Student Thinking in an Inquiry-Oriented Approach to Teaching Least Squares

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We present the results of a classroom teaching experiment for a recently designed unit for the Inquiry-Oriented Linear Algebra (IOLA) curriculum. The new unit addresses orthogonality and least squares using Realistic Mathematics Education design principles with the intent to implement the new unit in an IOI (Inquiry-Oriented Instruction)-style classroom. We present an analysis of students' written responses to characterize how they thought about the notion of shortest distance, travel vectors, orthogonality, and dot product in the "Meeting Gauss" context.

Keywords: Linear Algebra, Inquiry-Oriented Instruction, Least Squares Method

Least squares in linear algebra is often introduced as a method for finding the "best possible solution" when solving a system of linear equations, a vector equation, or a matrix equation that has no exact solution. As part of a larger research project involving designing new linear algebra task sequences for classrooms implementing inquiry-oriented instruction (IOI), we designed a task sequence, the "Meeting Gauss" unit, that would facilitate student exploration in \mathbb{R}^3 of the notion of "best possible solution" and lead to the reinvention of the approximation equation $A^T A \hat{x} = A^T b$, least squares solution \hat{x} , projection, and least squares error. In the Meeting Gauss task, students were asked if they could reach Gauss' location using three transportation vectors. Students determined they could not reach Gauss and were tasked with finding a location that is closest to Gauss' initial location. Gauss' location is represented by a vector, g, in \mathbb{R}^3 located off the plane that is the span of the transportation vectors. The shortest distance Gauss could travel corresponds to the magnitude of the orthogonal vector pointing from Gauss' location to the plane. We notate the shortest distance between Gauss and the meeting location as the length of e (error) to be minimized and the vector from the origin as e (Figure 1).

In this paper, we analyze two "snapshots" of student reasoning from the first and second day of a classroom teaching experiment (CTE) (Cobb, 2000) using students' reflection writings as data. Our research question is: *How do students interpret and use notions of shortest distance and travel vectors when learning least squares through an inquiry-oriented task in a linear algebra class?* We investigate various ways students interpreted finding the "shortest distance" (i.e., how students determined which direction Gauss should travel so that his trip is the shortest possible distance). Further, we characterize students' interpretations of the relationships (e.g., orthogonality, projection) that we perceive as important aspects for learning least squares.

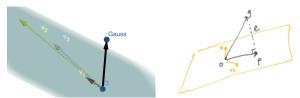


Figure 1. Meeting Gauss task setting in GeoGebra (Left) and Class instruction (Right)

Background Literature and Theoretical Framing

There is little literature on the teaching or learning of least squares or closely related topics within the context of linear algebra. Turgut (2013) created a lesson in which students used Mathematica to solve least squares problems involving finding lines or curves of best fit. He reflected on how students' use of these tasks incorporates Harel's (2000) concreteness, necessity, and generalizability principles. Donevska-Todorova (2015) used Hillel's (2000) three modes of description (arithmetic, geometric, and axiomatic) to explicate three definitions of dot product. Arithmetic referred to multiplying vector components and adding. Geometric referred to the cosine definition and the axiomatic definition was in terms of general properties. She also designed an applet to support students' understanding of dot product of vectors in a dynamic geometry environment (DGE). Cooley et al. (2014) created a module for teaching dot product using the cosine of the angle between two vectors. Their task sequence focused on comparing frequency vectors to determine whether or not the same author wrote two different texts.

Our instructional materials were designed to support students' ideas and support instructors in facilitating conversations around students' ideas. Students make mathematical progress as they participate in class and group discussion, ask questions, and explain their ideas. Instructors guide classroom activity by encouraging students to share their thoughts, asking their thinking about how and why they make decisions, and leveraging their ideas to move forward. Specifically, for developing our instructional sequence, we adopted the instructional design heuristics of Realistic Mathematics Education (RME) informed by Freudenthal (1991). When designing instructional task sequences, we leveraged the notion of guided reinvention to support students' transition towards more formal mathematics (Gravemeijer, 1999). From the students' perspective they are not re-inventing anything, our intent with the least squares task sequence is to support students in advancing their mathematical activity through symbolizing, algorithmatizing, and defining (Rasmussen et al, 2005). We designed the "Meeting Gauss" task in the least squares instructional sequence to be an experientially real starting point. This is similar to the Magic Carpet task in the first unit of the IOLA curriculum (Wawro et al., 2012; Wawro, Zandieh, et al., 2013). Students could engage in mathematical activity immediately, and their initial activity should constitute a basis for more formal mathematization.

