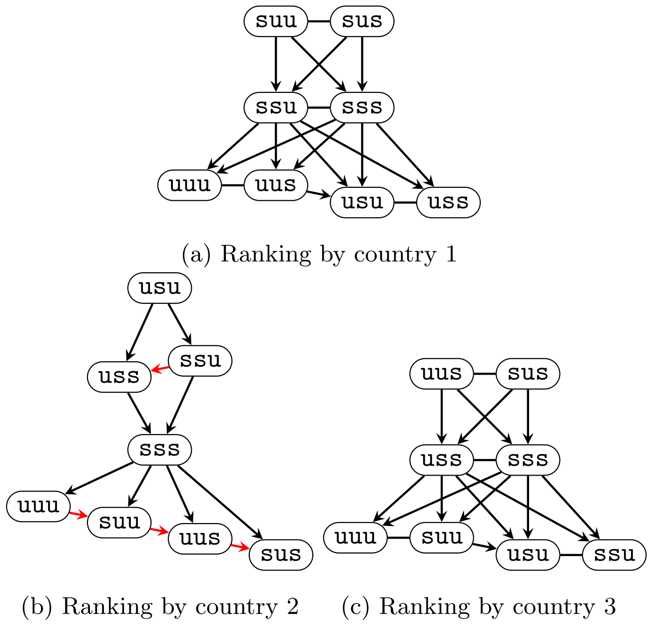


Fig. 3. A total order of the state vectors.

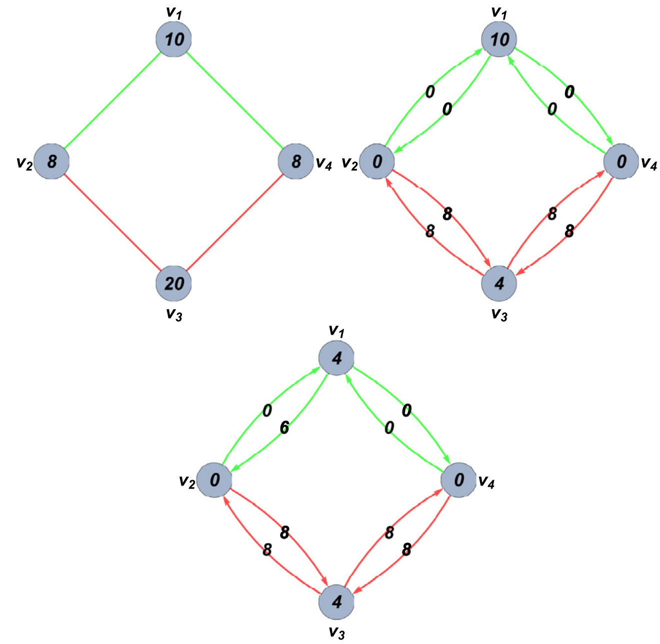


Fig. 4. The environment graph and two equilibria.

Assume the following for country 2's preferences (or utility). By Theorem 1, suppose country 2 derives a higher utility from having country 3 as an unsafe or precarious adversary than country 1 as such an adversary. If country 2 can only overwhelm one of these two adversaries, it will choose to attack country 3. Therefore, country 2 strictly prefers [ssu] to [uss], and [suu] to [uus]. As illustrated in Fig. 3, country 2's preference order is now a total order as well.

Equilibrium concept

The Nash equilibrium concept is naturally employed in the power allocation game to make predictions. Let country i's deviation from the strategy matrix U be an n-dimensional row vector d_i such that $u_i + d_i$ satisfies the total power constraint for country i. The deviation set $\mathcal{D}_i(U)$ is the set of all possible deviations of country i from the strategy matrix U. A strategy matrix U is a pure strategy Nash Equilibrium if no unilateral deviation in strategy by any country i is "profitable". In other words,

$$U + e_i d_i \preceq_i U, \qquad \text{for all} \quad d_i \in \mathcal{D}_i(U), \tag{9}$$

where e_i is the ith $n \times 1$ unit vector.

Equilibrium equivalence class

Denote by $\mathcal{U}^*$ the set of pure strategy Nash equilibria. Call $U \in \mathcal{U}^*$ *equilibrium equivalent* to $V \in \mathcal{U}^*$ if and only if $x(U) = x(V)$. The relation "equilibrium equivalence" is the equivalence kernel of the restriction of x to $\mathcal{U}^*$ and thus is an equivalence relation on $\mathcal{U}^*$. Let $[U]_*$ be the equilibrium equivalence class of $U \in \mathcal{U}^*$. The total number of equilibrium equivalence classes is at most 3^n, which in turn is the cardinality of the co-domain of x.

Example 2. Consider a power allocation game where

(1) Countries: $\mathbf{n} = \{1, 2, 3, 4\}$.
(2) Their total power: $p = [10\ 8\ 20\ 8]$.
(3) Relationships: $\mathcal{F}_1 = \{1, 2, 4\}$ and $\mathcal{A}_1 = \emptyset$; $\mathcal{F}_2 = \{1, 2\}$ and $\mathcal{A}_2 = \{3\}$; $\mathcal{F}_3 = \{3\}$ and $\mathcal{A}_3 = \{2, 4\}$; $\mathcal{F}_4 = \{1, 4\}$ and $\mathcal{A}_4 = \{3\}$.
(4) Preferences: assume the preference axioms.

Optimally, country 3 would hope to make both country 2 and country 4 unsafe or precarious. However, country 1 is a friend of both country 2 and country 4, and these three countries' total power exceeds that of country 3. For this reason, country 3 cannot make both of its adversaries, country 2 and country 4, unsafe. Then the best outcome country 3 can achieve in equilibrium is to make *at least* one of its two adversaries precarious. Fig. 4 suggests a pure strategy Nash equilibrium in which both country 2 and country 4 are precarious, and another pure strategy Nash equilibrium in which country 2 is safe, and country 4 is precarious. In fact, country 3 can make neither country 2 nor country 4 unsafe in any equilibrium of this game. Having to save enough resources in precautions against country 4 or 2's attacks, country 3 cannot attack country 2 or 4 with all its power. So when country 3 does make one of its adversaries unsafe, country 1 can always support this adversary to be at least precarious. Therefore, country 3 will settle by making both adversaries precarious. This game will have no pure strategy Nash equilibria if the preference axioms do not hold — when country 3 is no longer indifferent between the adversaries being unsafe and precarious. A cycle would ensue, with country 3 ever adding allocations to make its adversaries unsafe and country 1 ever adding allocations to make them survive.

Result I. Equilibrium existence

To the best of our knowledge, there are no general equilibrium existence results that can be directly applied to the power allocation game,[1] which is an infinite game (Fudenberg & Tirole, 1991) with a discontinuous utility representation of countries' preferences. Algorithm 1 establishes pure strategy Nash equilibrium existence for the normal form game by recursively constructing one such equilibrium.

[1] Nash (1950, 1951) have established the mixed strategy Nash equilibrium existence for any finite game. Prior work such as Monderer and Shapley (1996) and Reny (1999) has also shown that pure strategy Nash equilibrium may exist in games with particular properties.

Algorithm 1. Let q be the number of adversarial pairs in $\mathcal{R}_a$, and $\mathbf{q} = \{1, 2, \ldots, q\}$ be the set of distinct labels for elements in $\mathcal{R}_a$. By an ordering map is meant a bijection $\gamma : \mathcal{R}_a \to \mathbf{q}$; any such map determines an ordering of $\mathcal{R}_a$ with $\{i, j\}$ being the $\gamma(\{i, j\})$-th term in the ordering.

Let $z_i(k)$ be the ith entry in $z(k)$ and e_k be the kth $n \times 1$ unit vector. Consider the first recursion where countries' remaining power is updated during *a traversal of the adversarial relationships*. During the traversal of each relationship, the *smaller* remaining power of the two countries is subtracted from both countries' powers in Eq. (11):

$$z(0) = p \tag{10}$$

$$z(k) = z(k-1) - \min\{z_i(k-1), z_j(k-1)\}(e_i + e_j) \tag{11}$$

where $k \in \mathbf{q}$, $z(k) \in \mathbb{R}^{n\times 1}$, and $\{i, j\} = \gamma^{-1}(k-1)$.

