Contextual and Mathematical Conceptual Resources for Reasoning about Null Spaces

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Vector spaces are often taught with an axiomatic focus, but this has been shown to rely on knowledge many students have not yet developed. In this paper, we examine two students' conceptual resources for reasoning about null spaces drawing on data from a paired teaching experiment. The task sequence is set in the context of a school with one-directional hallways. Students' informal reasoning about paths that leave the room populations unchanged supported more formal reasoning about null spaces. We found that one student used context-based resources (such as 'loops' in hallway) to reason about null spaces, while the other student drew largely on previously formalized mathematical resources (e.g. free variables, linear dependence). The use of formal resources sometimes required recontextualization, which may function to constrain student sense-making or afford opportunities for broadening students' formal prior knowledge.

Keywords: linear algebra, subspaces, null spaces, conceptual resources

Axiomatic treatments of vector spaces are thought to be often inaccessible to students because they unify and formalize many ideas that many students have not yet developed (Dorier, 2000; Grenier-Boley, 2014). In their genetic decomposition of vector spaces, Parraguez and Oktaç (2010) identified the binary operations of scalar multiplication and vector addition, as well as the closure of sets under linear combinations of these operations, as the critical constructs for consideration in regard to students' learning about vector spaces. We thus argue that subspaces function as a more accessible entry point for supporting students' learning about vector spaces.

Given the increasingly important role of linear algebra in real world applications, more work is needed to support students in developing robust conceptualizations for subspaces, and particularly null spaces, given their applicability to a wide range of closed systems problems. In this paper, we examine data from an instructional sequence on subspaces whose design is informed by the principles of Realistic Mathematics Education (RME; van den Heuvel-Panhuizen & Drijvers, 2020). In particular, we address the research question: What are students' conceptual resources for reasoning about null spaces in the context of an RME-based instructional sequence focused on subspaces?

Literature and Theory

A limited number of studies target students' early reasoning about vector spaces, subspaces, and especially null and/or column spaces (Stewart et al., 2019). Two recent exceptions in the area of subspaces are the work of Wawro et al. (2011) and Caglayan (2019). Wawro et al. (2011) documented 8 undergraduate students' concept images for subspaces in relation to how they reasoned about the concept definition -- identifying geometric and algebraic interpretations, and the critical role of reasoning about a subspace as a part of a whole for making sense of the formal textbook definition presented to students. Caglayan (2019) interviewed 14 undergraduate math majors who had recently taken linear algebra, asking them to use a digital geometry system

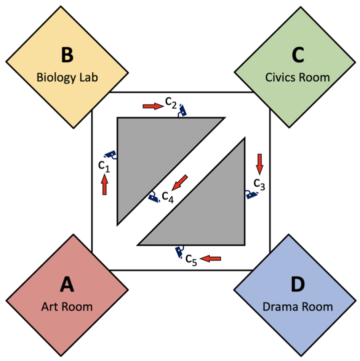
(Matlab) to classify 57 subsets of 7 different vector spaces as subspaces or not -- particularly highlighting ways in which students used the zero vector to show the set was not closed. These studies and others (e.g. Açikyildiz & Kösa, 2021; Dogan, 2018) highlight the potential value of geometry for introducing ideas related to vector spaces. Our study differs as we examine an experientially real context for engaging students in reasoning about ideas related to subspaces and null spaces, but do not leverage the geometry of two and three dimensional space.

We draw on diSessa's (2018) Knowledge in Pieces (KiP) view of learning and cognition, which is influenced heavily by constructivist and cognitivist traditions. We value this as a lens to theorize conceptual resources (Taber, 2008) and adopt anti-deficit views of students and their learning (Adiredja, 2019). KiP emphasizes the complex, continuously shifting, and contextually-bound nature of knowledge, where learning involves reconfiguration of "naive" knowledge into more normative understandings (diSessa, 2018). For example, the notion that 'multiplication makes bigger' is a concept that is considered normatively correct in some contexts (numbers larger than 1) and not others. It is a "small knowledge structure" that can either be cast as an unuseful idea from a deficit perspective or a useful idea that can be more explicitly contextualized and reconfigured to reflect broader and more normatively accepted understandings.

