

# Scalable Zonotopic Under-approximation of Backward Reachable Sets for Uncertain Linear Systems

Liren Yang

Necmiye Ozay

**Abstract**—Zonotopes are widely used for over-approximating forward reachable sets of uncertain linear systems for verification purposes. In this paper, we use zonotopes to achieve more scalable algorithms that under-approximate backward reachable sets of uncertain linear systems for control design. The main difference is that the backward reachability analysis is a two-player game and involves Minkowski difference operations, but zonotopes are not closed under such operations. We under-approximate this Minkowski difference with a zonotope, which can be obtained by solving a linear optimization problem. We further develop an efficient zonotope order reduction technique to bound the complexity of the obtained zonotopic under-approximations. The proposed approach is evaluated against existing approaches using randomly generated instances and illustrated with several examples.

## I. INTRODUCTION

For autonomous control systems, the control objectives need to be achieved robustly against system uncertainties. Central to many control synthesis techniques for uncertain systems is backward reachability analysis. Given an uncertain control system and a set  $X_0$  of target states, the backward reachable set (BRS) consists of the states that can be steered into  $X_0$  in finite time, regardless of the system uncertainties. Being able to compute such sets is important to design controllers with safety or reachability objectives [4], [19], and is one building block for achieving more complicated control tasks [6]. Whenever the exact computation is hard, an under-approximation can be still used to define a conservative control law. A variety of approaches exists in the literature, including polyhedral computation [5], interval analysis [17], HJB method [21] and polynomial optimization [15], just to name a few. For linear dynamics with linear constraints, polyhedra can be used to represent the BRSs as they are closed under linear transformation, Minkowski addition and subtraction, and can be computed leveraging linear optimization tools. However, it is limited to low dimensional systems (typically, state dimension  $\leq 4$ ) due to an expensive quantifier elimination step.

One related problem is the forward reachability analysis, where we deal with uncertain system with *no control inputs* (e.g., closed-loop systems), and compute the set of states that can be visited in the future from some initial state in a given set  $X_0$ . Such forward reachable sets can be computed offline for verification and online for state prediction [3]. Often times, the forward reachable sets are overestimated for robustness. For linear systems, a special polyhedron called zonotope is widely used to represent forward reachable

sets thanks to the favorable complexity of applying linear transformations (for forward state evolution) and Minkowski additions (to account for additive uncertainty) to zonotopes (see, e.g., [1], [8]). Algorithms that compute zonotopic forward reachable sets are more scalable than those dealing with general polyhedra.

One natural question is: for *uncertain* linear dynamics, is there a way to reverse the time so that the efficient zonotopic set computation for forward reachability analysis can be directly adopted to compute BRSs? Unfortunately, this is not the case. The main reason is that there lacks a meaningful notion of two-player game in forward reachability analysis. In the forward case, there is only one player (i.e., the environment) picking the initial state and the system uncertainty, whereas in the backward case, there are two players (i.e., the controller and the environment) picking the control input and the uncertainty in turn (see Section 4.2 of [20]). Particularly, the existence of the environment player leads to a Minkowski subtraction step in the sequential BRS computation, but zonotopes are not closed under Minkowski subtraction [2]. Therefore, combining the idea of time-reversing and efficient computational tools for forward reachability (e.g., based on zonotopes [18], [10] or polynomial zonotopes [12]) were explored only for *deterministic* systems, but using zonotopes for *uncertain* systems' backward reachability, to the best of our knowledge, is still missing.

In this paper, we use zonotopes to represent and compute BRSs for uncertain linear systems. The key ingredient is an efficient way to under/over-approximate the Minkowski difference of two zonotopes by solving convex optimization problems. While the under approximation allows us to efficiently compute a subset of the BRS without polyhedral projection, the over-approximation can be used to quantify how conservative this subset is. Different from [2], which manipulates a hyperplane-representation, our approach only deals with the generator-representations of zonotopes, and hence is more efficient and suitable for sequential computation, but at the cost of accuracy. The accuracy issue, however, is mitigated by the fact that our subtrahend zonotope represents the impact of uncertainties and is usually small compared to the minuend zonotope. Moreover, [2] does not guarantee if the approximation is an inner one or an outer one. We also leverage the linear encoding of zonotope-containment problems [23] and derive an alternative approach for under-approximating the Minkowski difference between zonotopes. Theoretical analysis and experiments show that our approach scales differently from this alternative. In order to upper bound the complexity of each step of the computation, we

further present a way to reduce the order of the obtained zonotopic BRSs. Zonotope order reduction is extensively studied (e.g., see [13], [26] and the references therein), but our approach is different: we search for a lower order zonotope *enclosed* by the given zonotope, whereas existing techniques, focusing on forward reachability analysis, all look for outer approximations. Our approach is evaluated with randomly generated zonotopes with different dimensions and orders, and its efficacy is illustrated with several examples.

## II. PRELIMINARIES

Let  $G = [g_1, g_2, \dots, g_N] \in \mathbb{R}^{n \times N}$  be a set of generators, and  $c \in \mathbb{R}^n$  be a center vector. A zonotope  $Z$  with generator-representation (or G-rep)  $(G, c)$  is defined to be the set  $\{c + \sum_{i=1}^N \theta_i g_i \mid \theta_i \in [-1, 1], i = 1, 2, \dots, N\}$ . With a slight abuse of notation, we will write  $Z = (G, c)$ . Let  $H \in \mathbb{R}^{L \times n}$  and  $h \in \mathbb{R}^L$ , a *polyhedron* with hyperplane-representation (or H-rep)  $(H, h)$  is the set  $\{x \in \mathbb{R}^n \mid Hx \leq h\}$ . If polyhedron  $X$  is bounded,  $X$  is called a polytope. A set  $V = \{x_1, x_2, \dots, x_M\} \subseteq \mathbb{R}^n$  is called a vertex-representation (or V-rep) of a polytope  $X$  if  $X$  is the convex hull of  $V$ , i.e.,  $X = \text{cvxh}(V) := \{\sum_{j=1}^M \lambda_j x_j \mid \sum_{j=1}^M \lambda_j = 1, \lambda_j \in [0, 1], j = 1, 2, \dots, M\}$ , where  $\text{cvxh}$  denotes the convex hull. Let  $A \in \mathbb{R}^{L \times n}$  and  $X \subseteq \mathbb{R}^n$  be a set,  $AX$  denotes the set  $\{Ax \mid x \in X\}$ .

Let  $X, Y \subseteq \mathbb{R}^n$  be two sets, the Minkowski sum of  $X$  and  $Y$ , denoted by  $X \oplus Y$ , is the set  $\{x + y \mid x \in X, y \in Y\}$ . Whenever  $X = \{x\}$  is a singleton set, we will write  $x + Y$  for  $X \oplus Y$ . The Minkowski difference of  $X$  and  $Y$ , denoted by  $X \ominus Y$ , is defined to be  $\{z \in \mathbb{R}^n \mid z + Y \subseteq X\}$ . For the Minkowski arithmetics, we assume that the operations are done in order from left to right, except as specified otherwise by parentheses. The following lemmas will be useful.

