JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 4, October 2021, Pages 909–936
https://doi.org/10.1090/jams/967
Article electronically published on June 7, 2021

ON THE CONSTANT SCALAR CURVATURE KÄHLER METRICS (I)—A PRIORI ESTIMATES

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Dedicated to Sir Simon Donaldson for his 60th birthday

1. Introduction

This is the first of two papers in the study of constant scalar curvature Kähler metrics (cscK metrics), following a program outlined in [14]. In this paper, we focus on establishing a priori estimates for cscK metrics on compact Kähler manifolds without boundary. Our estimates can be easily adapted to extremal Kähler metrics and for simplicity of presentations, we leave such an extension to the interested readers except to note that for extremal Kähler metrics, its scalar curvature is apriori bounded depending on Kähler class. In the subsequent two papers, we will use these estimates (and their generalizations) to study the Calabi-Donaldson theory on the geometry of extremal Kähler metrics, and in particular, to establish the celebrated conjecture of Donaldson on geodesic stability (the L^1 version) as well as the well known properness conjecture relating the existence of cscK metrics with the properness of K-energy functional.

Let us recall a conjecture made earlier by the first named author (c.f. [20]).

Conjecture 1.1. *Let $(M, [\omega_0])$ be any compact Kähler manifold without boundary. Suppose ω_φ is a constant scalar curvature Kähler metric. If φ is uniformly bounded, then any higher derivatives of φ are also uniformly bounded.*

It is worthwhile to give a brief review of the history of this subject and hopefully, this will make it self-evident why this conjecture is interesting. A special case of constant scalar curvature Kähler metric is the well known KE metric which has been the main focus of Kähler geometry since the inception of the celebrated Calabi conjecture [7] on Kähler Einstein metrics in 1950s. In 1958, E. Calabi published the fundamental C^3 estimate for Monge-Ampère equations [8] which later played a crucial role in Yau's seminal resolution of Calabi conjecture [48] in 1976 when the first Chern class is either negative or zero (In negative case, T. Aubin [1] has an independent proof) . This work of Yau is so influential that generations of experts in Kähler geometry afterwards largely followed the same route: Securing a C^0 estimate first, then move on to obtain C^2, C^3 estimates etc. In the case of positive first Chern class, G. Tian proved Calabi conjecture in 1989 [44] for Fano surfaces when the automorphism group is reductive. It is well known that

Received by the editors February 27, 2018, and, in revised form, February 24, 2020, June 3, 2020, and July 26, 2020.

2020 *Mathematics Subject Classification.* Primary 53C21, 53C55; Secondary 35J30, 35J60, 35J96.

The first named author was partially supported by NSF grant DMS-1515795 and Simons Foundation grant 605796.

there are obstructions to the existence of KE metrics in Fano manifolds; around 1980s, Yau proposed a conjecture which relates the existence of Kähler Einstein metrics to the stability of underlying tangent bundles. This conjecture was settled in 2012 through a series of work CDS [16] [17] [18] and we refer interested readers to this set of papers for further references in the subject of KE metrics. The proof of CDS's work is itself quite involved as it sits at the intersection of several different subjects: algebraic geometry, several complex variables, geometry analysis and metric differential geometry etc.

To move beyond CDS's work on Kähler Einstein metrics, one direction is the study of the existence problem of cscK metrics which satisfy a 4th order PDE. The following is a conjecture which is a refinement of Calabi's original idea that every Kähler class must have its own best, canonical representatives.

Conjecture 1.2 (Yau-Tian-Donaldson). *conj1.2 Let $[\omega_0]$ be an integral class induced by a line bundle $L \to M$. There exists a cscK metric in $[\omega_0]$ if and only if (L, M) is K-stable.*

One conspicuous and memorable feature of CDS's proof is the heavy use of Cheeger-Colding theory on manifolds with Ricci curvature bounded from below. The apriori bound on Ricci curvature for KE metrics makes such an application of Cheeger-Colding theory seamlessly smooth and effective. However, if we want to attack this general conjecture, there will be a dauntingly high wall to climb since there is no apriori bound on Ricci curvature. Therefore, the entire Cheeger-Colding theory needs to be re-developed if it is at all feasible. On the other hand, there is a second, less visible but perhaps even more significant feature of CDS's proof is: The whole proof is designed for constant scalar curvature Kähler metrics and the use of algebraic criteria and Cheeger-Colding theory is to conclude that the a C^0 bound holds for Kähler potential so that we can apply the apriori estimates for complex KE metrics developed by Calabi, Yau and others. Indeed, this is exactly how we make use of Cheeger-Colding theory and stability condition in CDS's proof to nail down a C^0 estimate on potential. Unfortunately, such an estimate is missing in this generality for a 4th order fully nonlinear equation. Indeed, as noted by other famous authors in the subject as well, the difficulties permeating the cscK theory are two folds: one cannot use maximal principle from PDE point of view and one cannot have much control of metric from the bound of the scalar curvature.

In this paper, we want to tackle this challenge and we prove:

Theorem 1.1. *If (M, ω_0) be a Kähler manifold such that the class $[\omega_0]$ admits a cscK metric $\omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$. Let φ be normalized so that $\sup_M \varphi = 0$, then all derivatives of the Kähler potential φ can be estimated in terms of an upper bound of $\int_M \log\big(\frac{\omega_\varphi^n}{\omega_0^n}\big)\omega_\varphi^n$.*

Conjecture 1.1 follows from this theorem. To see this, note that any cscK metric minimizes K-energy. This combined with a bound for $||\varphi||_0$ implies an upper bound for the entropy functional $\int_M \log\big(\frac{\omega_\varphi^n}{\omega_0^n}\big)\omega_\varphi^n$. Hence Conjecture 1.1 follows from Theorem 1.1. This is explained in more detail in section 5.

With later applications in mind, we study equation of general type:

$$R_\varphi = f + \Delta_\varphi\, \eta.$$

In the above, f is a given smooth function and η is a given smooth real $(1,1)$ form. In local coordinates, $\eta = \sqrt{-1}\eta_{i\bar{j}}dz_i \wedge d\bar{z}_j$. We remark that when $\eta = Ric_g$

and $f = \underline{R}$,[1] this gives rise to the well known constant scalar curvature Kähler metric equation. We can re-write this 4th order equation in a coupled second order equations:

$$\log\det(g_{\alpha\bar{\beta}} + \varphi_{\alpha\bar{\beta}}) = F + \log\det(g_{\alpha\bar{\beta}}), \tag{1.1}$$

$$\Delta_\varphi F = -f + tr_\varphi \eta. \tag{1.2}$$

Theorem 1.1 can be extended to a more general version.

Theorem 1.2. *Suppose (M, ω_φ) satisfy the coupled equations (1.1), (1.2). Let φ be normalized so that $\sup_M \varphi = 0$, then all derivatives of the Kähler potential φ can be estimated in terms of an upper bound of $\int_M \log\big(\frac{\omega_\varphi^n}{\omega_0^n}\big)\omega_\varphi^n$ and $|f|_{L^\infty}$, $|\eta|_{\omega_0}$.*

Now we present technical theorems which lead to this main theorem. Indeed, these technical theorems are interesting in its own right and may be used in other applications.

Theorem 1.3 (Corollary 5.1)**.** *Let φ be a smooth solution to (1.1), (1.2), then for any $1 < p < \infty$, there exists a constant C, depending only on the background Kähler metric (M, g), an upper bound of $\int_M e^F F dvol_g$, and p, such that*

$$||e^F||_{L^p(dvol_g)} \le C, \ ||\varphi||_0 \le C. \tag{1.3}$$

The constants C in the theorems below can change from line to line. More generally, throughout this paper, the "C" without subscript may change from line to line, while if there is subscript, then it is some fixed constant.

Theorem 1.4 (Corollary 5.2)**.** *Let φ be a smooth solution to (1.1), (1.2), then there exists a constant C, depending only on the background metric (M, g) and an upper bound for $\int_M e^F F dvol_g$, such that*

$$e^F \le C. \tag{1.4}$$

Theorem 1.5 (Theorem 2.1)**.** *Let φ be a smooth solution to (1.1) and (1.2), then there exists a constant C, depending only on $||\varphi||_0$, and the background metric g, such that*

$$\max_M e^{\frac{F}{n}} \le C \max_M |\nabla\varphi|^2. \tag{1.5}$$

For the second order estimate, Chen-He[20] established an apriori bound on $n + \Delta\varphi$ in terms of $|\nabla F|_{L^p}(p > 2n)$ via integral estimates, in the absence of (1.2). Inspired by this paper [20] and utilizing the additional equation (1.2), we are able to obtain a $W^{2,p}$ estimate for any $p > 0$, using only $||F||_0$. Theorem 1.9 is used essentially in this estimate.

Theorem 1.6 (Theorem 3.1, Corollary 3.1)**.** *Let φ be a smooth solution to (1.1), (1.2), then for any $1 < p < \infty$, there exists a constant $\alpha(p) > 0$, depending only on p, and another constant C, depending only on $||\varphi||_0$, the background metric g, and p, such that*

$$\int_M e^{-\alpha(p)F}(n + \Delta\varphi)^p \le C. \tag{1.6}$$

In particular, $||n + \Delta\varphi||_{L^p(dvol_g)} \le C'$, where C' has the same dependence as C in this theorem, but additionally on $||F||_0$.

[1] Here Ric_g is the Ricci curvature of background metric g and $\underline{R}$ is the average scalar curvature in Kähler class $[\omega]$.

If we can prove an upper bound for F, then the following theorem becomes very interesting.

Theorem 1.7 (Proposition 4.1)**.** *Let φ be a smooth solution to* (1.1), (1.2). *Then there exists $p_n > 1$, depending only on n, and a constant C, depending on $||\varphi||_0$, $||F||_0$, $||n+\Delta\varphi||_{L^{p_n}(dvol_g)}$, and the background metric g, such that*

$$n + \Delta\varphi \leq C. \tag{1.7}$$

We now discuss a direct application in Kähler geometry from Theorem 1.1 and delay more applications to our second and third paper.

Theorem 1.8. *The Calabi flow can be extended as long as the scalar curvature is uniformly bounded.*

Remark 1.1. This is a surprising development. With completely different motivations in geometry, the first named author has a similar conjecture on Ricci flow which states that the only obstruction to the long time existence of Ricci flow is the L^∞ bound of scalar curvature. There has been significant progress in this problem, first by a series of works of B. Wang (c.f. [46], [23]) and more recently by the interesting and important work of Bamler-Zhang [2] and M. Simons [42] in dimension 4.

Theorem 1.8 is a direct consequence of Theorem 1.1 and Chen-He short time existence theorem (c.f. Theorem 3.2 in [19]), where the authors proved the life span of the short time solution depends only on $C^{3,\alpha}$ norm of the initial Kähler potential and lower bound of the initial metric. By assumption, we know that $\partial_t\varphi$ remains uniformly bounded, hence φ is bounded on every finite time interval. On the other hand, since K-energy is decreasing along the flow, in particular K-energy is bounded from above along the flow. Due to a K-energy function decomposition formula in [12] and that φ is bounded, we see that the entropy is bounded as well. Following Theorem 5.3 with $\eta = 0$ and f bounded, the flow remains in a pre-compact subset of $C^{3,\alpha}(M)$ on every finite time interval, hence can be extended.

In light of Theorem 5.3 and a compactness theorem of Chen-Darvas-He [21], a natural question is if one can extend the Calabi flow assuming only an upper bound on Ricci curvature. A more difficult question is whether one-sided bound of the scalar curvature is sufficient for the extension of Calabi flow. Ultimately, the remaining fundamental question is

Conjecture 1.3 (Calabi, Chen)**.** *Initiating from any smooth Kähler potential, the Calabi flow always exists globally.*

Given the recent work by J. Streets [43], Berman-Darvas-Lu[4], the weak Calabi flow always exists globally. Perhaps one can prove this conjecture via improving regularity of weak Calabi flow. On the other hand, one may hope to prove this conjecture on Kähler classes which already admit constant scalar curvature Kähler metrics and prove the flow will converges to such a metric as $t \to \infty$. An important and deep result in this direction is Li-Wang-Zheng's work [35].

We also show that one can estimate the upper bound of F directly in terms of gradient bound of potential φ. This result is not directly needed for our main result, but we believe it is important for follow-up problems in more general settings where full C^2 estimate is not feasible.

Proposition 1.1 (Proposition 2.1). *Let φ be a smooth solution to* (1.1), (1.2), *then there exists a constant C, depending only on $||\varphi||_0$, such that*

$$F \geq -C. \tag{1.8}$$

Theorem 1.9 (Theorem 2.2). *Let φ be a smooth solution to* (1.1), (1.2), *then there exists a constant C, depending only on $||\varphi||_0$ and the background metric g, such that*

$$\frac{|\nabla\varphi|^2}{e^F} \leq C. \tag{1.9}$$

Finally we would like to explain the organization of this paper:

In section 2, we prove Proposition 1.1, Theorem 1.9 and Theorem 1.5.