Then in the second recursion, derive the final allocation matrix $U(q)$ by taking into account both the countries' remaining powers after the traversal and their allocations toward and from adversaries and friends. Countries' remaining powers are denoted using $U(q)$'s diagonal elements as in Eq. (12). Their allocations toward and from adversaries (as computed during the traversal) and friends (assumed to be 0) are denoted using $U(q)'s$ non-diagonal elements as in Eq. (13):

$$U(0) = \text{diagonal}\{z_1(q), z_2(q), \ldots, z_n(q)\} \tag{12}$$

$$U(k) = U(k-1) + \min\{z_i(k-1), z_j(k-1)\}(e_i e_j^T + e_j e_i^T) \tag{13}$$

where $k \in \mathbf{q}$, $U(k) \in \mathbb{R}^{n\times n}$, and $\{i, j\} = \gamma^{-1}(k-1)$[2]

Alternatively, the second recursion of Algorithm 1 can be restated as

$$\begin{aligned} U(0) &= \text{diagonal}\{z_1(0), z_2(0), \ldots, z_n(0)\} \\ &= \text{diagonal}\{p_1, p_2, \ldots, p_n\} \end{aligned} \tag{14}$$

$$U(k) = \text{diagonal}\{z_1(k), z_2(k), \ldots, z_n(k)\} + \sum_{r=1}^{k} \min\{z_i(r-1), z_j(r-1)\}(e_i e_j^T + e_j e_i^T) \tag{15}$$

where $k \in \mathbf{q}$, $U(k) \in \mathbb{R}^{n\times n}$, and $\{i, j\} = \gamma^{-1}(k-1)$.

Incidence Matrix for the Subgraph of All Adversary Pairs. The q adversary pairs make up a subgraph $\mathbb{E}'_E$ of the environment graph $\mathbb{E}_E$. The incidence matrix of $\mathbb{E}'_E$ is $B = [b_{ik}]_{n\times q}$, whose ith row is b_i. $b_{ik} = 1$ if country i is in the kth $(1 \le k \le q)$ adversary pair and 0 otherwise.

Lemma 1. *The normal form game has a pure strategy Nash Equilibrium if the vector of countries' power $p = [p_i]_{n\times 1}$ can be decomposed as*

$$p = Bd + c \tag{16}$$

where the following conditions are satisfied:

(1) B is an $n \times q$ incidence matrix for the subgraph $\mathbb{E}'_E$.
(2) d is a $q \times 1$ nonnegative-valued column vector, and c is an $n \times 1$ nonnegative-valued column vector.
(3) $\nexists\{i, j\} \in \mathcal{R}_a$, $c_i > 0$ and $c_j > 0$.

Proof of Lemma 1. Suppose a decomposition of the total power vector that satisfies the three conditions exists. We then derive a strategy matrix U such that for $i \in \mathbf{n}$:

(1) $u_{ij} = u_{ji} = d_i \ge 0$
(2) $u_{ii} = c_i \ge 0$
(3) $u_{ij} = u_{ji} = 0$, and $j \in \mathcal{F}_i - \{i\}$ as a consequence of (1) and (2).
(4) $u_{ij} = u_{ji} = 0$, and $j \notin \mathcal{A}_i \cup \mathcal{F}_i$ by default.

U is a valid strategy matrix because

$$\sum_{j=1}^{n} u_{ij} = \sum_{j\in\mathcal{A}_i} u_{ij} + u_{ii} = b_i d + c_i = p_i. \tag{17}$$

No country i with adversaries will unilaterally deviate from u_i, because it must fall into either case:

(1) $c_i = 0$. For country i, it must be that $x_j(U) =$ precarious, because $\sigma_i(U) = \tau_i(U) = p_i$. For any $j \in \mathcal{A}_i$, $u_{ij} = u_{ji}$, and $u_{ii} = 0$. Thus, i cannot deviate to make itself strictly better off due to power deficiency.
(2) $c_i > 0$. For country i, it must be that $\forall j \in \mathcal{A}_i$, $c_j = 0$, which means $x_j(U) =$ precarious, and that $\forall j \in \mathcal{F}_i$, $x_j(U) =$ safe or precarious. By the preference axioms, given any arbitrary $V \in \mathcal{U}$, country i must weakly prefer U to V and therefore does not need to deviate.

Any country i without adversaries will not unilaterally deviate from u_i, either, because the following holds:

(1) $p_i = u_{ii}$.
(2) $x_i(U) =$ safe or precarious, $j \in \mathcal{F}_i$.

By the preference axioms, given any arbitrary $V \in \mathcal{U}$, country i must weakly prefer U to V, and does not need to deviate, either.

Therefore, if the total power vector can be decomposed as $p = Bd + c$, for which the three requirements hold, we can derive a strategy matrix that is a pure strategy Nash equilibrium. □

Theorem 2. *In the normal form game, pure strategy Nash equilibrium always exists.*

Proof of Theorem 2. In what follows, we prove that Algorithm 1 guarantees to construct a set of allocations that satisfy the conditions in Lemma 1. We decompose the total power vector $p = Bd + c$ by defining d and c as follows based on Algorithm 1:

(1) let $d_k = \min\{z_i(k-1), z_j(k-1)\}$ for $\gamma^{-1}(k-1) = \{i, j\}$, $k \in \mathbf{q}$
(2) let $c_i = z_i(q)$, $i \in \mathbf{n}$.

The first two conditions about the decomposition in Lemma 1 are thus proven. First, d is a $q \times 1$ nonnegative-valued column vector. Second, c is an $n \times 1$ nonnegative-valued column vector. Next, we prove the third condition. By Algorithm 1, at the kth recursion where $\gamma^{-1}(k-1) = \{i, j\}$ is traversed, the remaining power of i and j, $z_i(k)$ and $z_j(k)$, cannot be both positive. $z_i(k)$ is also non-increasing with k. Therefore, $z_i(q)$ and $z_j(q)$ cannot be both positive. In other words,

$$\nexists\{i, j\} \in \mathcal{R}_a, c_i > 0 \text{ and } c_j > 0. \tag{18}$$

The decomposition is also valid, which means satisfying the total power constraint:

$$b_i d + z_i(q) = p_i, i \in \mathbf{n} \tag{19}$$

Therefore, the normal form game always has a pure strategy Nash equilibrium, which can be constructed by Algorithm 1. □

[2] Note that the first equilibrium in Example 2 can be regarded as having been constructed with Algorithm 1. The first recursion is: $z(1) = z(0) - 8[0\ 1\ 0\ 0]^T - 8[0\ 0\ 1\ 0]^T = [10\ 0\ 12\ 8]^T$, and $z(2) = z(1) - 8[0\ 0\ 1\ 0] - 8[0\ 0\ 0\ 1] = [10\ 0\ 4\ 0]$. And the second recursion is: $U(0) = \text{diagonal}\{10, 0, 4, 0\}$, $U(1) = U(0) + 8 * ([0\ 1\ 0\ 0]^T * [0\ 0\ 1\ 0] + [0\ 0\ 1\ 0]^T * [0\ 1\ 0\ 0])$, and $U(2) = U(1) + 8 * ([0\ 0\ 1\ 0]^T * [0\ 0\ 0\ 1] + [0\ 0\ 0\ 1]^T * [0\ 0\ 1\ 0])$.

Result II. Equilibrium set

Another algorithm is developed to generate the pure strategy Nash equilibrium set $\mathcal{U}^*$. Algorithm 2 is to establish the geometric property of the equilibrium set by deriving the set's algebraic representation.

Algorithm 2. The input of Algorithm 2 consists of the countries set, $\mathbf{n}$, the total power vector p, the friend set $\mathcal{F}_i$, the adversary set $\mathcal{A}_i$, and the preference order $\preceq_i$ of each country $i \in \mathbf{n}$. For country i, let a valid total order of the 3^n state vectors in $\mathcal{X}$ be

$$X_1 \succeq_i X_2 \cdots \succeq_i X_k \succeq_i \cdots \succeq_i X_{3^n} \tag{20}$$

Next we derive three sets of constraints the kth equilibrium equivalence class ($1 \leq k \leq 3^n$) must satisfy — namely, each country's *total power constraint*, *state constraint*, and *locally feasible best response constraint*.

Total Power Constraint: Country i's total power constraint $\mathcal{P}_i(k)$ is the intersection of the linear inequalities each dimension of i's allocations must satisfy in any strategy matrix U. The symbols $\bigcap$ and $\bigcup$ refer respectively to logical AND and logical OR.

$$\mathcal{P}_i(k) := \bigcap_{j\in\mathcal{F}_i\cup\mathcal{A}_i} (u_{ij} \geq 0) \bigcap (\sum_{j\in\mathcal{F}_i\cup\mathcal{A}_i} u_{ij} = p_i) \tag{21}$$

which gives the upper and lower bounds for i's each dimension of allocations.

