

Winning the War by (Strategically) Losing Battles: Settling the Complexity of Grundy-Values in Undirected Geography

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Abstract—We settle two long-standing complexity-theoretical questions—open since 1981 and 1993—in combinatorial game theory (CGT).

We prove that the *Grundy value* of UNDIRECTED GEOGRAPHY is PSPACE-complete to compute. This exhibits a stark contrast with a result from 1993 that UNDIRECTED GEOGRAPHY is *polynomial-time solvable*. By distilling to a simple reduction, our proof further establishes a *dichotomy theorem*, providing a sharp “phase transition to intractability”: The Grundy value of the game over any degree-three graph is polynomial-time computable, but over degree-four graphs—even when planar & bipartite—is PSPACE-hard. Additionally, we show, for the first time, how to construct UNDIRECTED GEOGRAPHY instances with Grundy value $\ast n$ and size polynomial in n .

We strengthen a result from 1981 showing that sums of tractable partisan games are PSPACE-complete in two fundamental ways. First, we extend the result to impartial games, a strict subset of partisan. Second, the 1981 construction is not built from a natural ruleset, instead using a long sum of tailored short-depth game positions. We use the sum of two UNDIRECTED GEOGRAPHY positions. Our result also has computational ramification to Sprague-Grundy Theory (1930s) which shows that the Grundy value of the *disjunctive sum* of any two impartial games can be computed—in polynomial time—from their Grundy values. In contrast, we prove that, assuming PSPACE is not equal to P, there is no general polynomial-time method to summarize two polynomial-time solvable impartial games to efficiently solve their disjunctive sum.

Our proof enables us to answer another long-term structural question in the field. We establish the following complexity independence: Unless $P = PSPACE$, there is no polynomial-time reduction from winnability in misère-play setting to the Grundy value, and vice versa (in UNDIRECTED GEOGRAPHY).

I. INTRODUCTION

Knowing how to win battles does not always translate into knowing how to win wars. More often than not, the victor must strategically lose some winnable battles in order to win the war. This timeless principle is elegantly captured in the celebrated Sprague-Grundy Theory [31], [18] from the 1930’s about impartial games. This theory introduces the concept of the *Grundy value* to characterize the winning strategy for the *disjunctive sum* of multiple “battlefield” games by extending Bouton’s constructive theory [6] on

NIM¹. In this paper, we resolve a long-standing complexity-theoretical question in combinatorial game theory (CGT)—open since 1981—concerning the *computational complexity of strategic losing* for the goal of winning the overall sum game (the disjunctive sum). As the main technical result of this paper, we settle another question in CGT—open since 1993—on the *complexity barrier* of Grundy values of a well-studied impartial graph-theoretical ruleset. Our theoretical work has also inspired new practical board games.

A. Games of Games: Disjunctive Sum

A combinatorial game is defined by a succinct *ruleset*, specifying the domain of *game positions*, and for each position, the set of *feasible options* each player can move the game to [3]. A ruleset is *impartial* if both players have the same options at every position. Games that aren’t impartial are known as *partisan*. In the *normal-play* setting, two players take turns advancing the game, and the player who is forced to start their turn on a *terminal position*—a position with no feasible options—loses the game. We combine the challenges of deciding the *winnability* and selecting a *winning move* (whenever one is available) into one term: *strategic determination*. (See also [15] for integrating the two tasks). If the player with a winning strategy can consistently solve the strategic determination problem, then they can play the game optimally and win.

For computational analysis, a *size* is associated with each game position—e.g. of bits encoding NIM—as the basis for measuring complexity [29], [15], [7]. The size measure is assumed to be *natural*² with respect to the key components of the ruleset. In particular, at each position with size n : (1) the space of feasible options can be identified in time polynomial in n , and (2) all positions reachable from the

¹A NIM game starts with a collection of piles of items. Two (or multiple players) takes turns picking at least one items from one of the piles. Under normal play, the player taking the last items wins the game. NIM was known in ancient China as Jian Shi Zi (picking pebbles).

²In other words, the naturalness assumption rules out rulesets with embedded hard-to-compute predicate like—as a slightly dramatized illustration—“If Riemann hypothesis is true, then the feasible options of a position include removing an item from the last NIM pile.”

position have size upper-bounded by a polynomial function in n . An impartial ruleset is said to be *polynomial-time solvable*—or simply, *tractable*—if there is a polynomial-time algorithm for its strategic determination. Multiple games can be combined into a new game:

Definition I.1 (Disjunctive Sum). *For any two games G and H (respectively, of rulesets R_1 and R_2), their disjunctive sum, $G + H$, is a game in which the next player chooses to make a move in exactly one of G and H , leaving the other alone. A sum game $G + H$ is terminal if and only if both G and H are terminal according to their own rulesets.*

B. Computational Questions About Sprague-Grundy Theory

In 1981, Morris [26] proved that the sum of tractable *partisan* games can be PSPACE-hard. His theorem elegantly encapsulates the fundamental intricacy of strategic interaction among (simple) games (even for introductory textbooks to the field [11]). Morris generates lists of individually-tractable partisan games that combine to create an intractable sum. It serves as a starting point for other versions, such as limiting the games to be depth 2 [35] and limiting the number of branches for each game to 3 [23]. A further adaptation is made to show that GO ENDGAMES are PSPACE-hard [33]. Morris’ theorem provides a framework for understanding other families of games. One important basic question has remained open since Morris’ construction:

Open Question 1 (Sum of Impartial Games). *Can the disjunctive sum of two (or more) polynomial-time solvable impartial games become intractable?*

This open question is fundamentally connected with Sprague-Grundy theory, a seminal part of CGT. Formulated in the 1930s, this theory provides a mathematical characterization for impartial games and their sums [31], [18], laying the foundation for modern CGT [11], [3]. We now highlight two fundamental aspects of this beautiful theory (and provide the background of our own work):

Concise Mathematical Summary of Impartial Games: Playing combinatorial games optimally usually requires deep strategic reasoning about long alternation down the last level of their *game trees*. Remarkably, Sprague and Grundy showed that the essence of every impartial game can be distilled into an “equivalent” single-pile NIM game. Its *Grundy value* (a.k.a. *nim-value* or *nimber*) is then the number of items in the equivalent single-pile NIM. The nim-value extends *winnability*: the current player has a winning strategy if and only if the Grundy value is not equal to zero. To win, it is sufficient to choose any feasible option with value zero. The Grundy value of a game provides a *succinct mathematical summary* of its game tree, whose size could be *exponential* in the number of options: the Grundy value is always *bounded above* by the number of options.

Systematic Framework for Combining Games: Sprague and Grundy’s pioneering theory establishes a systematic framework not only for combining games across different rulesets, but also for a complete characterization of strategic interaction among games in the overall sum, based on their concise summaries [3], [11]. Combined with Bouton’s theory on NIM [6], letting $\oplus$ denote the *bitwise xor* (the *nim-sum*), the theory establishes: for all impartial G and H :

$$\text{nimber}(G + H) = \text{nimber}(G) \oplus \text{nimber}(H) \quad (1)$$

In general, losing a winnable game may be necessary to win in the overall sum (true to the meaning of “losing a battle but winning the war”). Sprague Grundy theory contains a profound computational statement, made prior to the inception of P vs NP. Because the nim-sum is linear-time computable, if the Grundy values of the games are tractable, then the Grundy value—hence the winnability—of the overall sum game is also tractable. This contrasts with some values in partisan games, as exploited by Morris [26], where he constructs a CGT representation for each component, but proves one is unable to “add them up” in polynomial time (unless P = PSPACE). The following open question is intrinsically connected with Open Question 1:

Open Question 2. *Can the information captured in the Grundy value of an impartial game be more expensive to compute than its strategic determination?*

Our research has been influenced by the following two tightly related yet subtly different formulations concerning the algorithmic connection between Grundy values and strategic determination.