In section 3, we prove Theorem 1.6 by iteration, which is a crucial step towards the main result.

In section 4, we use iteration again to improve L^p bound of $n + \Delta\varphi$ for $p < \infty$ to an L^∞ bound of $n + \Delta\varphi$, proving Theorem 1.7. This estimate requires a bound for $||n + \Delta\varphi||_{L^p}$ for some $p > p_n$, [2] depending on $||F||_0$. The key ingredient is a calculation for $\Delta_\varphi(|\nabla_\varphi F|^2)$. We also explains how the L^∞ bound of $n + \Delta\varphi$ gives estimates for higher derivatives. This is not new and has been observed in [14].

In section 5, we prove Theorem 1.3 and Theorem 1.4. From these two results, we get estimate for $||F||_0$ and $||\varphi||_0$ depending only on an upper bound of entropy of F defined as $\int_M \log(\frac{\omega_\varphi^n}{\omega_0^n})\omega_\varphi^n$. The key ingredient is the use of α-invariant and the construction of a new test function. On the other hand, if we start with a bound for $||\varphi||_0$, and use the convexity of K-energy along $C^{1,1}$ geodesics, it is relatively easy to get an upper bound of entropy bound of F, hence all higher estimates follow.

2. The volume ratio $\frac{\omega_\varphi^n}{\omega^n}$ and C^1 bound on Kähler potential

The main theorem in this section is to prove that the first derivative of φ is pointwisely controlled by volume ratio e^F from above, assuming a bound for $||\varphi||_0$. Conversely, the bound for $||\nabla\varphi||_0$ can in turn control e^F. However, this control is much weaker since it is of global nature.

First we show that a C^0 bound for φ implies a lower bound for F.

Proposition 2.1. *Let (φ, F) be a smooth solution to* (1.1), (1.2), *then there exists a positive constant C_2 depending only on $||\varphi||_0$, $||f||_0$, $\max_M |\eta|_{\omega_0}$, such that $F \geq -C_2$.*

This proposition first appeared in [33]. However, for the convenience of the reader, we include a proof here.

Proof. This step is relatively easy. Let $p \in M$, we may choose a local normal coordinate in a neighborhood of p, such that

$$g_{i\bar{j}}(p) = \delta_{ij}, \quad \nabla g_{i\bar{j}}(p) = 0, \quad \text{and} \quad \varphi_{i\bar{j}}(p) = \varphi_{i\bar{i}}(p)\delta_{ij}. \tag{2.1}$$

In this paper, we will always work under this coordinate unless specified otherwise. Choose the constant $C_{2.1}$ to be $C_{2.1} = 2\max_M \max_i |\eta_{i\bar{i}}| + \frac{2||f||_0}{n} + 1$. Under this

[2] Here $p_n \leq (3n-3)(4n+1)$. But this most likely is not sharp.

coordinate, we can calculate:

$$\begin{aligned}\Delta_\varphi(F+C_{2.1}\varphi) &= -f+\sum_i\frac{\eta_{i\bar i}}{1+\varphi_{i\bar i}}+C_{2.1}\sum_i\frac{\varphi_{i\bar i}}{1+\varphi_{i\bar i}}\\ &= -f+C_{2.1}n-\sum_i\frac{C_{2.1}-\eta_{i\bar i}}{1+\varphi_{i\bar i}}\le -f+C_{2.1}n-\frac{nC_{2.1}}{2}e^{-\frac{F}{n}}\\ &\le 2C_{2.1}n-\frac{nC_{2.1}}{2}e^{-\frac{F}{n}}.\end{aligned}\tag{2.2}$$

In the second line above, we used the arithmetic-geometric mean inequality:

$$\frac{1}{n}\sum_i\frac{1}{1+\varphi_{i\bar i}}\ge \Pi_i(1+\varphi_{i\bar i})^{-\frac{1}{n}}=e^{-\frac{F}{n}}.$$

Now let p_0 be such that the function $F+C_{2.1}\varphi$ achieves minimum at p_0, then from (2.2), we see

$$0\le 2C_{2.1}n-\frac{nC_{2.1}}{2}e^{-\frac{F}{n}}(p_0).\tag{2.3}$$

This gives a lower bound for F, depending only on C^0 bound for φ. □

Next we move on to estimate the upper bound of F in terms of $\max_M|\nabla\varphi|$. Here and in the following, we denote $\nabla f\cdot\nabla h=Re(g^{i\bar j}f_ih_{\bar j})$, and $\nabla_\varphi\cdot_\varphi\nabla_\varphi f=Re(g_\varphi^{i\bar j}f_ih_{\bar j})$.

Theorem 2.1. *Let (φ,F) be smooth solutions to cscK, then there exists a constant $C_{2.2}$, depending only on $||\nabla\varphi||_0$, $||f||_0$, $\max_M|\eta|_{\omega_0}$ and lower bound of bisectional curvature of the background metric g, such that $F\le C_{2.2}$.*

Proof. The argument uses maximum principle again. This time we will calculate $\Delta_\varphi(e^{F-\lambda\varphi}(K+|\nabla\varphi|^2))$ where $\lambda,K>0$ are constants to be determined later. We choose a normal coordinate (equation (2.1)) and do the following calculations. We have

$$\begin{aligned}\Delta_\varphi(e^{F-\lambda\varphi}(K+|\nabla\varphi|^2)) &= \Delta_\varphi(e^{F-\lambda\varphi})(K+|\nabla\varphi|^2)+e^{F-\lambda\varphi}\Delta_\varphi(K+|\nabla\varphi|^2)\\ &\quad+e^{F-\lambda\varphi}\cdot\frac{(F_i-\lambda\varphi_i)(|\nabla\varphi|^2)_{\bar i}+(F_{\bar i}-\lambda\varphi_{\bar i})(|\nabla\varphi|^2)_i}{1+\varphi_{i\bar i}}.\end{aligned}\tag{2.4}$$

We can first calculate:

$$\begin{aligned}\Delta_\varphi(e^{F-\lambda\varphi}) &= e^{F-\lambda\varphi}\frac{|F_i-\lambda\varphi_i|^2}{1+\varphi_{i\bar i}}+e^{F-\lambda\varphi}(\Delta_\varphi F-\frac{\lambda\varphi_{i\bar i}}{1+\varphi_{i\bar i}})\\ &= e^{F-\lambda\varphi}\frac{|F_i-\lambda\varphi_i|^2}{1+\varphi_{i\bar i}}+e^{F-\lambda\varphi}(-f-\lambda n+\frac{\lambda+\eta_{i\bar i}}{1+\varphi_{i\bar i}}).\end{aligned}\tag{2.5}$$

By differentiating equation (1.1) in z_α direction, we obtain

$$\sum_i\frac{\varphi_{i\bar i\alpha}}{1+\varphi_{i\bar i}}=F_\alpha\ \text{ and }\ \sum_i\frac{\varphi_{i\bar i\bar\alpha}}{1+\varphi_{i\bar i}}=F_{\bar\alpha}.\tag{2.6}$$

Then we calculate

$$
\begin{aligned}
\Delta_\varphi(|\nabla\varphi|^2) &= \frac{R_{\alpha\bar\beta i\bar i}\varphi_\alpha\varphi_{\bar\beta}}{1+\varphi_{i\bar i}} + \frac{|\varphi_{i\alpha}|^2}{1+\varphi_{i\bar i}} + \frac{\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}} + \frac{\varphi_\alpha\varphi_{\bar\alpha i\bar i} + \varphi_{\bar\alpha}\varphi_{\alpha i\bar i}}{1+\varphi_{i\bar i}} \\
&\geq -\frac{C_{2.21}|\nabla\varphi|^2}{1+\varphi_{i\bar i}} + \frac{|\varphi_{i\alpha}|^2}{1+\varphi_{i\bar i}} + \frac{\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}} + \varphi_\alpha F_{\bar\alpha} + \varphi_{\bar\alpha}F_\alpha \\
&= -\frac{C_{2.21}|\nabla\varphi|^2}{1+\varphi_{i\bar i}} + \frac{|\varphi_{i\alpha}|^2}{1+\varphi_{i\bar i}} + \frac{\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}} + \varphi_\alpha(F_{\bar\alpha} - \lambda\varphi_{\bar\alpha}) \\
&\quad + \varphi_{\bar\alpha}(F_\alpha - \lambda\varphi_\alpha) + 2\lambda|\varphi_\alpha|^2 \\
&\geq -\frac{C_{2.21}|\nabla\varphi|^2}{1+\varphi_{i\bar i}} + \frac{|\varphi_{i\alpha}|^2}{1+\varphi_{i\bar i}} + \frac{\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}} - \varepsilon(n+\Delta\varphi) \\
&\quad - \frac{|F_\alpha - \lambda\varphi_\alpha|^2|\varphi_\alpha|^2}{\varepsilon(1+\varphi_{\alpha\bar\alpha})} + 2\lambda|\varphi_\alpha|^2.
\end{aligned} \tag{2.7}
$$

Here $C_{2.21}$ depends only on lower bound of bisectional curvature of g.

For the last term in (2.4), we estimate in the following way:

$$
\begin{aligned}
&\frac{|(F_{\bar i} - \lambda\varphi_{\bar i})(|\nabla\varphi|^2)_i|}{1+\varphi_{i\bar i}} \\
= &\frac{|(F_{\bar i} - \lambda\varphi_{\bar i})(\varphi_\alpha\varphi_{\bar\alpha i} + \varphi_{\bar\alpha}\varphi_{\alpha i})|}{1+\varphi_{i\bar i}} \\
\leq &\frac{|(F_{\bar i} - \lambda\varphi_{\bar i})\varphi_i\varphi_{i\bar i}|}{1+\varphi_{i\bar i}} + \frac{|(F_{\bar i} - \lambda\varphi_{\bar i})\varphi_{\bar\alpha}\varphi_{\alpha i}|}{1+\varphi_{i\bar i}} \\
\leq &\frac{|F_i - \lambda\varphi_i|^2|\varphi_i|^2}{2\varepsilon(1+\varphi_{i\bar i})} + \frac{|F_i - \lambda\varphi_i|^2|\varphi_\alpha|^2}{2\varepsilon(1+\varphi_{i\bar i})} + \frac{\varepsilon\varphi_{i\bar i}^2}{2(1+\varphi_{i\bar i})} + \frac{\varepsilon|\varphi_{i\alpha}|^2}{2(1+\varphi_{i\bar i})}.
\end{aligned} \tag{2.8}
$$

The other conjugate term satisfies the same estimate as above. Combining above calculations, we obtain:

$$
\begin{aligned}
&\frac{\Delta_\varphi(e^{F-\lambda\varphi}(K+|\nabla\varphi|^2))}{e^{F-\lambda\varphi}} \\
\geq &\left(K + |\nabla\varphi|^2 - 3\varepsilon^{-1}|\nabla\varphi|^2\right)\frac{|F_i - \lambda\varphi_i|^2}{1+\varphi_{i\bar i}} \\
&+ \frac{(\lambda+\eta_{i\bar i})(K+|\nabla\varphi|^2) - C_{2.21}|\nabla\varphi|^2}{1+\varphi_{i\bar i}} + \frac{(1-\varepsilon)\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}} - \varepsilon(n+\Delta\varphi) \\
&+ \frac{|\varphi_{i\alpha}|^2(1-\varepsilon)}{1+\varphi_{i\bar i}} + (-\underline{R} - \lambda n)(K+|\nabla\varphi|^2).
\end{aligned} \tag{2.9}
$$

Now it's time to choose the constants ε, λ, and K appearing above.