**Lemma 1.** Let  $X, Y, Z \subseteq \mathbb{R}^n$ .

- i) [[16], Proposition 3.1, [25], Lemma 4]  $X \ominus Y \oplus Z \subseteq X \oplus Z \ominus Y$ , particularly,  $X \ominus Y \oplus Y \subseteq X \subseteq X \oplus Y \ominus Y$ .
- ii) [From [22]] If  $X, Y$  and  $Z$  are convex, compact and nonempty, then  $X \oplus Z = Y \oplus Z$  implies that  $X = Y$ .
- iii) If  $X, Y$  are convex, compact and nonempty, then  $X \oplus Y \ominus Y = X$ .

*Proof.* To prove iii), note that, by i),  $X \oplus Y \ominus Y \oplus Y = X \oplus Y$ . Then applying item ii) yields  $X \oplus Y \ominus Y = X$ .  $\square$

**Lemma 2.** [From [8]] Let  $Z = (G, c) \subseteq \mathbb{R}^n$  be a zonotope.

- i)  $Z = \bigoplus_{i=1}^N Z_i$  where  $Z_i = (\{g_i\}, c_i)$  s.t.  $\sum_{i=1}^N c_i = c$ .
- ii) Let  $A \in \mathbb{R}^{L \times n}$ ,  $AZ = (AG, Ac)$ .
- iii) Let  $Z' = (G', c')$ ,  $Z \oplus Z' = ([G, G'], c + c')$ .

## III. BACKWARD REACHABLE SETS

Consider a discrete-time system in the following form:

$$x_{t+1} = Ax_t + Bu_t + Ew_t + K, \quad (1)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u \in U \subseteq \mathbb{R}^{n_u}$  is the control input and  $w \in W \subseteq \mathbb{R}^{n_w}$  is the disturbance input. Given a set  $X_0$  of target states, we want to compute (or to

under-approximate, if exact computation is hard) the  $k$ -step backward reachable set  $X_k$  of set  $X_0$ , defined recursively as

$$X_{k+1} = \{x \in \mathbb{R}^{n_x} \mid \exists u \in U : \forall w \in W : Ax + Bu + Ew + K \in X_k\}, \quad k = 0, 1, 2, \dots \quad (2)$$

Set  $X_k$  contains the states from where it is possible to reach the target set  $X_0$  in *exactly*  $k$  steps. A weaker definition of the  $k$ -step BRS would require  $X_0$  to be reached in *no more than*  $k$  steps, whose formal definition is similar to Eq. (2) except for an extra “ $\cup X_k$ ” at the end of the formula. Here, we adopt the stronger definition in Eq. (2) for simplicity because the union operation may lead to non-convex sets. There exists slightly different notions of reachable sets [14], depending on the order of the quantifiers. We will focus on under-approximating the set defined by (2) while our approach applies in general.

Suppose that set  $U, W$ , and  $X_0$  are polytopes, and that the H-rep of  $U, X_0$  and the V-rep of  $W$  is known, one can compute  $X_k$  as a polytope in H-rep, i.e.,

$$X_{k+1} = \mathbf{Proj}_x(\{x \in \mathbb{R}^{n_x}, u \in U \mid \forall w_j \in V_W : Ax + Bu + Ew_j \in X_k\}), \quad k = 0, 1, 2, \dots, \quad (3)$$

where  $\mathbf{Proj}_x(S) = \{x \mid \exists u : (x, u) \in S\}$  is the projection operation (e.g., see [24], Proposition 1). However, polytope projection is time-consuming, which limits the use of this approach to low dimensional systems (typically  $n_x \leq 4$ ).

In this paper, we consider under-approximating the BRSs of system (1) instead under the following assumptions.

- A1. The target set is a zonotope (denoted by  $Z_0$  hereafter), whose G-rep is known.
- A2. The disturbance set  $W$  is a polytope, whose H-rep  $(H, h)$  and V-rep  $V$  are both known.
- A3. Matrix  $A \in \mathbb{R}^{n_x \times n_x}$  is invertible. This assumption is true whenever Eq. (1) is obtained by time-discretizing an underlying continuous-time linear dynamics.

Finding under-approximation of BRSs is useful in control problems with reachability objectives and falsification problems against safety requirements [7].

## IV. SOLUTION APPROACH

We explore the use of zonotopes in under-approximating the BRS  $X_k$ . This is based on i) the modest computational complexity of operations on zonotopes such as Minkowski addition and affine transformation, and ii) the fact that (2) can be re-written as follows using Minkowski arithmetic [14]:

$$X_{k+1} = \{x \in \mathbb{R}^{n_x} \mid Ax \in X_k \ominus EW \oplus -BU - K\}. \quad (4)$$

In Eq. (4), if  $W = \{0\}$  and the term “ $\ominus EW$ ” were not there, then one could show inductively that, under assumption A1-A3,  $X_{k+1}$  is a zonotope whose G-rep can be easily computed from the G-reps of  $X_k$  and  $U$  after Minkowski addition and linear transformation. Whenever  $W$  is not a singleton set, the key step is to efficiently under and over approximate the Minkowski difference in Eq. (4) with zonotopes in their G-reps. Whereas the former leads to an inner approximation of  $X_{k+1}$ , the latter one can be used to quantify the conservatism of this inner approximation.

### A. Zonotopic Inner/Outer Approximation of $Z \ominus EW$

Let  $Z = (G, c) \subseteq \mathbb{R}^{n_x}$  be a zonotope, where  $G = [g_1, g_2, \dots, g_N]$ . We formulate two optimization problems, one computes a zonotopic outer approximation  $\bar{\mathfrak{Z}}(Z, EW)$ , and the other computes a zonotopic inner approximation  $\underline{\mathfrak{Z}}(Z, EW)$ , of set  $EW$  using  $Z$  as a “template”. The obtained outer/inner approximation are also in G-reps. Particularly, their generators are scaled versions of  $Z$ ’s generators, i.e., in the form of  $\alpha_i g_i$  for some  $\alpha_i \in [0, 1]$  (see Definition 1). We then show that the Minkowski difference  $Z \ominus \bar{\mathfrak{Z}}(Z, EW)$  and  $Z \ominus \underline{\mathfrak{Z}}(Z, EW)$  can be done element-wise via generator subtraction. This leads to an efficient way to inner/outer approximate  $Z \ominus EW$  with zonotopes in G-reps. This technique will become our key ingredient of BRS under-approximation.

**Definition 1.** Let  $Z = (G, c)$  and  $Z' = (G', c')$  be zonotopes.  $Z'$  is aligned with  $Z$  if  $G = [g_1, g_2, \dots, g_N]$  and  $G' = [\alpha_1 g_1, \alpha_2 g_2, \dots, \alpha_N g_N]$  for some  $\alpha_i \in [0, 1]$ .