Rasmussen and Keene (2019) used a river journey metaphor to capture what many in the mathematics education research field know as hypothetical learning trajectories (Simon, 1995). We use the river journey metaphor intentionally to move away from the image of a learning trajectory as a singular path that a researcher hypothesizes as the only way to learn a particular idea. We agree with Rasmussen and Keene that the image of a learning trajectory as the only path to learn a mathematical idea is not representative of the learning process. Further, Rasmussen and Keene utilized the notion of waypoints (Corcoran, Mosher, & Rogat, 2009) from the learning progression literature as "islets" within their river journey metaphor, acknowledging that students may, or may not, visit each of the islets (waypoints) outlined in the river journey (learning progression). To be clear, Rasmussen and Keene's waypoint journey for student reasoning about ODEs is a hypothetical research tool stemming from seven semester-long classroom teaching experiments and years of design-based research. The waypoint journey presented here is more of a "first journey down the river" in the context of a new task being used in an Inquiry-Oriented Linear Algebra (IOLA) classroom for the first time.

Methodology and Research Setting

This study is a part of a larger linear algebra curricular development project funded by the NSF. We conducted a classroom teaching experiment (CTE) to study the first implementation of

our instructional sequence and to study how students' reasoning evolves throughout the task sequence (Cobb, 2000). The tasks implemented in the CTE used an *experientially real* situation designed to help students *reinvent* the least squares method. The CTE was conducted with STEM students in two Linear Algebra classes at a large public university of the Southeastern United States. Due to COVID-19, the course was taught synchronously online via Zoom. The course prerequisite a B or better in Calculus I or passing Calculus II. There were 33 students enrolled in one class and 38 students in the other. The CTE lasted four consecutive class days over the course of two weeks towards the end of semester. The fourth author was the course instructor and last author acted as a TA throughout the semester. The breakout group composition was kept as consistent as possible unless student absences necessitated the rearrangement of groups. In total, 22 students consented for their work to be used in research.

Description of the Task

The Meeting Gauss task begins with a callback to a previous IOLA task from earlier in the semester¹. The vectors v_1 , v_2 and v_3 are presented as three modes of transportation, Gauss lives at a location g in 3D, and students are trying to determine where they can meet Gauss. The Meeting Gauss task is comprised of two parts (Figure 2): (1) investigate if it is possible to reach Gauss at his specific location using the three modes of transportation, (2) explore the trips that you and Gauss should make given the fact that Gauss needs to meet you somewhere you can reach because his house (g) is not reachable using the given modes of transportation (i.e., the travel vectors).

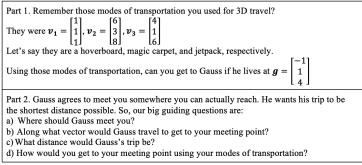


Figure 2. Statement of the Meeting Gauss task

On Day 1 of the CTE, students discussed their ideas related to Part 1 in breakout rooms. At the end of class, students were asked to write a reflection giving their initial thoughts, intuitions, or ideas for questions a)-d) of Part 2. On Day 2, the instructor began class by incorporating many of the students' reflection writings into her mini-lecture. The instructor highlighted students' reflections that incorporated Gauss' shortest trip distance and the span of the travel vectors. In the first breakout session of Day 2, students used two previously created GeoGebra applets to further explore the second part of the Meeting Gauss task. Many of the breakout groups used the applets to obtain estimates for Part 2, questions a)-d). Afterwards, students were called back to the whole class session to share their ideas with other groups and to listen to a set of minilectures about lengths of vectors and dot products. In addition to the given transportation vectors v_1, v_2, v_3 , and the Gauss vector g from Part 1, the instructor defined the vector Gauss travels

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¹ In particular, Task 3 from the "Magic Carpet Ride" unit, which introduced linear independence (Wawro et al., 2012).

along as e and the vector from the origin to the meeting point as e (as pictured in Figure 1). In the second breakout session, students were tasked with using the given information and mathematical relationships developed during class (see Figure 3) to provide exact solutions to a)-d). After Day 2, students wrote a reflection summarizing the progress either they or their group made in finding the exact answers for questions a)-d) along with something that was clear and something that they wondered about.

$$V_{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X_{1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + X_{2} \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix} = \bar{p}$$

$$V_{2} = \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix}$$

$$V_{2} = \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix}$$

$$V_{3} \cdot e = 0$$

$$V_{2} \cdot e = 0$$

$$V_{3} \cdot e = 0$$

$$V_{4} \cdot e = 0$$

$$V_{2} \cdot e = 0$$

$$V_{3} \cdot e = 0$$

$$V_{4} \cdot e = \bar{q}$$

Figure 3. The known information and mathematical relationships developed during class.