First we choose $\varepsilon = \frac{1}{4}$. With this choice, we have

$$
\begin{aligned}
\sum_i \frac{(1-\varepsilon)\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}} - \varepsilon(n+\Delta\varphi) &= \frac{3}{4}(n+\Delta\varphi) - \frac{3n}{2} + \frac{3}{4}\sum_i \frac{1}{1+\varphi_{i\bar i}} - \frac{1}{4}(n+\Delta\varphi) \\
&\geq \frac{1}{2}(n+\Delta\varphi) - \frac{3n}{2}.
\end{aligned} \tag{2.10}
$$

Then we choose λ so large that $\lambda - \max_M \max_i |\eta_{i\bar{i}}| > 1$. Finally, we choose K so large that

$$K > C_{2.21} \max_M |\nabla\varphi|^2 \tag{2.11}$$

$$K > 3\varepsilon^{-1} \max_M |\nabla\varphi|^2 = 12 \max_M |\nabla\varphi|^2. \tag{2.12}$$

With above choices for λ and K, we have

$$(\lambda + \eta_{i\bar{i}})(K + |\nabla\varphi|^2) - C_{2.21}|\nabla\varphi|^2 \geq K - C_{2.21}|\nabla\varphi|^2 > 0, \tag{2.13}$$

and also

$$K - 3\varepsilon^{-1}|\nabla\varphi|^2 > 0. \tag{2.14}$$

Hence we conclude from (2.9) that

$$\Delta_\varphi(e^{F-\lambda\varphi}(K+|\nabla\varphi|^2)) \geq e^{F-\lambda\varphi}\big(-(||f||_0+\lambda n)(K+\max_M |\nabla\varphi|^2) - C_{2.22} + \frac{1}{4}(n+\Delta\varphi)\big). \tag{2.15}$$

Denote $v = e^{F-\lambda\varphi}(K + |\nabla\varphi|^2)$, it is enough to show v has an upper bound. We see from (2.15) that there exists constants $C_{2.23} > 0$, $C_{2.24} > 0$, possibly depending on $\max_M |\nabla\varphi|^2$, such that

$$\Delta_\varphi(v) \geq v(-C_{2.23} + \frac{1}{C_{2.24}}(n + \Delta\varphi)). \tag{2.16}$$

Here we notice that $n + \Delta\varphi \geq ne^{\frac{F}{n}}$. Hence we obtain from (2.16) that

$$\Delta_\varphi(v) \geq v(-C_{2.23} + \frac{1}{C_{2.24}} e^{\frac{F}{n}}). \tag{2.17}$$

Let the maximum of v be achieved at point p, then we know $-C_{2.23} + \frac{e^{\frac{F}{n}(p)}}{C_{2.24}} \leq 0$. This gives an upper bound of F at p, hence an upper bound for v, where this bound depends on $\max_M |\nabla\varphi|$. Here we noted that $||\varphi||_0$ can be bounded in terms of $\sup_M ||\nabla\varphi||_0$ as long as $\sup_M \varphi = 0$ and that M is compact. $\square$

Conversely, we have the following key estimate, which will be needed when we do the $W^{2,p}$ estimates of φ.

Theorem 2.2. *There exists a constant $C_{2.3}$, depending only on $||\varphi||_0$, lower bound of bisectional curvature, $||f||_0$, $\max_M |\eta|_{\omega_0}$, such that*

$$\frac{|\nabla\varphi|^2}{e^F} \leq C_{2.3}.$$

Proof. We will consider $\Delta_\varphi(e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2}(|\nabla\varphi|^2 + K))$. Here $\lambda > 0$, $K > 0$ are constants to be determined below. Then we have

$$\begin{aligned}
&\Delta_\varphi(e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2}(|\nabla\varphi|^2 + K)) \\
= \; & \Delta_\varphi(e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2})(|\nabla\varphi|^2 + K) + e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2}\Delta_\varphi(|\nabla\varphi|^2) \\
&+ \frac{2e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2}}{1+\varphi_{i\bar{i}}} Re\big((-F_i - \lambda\varphi_i + \varphi\varphi_i)(|\nabla\varphi|^2)_{\bar{i}}\big).
\end{aligned} \tag{2.18}$$

For simplicity of notation, set

$$A(F, \varphi) = -(F + \lambda\varphi) + \frac{1}{2}\varphi^2.$$

Similar as before, we may calculate:

$$
\begin{aligned}
(2.19)\quad & \Delta_\varphi(e^{A(F,\varphi)}) \\
= \quad & e^A\frac{|-F_i-\lambda\varphi_i+\varphi\varphi_i|^2}{1+\varphi_{i\bar i}}+e^A\big(-\Delta_\varphi(F+\lambda\varphi)+\varphi\Delta_\varphi\varphi\big)+e^A\frac{|\varphi_i|^2}{1+\varphi_{i\bar i}} \\
= \quad & e^A\frac{|-F_i-\lambda\varphi_i+\varphi\varphi_i|^2}{1+\varphi_{i\bar i}}+e^A\left(f-\lambda n+n\varphi+\sum_i\frac{\lambda-\eta_{i\bar i}-\varphi}{1+\varphi_{i\bar i}}\right)+\frac{e^A|\varphi_i|^2}{1+\varphi_{i\bar i}}.
\end{aligned}
$$

Recall the calculation in (2.7):

$$
\begin{aligned}
\Delta_\varphi(|\nabla\varphi|^2) &= \frac{R_{i\bar i\alpha\bar\beta}\varphi_\alpha\varphi_{\bar\beta}}{1+\varphi_{i\bar i}}+\frac{|\varphi_{i\alpha}|^2}{1+\varphi_{i\bar i}}+\frac{\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}}+F_{\bar\alpha}\varphi_\alpha+F_\alpha\varphi_{\bar\alpha} \\
&\geq -C_{2.21}|\nabla\varphi|^2\sum_i\frac{1}{1+\varphi_{i\bar i}}+\frac{|\varphi_{i\alpha}|^2}{1+\varphi_{i\bar i}}+\frac{\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}} \qquad (2.20)\\
&\quad+(-2\lambda+2\varphi)|\nabla\varphi|^2+2Re\big((F_\alpha+\lambda\varphi_\alpha-\varphi\varphi_\alpha)\varphi_{\bar\alpha}\big).
\end{aligned}
$$

Again $C_{2.21}$ depends only on curvature bound of g. Also

$$(|\nabla\varphi|^2)_{\bar i}=\varphi_\alpha\varphi_{\bar\alpha\bar i}+\varphi_{\bar i}\varphi_{i\bar i},\ (|\nabla\varphi|^2)_i=\varphi_{\bar\alpha}\varphi_{\alpha i}+\varphi_i\varphi_{i\bar i}.$$

Hence if we plug in (2.19) and (2.20) back to (2.18), we obtain:

$$
\begin{aligned}
(2.21)\quad & \\
& \Delta_\varphi(e^A(|\nabla\varphi|^2+K))e^{-A} \\
& \geq |\nabla_\varphi(F+\lambda\varphi)-\varphi\nabla_\varphi\varphi|^2(|\nabla\varphi|^2+K)+|\nabla_\varphi\varphi|^2(|\nabla\varphi|^2+K) \\
& +\big(f-\lambda n+n\varphi+\sum_i\frac{\lambda-\eta_{i\bar i}-\varphi}{1+\varphi_{i\bar i}}\big)(|\nabla\varphi|^2+K)+\frac{-C_{2.21}|\nabla\varphi|^2+|\varphi_{i\alpha}|^2+\varphi_{i\bar i}^2}{1+\varphi_{i\bar i}} \\
& +(-2\lambda+2\varphi)|\nabla\varphi|^2+2Re\big((F_\alpha+\lambda\varphi_\alpha-\varphi\varphi_\alpha)\varphi_{\bar\alpha}\big) \\
& \qquad\qquad+\frac{2Re\big((-F_i-\lambda\varphi_i+\varphi\varphi_i)(\varphi_\alpha\varphi_{\bar\alpha\bar i}+\varphi_{\bar i}\varphi_{i\bar i})\big)}{1+\varphi_{i\bar i}}.
\end{aligned}
$$

We notice the following complete square in the above sum:

$$
\begin{aligned}
(2.22)\quad & \\
& \frac{1}{1+\varphi_{i\bar i}}|\varphi_{i\alpha}-\big(F_i+\lambda\varphi_i-\varphi\varphi_i\big)\varphi_\alpha|^2 \\
& =\frac{|\varphi_{i\alpha}|^2}{1+\varphi_{i\bar i}}+\frac{2Re\big((-F_i-\lambda\varphi_i+\varphi\varphi_i)\varphi_\alpha\varphi_{\bar\alpha\bar i}\big)}{1+\varphi_{i\bar i}}+\frac{|-F_i-\lambda\varphi_i+\varphi\varphi_i|^2|\nabla\varphi|^2}{1+\varphi_{i\bar i}}.
\end{aligned}
$$

We will drop this complete square in the following. Next we observe a crucial cancellation, which is the key point of this argument. We look at the last two terms in (2.21) and observe:

$$(2.23)\quad (F_\alpha+\lambda\varphi_\alpha-\varphi\varphi_\alpha)\varphi_{\bar\alpha}+\frac{(-F_i-\lambda\varphi_i+\varphi\varphi_i)\varphi_{\bar i}\varphi_{i\bar i}}{1+\varphi_{i\bar i}}=\frac{(F_i+\lambda\varphi_i-\varphi\varphi_i)\varphi_{\bar i}}{1+\varphi_{i\bar i}}.$$

Hence we have

(2.24)
$$\Delta_\varphi(e^A(|\nabla\varphi|^2+K))e^{-A} \geq K\frac{|-F_i-\lambda\varphi_i+\varphi\varphi_i|^2}{1+\varphi_{i\bar{i}}}+\frac{|\varphi_i|^2(|\nabla\varphi|^2+K)}{1+\varphi_{i\bar{i}}}$$
$$+\sum_i\frac{\lambda-\eta_{i\bar{i}}-\varphi}{1+\varphi_{i\bar{i}}}(|\nabla\varphi|^2+K)+\Big(f-\lambda n+n\varphi\Big)(|\nabla\varphi|^2+K)$$
$$-C_{2.21}|\nabla\varphi|^2\sum_i\frac{1}{1+\varphi_{i\bar{i}}}+\frac{\varphi_{i\bar{i}}^2}{1+\varphi_{i\bar{i}}}+(-2\lambda+2\varphi)|\nabla\varphi|^2+2Re\Big(\frac{(F_i+\lambda\varphi_i-\varphi\varphi_i)\varphi_{\bar{i}}}{1+\varphi_{i\bar{i}}}\Big).$$

Now we make the choices of λ, K. We choose $\lambda=10(\max_M\max_i|\eta_{i\bar{i}}|+||\varphi||_0+C_{2.21}+1)$ and $K=10$. With this choice, we now estimate the terms in (2.24), with various constants C_i which depends only on the curvature bound of g and $||\varphi||_0$.

$$\big(\underline{R}-\lambda n+n\varphi\big)(|\nabla\varphi|^2+K)\geq -C_{2.31}(|\nabla\varphi|^2+1). \tag{2.25}$$

$$(-2\lambda+2\varphi)|\nabla\varphi|^2\geq -C_{2.32}|\nabla\varphi|^2. \tag{2.26}$$

$$\begin{aligned}\frac{|(F_i+\lambda\varphi_i-\varphi\varphi_i)\varphi_{\bar{i}}|}{1+\varphi_{i\bar{i}}} &\leq \frac{1}{2}\frac{|F_i+\lambda\varphi_i-\varphi\varphi_i|^2}{1+\varphi_{i\bar{i}}}+\frac{1}{2}\frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}\\ &\leq \frac{1}{2}\frac{|F_i+\lambda\varphi_i-\varphi\varphi_i|^2}{1+\varphi_{i\bar{i}}}+\frac{1}{2}|\nabla\varphi|^2\sum_i\frac{1}{1+\varphi_{i\bar{i}}}.\end{aligned} \tag{2.27}$$

$$\sum_i\frac{\lambda-\eta_{i\bar{i}}-\varphi}{1+\varphi_{i\bar{i}}}(|\nabla\varphi|^2+K)-C_{2.21}|\nabla\varphi|^2\sum_i\frac{1}{1+\varphi_{i\bar{i}}}\geq 10|\nabla\varphi|^2\sum_i\frac{1}{1+\varphi_{i\bar{i}}}. \tag{2.28}$$

$$\frac{\varphi_{i\bar{i}}^2}{1+\varphi_{i\bar{i}}}\geq 0. \tag{2.29}$$

Combining all these estimates, we obtain from (2.24) that

(2.30)
$$\Delta_\varphi(e^A(|\nabla\varphi|^2+K))\geq e^A\big(\frac{|\varphi_i|^2|\nabla\varphi|^2}{1+\varphi_{i\bar{i}}}+9|\nabla\varphi|^2\sum_i\frac{1}{1+\varphi_{i\bar{i}}}-C_{2.33}(|\nabla\varphi|^2+1)\big).$$

Here $C_{2.33}$ depends only on curvature bound of g and $||\varphi||_0$. Using Young's inequality, we have, if $n\geq 2$,

$$\begin{aligned}|\nabla\varphi|^{\frac{2}{n}}e^{-\frac{F}{n}} &\leq \sum_i\frac{|\varphi_i|^{\frac{2}{n}}}{(1+\varphi_{i\bar{i}})^{\frac{1}{n}}}\cdot(1+\varphi_{i\bar{i}})^{\frac{1}{n}}e^{-\frac{F}{n}}\\ &\leq \frac{1}{n}\sum_i\frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}+\frac{n-1}{n}\sum_i(1+\varphi_{i\bar{i}})^{\frac{1}{n-1}}e^{-\frac{F}{n-1}}\\ &\leq (n-1)\big(\frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}+\frac{1}{n}\sum_i(1+\varphi_{i\bar{i}})^{\frac{1}{n-1}}e^{-\frac{F}{n-1}}\big)\\ &\leq (n-1)\big(\frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}+(n+\Delta\varphi)^{\frac{1}{n-1}}e^{-\frac{F}{n-1}}\big)\\ &\leq (n-1)\big(\frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}+\sum_i\frac{1}{1+\varphi_{i\bar{i}}}\big).\end{aligned}$$