Open Question 3 (Tractable Structures). *For any rule-set, does polynomial-time strategic determination imply polynomial-time Grundy-value computation?*

Open Question 4 (Efficient Reduction). *Is there a general polynomial-time reduction from Grundy-value computation to strategic determination?*

Questions 3 and 4 are directly related - a YES answer to the second affirms the first (and thus a NO answer to the first also refutes the second.) An efficient solution for these two would also provide a unified algorithm—based on Sprague-Grundy theory—for efficiently solving the disjunctive sum of tractable impartial games (and hence Open Question 1).

On the tractable spectrum, some polynomial-time solvable rulesets—including NIM, SUBTRACTION GAME [3], and many others—have *dual tractability*: their Grundy values are also polynomial-time computable. Open Question 3 focuses on whether strategic determination and Grundy values have common underlying mathematical structures for tractability beyond the fact that both can be obtained by evaluating the game tree [3]. This is relevant to the part of Fraenkel’s work [15], where he conceptualized a class called *games*

with an efficient strategy by combining the tractability of their own strategic determination with the tractability of their involvements in disjunctive sums and in misère-play (that is, the current player wins at any terminal position).

On the intractable spectrum, for any PSPACE-complete game with polynomial game-tree height—e.g., NODE-KAYLES [29], GENERALIZED GEOGRAPHY [29], [20], COL [11], [2] and many based on logic, topology, network sciences, *etc.*, [29], [10], [30], [17], [8]—the answer to Open Question 4 is always a YES. However, this complexity-theoretical polynomial-time reduction is not extendable from PSPACE-complete games to games with potentially lower complexity. In addition to tractable games, it remains open whether a polynomial-time reduction from Grundy values to strategic determination exists for intractable impartial games, whose complexity might be “strictly” in-between NP and PSPACE.³ Open Question 4 hypothesizes whether a unified algorithmic approach exists for Grundy-values using winnability testing & winning-move finding as subroutines.

As Fraenkel pointed out, winnability alone may not capture the whole picture of game’s tractability [15]. Recent progress on Poset games [30]⁴ highlights that aspect. It is well-known—by *strategy-stealing*—that the first player has a winning strategy in any POSET game where the underlying poset has a *greatest element* (e.g. in CHOMP), providing a straightforward answer to winnability. On the other hand, Bodwin and Grossman [4] prove that in this family, finding a winning move can be PSPACE-complete. Hence even in this special case, the Grundy-value and strategic determination are polynomial-time reducible to each other. The implication on the nimber-winnability complexity separation also has a caveat. POSET games with the greatest element may have reachable game positions without the greatest element: playing these special games requires later moves on “normal” POSET games, outside the greatest-element family. Indeed, Grier [17] proves that deciding winnability of normal POSET games is PSPACE-complete.

C. Battles of Geography Without Directions

As a game version of the *Seven Bridges of Königsberg*, GEOGRAPHY grew from a real-world “Word Chain” game—with cities as the category—into an abstract game on graphs, as suggested by Richard Karp [29]. This game, known as GENERALIZED GEOGRAPHY, became the main subject for complexity study in the landmark paper, “GO is polynomial-space hard” (1978) by Lichtenstein and Sipser [20]. In this

impartial game, a position is defined by a directed graph and a specified node (with the token). During the game, two players take turns moving the token to an outgoing neighbor and removing the node it just occupied. In the normal-play setting, the player who cannot make a move on their turn loses the game. GENERALIZED GEOGRAPHY was originally shown to be PSPACE-complete by Schaefer [29]; this was improved by Lichtenstein and Sipser [20] to be PSPACE-complete, even when the graph is planar, bipartite, and has a maximum degree of three. These properties are essential to their analysis of GO, whose game board is a 2D grid.

In 1993, Fraenkel, Scheinerman, and Ullman [16] added a new twist. They proved that UNDIRECTED (VERTEX) GEOGRAPHY—the special case over undirected graphs—is polynomial-time solvable. In 2015, Renault and Schmidt [28] revitalized interest in UNDIRECTED GEOGRAPHY by showing that it’s PSPACE-complete under misère-play instead of normal play. Both the edge variant [16] and short version [24] of UNDIRECTED GEOGRAPHY are also shown to be PSPACE-complete. Various extensions to UNDIRECTED GEOGRAPHY have been analyzed [32], [5], [22].

The Fraenkel-Scheinerman-Ullman solution is guided by an elegant matching theory and supported by efficient matching algorithms. For any $G = (V, E)$ and $s \in V$ satisfying $E \neq \emptyset$, the current player has a winning strategy at UNDIRECTED GEOGRAPHY position (G, s) iff s is in every maximum matching of G . However, this matching-based characterization appears to be limited to winnability. *Whether or not the Grundy value of UNDIRECTED GEOGRAPHY is polynomial-time computable had been elusive.*

Open Question 5. *Is the Grundy value for UNDIRECTED GEOGRAPHY computable in polynomial-time?*

UNDIRECTED GEOGRAPHY has thus become an exemplary tractable impartial game for which no efficient algorithm had been discovered for its Grundy-value computation. Others such as MOORE’S NIM [25], [3] and WYTHOFF’S GAME [34] are also wonderful examples [15].

Viewing Renault-Schmidt’s result [28] through Fraenkel’s framework [15] signifies the following structural question:

Open Question 6 (Complexity Independence). *Are there subfamilies of games (in UNDIRECTED GEOGRAPHY) for which the Grundy values are polynomial-time computable, but misère-Winnability is not, and vice versa?*

D. Our Contributions

In this paper, we settle these open questions.

A Dichotomy Theorem on Grundy Values: As our main technical result, we prove that computing the Grundy value of UNDIRECTED GEOGRAPHY is PSPACE-complete. The key step is to *impose a direction* over undirected edges, where game paths can travel across in either direction. The

³Is there a polynomial-time nimber-to-winnability reduction for impartial games—arising in quantum combinatorial game theory [7]—for which strategic determination is complete for a particular level of the polynomial-time hierarchy?

⁴A POSET game is a two-player impartial game over a *partially ordered set* (poset), in which each move—the selection of an element in the poset—removes it together with all elements that are greater. The POSET game generalizes the classical “chocolate-eating” game CHOMP [36] as well as NIM. A poset with the greatest element is a poset that contains an element greater than any other element in the poset.

complexity analysis has another intricacy - the polynomial-winnability is an obstacle. By distilling our original complex construction into a simpler reduction using Lichtenstein-Sipser, we are able to establish a *dichotomy theorem*—in Section III—providing a “phase transition to intractability” in Grundy-value computation, sharply characterized by a maximum degree of four:

Theorem I.1 (Geographical Dichotomy). *The Grundy value of UNDIRECTED GEOGRAPHY over degree-three graphs is polynomial-time computable but over degree-four graphs—even when planar & bipartite—is PSPACE-hard.*

Our polynomial-time algorithm in Theorem I.1 applies the Fraenkel-Scheinerman-Ullman algorithm [16] to navigate a branch-and-bound process for evaluating game trees.