State Constraint: Country i's *state constraint* $\mathcal{S}_i(k)$ is the intersection of the linear inequalities that i's allocations in its total support σ_i and total threat τ_i must satisfy in the kth equilibrium equivalence class.

$$\mathcal{S}_i(k) := \begin{cases} (\sigma_i(U) > \tau_i(U)) & x_i(U) = \text{safe} \\ (\sigma_i(U) = \tau_i(U)) & x_i(U) = \text{precarious} \\ (\sigma_i(U) < \tau_i(U)) & x_i(U) = \text{unsafe} \end{cases} \tag{22}$$

Locally Feasible Best Response Constraint: Country i's locally feasible best response constraint $\text{Br}_i(k)$ can be obtained by intersecting the total power constraint and the state constraint for the kth equilibrium equivalence class,

$$\text{Br}_i(k) = \bigcap_{j\in\mathcal{A}_i\cup\mathcal{F}_i} (\mathcal{S}_j(k) \cap \mathcal{P}_j(k)). \tag{23}$$

Every formula can be expanded into an equivalent conjunctive or disjunctive normal form ("CNF" or "DNF") computationally.[3] We represent the right-hand side of Eq. (23) in a conjunctive normal form, an intersection of $T_i \geq 0$ disjunctive clauses (Whitesitt, 2012)

$$\text{CNF}(\bigcap_{j\in\mathcal{A}_i\cup\mathcal{F}_i} \mathcal{S}_j(k) \cap \mathcal{P}_j(k)) = \bigcap_{1\leq t_i\leq T_i} \text{Clause}_{t_i} \tag{24}$$

If the t_ith clause ($0 \leq t_i \leq T_i$) contains only the linear inequalities of i's own allocations, denote it as $\text{Strat}_{t_i}(k)$, a *strategy clause* of i. Otherwise, denote it as $\text{Cond}_{t_i}(k)$, a *condition clause* of i.

Non-Deviation Condition: Given the intersection of countries' locally feasible best response constraints $\bigcap_{i\in\mathbf{n}} \text{Br}_i(k)$ for the kth equilibrium class, any country i cannot deviate to a different class by unilaterally changing its strategy. If this *non-deviation condition* holds, it means that the kth class is a valid equilibrium equivalence class. Denote $\bigcap_{i\in\mathbf{n}} \text{Br}_i(k)$ that satisfies the non-deviation condition as

$$\bigcap_{i\in\mathbf{n}} \text{Br}_i(k)^* \tag{25}$$

[3] A class of games based on Boolean algebra is called "Boolean Games" (e.g., Ding et al. (2021)).

Pure Strategy Nash Equilibrium Set: The pure strategy Nash equilibrium set $\mathcal{U}^*$ is the union of all valid equilibrium equivalence classes,

$$\mathcal{U}^* := \bigcup_{1\leq k\leq 3^n} \Big(\bigcap_{i\in\mathbf{n}} \text{Br}_i(k)^* \Big). \tag{26}$$

The output of Algorithm 2 is each equilibrium equivalence class in its algebraic representation. However, obtaining the exact algebraic representation of the classes using Algorithm 2 is a *constraint satisfaction problem* (CSP), which can be of high complexity. Therefore, we aim only to establish the geometric property of the equilibrium equivalence classes below, which then helps determine the equilibrium set's geometric property, and will do so through Theorems 3 and 4.

Half-space Representation of Convex Polytopes: A half-space representation of a polytope is

$$P = \{x \in \mathbb{R}^{n\times 1} : Ax \leq b\}, \tag{27}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^{m\times 1}$, and $m, n \in \mathbb{Z}$ $(m > n \geq 1)$.

Theorem 3. *The strategy space of the normal form game is a $2m$-dimensional convex polytope, where m is the number of pairs of distinct countries that are friends or adversaries.*

Proof of Theorem 3. Let the set of allocations between any pair of distinct countries which are friends or adversaries be $\bigcup_{i\in\mathbf{n}}\{u_{ij} : j \in \mathcal{F}_i \cup \mathcal{A}_i \setminus \{i\}\}$ whose cardinality is $2m$.

Label all allocations in the above set. The labeling set is $\mathbf{z} = \bigcup_{i\in\mathbf{n}} \mathbf{z}_i = \{1, 2, \ldots, 2m\}$, where $\mathbf{z}_i$ is the labeling set for the allocations in $\{u_{ij} : j \in \mathcal{F}_i \cup \mathcal{A}_i \setminus \{i\}\}$.

For each $n \times n$ allocation matrix U, a $2m$-dimensional vector $\hat{u}$ can be constructed such that $\forall i, j \in \mathbf{n}$ such that $i \neq j$, the k-entry $\hat{u}_k$ is equal to the kth allocation in $\mathbf{z}$. The projection $\pi : U \mapsto \hat{u}$ is bijective. As π is a bijection, $\hat{u}$ and U can be used interchangeably. The idea is that subject to the total power constraint, the number of independent nonzero entries in U is $2m$.

Let $A = [a_{ij}]_{n\times 2m}$ be a real matrix, whose ith row vector a_i is defined such that $a_{iq} = 1$ if and only if $q \in \mathbf{z}_i$, and 0 otherwise. And let $I = [I_{ij}]_{2\ m\times 2m}$ be an identity matrix and $0 = [0]_{2\ m\times 1}$. Eq. (41) yields both the total power constraint and the nonnegative requirement for all the allocations. $P = \left\{\hat{u} \in \mathbb{R}^{2\ m\times 1} : \left[\begin{smallmatrix} A \\ -I \end{smallmatrix}\right] \hat{u} \leq \left[\begin{smallmatrix} p \\ 0 \end{smallmatrix}\right]\right\}$. in the half-space representation of the strategy space. Therefore, the strategy space of the normal-form power allocation game is a $2m$-dimensional convex polytope. □

Theorem 4. *Each equilibrium equivalence class of the power allocation game is a convex polytope with at most $2m$ dimensions. The equilibrium set $\mathcal{U}^*$ is a collection of convex polytopes.*

Proof of Theorem 4. By Algorithm 2, if the non-deviation condition is not satisfied for the kth equilibrium equivalence class, the class will have been ruled out in the equilibrium set. If the non-deviation condition is satisfied, the kth class is a valid equilibrium equivalence class. Recall that the intersection of the locally feasible best response constraints $\bigcap_{i\in\mathbf{n}} \text{Br}_i(k)^*$ is a finite intersection of simple linear inequalities of countries' allocations for this class.

Since each linear inequality represents a half-space, each candidate equilibrium equivalence class is either empty or a nonempty convex polytope. Therefore, the equilibrium set $\mathcal{U}^*$ is a union of convex polytopes. □

2.2. Static power allocation game in extensive form

In contrast with the normal form game, the extensive form of the power allocation game assumes a sequencing of countries' moves and can be set up as follows. We also show that the extensive form game can be used for refining equilibrium predictions of the corresponding normal form game (Li & Morse, 2018c, 2018d).

Countries and power

There are a set of countries labeled from 1 to n in $\mathbf{n} = \{1, 2, \ldots, n\}$. At $t \in \{0, 1, 2, \ldots, T\}$, each country is endowed with a nonnegative quantity of total power $p_i(t)$. As an extensive form game is not strictly "dynamic", it can be reasonably assumed that $p_i(t)$ does not change over time.

A sequence of spanning subgraphs

The extensive form game takes place on a sequence of time-varying environment graphs. Denote Δ_E as the set of all spanning subgraphs of $\mathbb{E}_E$. Let a sequence of graphs $\mathbb{E}_E(t), \quad t \in \{0, 1, 2, \ldots, T\}$ from Δ_E be such that

$$\mathbb{E}_E(t) \in \Delta_E, t \in \{0, 1, 2, \ldots, T\}. \tag{28}$$

$\mathcal{F}_i(t)$ and $\mathcal{A}_i(t)$ are respectively the label sets of country i's friends and adversaries at time t.

Decision rule

At time $t \in \{0, 1, 2, \ldots, T\}$, every country i decides on how to allocate

$$p_i(t) = \sum_{j \in \mathcal{F}_i(t-1) \cup \mathcal{A}_i(t-1)} u_{ij}(t-1), \tag{29}$$

to its friends and adversaries at time t.

Country i's decision rule is the following.[4] i will keep constant the allocations to its neighbors that remain so at time t. And i will update the reserved power $u_{ii}(t-1)$ to be $u_{ii}(t)$ by adding back

$$\sum_{j \in \mathcal{F}_i(t-1) - \mathcal{F}_i(t)} u_{ij}(t-1) + \sum_{j \in \mathcal{A}_i(t-1) - \mathcal{A}_i(t)} u_{ij}(t-1) \tag{30}$$

which are its allocations to the neighbors that disappear at time t, and then subtracting

$$\sum_{j \in \mathcal{F}_i(t) - \mathcal{F}_i(t-1)} u_{ij}(t) + \sum_{j \in \mathcal{A}_i(t) - \mathcal{A}_i(t-1)} u_{ij}(t) \tag{31}$$

which are its allocations to the neighbors that newly appear at time t. Therefore, the following equality holds

$$u_{ii}(t) = u_{ii}(t-1) + (12) - (13). \tag{32}$$

Denote $U(t)$ as a strategy matrix and $\mathcal{U}(t) \subset \mathcal{U}$ as the set of power allocation matrices at time $t \in \{1, 2, \ldots, T\}$. Each country has perfect information of *the power allocation path* before time t, which is

$$U(0), U(1), \ldots, U(t-1) \in \mathcal{U}(0) \times \mathcal{U}(1) \times \cdots \times \mathcal{U}(t-1). \tag{33}$$

And at time T, the game outcome, the state vector $x(U(T)) \in \mathcal{X}$ whose ith component is country i's state, is realized. Country i orders the state vectors based on the preference axioms.