Study Context

This work was conducted as part of a broader research project focused on developing a series of research-based linear algebra materials. These materials were developed in alignment with RME principles of experientially real starting points, guidance from an instructional figure, and the critical role of model-of, model-for shifts in students' mathematical activity (Van den Heuvel-Panhuizen & Drijvers, 2020). In this manuscript, we leverage a KiP lens for examining students' mathematical activity. To our knowledge, KiP frameworks have not been heavily leveraged in the literature to analyze student reasoning in RME-inspired task sequences and we believe coordinating these perspectives may provide a meaningful contribution to the field.

In its current form, the sequence consists of three core tasks. In this manuscript, we focus our analysis on students' reasoning in the context of the second and third tasks. Our team's approach to subspaces was organized around the notion that subspaces are non-empty subsets of vector spaces that are closed under linear combinations. To support students' development of meaning in relation to this notion, we leverage the scenario of one-way hallways during a pandemic (Plaxco et al, 2021). The hallway scenario can be viewed as a coordination between two sets of quantities: the number of people who pass through each of the one-way hallways as observed by cameras and the change in the population of each classroom that occurs during a period of observation (Figure 1a). For example, assuming people do not linger in the hallway, if 3 people pass by Camera 1 and Camera 2 during a class changeover, then the net change of the number of people in the Biology Lab is zero. This induces a mapping of the set of n-tuples encoding of the number of people passing by the camera in each hallway ("*camera vectors*"; in the West Wing, vectors of the form $\langle c_1, c_2, c_3, c_4, c_5\rangle$) to the set of m-tuples showing the net change in each room's population over the period of hallway observations ("*room change vectors*"; in the West Wing, vectors of the form $\langle \Delta A, \Delta B, \Delta C, \Delta D\rangle = \langle c_4 + c_5 - c_1, c_1 - c_2, c_2 - c_3 - c_4, c_3 - c_5\rangle$). In Tasks 1 and 2, students are asked to represent journeys of students walking through halls using camera vectors, reason about the relationships between the *camera vectors* and the *room change vectors* within the problem context, and reflect on and make generalizations based on their reasoning. These activities are focused on supporting students to reason about the set of camera vectors that do not change the populations of the rooms (which corresponds to the null space of the matrix that maps camera vectors to room change vectors). This activity culminates in the students representing the problem context using a matrix transformation. During the third task, students are asked to reason about a new wing of the school (the East Wing; Figure 1b) based on a given matrix, rather than a given hallway diagram.



(a)

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}$$

(b)

Figure 1. Information about the West Wing (a) and East Wing (b) of Ida B. Wells High School

Participants, Data Sources, and Methods of Analysis

The study had two white male participants (who we refer to with pseudonyms Drew and Carson) at a predominantly minority public post-secondary institution in the Southeastern U.S. Drew and Carson were the only two volunteers satisfying the study constraints of their age being 18 and older and having taken an inquiry-oriented linear algebra course based on the IOLA curriculum. Both students had earned an “A” in the course. Our data sources for this analysis consist of video-recordings of a four-day paired teaching experiment (Steffe & Thompson, 2000), as well as any inscriptions created by participants or the teacher-researcher.

The paired-teaching experiment (PTE) consisted of four, 90-minute sessions. Each session was conducted and recorded on Microsoft teams and included two research team members. One was the teacher-researcher across all sessions; the other collected field notes and asked clarifying questions. Students were asked to think aloud and respond to questions regarding their work. Student work was captured by having students upload work, hold up their work to the screen, or work on a shared whiteboard. The research team debriefed after each interview session, noting mathematically significant aspects of student reasoning that emerged in the interview.

To identify students’ conceptual resources for reasoning about null spaces, our team began by first assigning two team members to review each session of the PTE and note themes in students’ mathematical reasoning that related to ideas about topics related to null spaces (including linear combinations of vectors, span, linear transformations, and solutions to homogeneous systems of linear equations as well as their applications in the problem context.) In this analysis, we focus on the final two days of the teaching experiment, when the definition of null spaces was formally introduced to students in relation to their prior work, and they were asked to extend their reasoning to a new context (one-directional hallways in relation to room locations in the East Wing of the high school, with information inscribed in a matrix rather than a map). Based on team members’ notes, videos were selectively transcribed, and we identified two broad themes that functioned as conceptual resources for reasoning about null spaces: closed loop reasoning and RREF reasoning.