Strategic Losing is Hard: We show distinguishing $*$ from $*2$ in UNDIRECTED GEOGRAPHY is PSPACE-hard, a detail crucial in our next theorem that settles Open Question I:

Theorem I.2 (Strategic Synergy). *The disjunctive sum of two tractable impartial games—e.g., two UNDIRECTED GEOGRAPHY games—can be PSPACE-hard to solve.*

Our result strengthens Morris’ 1981 result [26] in two fundamental aspects. First, we extend the PSPACE-hardness from the sum of partisan games to the sum of impartial games, shedding new light on the computational facet of the Sprague-Grundy characterization (more below). Second, Morris’ construction is built from a long sum of tailored short-depth game positions. In contrast, we use two games of a natural, well-studied UNDIRECTED GEOGRAPHY impartial ruleset to create our hard sum. Our construction is in fact robust: In our PSPACE-hard sum, one of the games—provided with non-zero Grundy value—can even be arbitrarily chosen, say by an adversary.

Sprague-Grundy theory presents a barrier to closely mimicking Morris’ construction in the realm of impartial games. Since nim sums are efficiently computable, one cannot present a long list of shallow impartial games where the winnability of the sum is intractable. We overcame this obstacle by instead summing two positions where the individual Grundy values (of at least one) are difficult to discern. We are curious whether there is a well-known, tractable, strictly-partisan ruleset where determining the winnability of the sum of two positions is computationally hard.

Mathematical-Computational Divergence in Sprague-Grundy Theory: Our complexity result on this concrete graph game has a wider computational ramification in connection with Sprague-Grundy Theory. The sharp contrast between the complexity of strategic determination and Grundy values in UNDIRECTED GEOGRAPHY illustrates a fundamental mathematical-computational divergence in Sprague-Grundy theory. When computational cost is no object, the Grundy values are *effective and concise* math-

ematical summaries of game trees for strategic reasoning in disjunctive sums. However, as we have shown, this elegant mathematical summary could be PSPACE-hard to obtain, even for polynomial-time solvable games. That is, assuming $\text{PSPACE} \neq \text{P}$, the Grundy values of combinatorial games capture provably richer and potentially hard-to-compute structures than just their solvability. In fact, Theorem I.2 implies a broader impossibility statement:

Theorem I.3 (Succinct Summarization is Hard). *Unless $\text{P} = \text{PSPACE}$, there is no general polynomial-time method to summarize two polynomial-time solvable impartial games to efficiently solve their disjunctive sum.*

Complexity Independence of Misère-Winnability and Grundy Values: For any impartial game, its normal-play winnability is polynomial-time computable from its Grundy value. In contrast, answering Open Question 6, we prove: Unless $\text{P} = \text{PSPACE}$, there is no polynomial-time reduction from winnability in misère-play setting to the Grundy value, and vice versa (in Undirected Geography).

Towards Practical Board Games: In Section IV, we apply Sprague-Grundy theory to resolve the complexity of several rulesets that generalize UNDIRECTED GEOGRAPHY. We first show very basic extensions, including MULTI-TOKEN UNDIRECTED GEOGRAPHY and UNDIRECTED GEOGRAPHY WITH PASSES. Then we demonstrate the versatility of the result, by showing that UNO SWAP, a minor modification of the tractable UNCOOPERATIVE UNO [12], is PSPACE-complete. These results have potential applications to the practical design of board games based on UNDIRECTED GEOGRAPHY, where the real world appreciates games with simple rules and positions, combined with deep strategic reasoning for winning moves [3], [8]. Thus, the removal of edge directions from GENERALIZED GEOGRAPHY, while also retaining its PSPACE-hard complexity opens up several possibilities. In the full paper on the archive [9], we discuss two further practical extensions using the standard GO or HEX game boards. For example, the web-version of one of our new games, BINARY UNDIRECTED GEOGRAPHY, can be played at <https://turing.plymouth.edu/~kgb1013/DB/combGames/twoBUG.html>.

Graphs with Polynomial-High Nimbers: In Section V, we give a constructive proof that, for any n , there exists a polynomial-sized UNDIRECTED GEOGRAPHY instance with Grundy value n . Logarithmic Grundy values are realizable by trees with recursive structures, and linear Grundy values can be achieved by directed graphs in GENERALIZED GEOGRAPHY. To the best of our knowledge, this is the first polynomial Grundy value construction for UNDIRECTED GEOGRAPHY, which also provides the technical support for proving Theorem III.2 that any classifier for *positive* Grundy values in UNDIRECTED GEOGRAPHY is PSPACE-hard.

II. THE VALUE OF GAMES BEYOND WINNING

In combinatorial game theory, a ruleset defines not just a single game, but many—possibly infinitely many—game instances (or *positions*). Playing games requires strategic reasoning of one’s own options as well as opponent’s subsequent options, to answer the key problem on *winnability*:

Definition II.1 (Strategic Determination). *Given a game G under a ruleset R , determine whether or not the current player in G has a winning option, and if YES, return a winning option of G .*

This fundamental problem—commonly involving *deep* alternation—has been the subject of intense mathematical and computational studies [3], [29], [14], [13], [27], [10], [20], [2]. A ruleset R defines a natural *game tree*, capturing this alternation for each of its positions, G , by recursively branching with feasible options. Thus, the game tree of G contains all *reachable positions* of G under ruleset R , with the leaves as the terminal positions.

The foundational Sprague-Grundy theory characterizes each impartial game G by a natural number, known as the *Grundy value* (a.k.a *nim-value* or *number*) of the game. Recursively, the Grundy value of G is:

Terminal Position: For any terminal G , $\text{number}(G) = 0$.

Non-Terminal Position: If $\{G_1, \dots, G_\Delta\}$ is the set of feasible options of G , then:

$$\text{number}(G) = \text{mex}(\{\text{number}(G_1), \dots, \text{number}(G_\Delta)\}) \quad (2)$$

where mex is the *minimum excluded value*, returning the smallest value of $\mathbb{Z}^+ \cup \{0\} \setminus \{\text{number}(G_1), \dots, \text{number}(G_\Delta)\}$.

We will use the notation standard in combinatorial game theory for Grundy values: $*k$ for k , except that $*$ is shorthand for $*1$ and 0 is shorthand for $*0$.⁵ By grouping all positions with non-zero Grundy values into a class called “Fuzzy”, impartial game positions can be partitioned into two *outcome classes*, characterizing winnability. (1) $\mathcal{N}$ (“Fuzzy”) - with positive Grundy values; the current (next) player always has a winning strategy. (2) $\mathcal{P}$ (“Zero”) - with zero Grundy value; the previous player always has a winning strategy.

III. A DICHOTOMY THEOREM

In this section, we prove Theorem I.1, setting up the Dichotomy Theorem of Grundy-value computation in UNDIRECTED GEOGRAPHY based on its degree. Because “Zero” ($\mathcal{P}$) is polynomial-time distinguishable from “Fuzzy” ($\mathcal{N}$) in UNDIRECTED GEOGRAPHY, to establish the hardness, we need to show that the “Fuzzy” region is PSPACE-hard to classify. By a (rather involved) reduction from *True Quantified Boolean Formula*, we proved that $*$ and $*2$ are PSPACE-hard to distinguish. While aiming for planar graphs, we distilled this construction, finding a simple gadget

⁵The reason for the $*0 = 0$ convention is that it is equivalent to the integer zero in CGT.