1) *Outer approximation of  $EW$* : Consider the following linear programming problem:

$$\begin{aligned} \min_{\theta, \alpha, c} \quad & \sum_{i=1}^N b_i \alpha_i \\ \text{s.t.} \quad & \forall w_j \in V : c + \sum_{i=1}^N \theta_{ij} g_i = E w_j, \quad (\text{min-out}) \\ & |\theta_{ij}| \leq \alpha_i \leq 1, \quad i = 0, 1, \dots, N \end{aligned}$$

where  $b_i > 0$  are constants and  $\theta$  and  $\alpha$  are vectors aggregated from  $\theta_{ij}$  and  $\alpha_i$  respectively. The V-rep  $V$  of the disturbance set  $W$ , which is available by Assumption A2, is used to formulate the above problem. Let  $N$  be the number of generators in the template zonotope  $Z$ ,  $n_x$  be the dimension of the ambient space, and  $M$  be the number of vertices in  $V$ . In the optimization problem (min-out),

$$\begin{aligned} \# \text{variables} &= \mathcal{O}(MN + n_x), \\ \# \text{constraints} &= \mathcal{O}(M(N + n_x)). \end{aligned} \quad (5)$$

**Proposition 1.** Let  $(\theta, \bar{\alpha}, \bar{c})$  be the minimizer of the optimization problem (min-out). Define  $\bar{\mathfrak{Z}}(Z, EW) = ([\bar{\alpha}_1 g_1, \bar{\alpha}_2 g_2, \dots, \bar{\alpha}_N g_N], \bar{c})$ . We have  $EW \subseteq \bar{\mathfrak{Z}}(Z, EW)$ .

*Proof.* By the conditions in (min-out), for any  $i$  and  $w_j \in V$ , there exist  $\theta_{ij} \in [-\bar{\alpha}_i, \bar{\alpha}_i]$  s.t.  $E w_j = \bar{c} + \sum_{i=1}^N \theta_{ij} g_i$ . Equivalently, there exist  $\theta'_{ij} \in [-1, 1]$  s.t.  $E w_j = \bar{c} + \sum_{i=1}^N \theta'_{ij} \bar{\alpha}_i g_i$ . Hence  $EW \subseteq \bar{\mathfrak{Z}}(Z, EW) = ([\bar{\alpha}_1 g_1, \bar{\alpha}_2 g_2, \dots, \bar{\alpha}_N g_N], \bar{c})$ . It then follows that  $EW = \text{Ecvxh}(V) = \text{cvxh}(EW) \subseteq \bar{\mathfrak{Z}}(Z, EW)$  from the convexity of zonotope  $\bar{\mathfrak{Z}}(Z, EW)$ .  $\square$

In general, there does not exist a unique minimal (in the set inclusion sense) zonotopic outer approximation of  $EW$  that aligns with the template zonotope  $Z$ . We hence minimize a weighted sum of  $\alpha_i$ ’s. The weights  $b_i > 0$  can be used for heuristic design to incorporate prior knowledge of disturbance set  $W$ . For example, when  $W$  is a hyper-rectangle and  $E \in \mathbb{R}^{n_x \times n_w}$  is full rank, we use  $b_i = \|T g_i\|_1 - \|T g_i\|_\infty$ , where  $T = (E^\top E)^{-1} E$  when  $n_x \geq n_w$  and  $T = E^\top (E E^\top)^{-1}$  otherwise. The idea is to encourage using generators that closely align with vector  $E e_p$ , where  $e_p$  is the  $p^{\text{th}}$  natural basis of vector space  $\mathbb{R}^{n_w}$ . A similar criteria was used for zonotope order reduction in [8].

2) *Inner approximation of  $EW$* : Consider the following optimization problem:

$$\begin{aligned} \max_{\alpha, c} \quad & \sum_{i=1}^N d_i \log(\alpha_i) \\ \text{s.t.} \quad & Hc + |HG|\alpha \leq h, \\ & 0 \leq \alpha \leq 1 \end{aligned} \quad (\text{max-in})$$

where  $d_i \geq 0$  are constants and  $|HG|$  is a matrix obtained by taking element-wise absolute value of matrix  $HG$ . The H-rep  $(H, h)$  of the disturbance set  $W$ , which is available by Assumption A2, is used to formulate the above problem. Suppose that  $H$  has  $L$  rows. In (max-in), we have

$$\begin{aligned} \# \text{ variables} &= \mathcal{O}(N + n_x), \\ \# \text{ constraints} &= \mathcal{O}(N + L). \end{aligned} \quad (6)$$

**Proposition 2.** Let  $(\underline{\alpha}, \underline{c})$  be the maximizer of optimization problem (max-in). Define  $\underline{\mathfrak{Z}}(Z, EW) = ([\underline{\alpha}_1 g_1, \underline{\alpha}_2 g_2, \dots, \underline{\alpha}_N g_N], \underline{c})$ . We have  $\underline{\mathfrak{Z}}(Z, EW) \subseteq EW$ .

*Proof.* We first show that, for  $\alpha \geq 0$  and any  $c$ ,  $Hc + |HG|\alpha \leq h$  if and only if

$$\forall \theta \in \prod_{i=1}^N [-\alpha_i, \alpha_i] : H(c + \sum_{i=1}^N \theta_i g_i) \leq h, \quad (7)$$

where  $\theta_i$  is the  $i^{\text{th}}$  element of  $\theta$ . Let  $H_\ell$  and  $h_\ell$  be the  $\ell^{\text{th}}$  row of  $H$  and  $h$  respectively. Eq. (7) is equivalent to

$$\forall \ell \in \{1, 2, \dots, L\} : \begin{aligned} \max_{\theta} \quad & H_\ell(c + G\theta) \leq h_\ell \\ \text{s.t.} \quad & \theta \in \prod_{i=1}^N [-\alpha_i, \alpha_i] \end{aligned} \quad (8)$$

$\Updownarrow$

$$\forall \ell \in \{1, 2, \dots, L\} : H_\ell c + |H_\ell G| \alpha \leq h_\ell. \quad (9)$$

Eq. (9) is equivalent to  $Hc + |HG|\alpha \leq h$ . Therefore the maximizer  $(\underline{\alpha}, \underline{c})$  satisfies Eq. (7), which implies

$$\forall \theta' \in \prod_{i=1}^N [-1, 1] : H(\underline{c} + \sum_{i=1}^N \theta'_i \underline{\alpha}_i g_i) \leq h. \quad (10)$$

That is,  $([\underline{\alpha}_1 g_1, \underline{\alpha}_2 g_2, \dots, \underline{\alpha}_N g_N], \underline{c}) \subseteq EW$ .  $\square$

Again, the maximal (in the set inclusion sense) inner approximation does not exist in general. Here we maximize the volume of a hyper-rectangle in  $\mathbb{R}^N$ , defined by  $d_i$  and  $\alpha$ . In particular, as a heuristic, we pick  $d_i = \|g_i\|$  for  $i = 1, 2, \dots, N$  throughout the paper.