Findings

Though the data presented in this analysis comes from the final day of the PTE, it is important to note that students developed important ways of reasoning about camera vectors, room change vectors, and relationships between them on the previous days. Namely, the students described the sets of camera vectors corresponding to possible paths a single person or 5 people could take from room A to room C, and from room C to room C. They identified the former sets as not closed under linear combinations, and they identified the latter sets as being closed under linear combinations (with vectors in these latter sets earning the designation of "closed loop" vectors). See Plaxco et al. (2021) for more detail on this. The students agreed that closed loop vectors left room populations unchanged and identified ways of expressing the relationship between camera vectors and room population change vectors. In this section, we highlight how closed loop reasoning emerged as the core conceptual resource for one student across a pair of questions, whereas linear dependence and RREF reasoning emerged as the core conceptual resource for the other student. (Though interviewed as a pair, the students rarely engaged in one another's reasoning so for clarity we discuss their reasoning in ways that are largely separate.)

Closed Loops as a Context-based Conceptual Resource

Early on the fourth day of the PTE, both students were then given the equation in Figure 2 and asked to interpret what it meant "for that four-tuple on the right-hand-side to be all zeros." Almost without hesitation, Drew responded, "there is no net change in the room population after all movement is complete" indicating a fluency between the two contexts of hallway movement and the matrix equation. The students were then asked if the set S of vectors **c** in R^5 such that A**c**=**0** (in correspondence with the matrix equation shown in Figure 2) is a subspace of R^5 (with the reminder that this would mean that if we sum of any two vectors in S the result must also be in S, and if we scale a vector in S the result must also be in S).

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Figure 2. Matrix equation given to students to reason about the null space

Drew initially argued that the sum of two vectors in S would be in S by stating "yes, because they both equal zero." Drew later elaborated:

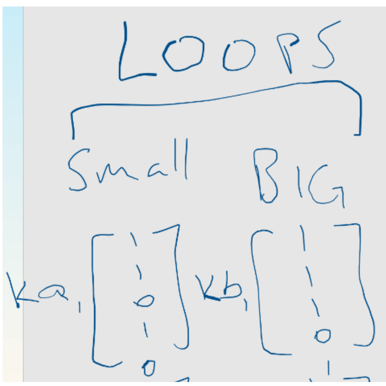
Drew: "Since both vectors are going to be equal to zero, the subspace will cover zero, so zero plus zero will still be in the subspace."

Interviewer: Are you saying that the camera vector has to be zero?

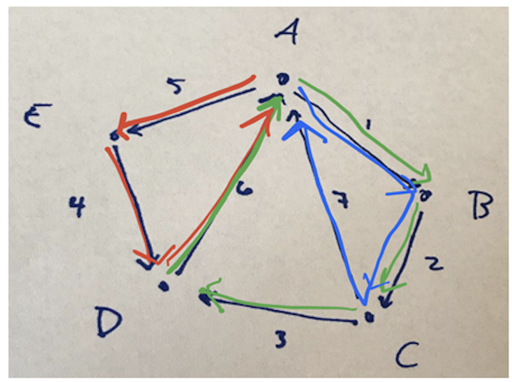
Drew: No, because they're both equal to zero after multiplying by the matrix. That's what we're saying… In order to get back to zero, we need to have a bunch of loops. That's how we get zero. If we keep adding more loops matter what and we just add the vectors of the loops together and we multiply it by that, it would still be zero, would it not? This is the way I'm seeing it. I'm saying 'cause like the way we get zeros is we just add a bunch of loops together. Uh, we just have a bunch of loops there and they're just constantly repeating. So if we had another vector, it's just another, butt ton of loops. We'd just have more loops which will get back to zero. The way I'm seeing it.

Drew later connected to the students' prior work describing paths from C to C:

> … it's like the linear combination that we did a while back, I think it's probably day one where it was like, like the big loops in the small loops and you could have as many of them as you wanted if you just continuously add more and more loops to it. It's still going to get you back to the same point. So, as long as we have loops like that one and the other one. It would, it would basically just be two different scalars… multiplied by those loops. And, that would be the combination of the vectors that we're adding together.