[4] It is worth noting that an extensive form game is not exactly equivalent to a dynamic game, but rather an alternative mathematical formulation of the normal form game by assuming this decision rule.

A sequence of subgames

Assume that countries have complete information about each other's power, strategies, game outcomes, and preferences for those outcomes. Then countries' allocations from $t = 0$ to $t = T$ can be represented using the following tree structure $\mathbb{T}$. At layer $t \in \{0, 1, \ldots, T\}$ of $\mathbb{T}$, each decision node represents the point at which countries decide on the allocations at time t. As the number of strategy matrices can be infinite, there grows an infinite number of branches from each decision node at layer t, whose pth branch represents a matrix $U_p(t)$ at time t. Each path from the root node to a terminal node of $\mathbb{T}$ represents a power allocation path from $t = 0$ to $t = T$, $U(0), U(1), \ldots, U(T)$.

A power allocation path from $t = 0$ to $t = T$ traverses a sequence of $T + 1$ *subgames*. Let $\kappa(t)$ be the set of subgames at time $t \in \{0, 1, \ldots, T\}$. A function η :

$$\mathcal{U}(0) \times \mathcal{U}(1) \ldots \times \mathcal{U}(T) \longrightarrow \kappa(0) \times \kappa(1) \ldots \times \kappa(T) \tag{34}$$

maps a power allocation path to the sequence of *subgames* it has traversed. Using backward induction, the ρth subgame of this sequence can be denoted as

$$\eta(U(0), U(1), \ldots, U(T))_\rho \tag{35}$$

and be represented by the subtree of $\mathbb{T}$, whose root node denotes the point where the following strategy matrix is made

$$U(T - \rho + 1), \rho \in \{1, 2, \ldots, T + 1\}. \tag{36}$$

Equilibrium concept

Let the power allocation path from $t = 0$ to $t = T$ be

$$U(0), U(1), \ldots, U(T), \tag{37}$$

And denote country i's deviation at time $T' \in \{0, 1, \ldots, T\}$ as

$$U(0), U(1), \ldots, U(T'-1), U^*(T') \ldots U^*(T). \tag{38}$$

By the Decision Rule and the one-shot deviation principle (Fudenberg & Tirole, 1991), country i's future allocations beyond time T' may have to change accordingly after the deviation at time T'.

A power allocation path $U(0), U(1), \ldots, U(T)$ is a *subgame perfect Nash equilibrium* if no unilateral deviation in strategy at any time by any country i is "profitable". This means that given i's deviation at time $T' \in \{0, 1, \ldots, T\}$,

$$U^*(T) \preceq_i U(T) \tag{39}$$

Theorem 5. *In the extensive form game, subgame perfect Nash equilibrium always exists.*

Proof of Theorem 5. A traversal of the adversarial relationships using Algorithm 1 gives rise to a sequence of spanning subgraphs of $\mathbb{E}_E$ that reach $\mathbb{E}_E$ at the last step. The edge sets of these graphs in the sequence are

$$\mathcal{E}_E(0) = \mathcal{R}_\mathcal{F}, \tag{40}$$

$$\mathcal{E}(k) = \mathcal{R}_\mathcal{F} \cup r^{-1}(0) \cup \ldots\ldots \cup r^{-1}(k-1), k \in \mathbf{q}. \tag{41}$$

A sequence of strategy matrices can be derived by Algorithm 1

$$\begin{aligned} U(0) &= \text{diagonal}\{z_1(0), \ldots, z_n(0)\} \\ &= \text{diagonal}\{p_1, \ldots, p_n\} \end{aligned} \tag{42}$$

$$\begin{aligned} U(k) &= \text{diagonal}\{z_1(k), z_2(k), \ldots, z_n(k)\} + \\ &\quad \sum_{r=1}^{k} \min\{z_i(r-1), z_j(r-1)\}(e_i e_j^T + e_j e_i^T) \end{aligned} \tag{43}$$

where $k \in \mathbf{q}$, $U(k) \in \mathbb{R}^{n \times n}$, and $\{i, j\} = \gamma^{-1}(k-1)$. These allocations are consistent with the Decision Rule. A sequence of $q + 1$

subgames is thus derived. Proceeding backwardly, $\eta(U(0), U(1), \ldots, U(q))_h$ is the subgame where the path $U(q-h+1), U(q-h+2), \ldots, U(q)$ is chosen, $h \in \{1, 2, \ldots, q+1\}$.

In this extensive form game, no country i will want to deviate from its strategy $u_i(k)$, $k \in \{0, 1, 2, \ldots, q\}$:

(1) At $k = 0$, either i has no neighbors, or has at least a friend j. In either case, it does not need to deviate.
(2) At $k \in \mathbf{q}$, if $r^{-1}(k-1)$ is one of i's adversarial relationships, then either $z_i(k) > 0$ or $z_i(k) = 0$. In the former case, i does not have to deviate because all its adversaries have zero remaining power, and all its friends are either safe or precarious. In the latter case, i cannot deviate due to power deficiency. If $r^{-1}(k-1)$ is not one of i's adversarial relationships, i does not have an allocation strategy at time k.

Therefore, the sequence of allocations represents a subgame perfect Nash equilibrium. □

As in the normal form game, the power allocation path $U(0), U(1), \ldots, U(q)$ depends on the chosen ordering map γ, with the implication that different sequences of environment graphs may give rise to different subgame perfect Nash equilibria.

Example 3. Consider a normal form game where

(1) Countries: $\mathbf{n} = \{1, 2, 3\}$.
(2) Total power: $p = [8\ 6\ 4]$.
(3) Relations: $\mathcal{F}_1 = \{1\}$ and $\mathcal{A}_1 = \{2, 3\}$; $\mathcal{F}_2 = \{2\}$ and $\mathcal{A}_2 = \{1, 3\}$; $\mathcal{F}_3 = \{3\}$ and $\mathcal{A}_3 = \{1, 2\}$.
(4) Preferences: assume the preference axioms.

Assume three different sequences of graphs. The first sequence contains the following edge sets, $\mathcal{E}_E(0) = \{1, 2\}, \mathcal{E}(1) = \{1, 2\} \cup \{1, 3\}$, and $\mathcal{E}_E(2) = \{1, 2\} \cup \{1, 3\} \cup \{2, 3\}$. The second sequence contains the following edge sets: $\mathcal{E}_E(0) = \{2, 3\}, \mathcal{E}(1) = \{2, 3\} \cup \{1, 2\}$, and $\mathcal{E}_E(2) = \{2, 3\} \cup \{1, 2\} \cup \{1, 3\}$. The third sequence contains the following edge sets: $\mathcal{E}_E(0) = \{1, 3\}, \mathcal{E}(1) = \{1, 3\} \cup \{1, 2\}$, and $\mathcal{E}_E(2) = \{1, 3\} \cup \{1, 2\} \cup \{2, 3\}$.

We construct a subgame perfect Nash equilibrium for each of the three graph sequences by applying Algorithm 1 as in Theorem 5. The state vectors predicted by the three equilibria are, respectively, [precarious, precarious, safe], [safe, precarious, precarious], and [precarious, safe, precarious]. All of the three state vectors are possible predictions from the pure strategy Nash equilibria of the normal form game. In contrast, only one of the vectors can be predicted from the subgame perfect Nash equilibria in the extensive form game. Equilibrium refinement with regard to the normal form game is thus achieved.

2.3. Application

2.3.1. Application I. Survival problem

An application of the power allocation game is countries' survival problem. It is of natural interest to identify networked, adversarial environments where a country can survive in at least a pure strategy Nash equilibrium of the power allocation game.

For example, country i will survive in the power allocation game if its power is no smaller than that of its adversaries, or if it is involved in a set of countries, any two of which are friends and the total power of which is no smaller than that of their adversaries. That is to say, there exists a strategy matrix U such that $x_i(U) \in \{\text{precarious, safe}\}$ if either of the following conditions holds

$$p_i \geq \sum_{j \in \mathcal{A}_i} p_j \tag{44}$$

$$\begin{aligned} &\forall i, j \in \mathcal{S} \subset \mathbf{n}, j \in \mathcal{F}_i \\ &\text{and } \sum_{i \in \mathcal{S}} p_i \geq \sum_{j \in \mathcal{A}_\mathcal{S}} p_j \\ &\text{where } \mathcal{A}_\mathcal{S} = \textstyle\bigcup_{i \in \mathcal{S}} \mathcal{A}_i \end{aligned} \tag{45}$$

Moreover, when either (44) or (45) holds, country i will survive in any equilibrium of the power allocation game. Other possible environments where country i may survive in the power allocation game exist, and Li et al. (2017) contain several results along this line.