3) *Efficient Minkowski Difference between Aligned Zonotopes*: Next, we show that the Minkowski difference amounts to element-wise generator subtraction when the subtrahend zonotope is aligned with the minuend zonotope.

**Proposition 3.** Let  $Z = (G, c)$  and  $Z' = (G', c')$  be zonotopes and suppose that  $Z'$  is aligned with  $Z$ . Then  $Z \ominus Z' = ([ (1 - \alpha_1) g_1, (1 - \alpha_2) g_2, \dots, (1 - \alpha_N) g_N ], c - c')$ .

*Proof.* Let  $\Delta := ([ (1 - \alpha_1) g_1, (1 - \alpha_2) g_2, \dots, (1 - \alpha_N) g_N ], c - c')$ . By Lemma 1 iii),  $\Delta = \Delta \oplus Z' \ominus Z'$  as  $\Delta, Z'$  are convex, compact and nonempty. Also note that  $\Delta \oplus Z' = Z$ , hence  $\Delta = Z \ominus Z'$ .  $\square$

We summarize this part by the following proposition.

**Proposition 4.** Let  $Z$  be a zonotope and let  $\bar{\mathfrak{Z}}(Z, EW)$ ,  $\underline{\mathfrak{Z}}(Z, EW)$  be defined by solving (min-out), (max-in) respectively, then  $Z \ominus \bar{\mathfrak{Z}}(Z, EW) \subseteq Z \ominus EW \subseteq Z \ominus \underline{\mathfrak{Z}}(Z, EW)$ .

Particularly,  $Z \ominus \bar{\mathfrak{Z}}(Z, EW)$  and  $Z \ominus \underline{\mathfrak{Z}}(Z, EW)$  can be computed efficiently with generator-wise subtraction.

*Proof.* It follows from Proposition 1-3 and the fact that both  $\bar{\mathfrak{Z}}(Z, EW)$  and  $\underline{\mathfrak{Z}}(Z, EW)$  are aligned with  $Z$ .  $\square$

### B. Approximation of Backward Reachable Sets

We can compute a zonotopic over/under-approximation of the BRS  $X_k$  recursively as follows:

$$\underline{Z}_0 = \bar{Z}_0 = Z_0, \quad (11)$$

$$\underline{Z}_{k+1} = A^{-1}(\underline{Z}_k \ominus \bar{\mathfrak{Z}}(\underline{Z}_k, EW) \oplus -BU - K), \quad (12)$$

$$\bar{Z}_{k+1} = A^{-1}(\bar{Z}_k \ominus \underline{\mathfrak{Z}}(\bar{Z}_k, EW) \oplus -BU - K). \quad (13)$$

**Proposition 5.** Let  $X_k$  be defined by Eq. (2), and  $\underline{Z}_k, \bar{Z}_k$  be defined by Eq. (11)-(13), we have  $\underline{Z}_k \subseteq X_k \subseteq \bar{Z}_k$ .

*Proof.* We prove this by induction. When  $k = 0$ ,  $\underline{Z}_0 = \bar{Z}_0 = Z_0 = X_0$  by (11). Suppose that  $\underline{Z}_k \subseteq X_k \subseteq \bar{Z}_k$ , we have

$$\begin{aligned} \underline{Z}_k \ominus \bar{\mathfrak{Z}}(\underline{Z}_k, EW) &\subseteq \underline{Z}_k \ominus EW & (\text{Proposition 4}) \\ &\subseteq X_k \ominus EW. & (\underline{Z}_k \subseteq X_k) \end{aligned} \quad (14)$$

Combining Eq. (14), (11) and Eq. (4) yields  $\underline{Z}_{k+1} \subseteq X_{k+1}$ . Similarly, one can show  $X_{k+1} \subseteq \bar{Z}_{k+1}$ .  $\square$

Eq. (12), (13) only involve Minkowski addition, linear transformation of zonotopes and Minkowski difference where the subtrahend zonotope is aligned with the minuend zonotope. The above three operations can be done efficiently with G-rep manipulations. The time for computing  $\underline{Z}_k$  grows modestly with  $k$  because the number of  $\underline{Z}_k$ 's generators, denoted by  $N_k$ , increases linearly with  $k$ . In fact,  $N_{k+1} = N_k + N_U$  where  $N_U$  is the (constant) number of generators of the zonotopic control input set  $U$ . In what follows, we introduce an order reduction technique to upper bound the time complexity of computing  $\underline{Z}_k$ .

*1) Zonotope Order Reduction:* The order of an  $n$ -dimensional zonotope with  $N$  generators is defined to be  $N/n$ . Zonotope order reduction problem concerns approximating a given zonotope with another one with lower order. Most of the existing techniques focus on finding outer approximations because zonotopes are typically used to overestimate forward reachable sets. Whereas in this paper, we find inner approximations using the following fact.

**Proposition 6.** Let  $Z = (G = [g_1, g_2, \dots, g_N], c)$  be a zonotope. Define  $G_1$  to be the matrix after removing arbitrary two columns  $g_i, g_j$  from  $G$  and appending  $g_i + g_j$ , and define  $G_2$  to be the matrix after removing columns  $g_i, g_j$  from  $G$  and appending  $g_i - g_j$ . Then  $Z_1 = (G_1, c) \subseteq Z$  and  $Z_2 = (G_2, c) \subseteq Z$ .

*Proof.* Let  $l(g_k) := \{\theta g_k \mid \theta \in [-1, 1]\}$ , then  $Z = c + \bigoplus_{k=1}^N l(g_k)$ ,  $Z_1 = c + \bigoplus_{k \neq i, j}^N l(g_k) \oplus l(g_i + g_j)$ . Since  $l(g_i + g_j) = \{\theta g_1 + \theta g_2 \mid \theta \in [-1, 1]\} \subseteq \{\theta_1 g_1 + \theta_2 g_2 \mid \theta_1, \theta_2 \in [-1, 1]\} = l(g_i) \oplus l(g_j)$ ,  $Z_1 \subseteq Z$ . Similarly,  $Z_2 \subseteq Z$ .  $\square$

Note that, in Proposition 6, the number of generators of  $Z_1$  (or  $Z_2$ ) is fewer than that of  $Z$  by one. Our zonotope order

reduction procedure will keep replacing two generators  $g_i, g_j$  by their combination (either  $g_i + g_j$  or  $g_i - g_j$ ) until the order of the resulting zonotope is small enough. Particularly, we use the following heuristic to select  $g_i, g_j$ :

$$(i, j) = \arg \min_{1 \leq i < j \leq N} \|g_i\|_2 \|g_j - \hat{g}_i g_j^\top \hat{g}_i\|_2, \quad (15)$$

where  $\hat{g}_i = \frac{g_i}{\|g_i\|_2}$ . Then we will add  $g_i + g_j$  if  $\|(g_i + g_j)^\top G'\|_2 \geq \|(g_i - g_j)^\top G'\|_2$ , and add  $g_i - g_j$  otherwise, where  $G'$  is the transpose of the right inverse of the generator matrix after removing columns  $g_i, g_j$ . The idea is to combine two generators that are either closely aligned or small in 2-norm, and the combined generator should be larger and more perpendicular to the remaining generators.