Thus, for $n \geq 2$,

$$\frac{|\varphi_i|^2|\nabla\varphi|^2}{1+\varphi_{i\bar{i}}} + |\nabla\varphi|^2 \sum_i \frac{1}{1+\varphi_{i\bar{i}}} \geq \frac{1}{n-1}|\nabla\varphi|^{2+\frac{2}{n}} e^{-\frac{F}{n}}.$$

Whereas if $n = 1$, $|\nabla\varphi|^{2+\frac{2}{n}} e^{-\frac{F}{n}} = |\nabla\varphi|^4 e^{-F} = \frac{|\varphi_i|^2|\nabla\varphi|^2}{1+\varphi_{i\bar{i}}}$. In any case, we have

$$\frac{|\varphi_i|^2|\nabla\varphi|^2}{1+\varphi_{i\bar{i}}} + |\nabla\varphi|^2 \sum_i \frac{1}{1+\varphi_{i\bar{i}}} \geq \frac{1}{n}|\nabla\varphi|^{2+\frac{2}{n}} e^{-\frac{F}{n}}.$$

Hence we get from (2.30) that

(2.31)

$$\begin{aligned}\Delta_\varphi(e^A(|\nabla\varphi|^2+K)) &\geq e^{-\lambda\varphi+\frac{1}{2}\varphi^2}\left(e^{-(1+\frac{1}{n})F}|\nabla\varphi|^{2+\frac{2}{n}} - C_{2.33}e^{-F}|\nabla\varphi|^2 - C_{2.33}e^{-F}\right)\\ &= e^{-\lambda\varphi+\frac{1}{2}\varphi^2}\left((e^{-F}|\nabla\varphi|^2)^{1+\frac{1}{n}} - C_{2.33}e^{-F}|\nabla\varphi|^2 - C_{2.33}e^{-F}\right).\end{aligned}$$

Suppose that the function $e^{-(F+C\varphi)+\frac{1}{2}\varphi^2}(|\nabla\varphi|^2+K)$ achieves maximum at p. Then at point p, we have

$$0 \geq (e^{-F}|\nabla\varphi|^2)^{1+\frac{1}{n}} - C_{2.33}e^{-F}|\nabla\varphi|^2 - C_{2.33}e^{-F}. \tag{2.32}$$

Recall Proposition 2.1 gives an estimate for e^{-F} which depends only on $||\varphi||_0$ and the curvature bound of g. Therefore, we get a bound for $e^{-F}|\nabla\varphi|^2(p)$ with the same dependence. Hence we have a bound for $e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2}(|\nabla\varphi|^2+K)(p)$, with the dependence as stated in the theorem.

But this function achieves maximum at p, so we are done. □

3. The volume ratio $\frac{\omega_\varphi^n}{\omega_0^n}$ and $W^{2,p}$ bound on Kähler potential

In this section, we prove

Theorem 3.1. *For any $p > 0$, there exist constants $\alpha(p) > 0$, $C(p) > 0$, so that*

$$\int_M e^{-\alpha(p)F}(n+\Delta\varphi)^p dvol_g \leq C(p). \tag{3.1}$$

Here $\alpha(p)$ depends only on p(can be explicitly calculated). The constant C_p depends only on p, $||\varphi||_0$, $\max_M |\eta|_{\omega_0}$, $||f||_0$, lower bound of the bisectional curvature of g, and the volume of g.

Proof. One starts by calculating:

(3.2)

$$\begin{aligned}\Delta_\varphi(e^{-\alpha(F+\lambda\varphi)}(n+\Delta\varphi)) &= \Delta_\varphi(e^{-\alpha(F+\lambda\varphi)})(n+\Delta\varphi) + e^{-\alpha(F+\lambda\varphi)}\Delta_\varphi(n+\Delta\varphi)\\ &\quad + e^{-\alpha(F+\lambda\varphi)}(-\alpha)\frac{(F_i+\lambda\varphi_i)(\Delta\varphi)_{\bar{i}} + (F_{\bar{i}}+\lambda\varphi_{\bar{i}})(\Delta\varphi)_i}{1+\varphi_{i\bar{i}}}.\end{aligned}$$

If we choose $\lambda > 2\max_M \max_i |\eta_{i\bar{i}}|$, then

(3.3)

$$\begin{aligned}\Delta_\varphi(e^{-\alpha(F+\lambda\varphi)}) &= \frac{\alpha^2|F_i+\lambda\varphi_i|^2}{1+\varphi_{i\bar{i}}}e^{-\alpha(F+\lambda\varphi)} + \alpha e^{-\alpha(F+\lambda\varphi)}(f-\lambda n + \sum_i \frac{\lambda-\eta_{i\bar{i}}}{1+\varphi_{i\bar{i}}})\\ &\geq \frac{\alpha^2|F_i+\lambda\varphi_i|^2}{1+\varphi_{i\bar{i}}}e^{-\alpha(F+\lambda\varphi)} + \alpha e^{-\alpha(F+\lambda\varphi)}(f-\lambda n) + \frac{\lambda\alpha}{2}e^{-\alpha(F+\lambda\varphi)}\sum_i \frac{1}{1+\varphi_{i\bar{i}}}.\end{aligned}$$

For the term $\Delta_\varphi(n+\Delta\varphi)$, we choose a normal coordinate (c.f. equation (2.1)) and then follow Yau's calculation [48]. First, note that

$$\Delta_\varphi(n+\Delta\varphi)=\frac{1}{1+\varphi_{k\bar{k}}}\Big(g^{i\bar{j}}\varphi_{i\bar{j}}\Big)_{k\bar{k}}=\frac{R_{i\bar{i}k\bar{k}}\varphi_{i\bar{i}}}{1+\varphi_{k\bar{k}}}+\frac{\varphi_{k\bar{k}i\bar{i}}}{1+\varphi_{k\bar{k}}}. \tag{3.4}$$

We wish to represent the 4th derivative of φ in terms of F. For this we take equation (1.1) and differentiate it twice in z_i, $z_{\bar{i}}$ and then sum over $i=1,2\cdots n$. We obtain:

$$\frac{\varphi_{k\bar{k}i\bar{i}}}{1+\varphi_{k\bar{k}}}-\frac{R_{k\bar{k}i\bar{i}}}{1+\varphi_{k\bar{k}}}-\frac{|\varphi_{k\bar{\beta}i}|^2}{(1+\varphi_{k\bar{k}})(1+\varphi_{\beta\bar{\beta}})}=F_{i\bar{i}}-R_{i\bar{i}}. \tag{3.5}$$

Hence

$$\begin{aligned}\Delta_\varphi(n+\Delta\varphi)&=\frac{R_{k\bar{k}i\bar{i}}(1+\varphi_{k\bar{k}})}{1+\varphi_{i\bar{i}}}+\frac{|\varphi_{p\bar{q}i}|^2}{(1+\varphi_{p\bar{p}})(1+\varphi_{q\bar{q}})}+\Delta F-R(g)\\&\geq -C_{3.1}(n+\Delta\varphi)\sum_i\frac{1}{1+\varphi_{i\bar{i}}}+\frac{|\varphi_{p\bar{q}i}|^2}{(1+\varphi_{p\bar{p}})(1+\varphi_{q\bar{q}})}+\Delta F-R(g).\end{aligned} \tag{3.6}$$

Here $C_{3.1}$ depends only on curvature bound of g and R is the scalar curvature of the background metric g. Plug in to equation (3.2) and we get

$$\begin{aligned}\Delta_\varphi(e^{-\alpha(F+\lambda\varphi)}(n+\Delta\varphi))&\geq e^{-\alpha(F+\lambda\varphi)}(\frac{\lambda\alpha}{2}-C_{3.1})(n+\Delta\varphi)\sum_i\frac{1}{1+\varphi_{i\bar{i}}}\\&\quad+\alpha e^{-\alpha(F+\lambda\varphi)}(f-\lambda n)(n+\Delta\varphi)+e^{-\alpha(F+\lambda\varphi)}(\Delta F-R(g)).\end{aligned} \tag{3.7}$$

Here we already drop the term:

$$\begin{aligned}&\frac{\alpha^2|F_i+\lambda\varphi_i|^2}{1+\varphi_{i\bar{i}}}(n+\Delta\varphi)+(-\alpha)\frac{(F_i+\lambda\varphi_i)(\Delta\varphi)_{\bar{i}}+(F_{\bar{i}}+\lambda\varphi_{\bar{i}})(\Delta\varphi)_i}{1+\varphi_{i\bar{i}}}\\&\quad+\frac{|\varphi_{p\bar{q}i}|^2}{(1+\varphi_{p\bar{p}})(1+\varphi_{q\bar{q}})}\\&\geq\frac{\alpha^2|F_i+\lambda\varphi_i|^2}{1+\varphi_{i\bar{i}}}(n+\Delta\varphi)-2\alpha Re\bigg(\frac{(F_i+\lambda\varphi_i)(\Delta\varphi)_{\bar{i}}}{1+\varphi_{i\bar{i}}}\bigg)+\frac{|(\Delta\varphi)_i|^2}{(n+\Delta\varphi)(1+\varphi_{i\bar{i}})}\\&=\frac{n+\Delta\varphi}{1+\varphi_{i\bar{i}}}|\alpha(F_i+\lambda\varphi_i)-\frac{(\Delta\varphi)_i}{n+\Delta\varphi}|^2\geq 0.\end{aligned}$$

From the first line to second line in the above, we observed that

$$\frac{|(\Delta\varphi)_i|^2}{1+\varphi_{i\bar{i}}}=\frac{|\sum_p\varphi_{p\bar{p}i}|^2}{1+\varphi_{i\bar{i}}}\leq\frac{|\varphi_{p\bar{p}i}|^2(n+\Delta\varphi)}{(1+\varphi_{i\bar{i}})(1+\varphi_{p\bar{p}})}\leq\frac{|\varphi_{p\bar{q}i}|^2(n+\Delta\varphi)}{(1+\varphi_{p\bar{p}})(1+\varphi_{q\bar{q}})}.$$

Set

$$u=e^{-\alpha(F+\lambda\varphi)}(n+\Delta\varphi)$$

and note that

$$(n+\Delta\varphi)\sum_i\frac{1}{1+\varphi_{i\bar{i}}}\geq e^{-\frac{F}{n-1}}(n+\Delta\varphi)^{1+\frac{1}{n-1}},$$

then we know from equation (3.7):

(3.8)

$$\begin{aligned}\Delta_\varphi u\geq &e^{-(\alpha+\frac{1}{n-1})F-\alpha\lambda\varphi}(\frac{\lambda\alpha}{2}-C_{3.1})(n+\Delta\varphi)^{1+\frac{1}{n-1}}-\alpha e^{-\alpha(F+\lambda\varphi)}(\lambda n-f)(n+\Delta\varphi)\\&+e^{-\alpha(F+\lambda\varphi)}(\Delta F-R(g)).\end{aligned}$$

We use the following equality, which holds for any $p \geq 0$:

$$\begin{aligned} \tfrac{1}{2p+1}\Delta_\varphi(u^{2p+1}) &= 2pu^{2p-1}|\nabla_\varphi u|_\varphi^2 + u^{2p}\Delta_\varphi u \\ &= 2pu^{2p-2}e^{-\alpha(F+\lambda\varphi)}(n+\Delta\varphi)|\nabla_\varphi u|_\varphi^2 + u^{2p}\Delta_\varphi u \\ &\geq 2pu^{2p-2}|\nabla u|^2 e^{-\alpha(F+\lambda\varphi)} + u^{2p}\Delta_\varphi u. \end{aligned}$$

Integrate with respect to $dvol_\varphi = e^F dvol_g$ and plug inequality (3.8) to get:

(3.9)

$$\begin{aligned} &\int_M 2pu^{2p-2}|\nabla u|^2 e^{(1-\alpha)F-\alpha\lambda\varphi}dvol_g \\ &\quad + \int_M e^{-(\alpha-\frac{n-2}{n-1})F-\alpha\lambda\varphi}(\frac{\lambda\alpha}{2} - C_{3.1})(n+\Delta\varphi)^{1+\frac{1}{n-1}}u^{2p}dvol_g \\ &\quad + \int_M e^{(1-\alpha)F-\alpha\lambda\varphi}u^{2p}\Delta F dvol_g \leq \int_M \alpha e^{(1-\alpha)F-\alpha\lambda\varphi}(\lambda n - f)(n+\Delta\varphi)u^{2p}dvol_g \\ &\quad + \int_M e^{(1-\alpha)F-\lambda\alpha\varphi}R(g)u^{2p}dvol_g. \end{aligned}$$