(a) Instructor's inscription of West Wing loops



<1,1,0,0,0,0,1> + <1,1,1,0,0,1,0> + <0,0,0,1,1,1,0>

(b) Drew's East Wing map and loop vectors

Figure 3. Inscriptions related to Drew's closed loop reasoning

As Drew spoke, the interviewer made the inscription shown in Figure 3a, as these corresponded to what students had described as the big and small loops on prior days of the PTE. The interviewer prompted, "if you just combine these two vectors together…." and Drew responded that we would "just have more loops." When asked by the interviewer if "more loops is also just a solution to this equation here," Drew again reverted to his hallway reasoning, "yeah, because they're just looping around." Drew similarly extended this line of reasoning when asked about scalar multiplication, saying in reference to figure 3a that, "if you multiply a_1 and b_1 by like ten each, it's just more loops."

When later presented with the matrix M in Figure 1b and asked to describe all the possible hallway flows that leave the number of students in each room unchanged, Drew generated the map shown in Figure 3b -- which is an appropriate translation of the matrix back into a hallway diagram -- and correctly identified three closed loop vectors (which mathematically speaking are a basis for the null space of the matrix M).

Drew's use of closed loops (or closed loop vectors) as a conceptual resource was characterized by four core ideas around which his reasoning was largely organized:

1. Closed loops are paths from a room back to itself
2. Closed loops leave room populations fixed
3. Linear combinations of closed loops are closed loops
4. Closed loop vectors are solutions to homogeneous equations

The first three ideas were developed on prior days of the PTE as indicated by Drew in his comments. However, Drew also connected these ideas to the matrix equation presented in Figure 3 - initially by connecting the zero vector on the right hand side of the equation to unchanging room populations, and then by further connecting these unchanging room populations to "closed loop" camera vectors (which sum and scale to closed loop vectors when scaling by non-negative integers). We argue that in this way, Drew's closed loop reasoning, as characterized and contextualized by this set of ideas, functioned as a robust conceptual resource for reasoning about null spaces in the hallways task sequence.

Multiple Solutions, Linear Dependence, and RREF as Formalized Conceptual Resources

In contrast with Drew, Carson did not leverage closed loops as a central conceptual resource in his reasoning on the tasks presented on the final day of the PTE. Rather, he drew heavily on more formal ideas from the linear algebra class but found that the hallways task setting required contextual reinterpretation of these ideas in ways that were not always straightforward.

When asked to reason about closure of the set S in relation to Figure 2, Carson did not immediately engage in the question set forth by the interviewer (namely, whether any two vectors that satisfy the given homogeneous equation would sum to a vector that also satisfies the homogeneous equation). Rather, he focused on interpreting the matrix equation, inquiring "So we're trying to solve the homogeneous equation, right?" After the interviewer restates his previous prompt (presumably to get Carson to focus on the closure aspect of the question), Carson continued. "There's a free variable, correct? … which means there is more than one solution to the homogeneous equation." Several minutes later, Carson seemed to continue this line of thought, saying "I have a question. If it wasn't linearly dependent, I don't think you'd be able to do it, be able to do loops and stuff. Is that correct?" The interviewer agreed that if the column vectors didn't form a linearly dependent set, then the only solution would be the trivial solution. The following exchange ensued.

Interviewer: Ok. So if those column vectors are linearly independent, then the loops would be what?

Drew: They wouldn't exist. They would be null, they wouldn't actually exist. They would get you a homogeneous answer.

Interviewer: Right. So the only way to get the classrooms to not change?

Drew: Would be to replace the c vector with zeros.

Interviewer: So what does that mean about student movement?

Carson: No one moves.

This episode highlights Carson's ideas about solutions to the homogeneous equations, free variables indicating multiple solutions, and linear dependence being important for the ability to "do loops." He then correctly reinterprets the trivial solution to the homogeneous vector equation to mean that no one moves in the hallway context.

When Carson is asked to find the hallways flows that leave room populations unchanged for the matrix shown in Figure 1b, he suggests "since it's dependent, find something that makes two of them zero. I make it, two of them, zero I guess? Try to find out the homogenous solution cause it wants to be unchanged… You could solve it using reduced row-echelon form is one way." Carson then used Maple to row reduce the matrix M. The conversation then turned to Drew's map (as discussed above) before the interviewer asked how Drew's loop vectors related to Carson's row reduced matrix. Drew argued that if you multiplied his vectors by Carson's matrix (excluding the final column of zeros), it should give you the zero vector, and verified it does. Carson commented that he “thought it would because it gives an absolute answer every single time. That’s why I like matrices. They give exact answers every time.” When asked if Drew’s closed loop vectors might be extracted from Carson RREF, Carson’s initial reaction was:

> It’s basically the idea of, there’s a free variable, right? So you could use any scalar to get to any other location that you wanted to by using scalars and adding them together. This is the part where I kind of get confused. I did all the steps, I just don’t know, I know each of these are column vectors that equal zero, zero, zero.