*2) Deriving Reachability Control Law using  $\underline{Z}_k$ :* Once zonotopic inner approximations  $\underline{Z}_k$  of the BRSs are computed, checking if a state  $x$  belongs to  $\underline{Z}_k$  amounts to solving a linear program. Moreover, for any state  $x \in \underline{Z}_{k+1}$ , we can find a control input  $u \in U(x, \underline{Z}_k)$  that brings  $x$  to  $\underline{Z}_k$  in one step, where  $U(x, \underline{Z}_k)$  is defined to be

$$\begin{aligned} &\{u \in U \mid \forall w \in W : Ax + Bu + Ew + K \in \underline{Z}_k\} \\ &= \text{Proj}_u \left\{ (u, \theta) \mid \begin{array}{l} Ax + Bu + K = \\ c^{(k)} + \sum_{i=1}^{N_k} \theta_i g_i^{(k)}, \\ u \in U, \theta_i \in [-1, 1] \end{array} \right\}, \end{aligned} \quad (16)$$

where  $([g_1^{(k)}, g_2^{(k)}, \dots, g_{N_k}^{(k)}], c^{(k)})$  is the G-rep of  $\underline{Z}_k \ominus \bar{\mathfrak{Z}}(\underline{Z}_k, EW)$ , which can be saved during the computation (see Eq. (12)). We do not need to explicitly perform the projection step in Eq. (16) as it is sufficient to find one  $u \in U(x, \underline{Z}_k)$  by solving a linear program. For any initial state  $x_0 \in \underline{Z}_k$ , iteratively applying  $u_t \in U(x_t, \underline{Z}_{k-t-1})$  yields a feedback control strategy, which generates a sequence  $u_0, u_1, \dots, u_{k-1}$  and drives the initial state  $x_0$  into the target set  $\underline{Z}_0 = Z_0$  in precisely  $k$ -steps, regardless of the disturbance inputs.

## V. EVALUATION & DISCUSSION

### A. Comparisons

We compare our approach for under-approximating  $Z \ominus EW$  with two other methods: one by Althoff [2] and one based on the work by Sadraddini and Tedrake [23]. Whenever the disturbance set  $W$  is a zonotope in its G-rep,  $Z \ominus EW$  can be estimated by [2], but the result is not guaranteed to be an under-approximation. This approach outperforms the exact computation but is still expensive due to an H-rep manipulation. Alternatively, using the linear encoding of zonotope-containment problems [23], we derive the linear program below that under-approximates  $Z \ominus EW$ :

$$\begin{aligned} &\max_{\Gamma, \gamma, \alpha, c} \sum_{i=1}^N \alpha_i \\ &\text{s.t. } [G_Z \text{diag}(\alpha), EG_W] = G_Z \Gamma \\ &\quad c_Z - (c + Ec_W) = G_Z \gamma \\ &\quad \|\Gamma, \gamma\|_\infty \leq 1, \quad 0 \leq \alpha \leq 1 \end{aligned} \quad (17)$$

where  $(G_W, c_W)$  and  $(G_Z, c_Z)$  are the G-reps of  $W$  and  $Z$  respectively. Similar to our approach, the solution of (17) also gives a zonotopic under-approximation  $(G_Z \text{diag}(\alpha), c)$  of  $Z \ominus EW$  that aligns with the template  $Z$ . The linear program (17) scales differently from (min-out), which dominates

the time of computing BRSs. Let  $N_W$  and  $N$  be the number of generators of  $W$  and  $Z$  respectively. For (17),

$$\begin{aligned} \# \text{variables} &= \mathcal{O}(N(N + N_W) + n_x), \\ \# \text{constraints} &= \mathcal{O}(N + n_x). \end{aligned} \quad (18)$$

The size of (17) is independent of the number of  $W$ 's vertices and grows with  $N_W$ , the number of generators of  $W$ . Thus (17) is more advantageous than (min-out) whenever  $W$  is a high dimensional zonotope with a small order. On the other hand, the number of variables in (min-out) is linear in  $N$ , whereas that in (17) is quadratic in  $N$ .

We randomly generate about 2000 test cases, each case consists of a zonotope  $Z \subseteq \mathbb{R}^{n_x}$ , a hyper-rectangular  $W \subseteq \mathbb{R}^{n_x}$  and a square matrix  $E \in \mathbb{R}^{n_x \times n_x}$ . The Minkowski difference  $Z \ominus EW$  is estimated using the three different methods. Fig. 1 shows the computation time w.r.t. the dimension and the order of zonotope  $Z$ . Each dot represents the time for a specific case, and the surface is plotted with averaged values. All the experiments are run on a 1.80 GHz laptop with 16 GB RAM. The computation time of Althoff's approach grows fast w.r.t. the order and the dimension of  $Z$  (in fact, we could not finish running any one of the higher-order cases after hours). Our approach scales better with the order of  $Z$ , but still grows relatively fast with the dimension  $n_x$  because the number of  $W$ 's vertices grows exponentially with  $n_x$  since we choose  $W$  to be hyper-rectangles in this example. Somewhat surprisingly, the computation time of Sadraddini's approach grows very slowly w.r.t. the order and the dimension of  $Z$ . This is consistent with the big-O analysis: in the largest test case,  $n_x = 10$  and  $N = 100$ , but  $W$  has about  $10^3$  vertices ( $M = 1000$ ). Hence (min-out) has approximately ten times more variables than (17). Another

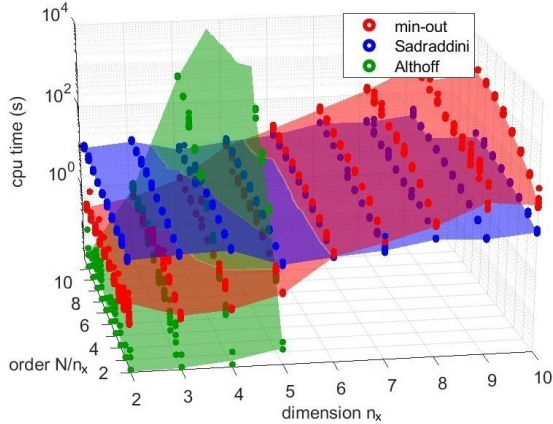


Fig. 1: Upper: computation time for estimating  $Z \ominus EW$ . Lower: volume ratio distribution.

metric is the size of the obtained estimation. The volumes of the obtained zonotopic estimations are comparable. Define  $r_1 = \left( \frac{V_{\text{Althoff}}}{V_{\text{min-out}}} \right)^{1/n_x}$  and  $r_2 = \left( \frac{V_{\text{Sadraddini}}}{V_{\text{min-out}}} \right)^{1/n_x}$ , the statistics of  $r_1, r_2$  are given in the table below.