Data Sources and Analytic Method

Our data sources include classroom videos, breakout group videos, Jamboard pages, and students' reflections. The whole class discussions and the breakout sessions were recorded via Zoom by the last two authors. The students used Jamboard to communicate and record their work in breakout rooms. Students' reflection writings submitted after Day 1 and Day 2 of the CTE were our main data source. The first two authors watched the whole class discussions and breakout group videos to gain a sense of how class was organized. The first two authors reviewed the Jamboard pages to see how students expressed their mathematical ideas during the breakout discussion. Consenting students were given pseudonyms, and their reflection writings were deidentified and transcribed by the last author. The first two authors engaged in open coding (Strauss & Corbin, 1990) of all four days of student reflections. A code book was created from attending to both (1) student thinking and (2) their descriptions of their methods/symbolization. We used our set of 26 initial codes to analyze students' Day 1 and Day 2 reflections line by line in a spreadsheet. After our initial pass at coding students' Day 1 and Day 2 reflections we identified several categories that emerged within and across our codes. Figure 4 displays our final categories and codes, which were agreed upon by the first three authors. Our process of creating an initial set of codes that identify trends within and across our data is consistent with an emergent coding method (Glaser & Strauss, 2017).

Results

Our findings can be thought of as two snapshots on the river journey. The first snapshot captured students' reflections at the end of Day 1. From their reflections, students were beginning to recognize a number of features of the Meeting Gauss scenario and how those are related. Day 1 reflections are grouped into two themes: (1) shortest distance and (2) the location of the transportation vectors. Students wrote about shortest distance as (1) a point-to-point trip, (2) an orthogonal/perpendicular direction, and as (3) ways to find the distance including the distance formula, the Pythagorean theorem, and trial and error. The second snapshot captures students' reflections at the end of Day 2. From their second reflection, students had moved down the river in various ways. We grouped their writing into (1) a more developed notion of the error vector, e, as perpendicular to the plane and transportation vectors, (2) the dot product as providing information about orthogonality and ways to calculate relationships between vectors,

and (3) the use of matrix multiplication and systems of linear equations (SLEs) to symbolize the relationships between the perpendicular vectors in ways that allow for finding the error vector.

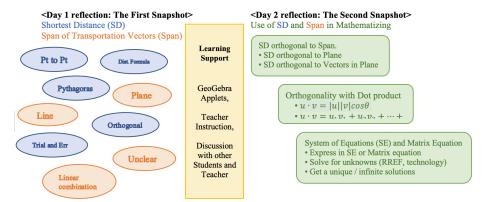


Figure 4. Overview of codes from students' Day 1 and Day 2 of the CTE reflection writing.

First Day of the CTE Snapshot Analysis

On Day 1, most students used row reduction to conclude that Gauss is not reachable using the transportation vectors (see Figure 5). Some students included geometric explanations in their reflections including descriptions of how the span of the transportation vectors was a plane.

Figure 5. A snippet of a group's Jamboard from working on Part 1 of the Meeting Gauss task.

Most students recognized that Gauss can travel around freely to meet the traveler, but the traveler's movement is restricted to use of the given transportation vectors. We will now present examples from students' reflections demonstrating our codes pictured in Figure 4.

Interpretation of shortest distance. Our analysis revealed that students' intuitive notions of Gauss' shortest travel distance included ideas such as *point-to-point distance* and *orthogonality*. Ten students answered that they needed to determine the shortest distance between two points: Gauss' location and the point on the plane closest to Gauss. For example, "We can *minimize the distance* from the *point to a point*," (Cecil), and "Gauss should take *the shortest distance from his point* (-1, 1, 4) *to the closest point on the plane* our transportation modes cover," (Iliana). Some of the students' thinking of the *point-to-point distance* indicated that they would apply the distance formula or a calculus-based methods to minimize the distance. Five students noted that Gauss's path is perpendicular to the plane. Some of those students also noted that the span of the travel vectors is the plane. For example, "If I had to imagine, Gauss would *travel in a straight line orthogonal to the plane* formed by the three vectors," (Liam) and "...the vector he travels along should be perpendicular to the final mode of transportation's vector" (Luka).

Some students mentioned how they would calculate the shortest distance. We view each of the following methods used for measuring the shortest distance as related to *point-to-point* distance thinking: distance formula, Pythagorean theorem, and trial and error. Three students explicitly mentioned that they would apply or did apply the distance formula. For example, "Not sure how to calculate distance, maybe plot the vectors and then use distance formula," (Damien). Tuan did apply the distance formula "Distance: $sqrt((-1-(-1))^2+(-1-(-2))^2+(-1-(-5))^2) =$

sqrt(0+1+16) = sqrt(17)," There were two students who mentioned they would use the Pythagorean theorem to find the shortest distance. For instance, "We could use *pythagroas theorem to find the distance* of the projection," (Anton).