For countries' survival problem, it is also worth exploring the kind of pure strategy Nash equilibria with particular properties, such as "Balanced Equilibrium (BE) (Li & Morse, 2018b)". A balanced equilibrium satisfies the following three requirements. First, any two adversaries balance out their offense toward each other. Second, any country with adversaries exhausts its power on offense. Third, any two friends invest zero toward each other. Consequently, any country with adversaries is precarious. That is to say, a strategy matrix U is a BE if three conditions below hold

$$\forall j \in \mathcal{A}_i, u_{ij} = u_{ji}, \tag{46}$$

$$\forall i \in \mathbf{n} \text{ s. t. } \mathcal{A}_i \neq \emptyset, p_i = \sum_{j \in \mathcal{A}_i} u_{ij}, \tag{47}$$

$$\forall j \in \mathcal{F}_i, u_{ij} = u_{ji} = 0. \tag{48}$$

The properties of BE may invoke a rationale for countries to play the "bait and bleed strategy (Mearsheimer, 2001)" in certain environments because a county would "have its hands free" if its adversaries "bloodlet (Mearsheimer, 2001) " — in other words, play a kind of BE among themselves. Li and Morse (2018b) contains a discussion of possible environments for the strategy's success — namely, for the existence of BE in the power allocation game. For example, if the adversarial relationships in the environment make up a complete graph, a necessary and sufficient condition for BE to exist is that the total power of *any* country in antagonisms does not exceed that of its adversaries,

$$\forall i, j \in \mathbf{n} \text{ s. t. } \mathcal{A}_i \neq \emptyset \text{ and } \mathcal{A}_j \neq \emptyset, j \in \mathcal{A}_i \tag{49}$$

BE exists if and only if the following holds

$$\forall i \in \mathbf{n} \text{ s. t. } \mathcal{A}_i \neq \emptyset, p_i \leq \sum_{j \in \mathcal{A}_i} p_j \tag{50}$$

And below we prove the condition for the BE to exist in games on structurally balanced graphs (Li & Morse, 2018b), which takes a similar form with that of the Hall's Maximum Matching Theorem (Hall, 1935).

Theorem 6. *A power allocation game on a structurally balanced graph (Harary et al., 1953), in other words, in an environment where the adversary pairs make up a bipartite graph, has a balanced equilibrium if and only if the following power condition holds for the countries with adversaries:*

(1) $\forall \mathcal{S} \subseteq \mathcal{L}, \sum_{j \in \mathcal{A}_\mathcal{S}} p_j \geq \sum_{i \in \mathcal{S}} p_i$
(2) $\forall \mathcal{S} \subseteq \mathcal{R}, \sum_{j \in \mathcal{A}_\mathcal{S}} p_j \geq \sum_{i \in \mathcal{S}} p_i \ (\mathcal{A}_\mathcal{S} = \bigcup_{i \in \mathcal{S}} \mathcal{A}_i)$

By definition, the two sets of nodes, $\mathcal{L}$ and $\mathcal{R}$, represent the two groups of countries in adversaries relationships, where each country in either set is only connected to countries in the other set. $\mathcal{S}$ is a subset of either set.

Proof of Theorem 6.

(1) Sufficient condition: The proof is by contradiction.
Suppose the power condition holds and a balanced equilibrium does not exist. Suppose then that for all symmetric

allocation matrices U of the bipartite graph, $\exists i \in \mathcal{L}$, the remaining power $z_i > 0$. Then $\|z_i\|_1 > 0$.
By the power condition, there must exist another node, $j \in \mathcal{R}$, with $z_j > 0$.
By the property of a connected graph, alternating paths exist between i and j, whose lengths should be odd. Since this is a directed graph with symmetric edges, the length of the path is given by the number of edges on it.
Below will be shown that there is at least an augmenting path P between i and j such that the positive $\|z\|_1$ can be further minimized. This would contradict the fact that when the power condition holds, $\|z_i\|_1$ is positive. Then a balanced equilibrium must exist.
Suppose on all alternating paths between $i \in \mathcal{L}$ and $j \in \mathcal{R}$ with $z_i > 0$ and $z_j > 0$, there must be an even edge in the path with zero allocations.
Then a case that contradicts the above statement is derived below:

(a) Initialize $\mathcal{L}' = \{i\}$, and $\mathcal{R}' = \emptyset$.
(b) Update $\mathcal{L}' = \mathcal{L}'$, and $\mathcal{R}' = \mathcal{R}' \cup \mathcal{A}_{\mathcal{L}'}$. This step generates the first odd edge of the paths and its symmetric edge.
(c) For each $m \in \mathcal{R}'$, if $\exists n \in \mathcal{A}_{\mathcal{R}'}$ such that $u_{nm} = u_{mn} > 0$, update $\mathcal{L}' = \mathcal{L}' \cup \{n\}$. Update $\mathcal{R}' = \mathcal{R}'$. This step gives the first even edge of the paths and its symmetric edge, on which the symmetric allocations are positive.
(d) Repeat step 2 and 3 until $\mathcal{L}'$ and $\mathcal{R}'$ stay fixed.

At the end of the above process, a subgraph $\mathbb{G}'$ of $\mathcal{L}'$ and $\mathcal{R}'$ is derived, where $\mathcal{R}' = \mathcal{A}_{\mathcal{L}'}$ but $\mathcal{L}' \neq \mathcal{A}_{\mathcal{R}'}$. Also in $\mathbb{G}'$, all the alternating paths starting at i have positive allocations on the even edges.
It has been supposed that for all alternating paths between two nodes with positive node allocations, there must be an even edge in them with zero allocations. $\mathbb{G}'$ does not have any alternating path starting with i with zero allocations on even edges. Also, by the property of the connected graph, any path that starts at i can end at any node in $\mathcal{R}'$. Then it must be that all nodes in $\mathcal{R}'$ have zero node allocations.
Given that $\mathcal{R}' = \mathcal{A}_{\mathcal{L}'}$, $\Sigma_{m\in\mathcal{L}'} p_m > \Sigma_{n\in\mathcal{R}'} p_n = 0$. Then the power condition does not hold for $\mathbb{G}'$. Contradiction. There will not be a balanced equilibrium as countries in $\mathcal{S}$ will not exhaust their power by allocating to their adversaries.

(2) Necessary condition: If the power condition does not hold, it means that there is a set of countries $\mathcal{S}$ whose total power exceeds that of their adversaries. There will not be a balanced equilibrium as countries in $\mathcal{S}$ will not exhaust their power by allocating to their adversaries. □

2.3.2. Application II. Prediction

Usually, there are multiple equilibrium equivalence classes, except in cases where no adversarial relationships exist. Estimating the probabilistic distribution of the equilibrium equivalence classes is a necessary task for understanding countries' *likelihood* of survival and success from power allocation. Naturally, the probability of an equilibrium equivalence class in the game can be associated with the class volume. However, the output of Algorithm 2 suggests the calculation of class volumes to be a challenging task. In Li, Yue, Liu, and Morse (2018), we have proposed an "update-rule-based" algorithm like in a networked evolutionary game (e.g., Cheng et al. (2015)) to obtain an approximation of the class volumes.

The algorithm's input is the parameters of the normal form game, which are countries' power, relationships, and preference axioms.

- At time 0, initialize the pure strategy Nash equilibrium set $\mathcal{U}$ to be empty and a strategy matrix $U(0)$.
- At time t, each country i updates its power allocation strategy $u_i(t-1)$ to have the best possible state vector based on a total order $\preceq_i$ on the set of state vectors $\mathcal{X}$, by assuming the strategies of all the others $u_j(t-1), j \neq i$ to be fixed and the total power constraint to be time constant,

$$p_i = \sum_{j\in\mathbf{n}} u_{ij}(t). \tag{51}$$

- Stop updating if reaching a pure strategy Nash equilibrium U^* or the maximum number of rounds $t = T$.
- Update the equilibrium set $\mathcal{U} \cup \{U^*\}$ and go back to initialize a different strategy matrix.

The output of the algorithm is the pure strategy Nash equilibrium set of the normal form game. In Li et al. (2018), we have computed the classes' likelihoods for a power allocation game assuming real-world data of countries including China, Russia, and the US, which can be used to calculate these countries' likelihoods of survival in different environments.