	mean	std.	min	max	confidence of [0.95, 1.05]
$r_1$	1.0017	0.0577	0.9900	1.3856	98.83%
$r_2$	0.9678	0.1891	0.8372	1.7498	95.10%

## B. Order Reduction

We evaluate our order reduction technique with 29000 randomly generated zonotopes with different dimensions and orders. The approach introduced in Section IV-B.1 is used to reduce the order of each testing zonotope by one. As shown in Fig. 2 (upper), the computation time grows modestly with the zonotope's dimension and order. The quality of the

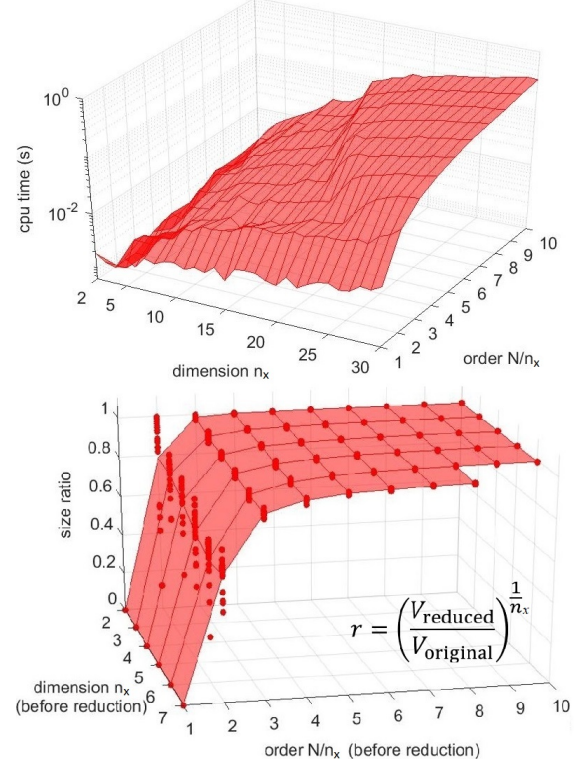


Fig. 2: Upper: averaged computation time for reducing a zonotope's order by one. Lower: volume ratio between the reduced-order zonotope and the one before reduction.

reduced-order zonotope is measured by the ratio between its volume and that of the original zonotope before reduction, defined in Fig. 2 (lower). We are able to run this evaluation for lower-dimensional cases because computing the exact volume of a zonotope is difficult for high-dimensional case due to the combinatorial complexity [9]. In Fig. 2 (lower), the volume ratio increases with the original zonotope's order because higher order means more freedom in selecting the generators to combine. In the presented cases, the ratio is close to one if the original zonotope's order is greater than three.

## VI. CASE STUDIES

### A. Aircraft Position Control.

With an aircraft position control system, we illustrate the overall BRS computation approach that combines Minkowski difference and order reduction to implement the iterations in Eq. (11)-(13). The linearized 6D lateral dynamics and the 6D longitudinal dynamics of the aircraft are in the form of Eq. (1), whose  $A, B$  matrices are given in Eq. (19). For both systems,  $E_{\text{lat}} = E_{\text{long}} = I$ . The



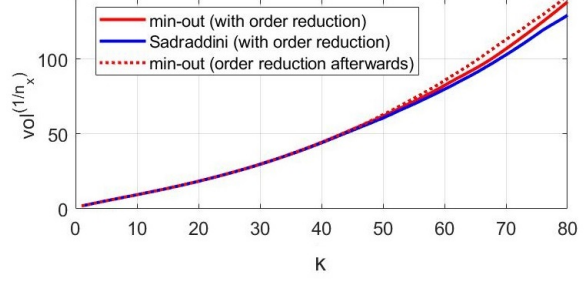
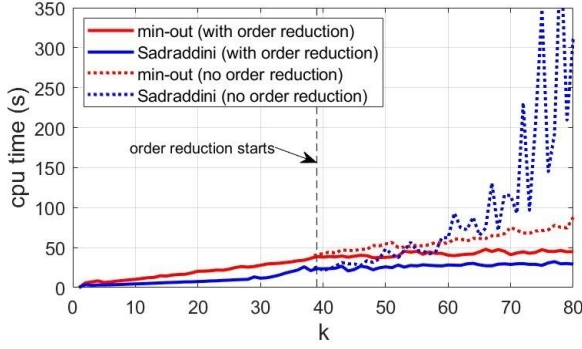


Fig. 3: Backward reachable set computation for lateral dynamics. Left: computation time. Right: set volume.

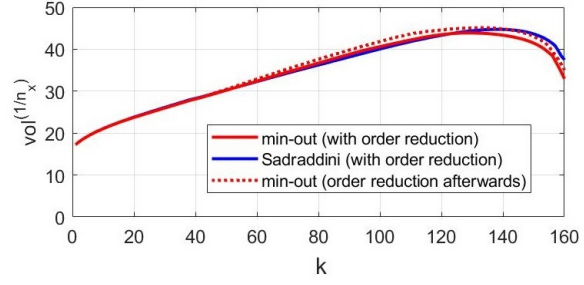
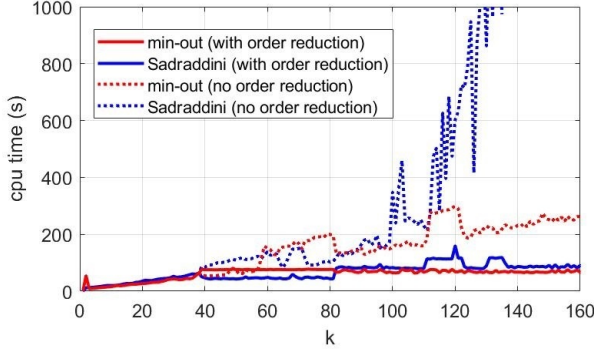


Fig. 4: Backward reachable set computation for longitudinal dynamics. Left: computation time. Right: set volume.

states of the lateral and longitudinal dynamics are  $x_{\text{lat}} = [v, p, r, \phi, \psi, y]^T$  and  $x_{\text{long}} = [u, w, q, \theta, x, h]^T$  respectively, and control inputs are  $u_{\text{lat}} = [\delta_a, \delta_r]^T$  and  $u_{\text{long}} = [\delta_e, \delta_t]^T$  respectively (see TABLE I and Fig. 5). We assume that the disturbance sets are hyper-boxes and their G-rep are  $W_{\text{lat}} = (\text{diag}([0.037, 0.00166, 0.0078, 0.00124, 0.00107, 0.07229]), 0)$  and  $W_{\text{long}} = (\text{diag}([0.3025, 0.4025, 0.01213, 0.006750, 1.373, 1.331]), 0)$ .