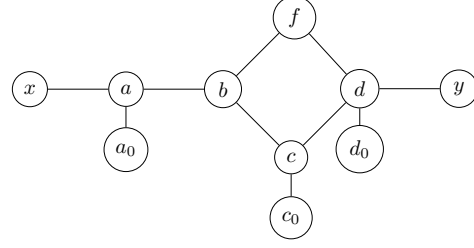


Figure 1. The gadget for each directed edge (x, y) .

(Figure I) to obtain a direct reduction from GENERALIZED GEOGRAPHY. For readers who may want to see more complex constructions for “direction control” in UNDIRECTED GEOGRAPHY, we refer them to Section V on number constructability. Replacing directed edges there is more intricate because high Grundy values cannot be truncated as in the complexity analysis below, and our attempts to simplify the proof haven’t yet produced the same outcome.

A. Complexity Separation of Winnability and Grundy Values

Theorem III.1 (PSPACE-Complete Grundy Values). *The Grundy value of polynomial-time solvable UNDIRECTED GEOGRAPHY is PSPACE-hard to compute.*

Our reduction, r , takes a GENERALIZED GEOGRAPHY position (G, s) and yields an UNDIRECTED GEOGRAPHY position $r(G, s) = (G', s)$ where:

$$(G', s) \begin{cases} = *, & \text{if } (G, s) = 0 \in \mathcal{P} \\ \in \mathcal{N} \setminus \{*\}, & \text{if } (G, s) \in \mathcal{N} \end{cases}$$

For readers unaccustomed to working with numbers, we provide another characterization: the (disjunctive) sum of (G', s) with a simple $*$ yields:

$$(G', s) + * \in \begin{cases} \mathcal{P}, & \text{if } (G, s) \in \mathcal{P} \\ \mathcal{N}, & \text{if } (G, s) \in \mathcal{N} \end{cases}$$

The reader can consider the winnability of (G, s) equivalent to $(G', s) + *$. The reduction itself contains two parts:

- 1) Modify G so that each vertex $v \in V$ is given an adjacent singleton vertex, v_0 , adjacent to no other vertices. In other words, $\forall v \in V$ we will add vertex v_0 and (undirected) edge (v, v_0) .
- 2) For each of the directed edges in G , (x, y) , we replace it with the gadget shown in Figure I.

The correctness of the reduction hinges on this gadget acting like a directed edge from x to y . We assert that with two lemmas. Here, we present proofs using number-based calculations. In the full version [9], we provide alternate proofs based on the winnability when added to $*$. The reader is welcome to use whichever reasoning they prefer to follow.

We will also use some notation to represent a graph, G , after moves have been made. For a subset $S \subset V$, we will use G_S be the graph obtained from G by removing S and all edges incident to S . Thus, from a position (G, s) , if a

player chooses to traverse edge (s, t) , the resulting position is $(G_{\{s\}}, t)$ (or (G_s, t) for short).

Lemma III.1 (Wrong Way). *Moving from y to any vertex d results in a value of $*2$ or $*3$, i.e., $(G'_y, d) = *2$ or $*3$.*

Proof: We prove this by examining the three options from d . Moving to d_0 is clearly a move to 0. Moving to c will be non-zero, because c_0 is zero. It remains to show that moving $d \rightarrow f$ results in a $*$ -position. We can see this by considering the following necessary move $f \rightarrow b$. Since both of b 's remaining neighbors, a and c , have terminal neighbors (a_0 and c_0), they are non-zero. Thus, the move to b must be a zero position, and the move to f must be $*$. ■

Lemma III.2 (Correct Way). *Moving from d to y results in a value of $*$ exactly when moving from x to a in the same gadget results in $*$.*

Proof: In both cases, we will use the fact that moving from b to f results in a 0-position, because d is always non-zero and it is f 's only neighbor. For the first case, assume that moving d to y results in $*$. This means that moving c to d has value $*2$, as options to f and d_0 are both 0. Thus, b to c has value $*$. Since b has options to both 0 and $*$, moving to b has value $*2$, and moving x to a has value $*$. In the other case, assume that moving d to y does not have value $*$. (Either it is $*2$ or above or y has already been removed.) Thus, moving c to d results in a value of $*$, because d 's other options are 0. b to c then has a value of $*2$, meaning that a to b has a value of $*$. This means that moving x to a has a value of $*2 \neq *$, completing the proof. ■

Proof: (of Theorem III.1) Determining the winnability of GENERALIZED GEOGRAPHY position (G, s) is PSPACE-hard. Thus, it remains to be shown that for the UNDIRECTED GEOGRAPHY position resulting from the reduction, $(G', s) \begin{cases} = *, & \text{if } (G, s) = 0 \in \mathcal{P} \\ \in \mathcal{N} \setminus \{*\}, & \text{if } (G, s) \in \mathcal{N} \end{cases}$

Consider any GENERALIZED GEOGRAPHY position (H, t) , k moves after (G, x) and the analogous UNDIRECTED GEOGRAPHY position $(H', t) = r(H, t)$, reached $5k$ moves after (G', s) by traversing the gadgets corresponding to the directed edges traversed to reach (H, t) . If there are no options from (H, t) , then (H', t) has options to t_0 , which has value 0; possibly to gadget vertices d , which have value either $*2$ or $*3$ by Lemma III.1; and possibly to gadget vertices a where the corresponding y vertex has already been removed, which have a non- $*$ value (specifically $*2$) by Lemma III.2. Thus, there is a move to zero and might be moves to $*2$ or $*3$. $(H', t) = *$. If there are options from (H, t) , then it is either in $\mathcal{P}$ or $\mathcal{N}$. We can complete our proof inductively by assuming that the theorem is true for all options of (H, t) and showing that it works for (H, t) . (1) If $(H, t) \in \mathcal{P}$, then each option, (H_t, p) is in $\mathcal{N}$. Thus, by our induction hypothesis, $(H'_t, p) = *z$, where $z \geq 2$.

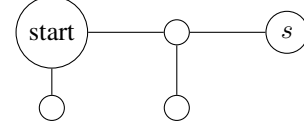


Figure 2. Prelude Gadget.

This means that (H', t) doesn't have any options equal to $*$. Since it does have a move to zero (t_0), $(H', t) = *$. $\checkmark$ (2) If $(H, t) \in \mathcal{N}$, then some option, $(H_t, p) \in \mathcal{P}$. Thus, $(H'_t, p) = *$. Since (H', t) has a move to zero (t_0) and $*$, the value is $*z$ where $z \geq 2$. ■

Currently our reduction creates positions where the initial value (at (G', s)) is either $*$ or higher. We can narrow this down so that the decision is on distinguishing between $*$ and $*2$, specifically, by appending a Prelude gadget (see Figure 2) before s and then asking what the value of the overall game position is when starting at the “start” vertex.

Corollary III.1. *Determining whether an UNDIRECTED GEOGRAPHY position equals $*$ or $*2$ is PSPACE-complete, even on bipartite, planar graphs with a maximum degree of four.*

Proof: Since the height of the game tree is at most n , the Grundy-value can be computed in polynomial-space using the standard DFS technique.

Adding our prelude gadget to the reduction from Theorem III.1, the value at vertex “start” will either be $*$ or $*2$. By calculating the number backtracking from the value at s , we see that it will be $*$ exactly when the value at s is $*$, and $*2$ for any of the other values of s .