3. The signed network formation game

We now design a "signed network formation game" as the second part of the games-on-signed graphs framework. While the power allocation game predicts countries' survivability in a given environment, the signed network formation game studies how they may change the environment to improve such survivability. Instead of allocating power on a given graph, countries' strategy would be now to change the graph itself.

3.1. Static signed network formation game

Countries and Strategies: Countries in the environment are labeled from 1 to n in $\mathbf{n} = \{1, 2, \ldots, n\}$. Country i's signed network formation strategy means choosing a friendly relationship, an adversarial relationship, or no specific relationship to be formed with every country in $\mathbf{n}$, and by default i is its own friend. That is to say, i picks a vector $l_i = [l_{ij}]_{1\times n}$, where $l_{ij} \in \{\text{friend}, \text{adversary}, \text{null}\}$ is the relationship intended by i to be formed with j and $l_{ii} = \text{friend}$. Each set of countries' signed network formation strategies $\{l_i : i \in \mathbf{n}\}$ determines a *linking matrix*, $L = [l_{ij}]$ whose ith row is l_i. Denote a *relation configuration* of the n countries as $R = [r_{ij}]_{n\times n}$ whose ij-th element represents the relationship between i and j.
A *relation function*

$$\tau : \mathcal{L} \to \mathcal{R} \tag{52}$$

maps a linking matrix L to a *relation configuration* R such that:

(1) $r_{ii} = \text{friend}$.
(2) $r_{ij} = \text{friend}$ if $(l_{ij} = \text{friend}) \wedge (l_{ji} = \text{friend})$.
(3) $r_{ij} = \text{adversary}$ if $(l_{ij} = \text{adversary}) \vee (l_{ji} = \text{adversary})$.
(4) $r_{ij} = \text{null}$ for the rest of the cases.

The set of all possible linking matrices is $\mathcal{L}$ and the set of all possible relation configurations is $\mathcal{R}$. The two sets' cardinalities are $|\mathcal{L}| = 3^{n(n-1)}$ and $|\mathcal{R}| = 3^{n(n-1)/2}$.

Basic Preference Axioms: Country i's preference for relation configurations depends importantly on its power allocation outcomes from the configurations. We formalize the linkage between the power allocation outcomes and countries' preferences below.

Given two relation configurations $R = [r_{ij}]_{n\times n}$ and $\hat{R} = [\hat{r}_{ij}]_{n\times n}$, let the pure strategy Nash equilibrium sets of the two corresponding power allocation games be respectively $\mathcal{U}^*$ and $\hat{\mathcal{U}}$. A sufficient condition for country i weakly preferring $\hat{R}$ to R, written as $R \preceq_i \hat{R}$, is that one of the following must be satisfied for any country $j \in \mathbf{n}$:

(1) if $r_{ij} = \hat{r}_{ij} =$ friend,
  (a) $(\forall \hat{U} \in \hat{\mathcal{U}}, x_j(\hat{U}) \in \{\text{safe, precarious}\}) \vee (\forall U^* \in \mathcal{U}^*, x_j(U^*) \in \{\text{unsafe, precarious}\})$.
(2) if $r_{ij} = \hat{r}_{ij} =$ adversary,
  (a) $(\forall \hat{U} \in \hat{\mathcal{U}}, x_j(\hat{U}) \in \{\text{unsafe, precarious}\}) \vee (\forall U^* \in \mathcal{U}^*, x_j(U^*) \in \{\text{safe, precarious}\})$.
(3) if $r_{ij} =$ friend and $\hat{r}_{ij} =$ null,
  (a) $\forall U^* \in \mathcal{U}^*, x_j(U^*) \in \{\text{unsafe, precarious}\}$.
(4) if $r_{ij} =$ null and $\hat{r}_{ij} =$ friend,
  (a) $\forall \hat{U} \in \hat{\mathcal{U}}, x_j(\hat{U}) \in \{\text{safe, precarious}\}$.
(5) if $r_{ij} =$ adversary and $\hat{r}_{ij} =$ null,
  (a) $\forall U^* \in \mathcal{U}^*, x_j(U^*) \in \{\text{safe, precarious}\}$.
(6) if $r_{ij} =$ null and $\hat{r}_{ij} =$ adversary,
  (a) $\forall \hat{U} \in \hat{\mathcal{U}}, x_j(\hat{U}) \in \{\text{unsafe, precarious}\}$.
(7) if $r_{ij} =$ null and $\hat{r}_{ij} =$ null,
  no condition is needed for j's state.

For the above scenarios, Axioms (1) - (7) contain the most intuitive criteria for evaluation. Other scenarios where country i may have to compare an environment in which country j is i's friend to another in which country j is i's adversary are much more ad hoc, usually requiring a case-by-case analysis. Axioms (8) and (9) provide an example of criteria for evaluating these ad hoc scenarios.

(8) if $r_{ij} =$ friend and $\hat{r}_{ij} =$ adversary,
  (a) $(\exists U^* \in \mathcal{U}^*, x_j(U^*) \in \{\text{unsafe, precarious}\}) \wedge$
    $(\forall \hat{U} \in \hat{\mathcal{U}}, x_j(\hat{U}) \in \{\text{unsafe, precarious}\})$, or
  (b) $(\forall U^* \in \mathcal{U}^*, x_j(U^*) \in \{\text{unsafe, precarious}\}) \wedge$
    $(\exists \hat{U} \in \hat{\mathcal{U}}, x_j(\hat{U}) \in \{\text{unsafe, precarious}\})$.
(9) if $r_{ij} =$ adversary and $\hat{r}_{ij} =$ friend,
  (a) $(\exists U^* \in \mathcal{U}^*, x_j(U^*) \in \{\text{safe, precarious}\}) \wedge$
    $(\forall \hat{U} \in \hat{\mathcal{U}}, x_j(\hat{U}) \in \{\text{safe, precarious}\})$, or
  (b) $(\forall U^* \in \mathcal{U}^*, x_j(U^*) \in \{\text{safe, precarious}\}) \wedge$
    $(\exists \hat{U} \in \hat{\mathcal{U}}, x_j(\hat{U}) \in \{\text{safe, precarious}\})$.

Equilibrium Concept: The Nash equilibrium concept is employed in the signed network formation game to make predictions. Let country i's deviation from the strategy matrix l_i be $\hat{l}_i$, and the linking matrix where i makes the unilateral deviation be $\hat{L}$. The deviation set $\mathcal{Z}_i(L)$ is the set of all possible deviations of country i's strategies from the linking matrix L. Therefore, a linking matrix L is a pure strategy Nash Equilibrium if no one deviation in strategy by any country i is "profitable". In other words,

$$\tau(\hat{L}) \preceq_i \tau(L), \ \ \forall l_i \in \mathcal{Z}_i(L). \tag{53}$$

3.2. Theoretical result

A utility function that satisfies the basic preference axioms exists. Nevertheless, a pure strategy Nash equilibrium for this signed network formation game exists independently of any preference axioms.

Theorem 7. *In the static signed network formation game, a pure strategy Nash equilibrium always exists.*

Proof of Theorem 7. For a linking matrix L, suppose that $\forall i \in \mathbf{n}$, $l_{ii} =$ friend and $\forall i, j \in \mathbf{n}$, $l_{ij} = l_{ji} =$ adversary. No country can unilaterally deviate to change any adversarial relationship. Therefore, L is always a pure strategy Nash equilibrium of the signed network formation game. □

3.3. Application: Optimal network design

An application of the signed network formation game is *optimal network design* by great powers. For instance, if a country is a "great power", then there exists a linking matrix realizing this relation configuration in an equilibrium of the signed network formation game (Li & Morse, 2017a). Great power is defined as a country whose power exceeds that of all the other countries combined and with an optimal relation configuration about the environment. That is to say, if for country i,

$$p_i \geq \sum_{j \in \mathbf{n} \setminus \{i\}} p_j, \tag{54}$$

$$\exists R \in \mathcal{R} \text{ and } \forall \hat{R} \in \mathcal{R} \text{ such that } \hat{R} \neq R, \hat{R} \preceq_i R. \tag{55}$$

then there exists L as a pure strategy Nash equilibrium where $\tau(L) = R$. This definition can somehow approximate the US's current status, and if certain conditions for a set of countries are met, they can be regarded indistinguishably from a great power. In Li and Morse (2017a), we call such a set of countries a "power bloc", which is necessarily an *alliance*, within which no adversarial relationships exist, and explain that the lack of great power conflicts in the unipolar, postwar world order is the product of optimal network design by a power bloc — NATO.

An alliance's reducibility

If an alliance $\mathcal{S} \subset \mathbf{n}$, which is a subset of nodes without any adversarial relationships, satisfies the above definition, then it is *reducible*. A condition for its reducibility is summarized in Theorem 8.