$$\begin{aligned}
 A_{\text{lat}} &= \begin{bmatrix} 1.004 & 0.1408 & 0.3095 & -0.3112 & 0 & 0 \\ 0.03015 & 1.177 & 0.6016 & -0.6029 & 0 & 0 \\ -0.02448 & -0.1877 & 0.3803 & 0.5642 & 0 & 0 \\ -0.01057 & -0.09588 & -0.3343 & 1.277 & 0 & 0 \\ 0.0003943 & 0.0095901 & -0.005341 & -0.007447 & 1 & 0 \\ -0.2579 & -23.32 & -51.03 & 61.35 & -37.86 & 1 \end{bmatrix}, \\
 A_{\text{long}} &= \begin{bmatrix} 0.9911 & -0.04858 & -0.01709 & -0.4883 & 0 & 0 \\ 0.0005870 & 0.9968 & 0.5168 & -0.0001398 & 0 & 0 \\ 0.0002070 & -0.001123 & 0.9936 & -5.092 \times 10^{-5} & 0 & 0 \\ 1.907 & -1.032 & 0.01832 & 1 & 0 & 0 \\ -0.04601 & 0.001125 & 0.0002638 & 0.01130 & 1 & 0 \\ -5.095 \times 10^{-5} & -0.1874 & -0.01185 & 4.004 & 0 & 1 \end{bmatrix}, \\
 B_{\text{lat}} &= \begin{bmatrix} -0.1189 & 0.007812 \\ -0.1217 & 0.2643 \\ 0.01773 & -0.2219 \\ -0.02882 & -0.09982 \\ -0.0005607 & 0.002437 \\ 0.1120 & -0.5785 \end{bmatrix}, \\
 B_{\text{long}} &= \begin{bmatrix} 1.504 & 7.349 \times 10^{-5} \\ -0.04645 & -3.421 \times 10^{-6} \\ -0.009812 & -1.488 \times 10^{-6} \\ -9.080 \times 10^{-5} & -1.371 \times 10^{-8} \\ -0.03479 & -1.700 \times 10^{-6} \\ 0.004171 & 2.913 \times 10^{-7} \end{bmatrix}. \quad (19)
 \end{aligned}$$

For both the lateral and longitudinal dynamics, we can efficiently compute their  $k$ -step BRSs using the proposed approach for reasonably large horizons  $k$ , whereas the computation gets stuck at  $k = 3$  using the exact Minkowski

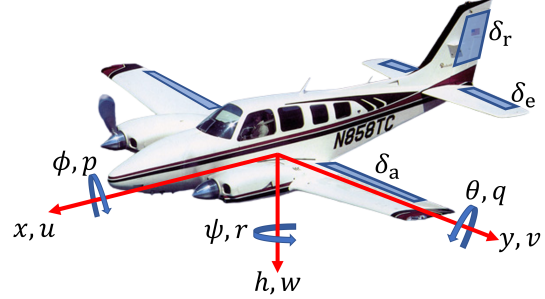


Fig. 5: Illustration of the states and control inputs.

difference provided by MPT3 [11], or the approximation function implemented in CORA. Fig. 3, 4 show the results for the lateral dynamics and the longitudinal dynamics, respectively. In each figure, the left (right, resp.) plot shows the cpu time for computing  $\underline{Z}_k$  (the size of  $\underline{Z}_k$ , resp.) versus  $k$ , the number of backward expansion steps. The red curves are for our approach and the blue ones for the approach using Sadraddini's zonotope containment encoding.

1) *Cpu time plots*: The solid (dotted, resp.) lines correspond to the computation time with (without, resp.) zonotope order reduction. Using the order reduction technique (activated at  $k = 39$ ), our approach and Sadraddini's approach give comparable results. Without order reduction, the computation time of Sadraddini's approach (dotted blue) grows faster w.r.t.  $k$  than ours (dotted red). This is consistent with the big-O analysis because in our approach, the time-dominant Minkowski difference step amounts to solving a linear pro-

TABLE I: Variables in the aircraft model

variable	physical meaning	range	unit
$v$	velocity	$[-1, 1]$	m/s
$p$	roll angular rate	$[-1, 1]$	rad/s
$r$	yaw angular rate	$[-1, 1]$	rad/s
$\phi$	roll angle	$[-\pi/5, \pi/5]$	rad
$\psi$	yaw angle	$[-\pi/5, \pi/5]$	rad
$y$	lateral deviation	$[-2, 2]$	m
$u$	velocity	$[40, 60]$	m/s
$w$	velocity	$[0, 10]$	rad/s
$q$	pitch angular rate	$[-0.1, 0.1]$	rad/s
$\theta$	pitch angle	$[-\pi, \pi]$	rad
$x$	horizontal displacement	$[0, 800]$	rad
$h$	altitude	$[260, 390]$	m
$\delta_a$	aileron deflection	$[-\pi, \pi]$	m
$\delta_r$	rudder deflection	$[-\pi, \pi]$	m
$\delta_e$	elevator deflection	$[-0.262, 0.524]$	m
$\delta_t$	throttle control	$[0, 10^4]$	m

gram whose number of variables is linear in  $k$  (proportional to the  $\underline{Z}_k$ 's order), whereas the number of variables is quadratic in  $k$  in Sadraddini's formulation. Although our approach scales well even without order reduction, order reduction is still important in efficiently storing the zonotopic BRSs and deriving the control law.

2) *Volume plots*: The solid lines correspond to the results with order reduction "in the loop" (i.e., in the  $k^{\text{th}}$  step,  $\underline{Z}_k$  is reduced to a certain order before  $\underline{Z}_{k+1}$  is computed). Whereas the dotted lines correspond to the results with order reduction after all  $\underline{Z}_k$ 's are computed (Ideally, we would like to compute  $\underline{Z}_k$ 's volume without order reduction at all, but this is impossible with the off-the-shelf volume computation tools in CORA because the complexity is combinatorial in  $\underline{Z}_k$ 's order). The two approaches give comparable results with order reduction. Moreover, since the dotted red line and the solid red line are close to each other, this indicates that the "wrapping effect" due to the order reduction in-the-loop is relatively small.

### B. Double Integrator with Uncontrollable Subspace

With a 10D system, we show the effectiveness of the reachability controller derived from the zonotopic BRSs as described in Section IV-B.2. The system consists of a double-integrator dynamics in the 3D space and a 4D uncontrollable subspace (the uncontrollable part affects the controllable part). The continuous-time dynamics is

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_7 + x_{10} + w_1, & \dot{x}_2 &= u_1 + w_2, \\
\dot{x}_3 &= x_4 - x_8 + w_3, & \dot{x}_4 &= u_2 + w_4, \\
\dot{x}_5 &= x_6 + x_9 + w_5, & \dot{x}_6 &= u_3 + w_6, \\
\dot{x}_7 &= -0.01x_7 + x_8 + w_7, & \dot{x}_8 &= -x_8 - 0.01x_7 + w_8, \\
\dot{x}_9 &= -10^{-4}x_7 + 2x_{10} + w_9, \\
\dot{x}_{10} &= -2x_9 - 10^{-4}x_{10} + w_{10}.
\end{aligned} \tag{20}$$

We discretize the above dynamics with a sampling period  $\Delta t = 0.5\text{s}$ , and define the disturbance set  $W$  so that  $w_{\{1,3,5\}} \in [-0.12, 0.12]$ ,  $w_{\{2,4,6\}} \in [-0.2, 0.2]$ ,  $w_{\{7,8,9,10\}} \in [-0.1, 0.1]$ , and the control set  $U = [-0.5, 0.5]^3$ . Starting from a randomly picked initial condition in  $\underline{Z}_{50}$ , our goal is to reach a final state for which

$x_i \in [9.5, 10.5]$  for  $i \in \{1, 3, 5\}$  and  $x_i \in [-0.5, 0.5]$  for the remaining  $i$ 's. We defined a controller as described in Section IV-B.2. Fig. 6 shows a closed-loop trajectory under random disturbances. The small target set is reached despite the oscillating uncontrollable dynamics.