From Lichtenstein and Sipser [20], we know that the winnability of GENERALIZED GEOGRAPHY is PSPACE-hard on bipartite, planar graphs with a maximum degree of three. Our reduction preserves the planarity, and, since there is no odd-cycle in the gadget, the bipartite property as well. We increase the degree by one because we add an extra vertex adjacent to the original vertices in V . Thus, we are still hard on graphs with a maximum degree of four. ■

The proof above also establishes that determining whether two “Fuzzy” games in UNDIRECTED GEOGRAPHY have the same Grundy value is PSPACE-hard. In Section V-C, we will use our “poly-high number constructor” to prove the following theorem, showing no polynomial-time classifier exists for Grundy values in UNDIRECTED GEOGRAPHY, beyond the well-known “Zero”-“Fuzzy” classifier, unless $P = PSPACE$.

Theorem III.2 (Too “Fuzzy” to Classify). *In UNDIRECTED GEOGRAPHY, determining whether or not the Grundy value of (G, s) , where s has degree Δ , is in any given set $S \subset [\Delta]$, is PSPACE-hard.*

B. Following the Winning Way in Branch-and-Bound

We show that for any undirected graph G with maximum-degree at most three, the Grundy value of UNDIRECTED GEOGRAPHY is polynomial-time computable. In this case, we present a polynomial-time reduction from the Grundy-value computation to decision of winnability.

For UNDIRECTED GEOGRAPHY at a position $G = (V, E)$ and $s \in V$, the *degree* of s in G is equal to the number of its feasible moves, and hence serves as a tight upper bound on the Grundy value of the position. Similarly, the maximum degree in G characterizes the maximum *branching factor* of the game tree at position (G, s) : If the maximum degree of G is Δ , then the branching factor of every node except the root is at most $\Delta - 1$ (the current geography path entering the node will take away at least one edge incident to the node). The root may have branching factor Δ but no more.

Theorem III.3 (Following the Winning Way in UNDIRECTED GEOGRAPHY). *For any undirected graph $G = (V, E)$ with maximum degree 3, and node $s \in V$, the Grundy value at the UNDIRECTED GEOGRAPHY position (G, s) can be computed in polynomial time in $n = |V|$.*

Proof: We focus on the case when the degree of s is 1 or 2. The proof naturally extends to the case of 3.

SINGLE OPTION: When the degree of s is 1, the Grundy value of UNDIRECTED GEOGRAPHY at (G, s) is $*$ if and only if (G, s) is a winning position. So, the Grundy-value can be directly reduced to the decision of winnability.

DOUBLE OPTIONS: When the degree of s is 2 (say with neighbors v_1 and v_2), the maximum branching factor of the game tree for position (G, s) is 2. Note that the degree of v_1 and v_2 in G_s is at most 2. We run the polynomial-time matching-based winnability algorithm to determine whether or not (G_s, v_1) and (G_s, v_2) are winning positions in UNDIRECTED GEOGRAPHY, and consider the four cases:

- 1) [“Fuzzy”, “Fuzzy”] - both (G_s, v_1) and (G_s, v_2) are winning positions: $\text{nimer}(G, s) = 0$.
- 2) [“Zero”, “Zero”] - both (G_s, v_1) and (G_s, v_2) are losing positions: $\text{nimer}(G, s) = *$.
- 3) [“Fuzzy”, “Zero”] - (G_s, v_1) is a winning position and (G_s, v_2) is a losing position:
 $\text{nimer}(G, s) = *(3 - \text{nimer}(G_s, v_1))$
- 4) [“Zero”, “Fuzzy”] - (G_s, v_1) is a losing position and (G_s, v_2) is a winning position:
 $\text{nimer}(G, s) = *(3 - \text{nimer}(G_s, v_2))$

In the last two cases, one of v_1 and v_2 is 0, and the other has value $x = *$ or $x = *2$. By the mex rule, (G, s) will be the other of those values ($*2$ or $*$, respectively) which is exactly $3 - x$ (or $3 \oplus x$), so the above derivation works.

In the first two cases, we find the Grundy value of (G, s) in polynomial time. Crucial to the tractability, in the last two cases, we reduce the Grundy-value computation of (G, s)

to a single Grundy-value computation of either (G_s, v_1) or (G_s, v_2) . Because G_s has one less node than G , the depth of the branch-and-bound process is $O(n)$. In total, we make $O(n)$ calls to the decision-of-winnability algorithm in order to compute the Grundy value of position (G, s) . ■

C. Misère-Play Winnability vs Grundy Values

We now demonstrate that the connection between misère rule and normal-play Grundy values is very subtle. Indeed, the strategy to play for termination positions and the strategy to play to avoid termination positions can be different. The following theorem provides an answer to Open Question 6.

Theorem III.4 (Complexity Independence). *There are non-empty sub-classes PP, HP, PH, and HH of UNDIRECTED GEOGRAPHY positions with the following properties: (1) For class PP, both misère-winnability and Grundy value are polynomial-time computable. (2) For class HP, Grundy values are polynomial-time computable, but misère-winnability is PSPACE-complete in the worst-case. (3) For class PH, misère-winnability is polynomial-time computable, but Grundy values are PSPACE-complete to compute in the worst-case (4) For class HH, both misère-winnability and Grundy values are PSPACE-complete in the worst-case.*

Proof: (High-level). UNDIRECTED GEOGRAPHY on trees is one of many examples of PP. By Theorem III.1 and Renault-Schmidt’s result [28], UNDIRECTED GEOGRAPHY itself is an example of HH.

We now show that Renault-Schmidt’s construction in [28] for the misère-play setting provides an example of HP. Thus, Renault-Schmidt’s construction can’t directly tell us about Grundy value-hardness. The main reason is that the Grundy value (in the normal setting) of the construction in [28] can be computed in polynomial time. More specifically, for a starting vertex u , the starting position has value $*$, $*2$, or $*3$, depending on the following properties (below we use the vertex naming conversion of the arc gadget of [28]): (1) If u is adjacent to some uv_1 and some ru_7 , then it has value $*3$. (2) If u is only adjacent to some uv_1 but no ru_7 , then it has value $*2$. (3) Otherwise, it has value $*$.

Our proof for Theorem III.1 provides an example for PH, because for all these games, the winnability in the misère setting is polynomial-time solvable. The main reason is that our gadget of Figure 1 has simple winnability for the current player at x or y in the misère setting without needing to get out the gadget (i.e. it localizes the decision). Particularly, (1) Any move to a is a winning move for the first player, since a_0 is a losing move for the second player (in misère setting), and thus they move to b , the first player to f , the second player to d , and then the first player to c . Then, the second player can only move to c_0 and lose the game. (2) Any move to d for the first player is a losing move, as the second player can move to f , then the first player must move to d , and the second player can move to c and thus

win, since the first player must move to c_0 and lose. (3) Any position with moves to neither a nor d must have no moves, and thus is a winning position for the first player. ■

IV. GAMES OF GAMES, SPRAGUE-GRUNDY CHARACTERIZATION, AND MATHEMATICAL-COMPUTATIONAL DIVERGENCE

Sprague-Grundy theory provides not only a unified theory for understanding diverse impartial rulesets, but also an elegant framework for their interactions. Because $\text{nimber}(G + H) = \text{nimber}(G) \oplus \text{nimber}(H)$, the Grundy value of $(G + H)$ can be reduced in polynomial-time to the Grundy values of G and H . In contrast, using our complexity result for UNDIRECTED GEOGRAPHY, we strengthen Morris' theorem [26] from strictly partisan to impartial games:

Theorem IV.1 (Beyond Winning Impartial Games). *If $P \neq PSPACE$, then the disjunctive sum of two polynomial-time tractable impartial games can be intractable.*

Thus, unlike Grundy-value, there is no general polynomial-time reduction from winnability of $(G + H)$ to strategic determination for G and H , unless $P = PSPACE$. This illustrates a striking view of the classical Sprague-Grundy characterization through the lens of computational complexity theory. The Grundy value and strategic determination are two different yet fundamental summaries of the game tree. Using the complexity gulf between P and $PSPACE$, our result demonstrates that the Grundy value is a significantly richer summary of game data than strategic determination.