We need to handle the term involving ΔF, which is done by integrating by parts:

(3.10)

$$\begin{aligned} &\int_M e^{(1-\alpha)F-\alpha\lambda\varphi}u^{2p}\Delta F dvol_g = \int_M (\alpha-1)e^{(1-\alpha)F-\alpha\lambda\varphi}u^{2p}|\nabla F|^2 dvol_g \\ &+ \int_M \alpha\lambda e^{(1-\alpha)F-\lambda\alpha\varphi}u^{2p}\nabla\varphi\cdot\nabla F dvol_g - \int_M 2pe^{(1-\alpha)F-\lambda\alpha\varphi}u^{2p-1}\nabla u\cdot\nabla F dvol_g. \end{aligned}$$

Also we can estimate the last term of (3.10)

$$u^{2p-1}\nabla u\cdot\nabla F \leq \frac{1}{2}u^{2p-2}|\nabla u|^2 + \frac{1}{2}u^{2p}|\nabla F|^2. \tag{3.11}$$

Then we estimate the second to last term of (3.10) and obtain:

$$\begin{aligned} &\alpha\lambda e^{(1-\alpha)F-\lambda\alpha\varphi}u^{2p}\nabla\varphi\cdot\nabla F \\ \leq &\frac{\alpha-1}{2}e^{(1-\alpha)F-\lambda\alpha\varphi}u^{2p}|\nabla F|^2 + \frac{\alpha^2\lambda^2}{2(\alpha-1)}u^{2p}|\nabla\varphi|^2 e^{(1-\alpha)F-\lambda\alpha\varphi} \\ \leq &\frac{\alpha-1}{2}e^{(1-\alpha)F-\lambda\alpha\varphi}u^{2p}|\nabla F|^2 + C_{2.3}\frac{\alpha^2\lambda^2}{2(\alpha-1)}u^{2p}e^{(2-\alpha)F-\lambda\alpha\varphi}. \end{aligned} \tag{3.12}$$

When estimating $|\nabla\varphi|^2$ above, we used Theorem 2.2, and $C_{2.3}$ is the constant given by that theorem. Plug (3.11), (3.12) back into (3.10), we obtain

(3.13)

$$\begin{aligned} &\int_M e^{(1-\alpha)F-\alpha\lambda\varphi}u^{2p}\Delta F dvol_g \geq \int_M (\frac{\alpha-1}{2} - p)e^{(1-\alpha)F-\alpha\lambda\varphi}u^{2p}|\nabla F|^2 dvol_g \\ &- \int_M C_{2.3}\frac{\alpha^2\lambda^2}{2(\alpha-1)}e^{(2-\alpha)F-\lambda\alpha\varphi}u^{2p}dvol_g - \int_M pe^{(1-\alpha)F-\lambda\alpha\varphi}u^{2p-2}|\nabla u|^2 dvol_g. \end{aligned}$$

Plug (3.13) back to (3.9), we see

$$\begin{aligned}
&\int_M pe^{(1-\alpha)F-\lambda\alpha\varphi}u^{2p-2}|\nabla u|^2 dvol_g + \int_M (\frac{\alpha-1}{2}-p)e^{(1-\alpha)F-\alpha\lambda\varphi}u^{2p}|\nabla F|^2 dvol_g \\
&\qquad + \int_M e^{-(\alpha-\frac{n-2}{n-1})F-\alpha\lambda\varphi}(\frac{\lambda\alpha}{2}-C_{3.1})(n+\Delta\varphi)^{1+\frac{1}{n-1}}u^{2p}dvol_g \\
&\leq \int_M \alpha e^{(1-\alpha)F-\alpha\lambda\varphi}(\lambda n+||f||_0)(n+\Delta\varphi)u^{2p}dvol_g \\
&\qquad + C_{2.3}\frac{\alpha^2\lambda^2}{2(\alpha-1)}\int_M e^{(2-\alpha)F-\lambda\alpha\varphi}u^{2p}dvol_g \\
&\qquad + \int_M e^{(1-\alpha)F-\lambda\alpha\varphi}R(g)u^{2p}dvol_g.
\end{aligned} \tag{3.14}$$

Now let $\alpha > 2p+1$ and $\lambda\alpha \geq 2C_{3.1}+1$, note that $n+\Delta\varphi \geq ne^{\frac{F}{n}}$ and F has positive lower bound, according to Proposition 2.1, then we find from above:

$$\begin{aligned}
&\int_M e^{-(\alpha-\frac{n-2}{n-1})F-\alpha\lambda\varphi}(n+\Delta\varphi)^{1+\frac{1}{n-1}}u^{2p}dvol_g \\
&\quad \leq C_{3.2}\alpha\int_M e^{(1-\alpha)F-\alpha\lambda\varphi}(n+\Delta\varphi)u^{2p}dvol_g + C_{3.2}\frac{\alpha^2}{\alpha-1}\int_M e^{(2-\alpha)F-\lambda\alpha\varphi}u^{2p}dvol_g.
\end{aligned} \tag{3.15}$$

Recall the definition of u, this means for any $p \geq 0$, $\alpha \geq 2p+2$:

$$\begin{aligned}
&\int_M \exp(-(2p+1)\alpha F + \frac{n-2}{n-1}F - \lambda\alpha(2p+1)\varphi)(n+\Delta\varphi)^{2p+1+\frac{1}{n-1}}dvol_g \\
&\leq C_{3.2}\alpha\int_M \exp(-(2p+1)\alpha F + F - (2p+1)\alpha\lambda\varphi)(n+\Delta\varphi)^{2p+1}dvol_g \\
&\quad + C_{3.2}\frac{\alpha^2}{\alpha-1}\int_M \exp(-(2p+1)\alpha F + 2F - (2p+1)\alpha\lambda\varphi)(n+\Delta\varphi)^{2p}dvol_g.
\end{aligned} \tag{3.16}$$

Hence for some constant $C_{3.3}$ which depends on $||\varphi||_0$, α, and p, we get:

$$\begin{aligned}
&\int_M \exp(-(2p+1)\alpha F + \frac{n-2}{n-1}F)(n+\Delta\varphi)^{2p+1+\frac{1}{n-1}}dvol_g \\
&\leq C_{3.3}\Big(\int_M \exp(-(2p+1)\alpha F + F)(n+\Delta\varphi)^{2p+1}dvol_g \\
&\quad + \int_M \exp(-(2p+1)\alpha F + 2F)(n+\Delta\varphi)^{2p}dvol_g\Big).
\end{aligned} \tag{3.17}$$

Take $p=0$, and $\alpha=2$ in (3.17), one obtains from (3.17) that:

$$\begin{aligned}
\int_M e^{-\frac{n}{n-1}F}(n+\Delta\varphi)^{\frac{n}{n-1}}dvol_g &\leq C_{3.3}\big(\int_M e^{-F}(n+\Delta\varphi)dvol_g + \int_M dvol_g\big) \\
&\leq C_{3.3}\big(n||e^{-F}||_0 vol(M) + vol(M)\big).
\end{aligned} \tag{3.18}$$

Since we obtained in Proposition 2.1 a bound for e^{-F} depending only on $||\varphi||_0$, $||f||_0$, $\max_M |\eta|_{\omega_0}$, and curvature bound of g. Hence we get a bound for $\int_M e^{-\frac{n}{n-1}F}(n+\Delta\varphi)^{\frac{n}{n-1}}dvol_g$.

We now claim that there exists a sequence of pair of positive numbers (p_k, γ_k) where $p_k \to \infty$ such that

$$\int_M e^{-\gamma_k F}(n+\Delta\varphi)^{2p_k+1}\, dvol_g < \infty$$

for all $k = 1, 2\cdots$. Now we explain how we choose this sequence of pairs of positive numbers successively: In general, suppose we already choose (p_k, γ_k) such that the preceding inequality holds. Choose α_{k+1} sufficiently large such that

$$\alpha_{k+1} \geq 2p_k + 2, \qquad \text{and} \qquad -(2p_k+1)\alpha_{k+1} + 2 \leq -\gamma_k.$$

Set $\alpha = \alpha_{k+1}$, $p = p_k$ in (3.17), we obtain

$$\begin{aligned} &\int_M \exp(-(2p_k+1)\alpha_{k+1}F + \frac{n-2}{n-1}F)(n+\Delta\varphi)^{2p_k+1+\frac{1}{n-1}} dvol_g \\ &\leq C_{3.31}\left(\int_M e^{-\gamma_k F}(n+\Delta\varphi)^{2p_k+1} dvol_g + \int_M e^{-\gamma_k F}(n+\Delta\varphi)^{2p_k} dvol_g\right) \\ &\leq C_{3.32}\int_M e^{-\gamma_k F}(n+\Delta\varphi)^{2p_k+1} dvol_g. \end{aligned} \tag{3.19}$$

In the second inequality above, we again used the fact that $n + \Delta\varphi \geq ne^{\frac{F}{n}}$, and e^F is bounded from below. Set

$$\gamma_{k+1} = (2p_k+1)\alpha_{k+1} - \frac{n-2}{n-1} \qquad \text{and} \qquad p_{k+1} = p_k + \frac{1}{2(n-1)}.$$

Then

$$\int_M e^{-\gamma_{k+1}F}(n+\Delta\varphi)^{2p_{k+1}+1} dvol_g \leq C\int_M e^{-\gamma_k F}(n+\Delta\varphi)^{2p_k+1},$$

where the constant depends on $||\varphi||_0$ and the background metric g. Our claim is then verified.

By induction, we then get a bound for $\int_M e^{-\gamma_p F}(n+\Delta\varphi)^p dvol_g$ for any $p > 0$ and some constant $\gamma_p > 0$. Here γ_p grows like p^2 as $p \to \infty$. □

Remark 3.1. By a more careful inspection of above argument, one sees that it is possible to choose $\gamma_p = \max((p+1)(p+2), np)$, but this is probably not sharp.

As an immediate consequence, we have the following $W^{2,p}$ estimate of φ in terms of $||F||_0$.

Corollary 3.1. *For any $1 < p < \infty$, there exist constants $\tilde{C}(p) > 0$, depending on $||\varphi||_0$, $||F||_0$, the background metric g, $||f||_0$, $\max_M |\eta|_{\omega_0}$ and p, such that $||n + \Delta\varphi||_{L^p} \leq \tilde{C}(p)$.*

4. $C^{1,1}$ bound of the Kähler potential in terms of its $W^{2,p}$ bound

In this section, we want to prove

Theorem 4.1. *There exists a constant C_4, depending only on $||\varphi||_0$, $||F||_0$, $||f||_0$, $\max_M |\eta|_{\omega_0}$, and the background metric g, such that $n + \Delta\varphi \leq C_4$.*

Note that this theorem is trivial when $n = 1$, since $1 + \Delta\varphi = e^F$ when $n = 1$. Hence we can assume $n \geq 2$ throughout this section. In view of Theorem 2.1, we have the following immediate consequence:

Corollary 4.1. *There exists a constant $C_{4.1}$, depending only on $||\varphi||_0$, $||\nabla\varphi||_0$, $||f||_0$, $\max_M |\eta|_{\omega_0}$ and the background metric g, such that $n+\Delta\varphi\le C_{4.1}$.*

With this assumption, we know from Corollary 3.1 that for any $p>0$, there exists constants C_p, depending on $||\varphi||_0$, $||F||_0$, and the background metric g, such that

$$||n+\Delta\varphi||_{L^p(M)}\le\tilde{C}(p). \tag{4.1}$$

Hence it suffices to prove the following statement:

Proposition 4.1. *Let (φ,F) be a smooth solution (1.1), (1.2), then there exists $p_n>0$, depending only on n, such that*

$$\max_M|\nabla_\varphi F|_\varphi+\max_M(n+\Delta\varphi)\le C_{4.2}. \tag{4.2}$$

Here C_4 depends only on $||F||_0$, $||n+\Delta\varphi||_{L^{p_n}(M)}$, and metric g(in the way described in Theorem 4.1).

Remark 4.1. From the argument below, one can explicitly get an upper bound for

$$p_n\le(2n-2)(4n+1).$$

This upper bound is probably not sharp.