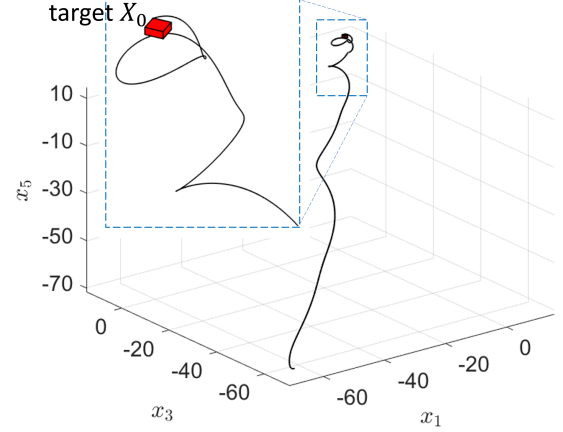


Fig. 6: A closed-loop trajectory for the double-integrator dynamics. The red box is the target set.

## VII. CONCLUSION

In this paper, we develop an approach that under-approximates the backward reachable sets for uncertain linear systems using zonotopes. The main technical ingredients are i) under-approximating the Minkowski difference between two zonotopes and ii) an order reduction technique tailored to enclosed zonotopes. These developments were evaluated with randomly generated instances and two case studies. Experiments show that our method is more scalable than the off-the-shelf tools (MPT3, CORA) and scales differently from the approach based on Sadraddini's zonotope-inclusion technique. In our method, the dominant Minkowski subtraction step requires solving a linear program whose size is linear in the zonotope's order, while that dependency is quadratic in Sadraddini's approach. We will investigate extending our approach to nonlinear systems in the future.

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## REFERENCES

- [1] M. Althoff. An introduction to CORA 2015. In *1st and 2nd Intl. Workshop on ARCH*, pages 120–151, 2015.
- [2] M. Althoff. On computing the minkowski difference of zonotopes. *arXiv preprint arXiv:1512.02794*, 2015.
- [3] M. Althoff, G. Frehse, and A. Girard. Set propagation techniques for reachability analysis. *Annual Review of Control, Robotics, and Autonomous Systems*, 4:369–395, 2021.
- [4] D. Bertsekas. Infinite time reachability of state-space regions by using feedback control. *IEEE TAC*, 17(5):604–613, 1972.
- [5] F. Blanchini and S. Miani. *Set-theoretic methods in control*. Springer, 2008.
- [6] M. Chen, Q. Tam, S. C. Livingston, and M. Pavone. Signal temporal logic meets reachability: Connections and applications. In *Intl. WAFR*, pages 581–601. Springer, 2018.

- [7] G. Chou, Y. E. Sahin, L. Yang, K. J. Rutledge, P. Nilsson, and N. Ozay. Using control synthesis to generate corner cases: A case study on autonomous driving. *IEEE TCAD*, 37(11):2906–2917, 2018.
- [8] A. Girard. Reachability of uncertain linear systems using zonotopes. In *Intl. Workshop on HSCC*, pages 291–305. Springer, 2005.
- [9] E. Gover and N. Krikorian. Determinants and the volumes of parallelotopes and zonotopes. *Linear Algebra Its Appl.*, 433(1):28–40, 2010.
- [10] D. Han, A. Rizaldi, A. El-Guindy, and M. Althoff. On enlarging backward reachable sets via zonotopic set membership. In *ISIC*, pages 1–8. IEEE, 2016.
- [11] M. Herceg, M. Kvasnica, C. Jones, and M. Morari. Multi-Parametric Toolbox 3.0. In *Proc. of ECC*, pages 502–510, Zürich, Switzerland, July 17–19 2013. <http://control.ee.ethz.ch/~mpt>.
- [12] N. Kochdumper and M. Althoff. Computing non-convex inner-approximations of reachable sets for nonlinear continuous systems. In *59th CDC*, pages 2130–2137. IEEE, 2020.
- [13] A.-K. Kopetzki, B. Schürmann, and M. Althoff. Methods for order reduction of zonotopes. In *56th CDC*, pages 5626–5633. IEEE, 2017.
- [14] A. A. Kurzhanskiy and P. Varaiya. Reach set computation and control synthesis for discrete-time dynamical systems with disturbances. *Automatica*, 47(7):1414–1426, 2011.
- [15] J. B. Lasserre. Tractable approximations of sets defined with quantifiers. *Mathematical Programming*, 151(2):507–527, 2015.
- [16] Y. Li. *Robustly complete temporal logic control synthesis for nonlinear systems*. PhD thesis, University of Waterloo, 2019.
- [17] Y. Li and J. Liu. Invariance control synthesis for switched nonlinear systems: An interval analysis approach. *IEEE TAC*, 63(7):2206–2211, 2017.
- [18] L. Liebenwein, W. Schwarting, C.-I. Vasile, J. DeCastro, J. Alonso-Mora, S. Karaman, and D. Rus. Compositional and contract-based verification for autonomous driving on road networks. In *Robotics Research*, pages 163–181. Springer, 2020.
- [19] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3):349–370, 1999.
- [20] I. M. Mitchell. Comparing forward and backward reachability as tools for safety analysis. In *Intl. Workshop on HSCC*, pages 428–443. Springer, 2007.
- [21] I. M. Mitchell, A. M. Bayen, and C. J. Tomlin. A time-dependent hamilton-jacobi formulation of reachable sets for continuous dynamic games. *IEEE TAC*, 50(7):947–957, 2005.
- [22] L. Montejano. Some results about minkowski addition and difference. *Mathematika*, 43:265–273, 1996.
- [23] S. Sadraddini and R. Tedrake. Linear encodings for polytope containment problems. In *58th CDC*, pages 4367–4372. IEEE, 2019.
- [24] S. W. Smith, P. Nilsson, and N. Ozay. Interdependence quantification for compositional control synthesis with an application in vehicle safety systems. In *55th CDC*, pages 5700–5707. IEEE, 2016.
- [25] L. Yang and N. Ozay. Efficient safety control synthesis with imperfect state information. In *59th CDC*, pages 874–880. IEEE, 2020.
- [26] X. Yang and J. K. Scott. A comparison of zonotope order reduction techniques. *Automatica*, 95:378–384, 2018.