Interpretation of transportation vectors. Our analysis further revealed that students thought about the transportation vectors in various ways. Some students tended explicitly to the notion of span such as *vectors generate the plane* or *vectors generate a line*. While 13 students mentioned the transportation vectors generate the plane, not every student explicitly mentioned where the plane came from. Other students explicitly used the terms span or linear combination in their descriptions. For example, "The way I am thinking about this is to locate *a spot that is within the area that the 3 vectors span*," (Ani), "Since Gauss is getting to *the plane that is the span of the three modes of transportation*, a *linear combination* of the three vectors will get us there as well," (Lureyna). Lureyna's response tends specifically to both the idea that the span of the transportation vectors creates the plane, and the traveler must use those modes of transportation to reach the meeting point with Gauss.

Second Day of the CTE Snapshot Analysis

In the following three subsections, we briefly outline the various sections of the river that students went down in the second snapshot from Day 2 of the CTE. Each subsection corresponds to one of three groupings of students' responses outlined in Figure 4.

The 'e' (error) vector is orthogonal to the plane. Our analysis revealed that students' description of orthogonality between the vector e and the plane is multifaceted. We categorized their descriptions into two ways: (1) The e vector is orthogonal to the plane, (2) The e vector is orthogonal to the vectors that constitute the plane. For example, "...We knew that the line of travel from Gauss to the plane must be orthogonal to the plane..." (Bianca). From Bianca's response, we see that she made a connection between the plane and the orthogonal path to said plane. On the other hand, some students stated the e vector is orthogonal all vectors parallel to the plane. For instance, "I know that e has to be orthogonal to p and v1, v2," (Annalisa).



Figure 6. Lucia's pictures from her CTE Day 2 reflection

Dot product and orthogonality. After students returned from the first breakout group session, the instructor gave a set of mini-lectures deriving the lengths of vectors and defining dot product. Students' written descriptions about how they thought about dot product was subtle and sometimes difficult to characterize. For example, Lucia stated (Figure 6, right), "The vectors p and e create vector g, from origin to Gauss, and the dot product of p and e is zero because they are perpendicular,". Lucia seemed aware that orthogonality is related to dot product. She also mentioned "We had tried to combine the statements $p \cdot e = 0$, $e \cdot v \cdot l = 0$, and $e \cdot v \cdot 2 = 0$ into one big equation, and I think you can eliminate the ||e|| from all sides, but the issue is that I do not know how to get the angles between the different vectors and such," (Lucia). From Lucia's complete response, it was possible to determine she was leveraging the cosine definition of the dot product in her thinking. Further, some students expressed the orthogonal relationship between the error vector (e) into a system of dot product equations (Figure 7, left). Other students mentioned that

they knew they could use the dot product equations in some manner, but it was unclear if they were thinking about orthogonality. There were some students who seemed to try connecting dot product with the transportation vectors and the e vector but were not comfortable with the transition towards a more formal mathematical expression. For example, "…looked at *plugging* in the v1 and v2 vectors into their dot product equations with e …using the coefficients we found as a matrix, whose dot product with e was 0. This didn't make much sense to me…" (Iliana).

Systems of equations and the matrix equation. Some students appeared to be moving towards formalizing the notion that the e (error) vector was orthogonal to each travel vector to create a matrix equation $A^Te = 0$ using their system of dot product equations. For example, "From the dot product of e and e and

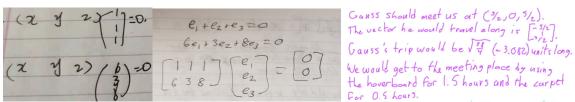


Figure 7. Min's(Left), Cecil's (Middle), Vaki's (Right) written solution.

Discussion

In the first journey down the river of developing the least squares solution method, we see that some of the students leveraged their intuitive notions of shortest distance and the span of the transportation vectors towards a more formal symbolization using the dot product. Students traveled down the river in their own way, spending different amounts of time on each piece of the river journey as seen in the variety of answers within both snapshots. One takeaway is that the least squares river journey may require more time in more areas than others. Vaki was one of the only students to interpret \hat{x} as the amounts to travel on each mode of transportation to meet Gauss. This meaning of \hat{x} is what we as experts recognize as the least squares solution to the Meeting Gauss context. This paper is limited to an analysis of the first two days of the CTE. Future work may investigate students' progress on the final two days in formalizing their thinking towards reinventing the least squares solution $\hat{x} = (A^T A)^{-1} A^T b$. On our next journey down the least squares river, we intend to reflect on where students ended at the end of each day of the CTE to inform future revisions of the Meeting Gauss context within the IOLA curriculum.

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