Proof. Let us first calculate $\Delta_\varphi(|\nabla_\varphi f|^2_\varphi)$ for any smooth function f in M. First we do the calculation under an orthonormal frame g_φ.

$$\begin{aligned}\Delta_\varphi|\nabla_\varphi f|^2 &= (f_i f_{\bar i})_{,j\bar j}\\ &= f_{,ij\bar j}f_{\bar i}+f_i f_{,\bar i j\bar j}+|f_{,ij}|^2_\varphi+|f_{,i\bar j}|^2_\varphi\\ &= f_{,ji\bar j}f_{\bar i}+f_i f_{,j\bar i\bar j}+|f_{,ij}|^2_\varphi+|f_{,i\bar j}|^2_\varphi\\ &= (\Delta_\varphi f)_i f_{\bar i}+f_i(\Delta_\varphi f)_{\bar i}+Ric_{\varphi,i\bar j}f_j f_{\bar i}+|f_{,ij}|^2_\varphi+|f_{,i\bar j}|^2_\varphi.\end{aligned}$$

In the above, $f_{,ij\dots}$ denote covariant derivatives under the metric g_φ. Let $B(\lambda):\mathbb{R}\to\mathbb{R}$ be a smooth function, now we calculate $\Delta_\varphi(e^{B(f)}|\nabla_\varphi f|^2_\varphi)$.

$$\begin{aligned}&e^{-B(f)}\cdot\Delta_\varphi(e^{B(f)}|\nabla_\varphi f|^2_\varphi)\\ &=\Delta_\varphi(|\nabla_\varphi f|^2_\varphi)+B'(f_i(|\nabla_\varphi f|^2_\varphi)_{\bar i}+f_{\bar i}(|\nabla_\varphi f|^2_\varphi)_i)\\ &\qquad+\left((B'^2+B'')|\nabla_\varphi f|^2+B'\Delta_\varphi f\right)|\nabla_\varphi f|^2_\varphi\\ &=(\Delta_\varphi f)_i f_{\bar i}+f_i(\Delta_\varphi f)_{\bar i}+Ric_{\varphi,i\bar j}f_j f_{\bar i}+|f_{,ij}|^2_\varphi+|f_{,i\bar j}|^2_\varphi\\ &+B'\left(f_i f_j f_{,\bar j\bar i}+f_i f_{,j\bar i}f_{\bar j}+f_{\bar i}f_j f_{,\bar j i}+f_{\bar i}f_{,ji}f_{\bar j}\right)\\ &\qquad\qquad+\left((B'^2+B'')|\nabla_\varphi f|^2_\varphi+B'\Delta_\varphi f\right)|\nabla_\varphi f|^2_\varphi\\ &\ge(\Delta_\varphi f)_i f_{\bar i}+f_i(\Delta_\varphi f)_{\bar i}+Ric_{\varphi,i\bar j}f_j f_{\bar i}+|f_{,i\bar j}|^2_\varphi\\ &+B'\left(f_i f_{,j\bar i}f_{\bar j}+f_{\bar i}f_j f_{,\bar j i}\right)+\left(B''|\nabla_\varphi f|^2_\varphi+B'\Delta_\varphi f\right)|\nabla_\varphi f|^2_\varphi.\end{aligned} \tag{4.3}$$

In the inequality above, we noticed and dropped the following complete square:

$$B'^2|\nabla_\varphi f|^4_\varphi+B'f_i f_j f_{,\bar j\bar i}+B'f_{\bar i}f_{\bar j}f_{,ij}+|f_{,ij}|^2_\varphi=|f_{,ij}+B'f_i f_j|^2_\varphi.$$

We apply above calculation to F. Notice that

$$Ric_{\varphi,i\bar j}=R_{i\bar j}-F_{i\bar j}.$$

Set $B'=\frac{1}{2}$, and we switch to normal coordinate of g (c.f. (2.1)), then we have

$$
(4.4)\quad \begin{aligned} e^{-\frac{F}{2}}\Delta_\varphi(e^{\frac{F}{2}}|\nabla_\varphi F|^2) \geq 2\nabla_\varphi F\cdot_\varphi \nabla_\varphi \Delta_\varphi F + \frac{R_{j\bar{i}}F_i F_{\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \\ + \frac{|F_{i\bar{\alpha}}|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{\alpha\bar{\alpha}})} + \frac{1}{2}\Delta_\varphi F|\nabla_\varphi F|^2_\varphi. \end{aligned}
$$

Notice that there will be no more terms like $\frac{F_{j\bar{i}}F_i F_{\bar{j}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})}$, because the choice $B' \equiv \frac{1}{2}$ makes such terms exactly cancel out. Next we wish to use the equation satisfied by F:

$$\Delta_\varphi F = -f + tr_\varphi \eta.$$

Hence in (4.4), the last term satisfies:

$$
(4.5)\quad \frac{1}{2}\Delta_\varphi F|\nabla_\varphi F|^2_\varphi \geq -\frac{1}{2}(||f||_0 + \max_M |\eta|_{\omega_0} tr_\varphi g)|\nabla_\varphi F|^2_\varphi.
$$

Also

(4.6)

$$
\frac{|R_{i\bar{j}}F_{\bar{i}}F_j|}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \leq \max_M |Ric(\omega_0)|_{\omega_0}\frac{|F_i F_{\bar{j}}|}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \leq \max_M |Ric(\omega_0)||\nabla_\varphi F|^2_\varphi tr_\varphi g.
$$

Also we observe that

$$
(4.7)\quad tr_\varphi g = \sum_i \frac{1}{1+\varphi_{i\bar{i}}} \leq ne^{-F}(n+\Delta\varphi)^{n-1} \leq C_{4.6}(1+\Delta\varphi)^{n-1}.
$$

Hence we conclude, after combining (4.4)–(4.7):

$$
(4.8)\quad \begin{aligned} \Delta_\varphi(e^{\frac{1}{2}F}|\nabla_\varphi F|^2_\varphi) \geq 2e^{\frac{1}{2}F}\nabla_\varphi F\cdot_\varphi \nabla_\varphi\Delta_\varphi F - C_{4.7}((n+\Delta\varphi)^{n-1}+1)|\nabla_\varphi F|^2_\varphi \\ + \frac{1}{C_{4.7}}\frac{|F_{i\bar{\alpha}}|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{\alpha\bar{\alpha}})}. \end{aligned}
$$

Using (3.6) and (4.7), we find

$$
(4.9)\quad \Delta_\varphi(n+\Delta\varphi) \geq -C_{4.9}(n+\Delta\varphi)^n + \Delta F - C_{4.8}.
$$

We combine (4.8), (4.9), and conclude:

(4.10)

$$
\begin{aligned} \Delta_\varphi(e^{\frac{1}{2}F}|\nabla_\varphi F|^2_\varphi + (n+\Delta\varphi)) \geq -C_{4.7}\big((n+\Delta\varphi)^{n-1}+1\big)|\nabla_\varphi F|^2_\varphi + 2e^{\frac{1}{2}F}\nabla_\varphi F\cdot_\varphi \nabla_\varphi\Delta_\varphi F \\ + \frac{1}{C_{4.7}}\frac{|F_{i\bar{\alpha}}|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{\alpha\bar{\alpha}})} + \Delta F - C_{4.8} - C_{4.9}(n+\Delta\varphi)^n. \end{aligned}
$$

To estimate ΔF, we may calculate:

(4.11)

$$
|\Delta F| \leq \frac{1}{2C_{4.7}}\frac{|F_{i\bar{i}}|^2}{(1+\varphi_{i\bar{i}})^2} + \frac{C_{4.7}}{2}(1+\varphi_{i\bar{i}})^2 \leq \frac{1}{2C_{4.7}}\frac{|F_{i\bar{i}}|^2}{(1+\varphi_{i\bar{i}})^2} + \frac{nC_{4.7}}{2}(n+\Delta\varphi)^2.
$$

Combining (4.10), (4.11), we conclude that there exists a constant $C_{4.91}$, with the same dependence as said above, such that

$$
(4.12)\quad \begin{aligned} \Delta_\varphi(e^{\frac{1}{2}F}|\nabla_\varphi F|^2_\varphi + (n+\Delta\varphi)) \geq -C_{4.91}e^{-\frac{1}{2}F}(n+\Delta\varphi)^{n-1}\big(e^{\frac{1}{2}F}|\nabla_\varphi F|^2_\varphi \\ + (n+\Delta\varphi)\big) + 2e^{\frac{1}{2}F}\nabla_\varphi F\cdot_\varphi \nabla_\varphi\Delta_\varphi F - C_{4.91}. \end{aligned}
$$

Set

$$u = e^{\frac{1}{2}F}|\nabla_\varphi F|^2_\varphi + (n+\Delta\varphi) + 1,$$

we obtain the key estimate from here:

$$\Delta_\varphi u \geq -C_{4.92}(n+\Delta\varphi)^{n-1}u + 2e^{\frac{1}{2}F}\nabla_\varphi F \cdot_\varphi \nabla_\varphi \Delta_\varphi F. \tag{4.13}$$

Next we plan to do iteration, using (4.13). Notice that for any $p > 0$:

$$\frac{1}{2p+1}\Delta_\varphi(u^{2p+1}) = u^{2p}\Delta_\varphi u + 2pu^{2p-1}|\nabla_\varphi u|_\varphi^2. \tag{4.14}$$

Integrate over M, we obtain:

(4.15)

$$\begin{aligned}\int_M 2pu^{2p-1}|\nabla_\varphi u|_\varphi^2 dvol_\varphi &= \int_M u^{2p}(-\Delta_\varphi u)dvol_\varphi \\ &\leq \int_M C_{4.92}(n+\Delta\varphi)^{n-1}u^{2p+1}dvol_\varphi - 2\int_M e^{\frac{1}{2}F}\nabla_\varphi F \cdot_\varphi \nabla_\varphi(\Delta_\varphi F)u^{2p}dvol_\varphi.\end{aligned}$$

We need to integrate by parts in the last term above, then we have

(4.16)

$$\begin{aligned}-\int_M 2e^{\frac{1}{2}F}\nabla_\varphi F \cdot_\varphi \nabla_\varphi(\Delta_\varphi F)u^{2p}dvol_\varphi &= \int_M 4pu^{2p-1}e^{\frac{1}{2}F}\Delta_\varphi F \nabla_\varphi F \cdot_\varphi \nabla_\varphi u dvol_\varphi \\ &+ \int_M 2u^{2p}e^{\frac{1}{2}F}(\Delta_\varphi F)^2 dvol_\varphi + \int_M u^{2p}e^{\frac{1}{2}F}|\nabla_\varphi F|_\varphi^2 \Delta_\varphi F dvol_\varphi.\end{aligned}$$

We wish to estimate the three terms on the right hand side of (4.16) from above. First,

$$\begin{aligned}\int_M 4pu^{2p-1}e^{\frac{1}{2}F}\Delta_\varphi F\nabla_\varphi F \cdot_\varphi \nabla_\varphi u dvol_\varphi &\leq \int_M pu^{2p-1}|\nabla_\varphi u|_\varphi^2 dvol_\varphi \\ &+ 4\int_M pu^{2p-1}e^F(\Delta_\varphi F)^2|\nabla_\varphi F|_\varphi^2 dvol_\varphi \\ &\leq \int_M pu^{2p-1}|\nabla_\varphi u|_\varphi^2 dvol_\varphi + 4\int_M pu^{2p}e^{\frac{1}{2}F}(\Delta_\varphi F)^2 dvol_\varphi.\end{aligned} \tag{4.17}$$

Also it is clear that

$$\int_M u^{2p}e^{\frac{1}{2}F}|\nabla_\varphi F|_\varphi^2 \Delta_\varphi F dvol_\varphi \leq \int_M u^{2p+1}|\Delta_\varphi F|dvol_\varphi. \tag{4.18}$$

Combining (4.16), (4.17) and (4.18), we see

$$\begin{aligned}-\int_M 2e^{\frac{1}{2}F}\nabla_\varphi F \cdot_\varphi \nabla_\varphi(\Delta_\varphi F)u^{2p}dvol_\varphi &\leq \int_M pu^{2p-1}|\nabla_\varphi u|_\varphi^2 dvol_\varphi \\ &+ \int_M (4p+2)u^{2p}e^{\frac{1}{2}F}(\Delta_\varphi F)^2 dvol_\varphi + \int_M u^{2p+1}|\Delta_\varphi F|dvol_\varphi.\end{aligned} \tag{4.19}$$

Combined with (4.15), we obtain

(4.20)

$$\begin{aligned}\int_M pu^{2p-1}|\nabla_\varphi u|_\varphi^2 dvol_\varphi &\leq \int_M C_{4.93}(n+\Delta\varphi)^{n-1}u^{2p+1}dvol_\varphi \\ &+ \int_M u^{2p+1}|\Delta_\varphi F|dvol_\varphi + \int_M (4p+2)u^{2p}e^{\frac{1}{2}F}(\Delta_\varphi F)^2 dvol_\varphi.\end{aligned}$$

In the above, we can estimate

$$\begin{aligned}|\Delta_\varphi F| \leq |f| + |tr_\varphi \eta| &\leq (||f||_0 + \max_M |\eta|_{\omega_0})(1 + tr_\varphi g) \\ &\leq C_{4.935}(1 + ne^{-F}(n+\Delta\varphi)^{n-1}).\end{aligned} \tag{4.21}$$