We interpret this comment to reflect Carson’s effort to brainstorm information he can glean from row-reduced matrices: the presence of free variables, information about span (e.g. places you can “get to”) -- but indicating that the connection between his row-reduced matrix and Drew’s closed loop vectors was not obvious.

We argue that Carson’s use of multiple solutions, linear dependence, and row reduction as conceptual resource was characterized by three core ideas around which his reasoning was largely organized:

1. Free variables indicate multiple solutions
2. Linear dependence is needed for (closed) loops
3. Row reducing matrices gives exact answers

Carson productively leveraged the first two ideas to reason about some aspects of null spaces, but the final p-prim did not provide sufficient detail to extract the desired camera vectors. Potentially complicating this issue is the fact that, in cases with infinitely many solutions, row reduction produces solution sets with free variables that can presumably range through all real numbers -- yet the hallways problem context requires limiting to solutions in which all entries are non-negative integers.

Discussion

The contextually developed conceptual resources organized around the idea of closed loops provided a productive entry point for several aspects of reasoning about null spaces (e.g. specifying elements, reasoning about closure, and identifying a basis), and both students had ideas related to closed loops. Previously formalized mathematical resources also offered useful insights into some aspects of null spaces, but were more useful in connecting to the hallways context at some times (e.g. connecting multiple solutions to linear dependence and closed loops) than at others (trying to extract camera vectors that leave room populations unchanged based on the RREF of a given matrix).

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References

Açikyildiz, G., & Kösa, T. (2021). Creating design principles of a learning environment for teaching vector spaces. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, *12*(1), 244-289.

Adiredja, A. P. (2019). Anti-deficit narratives: Engaging the politics of research on mathematical sense making. *Journal for Research in Mathematics Education*, *50*(4), 401-435.

Caglayan, G. (2019). Is it a subspace or not? Making sense of subspaces of vector spaces in a technology-assisted learning environment. *ZDM*, *51*(7), 1215-1237

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and instruction*, *10*(2-3), 105-225.

diSessa, A. A. (2018). A friendly introduction to "knowledge in pieces": Modeling types of knowledge and their roles in learning. In *Invited lectures from the 13th international congress on mathematical education* (pp. 65-84). Springer International Publishing.

Dogan, H. (2018). Mental schemes of linear algebra visual constructs. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and Strategies in teaching linear algebra* (pp. 219-239). Springer, Cham.

Dorier, J. L. (Ed.). (2000). *On the teaching of linear algebra* (Vol. 23). Springer Science & Business Media.

Freudenthal, H. (2006). *Revisiting mathematics education: China lectures* (Vol. 9). Springer Science & Business Media.

Grenier-Boley, N. (2014). Some issues about the introduction of first concepts in linear algebra during tutorial sessions at the beginning of university. *Educational Studies in Mathematics*, *87*(3), 439-461.

Parraguez, M., & Oktaç, A. (2010). Construction of the vector space concept from the viewpoint of APOS theory. *Linear algebra and its applications*, *432*(8), 2112-2124.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

Plaxco, D., Andrews-Larson, C. J., Smith, J., Kim, M., Mauntel, M., & Watford, M. (in press). Introducing an RME-based task sequence to support the guided reinvention of vector spaces. In *23rd Annual RUME Conference on Research in Undergraduate Mathematics Education*. MAA.

Stewart, S., Andrews-Larson, C., & Zandieh, M. (2019). Linear algebra teaching and learning: themes from recent research and evolving research priorities. *ZDM*, *51*(7), 1017-1030.

Taber, K. S. (2008). Conceptual resources for learning science: Issues of transience and grain-size in cognition and cognitive structure. *International Journal of Science Education*, *30*(8), 1027-1053.

van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. *Encyclopedia of mathematics education*, 713-717.

Wawro, M., Sweeney, G. F., & Rabin, J. M. (2011). Subspace in linear algebra: Investigating students' concept images and interactions with the formal definition. *Educational Studies in Mathematics*, *78*(1), 1-19.