Theorem 8. *Suppose that there are two power allocation games with the sets of countries' labels respectively being* $\mathbf{n}$ *and* $\tilde{\mathbf{n}}$, *and that the former game has an alliance* $\mathcal{S}$. *The game with* $\tilde{\mathbf{n}}$ *is a* reduced form *of the game with* $\mathbf{n}$ *if the following conditions hold for the alliance* $\mathcal{S}$,

(1) Any two members are allies: $\forall i, j \in \mathcal{S}$, $r_{ij} =$ *friend.*
(2) Members sharing the external relations with outsiders: $\forall i, j \in \mathcal{S}$ *and* $\forall k \in \mathbf{n} \setminus \mathcal{S}$, $r_{ik} = r_{jk}$.

Proof of Theorem 8. When an alliance $\mathcal{S}$ satisfying the above two conditions exists, a surjection $f : \mathbf{n} \to \tilde{\mathbf{n}}$ can be constructed such that

(1) $\forall i \in \mathcal{S}, f(i) = s$.
(2) $\forall j \in \mathbf{n} \setminus \mathcal{S}, f(j) = j$.
(3) $\forall i, j \in \mathcal{S}, h \in \mathbf{n} \setminus \mathcal{S}, r_{ih} = r_{jh} = \tilde{r}_{sh}$.
(4) $\sum_{i \in \mathcal{S}} p_i = \tilde{p}_s$.

where conditions (1) and (4) in the definition of a game's reduced form are easily satisfied.

Conditions (2) and (3) are also satisfied because for any equilibrium U in the game with $\mathbf{n}$, let $\tilde{U}$ be such that $\forall p, q \in \tilde{\mathbf{n}}$, $\tilde{u}_{pq} = \sum_{i \in f^{-1}(p)} \sum_{i \in f^{-1}(q)} u_{ij}$. Then we have

$$\sum_{i \in \mathcal{S}} \sigma_i(U) = \sum_{i \in \mathcal{S}} \Big(\sum_{j \in \mathcal{F}_i} u_{ji} + \sum_{j \in \mathcal{A}_i} u_{ij}\Big) = \sum_{p \in \tilde{\mathcal{F}}_s} \tilde{u}_{ps} + \sum_{q \in \tilde{\mathcal{A}}_s} \tilde{u}_{sq} = \tilde{\sigma}_s(\tilde{U}) \tag{56}$$

$$\sum_{i \in \mathcal{S}} \tau_i(U) = \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{A}_i} u_{ji} = \sum_{k \in \tilde{\mathcal{A}}_s} \tilde{u}_{ks} = \tilde{\tau}_s(\tilde{U}) \tag{57}$$

For $i \in \mathcal{S}$, either $\sigma_i(U) \geq \tau_i(U)$ or $\sigma_i(U) \leq \tau_i(U)$ holds. In the former case, $\tilde{\sigma}_s(\tilde{U}) \geq \tilde{\tau}_s(\tilde{U})$. Therefore, $x_i(U), x_s(\tilde{U}) \in \{\text{safe, precarious}\}$. In the latter case, $\tilde{\sigma}_s(\tilde{U}) \leq \tilde{\tau}_s(\tilde{U})$. Therefore, $x_i(U), x_s(\tilde{U}) \in \{\text{unsafe, precarious}\}$.

In addition, $\forall j \in \mathbf{n} \setminus \mathcal{S}, \sigma_j(U) = \tilde{\sigma}_j(\tilde{U})$ and $\tau_j(U) = \tilde{\tau}_j(\tilde{U})$. Just as no country hopes to deviate from U, no country has any incentives to deviate from $\tilde{U}$. Then $\tilde{U}$ is an equilibrium of the game with $\tilde{\mathbf{n}}$. The alliance $\mathcal{S}$ is thus reducible into node s under f. In turn, the game with $\mathbf{n}$ is reducible to the game with $\tilde{\mathbf{n}}$. $\square$

The goal of reducibility could be unrealistic for many alliances. Next, we explore how a "power bloc" may form in an equilibrium of the signed network formation game, which achieves "reducibility" for itself and an "optimal network design" of the international environment. This speaks to the literature of unipolar politics (e.g., Monterio (2014)) in the world system (e.g., Deutsch and Singer (1964)).

Power blocs and optimal network design

Suppose there are $k \in \mathbb{N}$ "power blocs", meaning k alliances, where bloc $\mathbf{n}_\phi \subset \mathbf{n}, 1 \le \phi \le k$, satisfies the following four properties:

(1) Conditions for the bloc's power preponderance: $\sum_{i\in\mathbf{n}_\phi} p_i \ge \sum_{j\in\mathbf{n}\setminus\mathbf{n}_\phi} p_j$.
(2) Conditions for members' internal relations: $\forall i, j \in \mathbf{n}_\phi$, $r_{ij} = r_{ji} =$ friend.
(3) Conditions for members' relations with outsiders: $\forall i, j \in \mathbf{n}_\phi$ and $\forall q \in \mathbf{n} \setminus \mathbf{n}_\phi$, $r_{iq} = r_{jq}$.
(4) Conditions for members' preferences: let countries in $\mathbf{n}_\phi$ have a nonempty intersection of their sets of "optimal" relation configurations,

$$\bigcap_{i\in\mathbf{n}_\phi} \mathcal{R}_i = \mathcal{R}_\phi, \tag{58}$$

where $\mathcal{R}_i \subset \mathcal{R}$ is country i's set of "optimal" relation configurations. By "optimality" is meant that for $i \in \mathbf{n}_\phi$,

(a) It is indifferent between any two relation configurations which are elements in $\mathcal{R}_i$:

$$\forall \hat{R}, \bar{R} \in \mathcal{R}_i, \hat{R} \sim_i \bar{R}. \tag{59}$$

(b) It weakly prefers any relation configuration which is an element in $\mathcal{R}_i$ to another relation configuration which is not an element in $\mathcal{R}_i$:

$$\forall \hat{R} \in \mathcal{R}_i \text{ and } \forall R \in \mathcal{R} \setminus \mathcal{R}_i, R \preceq_i \hat{R}. \tag{60}$$

In each relation configuration in $\mathcal{R}_\phi$, the relations of countries in $\mathbf{n}_\phi$ satisfy conditions (2) and (3).

Condition (4) suggests a power bloc's political dimension, which is the homogeneous preference structures of its members. A power bloc also has a command and control dimension, which primarily consists in its "optimal network design" of the world by its preferences.

Theorem 9. *If at least a power bloc exists and Axioms (1) – (9) hold, there exists a pure strategy Nash equilibrium of the signed network formation game that realizes an optimal relation configuration for the bloc $\mathbf{n}_\phi$.*

Proof of Theorem 9. By Condition (1) in the definition of a power bloc, any adversary of countries in $\mathbf{n}_\phi$ will always be unsafe or precarious, and any friend of them will always be safe or precarious in power allocation. By Axioms (4) and (6), each country in $\mathbf{n}_\phi$ weakly prefers either a friend or an adversary relationship with every outsider in $\mathbf{n} \setminus \mathbf{n}_\phi$ to having no relationships with them.

Then there are two sets of outsiders to consider in $\mathbf{n} \setminus \mathbf{n}_\phi$, with which countries in $\mathbf{n}_\phi$ are friends $(\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{F}$ or adversaries $(\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{A}$. Let a linking matrix L be:

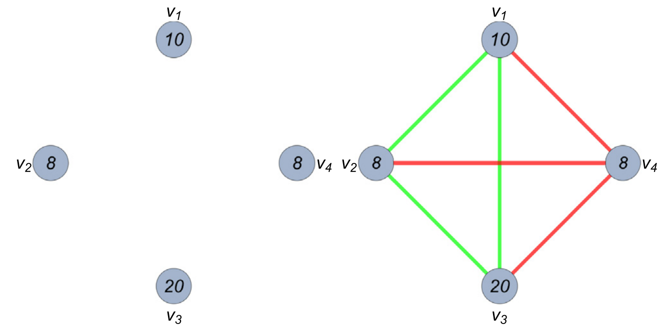


Fig. 5. The equilibrium of the signed network formation game.

(1) $\forall i \in \mathbf{n}$, $l_{ii} =$ friend.
(2) $\forall i, j \in \mathbf{n}_\phi$ and $\forall p, q \in (\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{F}$, $l_{ip} = l_{jp} = l_{pi} = l_{pj} = l_{pq} = l_{qp} =$ friend.
(3) $\forall i, j \in \mathbf{n}_\phi$ and $\forall g, h \in (\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{A}$, $l_{ig} = l_{jg} = l_{gi} = l_{gj} = l_{gh} = l_{hg} =$ adversary.
(4) $\forall p \in (\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{F}$ and $g \in (\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{A}$, $l_{pg} = l_{gp} =$ adversary.