Recall that $n+\Delta\varphi$ is bounded from below in terms of $||F||_0$ and u is bounded below by 1, we obtain from (4.20) that

$$\int_M pu^{2p-1}|\nabla_\varphi u|_\varphi^2 dvol_\varphi \le \int_M C_{4.94}(p+1)(n+\Delta\varphi)^{2n-2}u^{2p+1}dvol_\varphi. \tag{4.22}$$

Here $C_{4.94}$ depends only on $||F||_0$, the background metric (M,g), $||f||_0$, and $\max_M|\eta|_{\omega_0}$. Above implies

$$\int_M |\nabla_\varphi(u^{p+\frac{1}{2}})|_\varphi^2 dvol_g \le \frac{C_{4.94}(p+\frac{1}{2})^2(p+1)}{p}\int_M C_{4.94}(n+\Delta\varphi)^{2n-2}u^{2p+1}dvol_g. \tag{4.23}$$

Here we used that $dvol_\varphi = e^F dvol_g$ and that F is bounded. Fix $0<\varepsilon<2$ to be determined, we estimate the right hand side of (4.23):

$$\begin{aligned}\int_M (n+\Delta\varphi)^{2n-2}u^{2p+1}dvol_g \\ \le \left(\int_M u^{(p+\frac{1}{2})(2+\varepsilon)}dvol_g\right)^{\frac{2}{2+\varepsilon}}\left(\int_M (n+\Delta\varphi)^{\frac{(2n-2)(2+\varepsilon)}{\varepsilon}}dvol_g\right)^{\frac{\varepsilon}{2+\varepsilon}}.\end{aligned} \tag{4.24}$$

Denote $v=u^{p+\frac{1}{2}}$, then (4.23) implies

$$\int_M|\nabla_\varphi v|_\varphi^2 dvol_g \le \frac{C_{4.94}(p+\frac{1}{2})^2}{2p}\left(\int_M (n+\Delta\varphi)^{\frac{(2n-2)(2+\varepsilon)}{\varepsilon}}dvol_g\right)^{\frac{\varepsilon}{2+\varepsilon}}\left(\int_M v^{2+\varepsilon}dvol_g\right)^{\frac{2}{2+\varepsilon}}. \tag{4.25}$$

Next we wish to estimate the left hand side of (4.25) from below:

$$\begin{aligned}|\nabla v|^{2-\varepsilon} &\le \sum_i |v_i|^{2-\varepsilon} = \sum_i \frac{|v_i|^{2-\varepsilon}}{(1+\varphi_{i\bar i})^{\frac{2-\varepsilon}{2}}}\cdot(1+\varphi_{i\bar i})^{\frac{2-\varepsilon}{2}} \\ &\le \left(\sum_i \frac{|v_i|^2}{1+\varphi_{i\bar i}}\right)^{\frac{2-\varepsilon}{2}}\left(\sum_i (1+\varphi_{i\bar i})^{\frac{2-\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{2}} \le |\nabla_\varphi v|_\varphi^{2-\varepsilon} n^{\frac{\varepsilon}{2}}(n+\Delta\varphi)^{\frac{2-\varepsilon}{2}}.\end{aligned} \tag{4.26}$$

In the last inequality above, we estimated each $1+\varphi_{i\bar i}$ from above by $n+\Delta\varphi$ and there are n terms in $\sum_i$. Integrate and use Hölder inequality, we get:

$$\begin{aligned}\int_M |\nabla v|^{2-\varepsilon}dvol_g &\le n^{\frac{\varepsilon}{2}}\int_M |\nabla_\varphi v|_\varphi^{2-\varepsilon}(n+\Delta\varphi)^{\frac{2-\varepsilon}{2}}dvol_g \\ &\le n^{\frac{\varepsilon}{2}}\left(\int_M |\nabla_\varphi v|_\varphi^2\right)^{\frac{2-\varepsilon}{2}}\left(\int_M (n+\Delta\varphi)^{\frac{2-\varepsilon}{\varepsilon}}dvol_g\right)^{\frac{\varepsilon}{2}}.\end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27), we see:

$$\begin{aligned}\left(\int_M |\nabla v|^{2-\varepsilon}dvol_g\right)^{\frac{2}{2-\varepsilon}} &\le n^{\frac{\varepsilon}{2-\varepsilon}}\left(\int_M (n+\Delta\varphi)^{\frac{2-\varepsilon}{\varepsilon}}dvol_g\right)^{\frac{\varepsilon}{2-\varepsilon}}\int_M |\nabla_\varphi v|_\varphi^2 dvol_g \\ &\le C_{4.95}pK_\varepsilon\left(\int_M v^{2+\varepsilon}dvol_g\right)^{\frac{2}{2+\varepsilon}}.\end{aligned} \tag{4.28}$$

Here

$$K_\varepsilon = n^{\frac{\varepsilon}{2-\varepsilon}}\cdot\left(\int_M (n+\Delta\varphi)^{\frac{2-\varepsilon}{\varepsilon}}dvol_g\right)^{\frac{\varepsilon}{2-\varepsilon}}\cdot\left(\int_M (n+\Delta\varphi)^{\frac{(2n-2)(2+\varepsilon)}{\varepsilon}}\right)^{\frac{\varepsilon}{2+\varepsilon}}. \tag{4.29}$$

Apply the Sobolev embedding with exponent $2-\varepsilon$, and denote $\theta = \frac{2n(2-\varepsilon)}{2n-2+\varepsilon}$ to be the improved integrability, we get

$$||v||_{L^\theta(dvol_g)} \le C_{sob}(||\nabla v||_{L^{2-\varepsilon}(dvol_g)} + ||v||_{L^{2-\varepsilon}(dvol_g)}).$$

Recall that $v = u^{p+\frac{1}{2}}$, this means:

$$\begin{aligned}
\Big(\int_M u^{(p+\frac{1}{2})\theta} dvol_g\Big)^{\frac{2}{\theta}} &\le C_{sob}\Big(\big(\int_M |\nabla(u^{p+\frac{1}{2}})|^{2-\varepsilon} dvol_g\big)^{\frac{2}{2-\varepsilon}} \\
&\quad + \big(\int_M u^{(p+\frac{1}{2})(2-\varepsilon)} dvol_g\big)^{\frac{2}{2-\varepsilon}}\Big) \\
&\le C_{sob}\Big(C_{4.95} p K_\varepsilon \big(\int_M u^{(p+\frac{1}{2})(2+\varepsilon)} dvol_g\big)^{\frac{2}{2+\varepsilon}} \\
&\quad + \big(\int_M u^{(p+\frac{1}{2})(2-\varepsilon)} dvol_g\big)^{\frac{2}{2-\varepsilon}}\Big) \\
&\le C_{4.96,\varepsilon} p \big(\int_M u^{(p+\frac{1}{2})(2+\varepsilon)} dvol_g\big)^{\frac{2}{2+\varepsilon}}.
\end{aligned} \tag{4.30}$$

Here $C_{4.96,\varepsilon}$ has the same dependence as C_i's above, but with additional dependence on ε. From the 1st line to 2nd line, we used (4.28). Now choose $\varepsilon > 0$ small so that $\theta > 2+\varepsilon$, then above estimate indeed improves integrability, namely we need

$$\frac{2n(2-\varepsilon)}{2n-2+\varepsilon} > 2+\varepsilon. \tag{4.31}$$

We fix ε and (4.30) gives for $p \ge \frac{1}{2}$:

$$||u||_{L^{(p+\frac{1}{2})\theta}} \le \big(C_{4.97} p\big)^{\frac{1}{2p+1}} ||u||_{L^{(p+\frac{1}{2})(2+\varepsilon)}}. \tag{4.32}$$

Denote $\chi = \frac{\theta}{2+\varepsilon} > 1$, and choose $p + \frac{1}{2} = \chi^i$, for $i \ge 0$. Then we obtain:

$$||u||_{L^{(2+\varepsilon)\chi^{i+1}}} \le \big(C_{4.97}\chi^i\big)^{\frac{1}{2\chi^i}} ||u||_{L^{(2+\varepsilon)\chi^i}}. \tag{4.33}$$

It follows that

$$||u||_{L^\infty} \le C_{4.97}^{\sum_{i\ge 0}\frac{1}{2\chi^i}} \cdot \chi^{\sum_{i\ge 0}\frac{i}{2\chi^i}} ||u||_{L^{2+\varepsilon}} \le C_{4.97}^{\sum_{i\ge 0}\frac{1}{2\chi^i}} \cdot \chi^{\sum_{i\ge 0}\frac{i}{2\chi^i}} ||u||_{L^1}^{\frac{1}{2+\varepsilon}} ||u||_{L^\infty}^{\frac{1+\varepsilon}{2+\varepsilon}}. \tag{4.34}$$

From above we get estimate of $||u||_{L^\infty}$ in terms of $||u||_{L^1}$. But recall $u = e^{\frac{1}{2}F}|\nabla_\varphi F|^2 + (n+\Delta\varphi) + 1$, so L^1 estimate is available.

Indeed, it is clear that $n + \Delta\varphi \in L^1$. To see $e^{\frac{1}{2}F}|\nabla_\varphi F|_\varphi^2 \in L^1$, we just need to show $|\nabla_\varphi F|_\varphi^2 \in L^1$ since F is now assumed to be bounded. Then we can calculate:

$$\Delta_\varphi(F^2) = 2|\nabla_\varphi F|_\varphi^2 + 2F\Delta_\varphi F = 2|\nabla_\varphi F|_\varphi^2 + 2F(-f + tr_\varphi \eta). \tag{4.35}$$

Integrate with respect to $dvol_\varphi = e^F dvol_g$, we see

$$\begin{aligned}
\int_M e^F |\nabla_\varphi F|_\varphi^2 dvol_g &= \int_M e^F F(f - tr_\varphi \eta) dvol_g \\
&\le C_{4.98} \int_M (1 + tr_\varphi g \max_M |\eta|_{\omega_0}) dvol_\varphi \le C_{4.99} vol(M).
\end{aligned} \tag{4.36}$$

Here $C_{4.98}$, $C_{4.99}$ may depend on $||F||_0$. To see the range of p_n asserted in the Remark 4.1, we notice the choice of $\varepsilon = \frac{1}{2n}$ verifies the requirement in (4.31). With this choice, the highest power of $n+\Delta\varphi$ appearing in (4.29) is exactly $(2n-2)(4n+1)$. Once we have control over K_ε, the rest of the proof goes through. $\square$

For completeness, we present Proposition 4.2 which might be well known to experts (c.f. [14]).

Proposition 4.2. *If $\frac{1}{C}\omega_0 \le \omega_\varphi \le C\omega_0$, for some constant $C > 0$, then all higher derivatives can be estimated in terms of C.*

Proof. The proof of this corollary is essentially the combination of several classical elliptic estimates, as we explain below. The assumption that $\frac{1}{C_0}\omega_0 \le \omega_\varphi \le C_0\omega_0$ implies (1.2) is now uniformly elliptic with bounded right hand side. Moreover, we see that (1.2) can be put in the divergence form:

$$Re\big(\partial_i(\det(g_{\alpha\bar\beta}+\varphi_{\alpha\bar\beta})g_\varphi^{i\bar j}F_{\bar j})\big) = (-f+tr_\varphi\eta)\det(g_{\alpha\bar\beta}+\varphi_{\alpha\bar\beta}). \tag{4.37}$$

From this we immediately know $||F||_{\alpha'} \le C_{0.1}$, where α' and $C_{0.1}$ has the said dependence. Then we go back to (1.1), we can then conclude from Evans-Krylov theorem that $||\varphi||_{2,\alpha''} \le C_{0.2}$ for any $\alpha'' < \alpha'$(see [24], [47] for details on extension of Evans-Krylov to complex setting). Then we go back to (4.37) and see that the coefficients on the left hand side are in $C^{\alpha''}$, while the right hand side is bounded. Hence we may conclude $||F||_{1,\alpha''} \le C_{0.3}$, from [30], Theorem 8.32.

Then from (1.1), by differentiating both sides of the equation, we see that the first derivatives of φ solve a linear elliptic equation with $C^{\alpha''}$ coefficient and right hand side, hence Schauder estimate applies and we conclude $\varphi \in C^{3,\alpha''}$([30], Theorem 6.2). But then we go back to (4.37) one more time, the coefficients are in C^α for any $0 < \alpha < 1$ with bounded right hand side, hence we conclude $F \in C^{1,\alpha}$ for any $0 < \alpha < 1$. Now the equation solved by the first derivatives of φ will have coefficients on the right hand side in C^α for any $0 < \alpha < 1$. Therefore $\varphi \in C^{3,\alpha}$ for any $0 < \alpha < 1$.