Theorem IV.2 (Intractability of Game Summary). *Unless $P = PSPACE$, there is no general polynomial-time method to summarize two given impartial games (say G and H) to efficiently solve the game of their sum $(G + H)$.*

Sprague-Grundy theory establishes that such concise summaries—in the form of Grundy values—of game data always exist when computational cost is no object. This work highlights a subtle yet fundamental contrast between the mathematical and computational facets of CGT. Applying Sprague-Grundy theory, our dichotomy theorem also enables us to settle the solvability of several families of games extending UNDIRECTED GEOGRAPHY.

MULTI-FIELD UNDIRECTED GEOGRAPHY - The disjunctive sum of multiple UNDIRECTED GEOGRAPHY games.

MULTI-TOKEN UNDIRECTED GEOGRAPHY - This game is played on an undirected graph, in which a game position is defined by a graph $G = (V, E)$ and a set $S \subset V$. Each node in S has a token, and in each turn, exactly one of the tokens can be moved to an adjacent *unoccupied node*, and the node of its previous location is removed from the graph. In MULTI-TOKEN UNDIRECTED GEOGRAPHY, alternating moves by two players create multiple node-disjoint exploring

paths in G , one by each token. The game ends when no valid extension exists to any of these paths.

UNDIRECTED GEOGRAPHY WITH PASSES - this natural extension of a game is to allow players to pass their turn: For $k \geq 0$, we consider UNDIRECTED GEOGRAPHY with k -Total Passes, which augments the feasible moves by allowing players to pass their turn, provided that the total number of passes taken so far (by both players) is less than k .

SWAP UNO - This game is inspired by a generalization of Uno, was shown by Domaine *et al* [12] to be in P via reduction to UNDIRECTED GEOGRAPHY. In this game, there are two hands, H_1 and H_2 , which each consist of a set of cards. This is a perfect information game, so both players may see each other's hands. Each card has two attributes, a color c and a rank r and can be represented as (c, r) . A card can only be played in the center (shared) pile if the previous card matches either the c of the current card or the r of the current card. Finally, for the special part that makes this "Swap" Uno, either player may, once a game, decide to use their turn to swap their hand for their opponent's rather than playing in a pile. Once a single player swaps, the other player may not swap.

We now prove that, although UNDIRECTED GEOGRAPHY is polynomial-time solvable, these basic extensions of UNDIRECTED GEOGRAPHY can be more challenging computationally.

Theorem IV.3 (The War of GEOGRAPHY Battles). *Deciding whether or not the current player has a winning strategy in the sum of two UNDIRECTED GEOGRAPHY games, and consequently, in TWO-TOKEN UNDIRECTED GEOGRAPHY and SWAP UNO games, is $PSPACE$ -complete. Furthermore, UNDIRECTED GEOGRAPHY WITH k -TOTAL PASSES is $PSPACE$ -complete to solve for odd k , and polynomial-time solvable for even k .*

Proof: Let's start with the complexity analysis of UNDIRECTED GEOGRAPHY WITH k -TOTAL PASSES. We consider a trivial game, called PASS. Each position in PASS is defined by an integer k . The terminal position is the one with $k = 0$. For any $k > 0$, there is a single move at position k to position $k - 1$. PASS with $k = 1$ is isomorphic to NIM with a single pile of one item. In general, the PASS position k is isomorphic to NIM with k piles, each containing a single item. The Grundy value of PASS at position k is zero if k is even and $*$ if k is odd. For any undirected graph G and positive integer k , UNDIRECTED GEOGRAPHY WITH k -TOTAL PASSES at position $((G, s), k)$ is isomorphic to the game defined by the disjunctive sum of two battlefield games: (1) UNDIRECTED GEOGRAPHY at position (G, s) and (2) PASS at position k . Therefore:

- When k is odd (e.g., $k = 1$), by the Sprague-Grundy theory, the Grundy value of position $((G, s), k)$ is equal to $\text{nimber}((G, s)) \oplus *$. The current player in this

game has NO winning strategy iff $\text{nimber}((G, s)) = *$ in UNDIRECTED GEOGRAPHY. We conclude that the winnability of this game is PSPACE-complete to solve, because deciding whether or not $\text{nimber}((G, s))$ is equal to $*$ (or $*2$) is PSPACE-complete (Theorem III.2).

- When k is even, the Grundy value of position $((G, s), k)$ is equal to $\text{nimber}((G, s))$, for which we can distinguish “Zero” from “Fuzzy” in polynomial time.

We can similarly characterize the complexity of the sum of two UNDIRECTED GEOGRAPHY games (see the full version). Because the sum of two UNDIRECTED GEOGRAPHY games is a special case of TWO-TOKEN UNDIRECTED GEOGRAPHY, the PSPACE-hardness extends, and, in fact, even when we require that the underlying graph is connected. In the full version, we show that UNO bipartite graphs have the structural property needed to encode the hard instances for Grundy value computation in UNDIRECTED GEOGRAPHY, as required in our proof for Theorem III.1. Thus, the PSPACE-hardness of SWAP UNO follows from that of UNDIRECTED GEOGRAPHY WITH ONE PASS. ■

Theorems IV.1 and IV.2 then follow directly from Theorem IV.3 on the PSPACE-hardness of the disjunctive sum of two UNDIRECTED GEOGRAPHY games, and the sum of NIM and UNDIRECTED GEOGRAPHY.

Proposition IV.1. *The sum of UNDIRECTED GEOGRAPHY over degree-three graphs are polynomial-time solvable.*

V. GRAPHS WITH POLYNOMIAL GRUNDY VALUES

A fundamental problem in CGT is that of *number constructability*. That is to say—when specialized to the game of our focus—the question of whether a game of UNDIRECTED GEOGRAPHY can actually have a certain Grundy value (equivalent to determining the habitat for impartial games), and if it can, whether it can be succinctly encoded. The existence is important primarily from a pure mathematical standpoint. The succinct encoding is needed for sums of games with high Grundy values to actually be shown intractable. In Section V-C, we highlight the use of the (polynomially) *succinct encoding* of high numbers to support our complexity analyses.

A. Logarithmic Intuition and Polynomial Challenge

The habitat going up to the maximum degree in the graph is simple. We will present it in the next construction to motivate our more advanced proof.