No country has incentives to deviate from L:

(1) Any country in $\mathbf{n}_\phi$ has no incentives to deviate from the optimal relation configuration $\tau(L)$.
(2) Any country in $(\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{F}$ has no incentives to change the friend relations with any country in or outside of $\mathbf{n}_\phi$ to otherwise.
   (a) By Axiom (4), they do not hope to change any friend relation to a "null" relation.
   (b) By Axiom (9a), they do not hope to change any friend relation to an "adversary" relation.
(3) They also cannot unilaterally change the adversary relations with countries in $(\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{A}$.
(4) Countries in $(\mathbf{n} \setminus \mathbf{n}_\phi)_\mathcal{A}$ cannot unilaterally change any adversary relation. $\square$

Example 4. Consider a relation formation game where

(1) Countries: $\mathbf{n} = \{1, 2, 3, 4\}$.
(2) Their total power: $p = [10\ 8\ 20\ 8]$.
(3) Preferences: assume that countries 1 and 3 form a power bloc, and in their optimal relation configuration, they are friendly with country 2 but antagonistic against country 4.

Fig. 5 shows the only equilibrium of the signed network formation game, which realizes countries 1 and 3's optimal relation configuration. Given the power preponderance of countries 1 and 3, it is *optimal* for country 2 to have country 4 as an adversary by Axioms (8)–(9).

Structural balance and "buffer states"

Suppose that there are k power blocs, which are not necessarily disjoint. If $k > 1$, the *only* case where those sets are disjoint is when $k = 2$, and the total power of either bloc has to be equal, $\sum_{i\in\mathbf{n}_1} p_i = \sum_{j\in\mathbf{n}_2} p_j$. In this case, "structural balance (Harary et al., 1953)" can be achieved in equilibrium. In this equilibrium, two blocs will be formed, with each country in one bloc antagonistic with each country in the other bloc. In other cases, as long as $k > 2$, "buffer states" must exist — country i is a *buffer state* in between blocs if it is indifferent being in each bloc

$$i \in \mathbf{n}_\phi \cap \mathbf{n}_\psi, \text{ where } \phi \neq \psi, \text{ and } \phi, \psi \in \{1, 2, \ldots, k\}. \tag{61}$$

Then the equilibrium relation configuration in Theorem 9 depends crucially on the preferences and actions of these buffer states, as in the Cold War era, the current diplomatic competition over the Indo-Pacific region between the Biden administration and China, and many others.

4. Concluding remarks

What we regard as several minimal fundamentals about countries' behaviors have motivated the development of the games-on-signed graphs framework. First, each country faces two realistic constraints from a given security environment – its power and relationships. Second, with the constraints, a country pursues survival and success in a constant or a changing environment, and may bring about some of the changes to the environment itself. These motivate the usage of a signed graph (''the environment graph'') to describe the environment and the development of the ''games on signed graphs'' to model countries' strategies therein.

Due to the scope of the paper, we leave out two discussions: the first is the extensions of the games to dynamic horizons, preferably with incomplete information as well, as applicable to the particular setting in international relations. One example is how, without complete knowledge about others, countries can best control their power allocation strategies over time or across multiple layers of the sea, land, and air, where their powers and relations may change based on an evolutionary law. This may lead to an extension of networked evolutionary games like Cheng et al. (2015). The second is a non-technical study of countries' decision-making, focusing on the game parameters like countries' powers and decision makers' preferences in historical and contemporary contexts and the outcomes thereof. We will present them in our future work.

References

Altafini, C. (2012). Consensus problems on networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, *58*(4), 935–946.

Bauso, D., Tembine, H., & Başar, T. (2016). Opinion dynamics in social networks through mean-field games. *SIAM Journal on Control and Optimization*, *54*(6), 3225–3257.

Bernheim, B. D., Peleg, B., & Whinston, M. D. (1987). Coalition-proof Nash equilibria I. Concepts. *Journal of Economic Theory*, *42*(1), 1–12.

Bryen, S. D. (2012). *The Application of Cybernetic Analysis to The Study of International Politics*. Springer Science & Business Media.

Cheng, D., He, F., Qi, H., & Xu, T. (2015). Modeling, analysis and control of networked evolutionary games. *IEEE Transactions on Automatic Control*, *60*(9), 2402–2415.

Cranmer, S. J., Menninga, E. J., & Mucha, P. J. (2015). Kantian fractionalization predicts the conflict propensity of the international system. *Proceedings of the National Academy of Sciences*, *112*(38), 11812–11816.

Deutsch, K. W. (1966). *The Nerves of Government: Models of Political Communication and Control: With A New Introduction, vol. 90729*. Free Press of Glencoe.

Deutsch, K. W., & Singer, J. D. (1964). Multipolar power systems and international stability. *World Politics*, *16*(03), 390–406.

Ding, X., Li, H., Lu, J., & Wang, S. (2021). Optimal strategy estimation of random evolutionary boolean games. *IEEE Transactions on Cybernetics*.

Fudenberg, D., & Tirole, J. (1991). *Game Theory*. The MIT Press.

Goyal, S., Vigier, A., & Dziubinski, M. (2015). Conflict and networks. In *The Oxford Handbook of the Economics of Networks*.

Hall, P. (1935). On representatives of subsets. *Journal of the London Mathematical Society*, *1*(1), 26–30.

Harary, F., et al. (1953). On the notion of balance of a signed graph. *The Michigan Mathematical Journal*, *2*(2), 143–146.

Jackson, M. O., & Nei, S. (2015). Networks of military alliances, wars, and international trade. *Proceedings of the National Academy of Sciences*, *112*(50), 15277–15284.

Li, Y. (2018). *A Network Approach to International Relations* (Ph.D. Thesis), Yale University.

Li, Y., & Morse, A. (2017a). The countries' relation formation problem: I and II'. In *Proceedings of International Federation of Automatic Control World Congress*.

Li, Y., & Morse, A. (2017b). Game of power allocation on networks. In *Proceedings of American Control Conference*.

Li, Y., & Morse, A. S. (2018a). The power allocation game on a network: A paradox. *IEEE/CAA Journal of Automatica Sinica*.

Li, Y., & Morse, A. S. (2018b). The power allocation game on a network: balanced equilibrium. In *Proceedings of the American Control Conference*.

Li, Y., & Morse, A. S. (2018c). The power allocation game on a network: equilibrium selection. In *Proceedings of the 23rd Mathematical Theory of Networks and Systems*.

Li, Y., & Morse, A. S. (2018d). The power allocation game on a network: subgame perfection. In *Proceedings of the IEEE Conference on Decision and Control*.

Li, Y., Morse, A., Liu, J., & Başar, T. (2017). Countries' survival in networked international environments. In *Proceedings of IEEE Conference on Decision and Control*.

Li, Y., Yue, J., Liu, F., & Morse, A. S. (2018). The power allocation game on a network: computation issue. In *Proceedings of the 7th IFAC Necsys workshop*.

Mearsheimer, J. J. (2001). *The Tragedy of Great Power Politics*. WW Norton & Company.

Mock, W. B. (2011). Pareto Optimality. *Encyclopedia of Global Justice*, 808–809.

Monderer, D., & Shapley, L. S. (1996). Potential games. *Games and Economic Behavior*, *14*(1), 124–143.

Monterio, N. (2014). *The Theory of Unipolar Politics*. Cambridge University Press.

Myerson, R. B. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, *2*(3), 225–229.

Nash, J. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, *36*(1), 48–49.

Nash, J. (1951). Non-cooperative games. *Annals of Mathematics*, 286–295.

Neumann, J. V., & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press.

Reny, P. J. (1999). On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica*, *67*(5), 1029–1056.

Roberson, B. (2011). Allocation games. In *Wiley Encyclopedia of Operations Research and Management Science*.

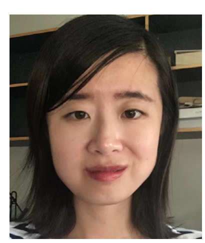
Stier-Moses, N. E. (1958). *Selfish Versus Coordinated Routing in Network Games* (Ph.D. Thesis), Massachusetts Institute of Technology.

Waltz, K. N. (1979). *Theory of International Politics*. New York: McGraw-Hill.

Whitesitt, J. E. (2012). *Boolean Algebra and Its Applications*. Courier Corporation.

Ye, M., Hu, G., Lewis, F. L., & Xie, L. (2019). A unified strategy for solution seeking in graphical N-coalition noncooperative games. *IEEE Transactions on Automatic Control*, *64*(11), 4645–4652.

Zhao, G., Wang, Y., & Li, H. (2016). A matrix approach to the modeling and analysis of networked evolutionary games with time delays. *IEEE/CAA Journal of Automatica Sinica*, *5*(4), 818–826.



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