Observation V.1 (Logarithmic Nimber). *There is a tree-based UNDIRECTED GEOGRAPHY position with nimber $*n$, highest degree n , and 2^n vertices.*

Proof: Recursively, we define a tree $t(n)$ with moves to $t(n-1), t(n-2) \dots t(0)$. For the base case, $t(0)$ is a single isolated vertex (with nimber 0). Inductively, we assume each

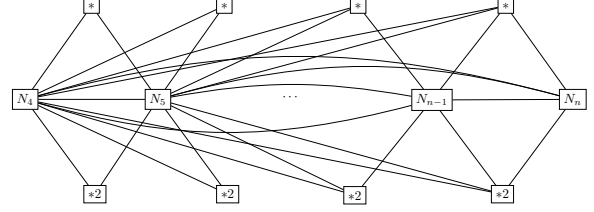


Figure 3. The overall schema for Nimber Constructability. The value of the position with the token at N_n is $*n$.

of the smaller $t(i)$ have 2^i vertices, so we have 2^{k+1} be $2^0 + 2^1 + \dots + 2^k + 1$, where the final 1 is the new root. ■

Thus, we can get a poly-log nimber using a polynomial number of vertices (and certainly any constant number, which we will use for gadgets up to $*3$).

The exponential size comes from the fact that we repeat each tree in each subtree. This is necessary, since if we attempt to combine the subtrees, being able to move “back up” those trees could change the Grundy values. In GENERALIZED GEOGRAPHY, one can use directed edges to prevent undesired “up” moves to share the lower nimber nodes. Thus, one can achieve nimber n with $n+1$ vertices. We can’t just replace these with our directed-edge gadget from Figure 1 because the inner degree on those is constant and will prevent arbitrarily large numbers. We need a more sophisticated mechanism to get numbers of any size.

B. Polynomial-High Nimber Constructability

To attain nimber n , we create n vertices $N_1, \dots, N_n$, which exist in a clique, as in Figure 3. Each N_i has nimber $*i$ so long as all N_k with $k < i$ remain. (These vertices are said to have a lower *rank*.) We argue that starting with the token on vertex N_n is a $*n$ -position. (We do not have N_0 , N_1 , N_2 , or N_3 , since we use 0 through $*3$ as mechanisms to ensure the player is unable to move “up” in rank.)

After any move from N_n to N_k , we no longer want vertices with higher-rank than k to retain their nimber value. To attain this, we create $*$ and $*2$ gadgets for each N_i , which

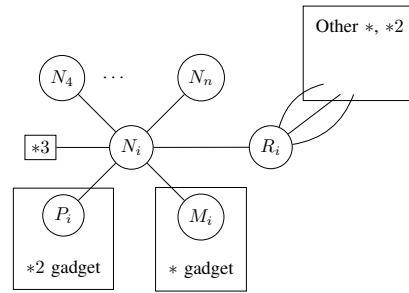


Figure 4. Each vertex N_i is connected to all other N -vertices, as well as its own $*3$ gadget, its own $*2$ gadget, its own $*$ gadget, and its own R_i vertex (with value 0). R_i is also connected to the P_k and M_k gadgets where $k > i$ as shown in the following figures.

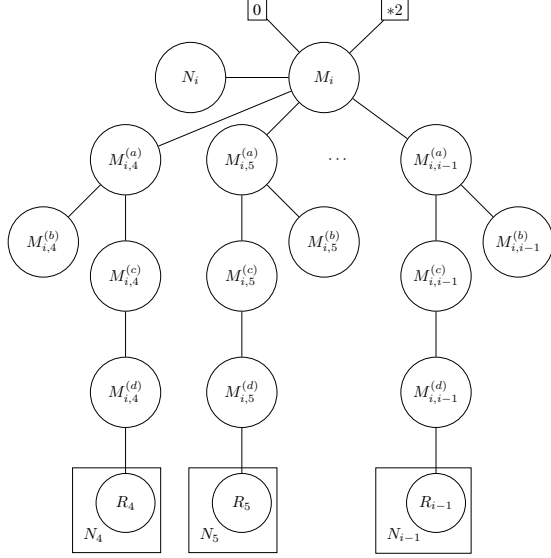


Figure 5. A $*$ gadget, which has value $*$ unless a lower-rank- N vertex is removed. If all of N_k vertices exist off the bottom, then moving to any one of them from R_k results in a 0-board (by Lemma V.2). This causes the move $M_i \rightarrow M_{i,k}^{(a)}$ to be equal to $*2$. Otherwise, one R_k has value 0, so $M_i \rightarrow M_{i,k}^{(a)}$ is a move to $*$, so M_i instead has value $*3$.

have their named values if and only if no vertex of a lower rank has been removed. As such, a later move to a higher-rank N_i will have value either $*$ or $*2$ instead of $*i$.

We present these designs in Figures 4, 5, and 6.

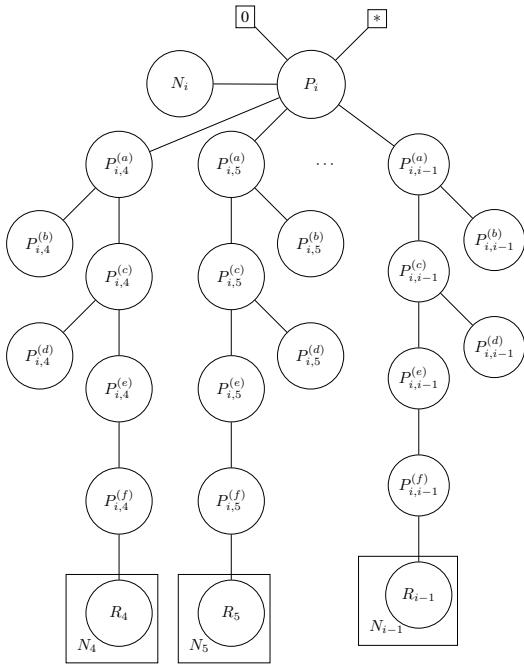


Figure 6. The $*2$ gadget

Lemma V.1 (Grounded). *In a game where the only vertices removed are some N_i vertices along with some of their associated M_i and M_{ip} or P_i and P_{ip} vertices, then traversing edge (N_k, R_k) always results in a move to 0.*

Proof: Any move to an $M_{i,k}^{(a)}$ vertex has number at least $*$ since it has a move to $M_{i,k}^{(b)}$ which is 0. Thus, moving $M_{i,k}^{(c)}$ from $M_{i,k}^{(d)}$ results in a 0, so all $M_{i,k}^{(d)}$ moves from R_k result in $*$. The same is true of moving from R_k to $P_{i,k}^{(f)}$ because $P_{i,k}^{(c)}$ is also non-zero. Since R_k only has $*$ -options, it's value is zero when moving from N_k . ■

Lemma V.2. *In a game where the only vertices removed are some N_i vertices along with some of their associated M_i and M_{ip} vertices or P_i and P_{ip} vertices, then traversing the edge (R_p, N_p) is a move to 0.*

Proof: There is a move to $*3$ (and, if $p = 4$, $*2$ and $*$), moves to M_p and P_p , which both have a move to 0 by construction, and to various other N_i , which have moves to R_i , which are moves to 0 by Lemma V.1. ■

Lemma V.3 (Skip $*2$). *So long as only N_i vertices are removed from the graph, the position from moving from N_k to M_k has nim-value $*$ if and only if no N_i have been removed with $i < k$. Otherwise, it is $*3$.*

Proof: Consider the result of moving $M_{i,k}^{(d)} \rightarrow R_k$. All moves to other $M_{j,k}^{(d)}$ vertices are losing moves as established in Lemma V.1. There is only a winning move if N_k still exists, so the position at $R_k \neq 0$ iff N_k still exists. Let's consider these two cases: (1) If N_k exists, then moving to $R_k \neq 0$. Thus, moving to $M_{i,k}^{(d)}$ yields 0, so moving to $M_{i,k}^{(c)}$ yields $*$, and moving to $M_{i,k}^{(a)}$ yields $*2$. If all N_k exist, then M_i does not have a move to $*$, so it's value is $*$ from N_i . (2) On the other hand, if N_k does not exist, then moving to R_k from $M_{i,k}^{(d)}$ yields 0. Thus, moving to $M_{i,k}^{(d)}$ yields $*$, so moving to $M_{i,k}^{(c)}$ yields 0, and moving to $M_{i,k}^{(a)}$ yields $*$. Since M_i has an $*$ -option, moving there from N_i now yields a position with value $*3$. ■

The proofs for the next three lemmas are in the full version.

Lemma V.4 (Skip $*$). *So long as only N_i vertices are removed, a token on P_k has nim-value $*2$ iff no N_i have been removed with $i < k$. If the value is not $*2$, it is $*3$.*

Lemma V.5. *If the token is on N_k , and only N_i with $i > k$ have been removed, then the number must be at least $*4$.*

Lemma V.6 (Parity). *If the only vertices removed are N_i vertices, the token is currently on N_k , N_j is of lower rank than N_k and is the lowest rank that has been removed, and if there are an odd number of N_p vertices remaining, where N_p are higher rank than N_j , then the nim-value of the game is $*$. If there are an even number of those vertices remaining,*

then the game has value $*2$.

Theorem V.1 (Right Amount of Stars). *When the token is on N_n , the resulting game has nim-value $*n$.*

Proof: For 0 through $*3$, we build a tree as described by Observation V.1. For larger Grundy values, we have the token on vertex N_n . We will prove this has value $*n$ through induction on the values of a starting token on N_i .

Base Case: As long as the only vertices removed from the graph are N_i vertices, the token on N_4 has value $*4$.

To establish this: there are moves to $*$, $*2$, and $*3$, each by construction. There is a move to 0 through R_4 by Lemma V.1. The only other available moves are some subset of of the N_j , which by Lemma V.6, have value $*$ or $*2$.

Inductive Hypothesis: As long as the only vertices removed from the graph are various N_i vertices where $i > k$, N_k has value $*k$.

Inductive Step: We need to show that as long as the only vertices removed from the graph are various N_i vertices where $i > k + 1$, N_{k+1} has value $*k + 1$.

To establish this: N_{k+1} has moves to $*3$ and $*4$, by construction, and to M_{k+1} , P_{k+1} , R_{k+1} , all of N_4 through N_k , and some of N_{k+2} to N_n . Moves to N_4 to N_k are $*4$ to $*k$ by induction. The move to R_{k+1} is a move to 0 by Lemma V.1. The move to M_{k+1} is $*$, by Lemma V.3 since all N_j remain. The move to P_{k+1} is $*2$, by Lemma V.4, again since no N_j is removed. ■

C. Complexity Implication

We now use Theorem V.1 to prove Theorem III.2, establishing that every classifier of UNDIRECTED GEOGRAPHY Grundy values, other than the polynomial-time “Zero”-“Fuzzy” classifier, is PSPACE-hard.

Proof: (of Theorem III.2) Recall that the proof of Corollary III.1 shows that distinguishing $*$ from $*2$ is PSPACE-hard. We will first use this to prove that distinguishing between $*(k - 1)$ and $*k$ is PSPACE-hard, for any $k \geq 2$.

We prove this by taking a position (G_2, v_2) that is hard to distinguish between $*$ and $*2$. We introduce a new vertex v_3 with moves to its own 0 and $*$ and add edge (v_3, v_2) to create G_3 . Then we will create a new vertex v_4 with moves to its own 0, $*$, $*2$, and connect (v_4, v_3) to create G_4 , and so on, until we create a vertex v_k with moves to its own 0 to $*(k - 1)$, and add edge (v_k, v_{k-1}) to create G_k . These vertices v_i and their associated gadgets have size polynomial in i due to Theorem V.1.

Now, if $(G_2, v_2) = *$, then (G_3, v_3) doesn’t have a move to $*2$, so $(G_3, v_3) = *2$. Similarly, $(G_4, v_4) = *3$, $(G_5, v_5) = *4, \dots, (G_k, v_k) = *(k - 1)$. If instead, $(G_2, v_2) = *2$, then $(G_3, v_3) = *3$, because there is a move to $*2$. Likewise, $(G_4, v_4) = *4, \dots, (G_k, v_k) = *k$. Thus, it is PSPACE-hard to distinguish between $*k$ and $*(k - 1)$.

Next, we prove that distinguishing between any $*k$ and $*p$ is PSPACE-hard. (We will assume $p > k$, without loss of

generality.) We first create a (G_k, v_k) where distinguishing $*(k - 1)$ from $*k$ is hard, then add a new vertex v_p which has moves to its own 0 to $*(k - 1)$, v_k , and $*(k + 1)$ to $*(p - 1)$. We name this graph G'_p ; the position (G'_p, v_p) has value $*p$ exactly when (G_k, v_k) has value $*k$. (G'_p, v_p) has value $*k$ otherwise. Thus, it is PSPACE-hard to distinguish between $*p$ and $*k$.

Finally, we use this to show that distinguishing between any possible fixed set of Grundy values is hard. For any possible set S , there must be at least one Grundy value $x \in S$ and one Grundy value in $y \in \bar{S} := [\Delta] \setminus S$. Then, we can, as described above, create a position where it’s PSPACE-hard to distinguish between $*x$ and $*y$. Thus, if one could classify the game to be within that set of Grundy values, one could solve a PSPACE-hard problem. ■

VI. MATH BEHIND BOARD GAMES: THEORY AND PRACTICE

“My experiences also strongly confirmed my previous opinion that the best theory is inspired by practice and the best practice is inspired by theory.” - Donald E. Knuth [19]

Combinatorial game theory is a fascinating field, where simplicity is valued, and both efficient methods for solving games and intriguing positions for challenging players are appreciated [3], [1]. Indeed, the magic smile on a six-year old’s face when they realize a winning trick (e.g. how to win two-pile NIM⁶ as introduced in Math Circle⁷) is as enchanting as the contemplative gaze [21] of CHESS, GO, and HEX champions. These are the polynomial-time smiles and PSPACE-hard gazes.

In this paper, we have proved that adding a single ‘PASS’—the smallest possible extension—to UNDIRECTED GEOGRAPHY transforms the game from polynomial-time solvable to PSPACE-hard intractable. And similarly, we showed that giving a single pass to the game of UNCOOPERATIVE UNO also had this same transformation from P to PSPACE. Characterizing the complexity impact of this small change to the ruleset has deepened and expanded our understanding of the foundational concept & characterization in combinatorial game theory. It has also added MULTI-TOKEN UNDIRECTED GEOGRAPHY to the collection of PSPACE-hard graph-based impartial games with simple rulesets.

In the full version [9], we present two practical board games inspired by our theoretical work and open questions inspired by these practical design.

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⁶when realizing the fact that NIM with two identical piles is a losing position can be used for finding a winning strategy for any two-pile NIM—including the decision to go first or second—so that they will never again lose to their parents.

⁷<https://mathcircles.org/>

19 pandemic, we are grateful to live in the Internet age with technologies—such as Zoom—that enable our weekly virtual meetings. This research was supported in part by the Simons Investigator Award for fundamental & curiosity-driven research and NSF grant CCF-1815254.

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