The second equation (1.2) now has $C^{1,\alpha}$ coefficient with bounded right hand side, then the classical L^p estimate gives $F \in W^{2,p}$ for any finite p([30], Theorem 9.11). Then differentiating the first equation (1.1) twice, we get a linear elliptic equation in terms of second derivatives of φ, which has C^α coefficients and L^p right hand side(we already have $F \in W^{2,p}$), it follows that $\varphi \in W^{4,p}$. $\square$

5. Entropy bound of the volume ratio and C^0 bound of Kähler potential

The main goal of this section is to show the C^0 bound of φ implies a bound for $\int_M e^F F dvol_g$ and vice versa:

Theorem 5.1. *Let (φ, F) be a smooth solution to* (1.1), (1.2)*, then $||\varphi||_0$ and $||F||_0$ can be bounded in terms of an upper bound for $\int_M Fe^F dvol_g$. Conversely, if ω_φ is a cscK metric, then a bound for $||\varphi||_0$ implies an upper bound for $\int_M e^F F dvol_g$.*

The most difficult part of above theorem is to show that an upper bound for $\int_M e^F F dvol_g$ implies a bound on $||\varphi||_0$ and $||F||_0$, which is the main focus of this section. That $||\varphi||_0$ implies a bound for $\int_M e^F F dvol_g$ when ω_φ is a cscK metric essentially follows from the fact that cscK are minimizers of K-energy. In particular, having a bound on $||\varphi||_0$ is enough to control $||F||_0$, hence estimates up to $C^{1,1}$, thanks to the results obtained in previous sections.

Actually we will see that in order to bound $||\varphi||_0$, $||F||_0$, it is enough to have a bound for $\int_M e^F\Phi(F)dvol_g$, where $\Phi(F)>0$ is coercive in F in the sense that

(1) $\lim_{t\to-\infty} e^t\cdot\Phi(t)=0$ and $\lim_{t\to\infty}\Phi(t)=\infty$

(2) $\lim_{t\to\infty}\frac{\Phi(t)}{t}<\infty$.

We want to show that, under these conditions, an upper bound for $\int_M e^F\Phi(F)dvol_g$ will imply a bound for $\int_M e^{qF}dvol_g$ for any $q<\infty$. This bound can then imply a bound for $||\varphi||_0$, due to the deep result by Kołodziej, [34], but an elementary argument which only uses Alexandrov maximum principle (Lemma 5.2) and avoids pluri-potential theory is also possible. This argument is due to Blocki (c.f. [5]). From Corollary 5.2, we obtain a bound for $||e^F||_0$. We have also shown in Proposition 2.1 that a C^0 bound of φ will imply a lower bound for F. Hence a bound for $||F||_0$ can be obtained this way. Then estimates in previous sections can be applied to obtain higher derivatives bound.

Define

$$P(M,g)=\{\phi\in C^2(M,\mathbb{R}):g_{i\bar{j}}+\frac{\partial^2\phi}{\partial z_i\partial\bar{z}_j}\geq 0,\sup_M\phi=0\}. \tag{5.1}$$

The following result of Tian is well known, whose proof may be found in [44], Proposition 2.1:

Proposition 5.1. *There exists two positive constant α, C_5, depending only on (M,g), such that*

$$\int_M e^{-\alpha\phi}dvol_g\leq C_5, \textit{for any } \phi\in P(M,g). \tag{5.2}$$

Here $\alpha=\alpha(M,[\omega])$ is the so called α-invariant. To start, we normalize φ so that $\sup_M\varphi=0$. We also need to consider the auxiliary Kähler potential $\psi\in\mathcal{H}$, which solves the following problem:

$$\det(g_{i\bar{j}}+\psi_{i\bar{j}})=\frac{e^F\Phi(F)\det(g_{i\bar{j}})}{\int_M e^F\Phi(F)dvol_g}, \tag{5.3}$$

$$\sup_M\psi=0. \tag{5.4}$$

The existence of such ψ follows from Yau's celebrated theorem on Calabi's volume conjecture (c.f. [48], Theorem 2) . Because of Proposition 5.1, we know that

$$\int_M e^{-\alpha\varphi}dvol_g\leq C_5,\qquad \int_M e^{-\alpha\psi}dvol_g\leq C_5.$$

We will show that the following estimate holds:

Theorem 5.2. *Given any $0<\varepsilon<1$, there exists a constant $C_{5.1}$, depending on ε, the background metric g, $||f||_0$, $\max_M|\eta|_{\omega_0}$, the choice of Φ, and the upper bound of $\int_M e^F\Phi(F)dvol_g$, such that*

$$F+\varepsilon\psi-2(1+\max_M|\eta|_{\omega_0})\varphi\leq C_{5.1}. \tag{5.5}$$

Corollary 5.1. *For any $0<q<\infty$, there exists a constant $C_{5.2}$, with the same dependence as $C_{5.1}$ above but additionally on q, such that*

$$\int_M e^{qF}dvol_g\leq C_{5.2},\ ||\varphi||_0\leq C_{5.2},||\psi||_0\leq C_{5.2}. \tag{5.6}$$

We will show this important corollary first.

Proof. First we derive the estimate for $\int_M e^{qF} dvol_g$ with $q > 1$.

From Theorem 5.2, we know

$$-\alpha\psi \geq \frac{\alpha}{\varepsilon}\big(F - 2(1 + \max_M |\eta|_{\omega_0})\varphi - C_{5.1}\big). \tag{5.7}$$

Hence

$$\begin{aligned} C_5 \geq \int_M e^{-\alpha\psi} dvol_g &\geq \int_M \exp\big(\frac{\alpha}{\varepsilon}(F - 2(1 + \max_M |\eta|_{\omega_0})\varphi - C_{5.1})\big) dvol_g \\ &\geq \int_M \exp\big(\frac{\alpha}{\varepsilon}(F - C_{5.1})\big) dvol_g. \end{aligned} \tag{5.8}$$

The last inequality holds because we normalized φ so that $\varphi \leq 0$. Choose $\varepsilon = \frac{\alpha}{q}$, then we immediately get the desired estimate for $\int_M e^{qF} dvol_g$. The claimed estimate for φ immediately follows from the estimate for $||e^F||_{L^q} (q > 2)$, given in Lemma 5.1. The bound for ψ follows in a similar way, using (5.3). $\square$

Lemma 5.1. *Let $\phi \in P(M, g)$ be such that $e^F = \frac{\omega_\phi^n}{\omega_0^n}$ with $e^F \in L^{2+s}(M, \omega_0)$, for some $s > 0$. Then $||\phi||_0 \leq C_{5.21}$, with $C_{5.21}$ depending only on the metric ω_0, $s > 0$ and $||e^F||_{L^{2+s}(M,\omega_0)}$.*

Note that this is a weaker result compared to the deep theorem of Kołodziej [34], which shows $e^F \in L^{1+s}(M, \omega_0)$ is already sufficient. However, the weaker result as stated above can be proved in an elementary way using Alexandrov maximum principle, discovered by Blocki [5], [6].

Combining Theorem 5.2 and Corollary 5.1, we immediately conclude:

Corollary 5.2. *There exists a constant $C_{5.2}$, depending only on the background metric g, the upper bound of $\int_M e^F F dvol_g$, such that*

$$F \leq C_{5.2}.$$

Proof. Choose $\Phi(t) = \sqrt{t^2 + 1}$ and observe that $\int_M e^F \sqrt{F^2 + 1} dvol_g$ is controlled in terms of an upper bound of $\int_M e^F F dvol_g$. Then the result follows from Theorem 5.2 and Corollary 5.1. $\square$

Now let's prove Theorem 5.2.

Proof of Theorem 5.2. Let $0 < \varepsilon < 1$ be given and fixed. Let d_0 be chosen so that for any $p \in M$, the geodesic ball $B_{d_0}(p)$ is contained in a single coordinate neighborhood, and under this coordinate, $\frac{1}{2}\delta_{ij} \leq g_{i\bar{j}} \leq 2\delta_{ij}$. For any $p \in M$, let $\eta_p : M \to \mathbb{R}_+$ be a cut-off function such that $\eta_p(p) = 1$, $\eta_p \equiv 1 - \theta$ outside the ball $B_{\frac{d_0}{2}}(p)$, with the estimate $|\nabla\eta_p|^2 \leq \frac{4\theta^2}{d_0^2}$, $|\nabla^2\eta_p| \leq \frac{4\theta}{d_0^2}$. Here $0 < \theta < 1$ is to be determined later. Let $0 < \delta < 1$, $\lambda > 0$ be constants to be determined. Assume the function $e^{\delta(F+\varepsilon\psi-\lambda\varphi)}$ achieves maximum at $p_0 \in M$. We now compute

(5.9)

$$\begin{aligned} &\Delta_\varphi\big(e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\eta_{p_0}\big) \\ &= \Delta_\varphi(e^{\delta(F+\varepsilon\psi-\lambda\varphi)})\eta_{p_0} \\ &\quad + e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\Delta_\varphi(\eta_{p_0}) + e^{\delta(F+\varepsilon\psi-\lambda\varphi)} 2\delta\nabla_\varphi(F + \varepsilon\psi - \lambda\varphi) \cdot_\varphi \nabla_\varphi\eta_{p_0} \\ &= e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\eta_{p_0}\big(\delta^2|\nabla_\varphi(F + \varepsilon\psi - \lambda\varphi)|_\varphi^2 + \delta\Delta_\varphi(F + \varepsilon\psi - \lambda\varphi)\big) \\ &\qquad + e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\Delta_\varphi(\eta_{p_0}) + e^{\delta(F+\varepsilon\psi-\lambda\varphi)} 2\delta\nabla_\varphi(F + \varepsilon\psi - \lambda\varphi) \cdot \nabla_\varphi\eta_{p_0}. \end{aligned}$$

First we can estimate

$$
\begin{aligned}
e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\Delta_\varphi\eta_{p_0} &\geq -e^{\delta(F+\varepsilon\psi-\lambda\varphi)}|\nabla^2\eta_{p_0}|tr_\varphi g \\
&\geq -e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\frac{4\theta}{d_0^2(1-\theta)}\eta_{p_0}tr_\varphi g
\end{aligned} \tag{5.10}
$$

$$
\begin{aligned}
2\delta\nabla_\varphi(F+\varepsilon\psi-\lambda\varphi)\cdot\nabla_\varphi\eta_{p_0} &\geq -\delta^2\eta_{p_0}|\nabla_\varphi(F+\varepsilon\psi-\lambda\varphi)|_\varphi^2 - \frac{|\nabla_\varphi\eta_{p_0}|_\varphi^2}{\eta_{p_0}} \\
&\geq -\delta^2\eta_{p_0}|\nabla_\varphi(F+\varepsilon\psi-\lambda\varphi)|_\varphi^2 - \frac{|\nabla\eta_{p_0}|^2 tr_\varphi g}{\eta_{p_0}} \\
&\geq -\delta^2\eta_{p_0}|\nabla_\varphi(F+\varepsilon\psi-\lambda\varphi)|_\varphi^2 - \frac{4\theta^2 tr_\varphi g}{d_0^2(1-\theta)}.
\end{aligned} \tag{5.11}
$$

Finally we compute

$$
\begin{aligned}
\Delta_\varphi(F+\varepsilon\psi-\lambda\varphi) &= -(f+\lambda n) + tr_\varphi\eta + \lambda tr_\varphi g + \varepsilon\Delta_\varphi\psi \\
&\geq (-f-\lambda n+\varepsilon n A_\Phi^{-\frac{1}{n}}\Phi^{\frac{1}{n}}(F)) + (\lambda-\varepsilon-\max_M|\eta|_{\omega_0})tr_\varphi g.
\end{aligned} \tag{5.12}
$$

Here $A_\Phi = \int_M e^F\Phi(F)dvol_g$. In the above calculation, we noticed that, using (5.3):

$$
\begin{aligned}
\Delta_\varphi\psi = g_\varphi^{i\bar{j}}(g_{i\bar{j}}+\psi_{i\bar{j}}) - tr_\varphi g &\geq n\big(\det(g_\varphi^{i\bar{j}})\det(g_{i\bar{j}}+\psi_{i\bar{j}})\big)^{\frac{1}{n}} - tr_\varphi g \\
&= n(e^{-F}e^F\Phi(F)A_\Phi^{-1})^{\frac{1}{n}} - tr_\varphi g.
\end{aligned}
$$

Plug (5.10), (5.11), (5.12) back into (5.9), we see

$$
\begin{aligned}
&\Delta_\varphi\big(e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\eta_{p_0}\big) \geq \delta\eta_{p_0}e^{\delta(F+\varepsilon\psi-\lambda\varphi)}(f-\lambda n+\varepsilon n A_\Phi^{-\frac{1}{n}}\Phi^{\frac{1}{n}}(F)) \\
&\quad + e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\big(\delta\eta_{p_0}(\lambda-\varepsilon-\max_M|\eta|_{\omega_0}) - \frac{4\theta}{d_0^2(1-\theta)}\eta_{p_0} - \frac{4\theta^2}{d_0^2(1-\theta)}\big)tr_\varphi g.
\end{aligned} \tag{5.13}
$$

Now we choose various constants δ, λ and θ appearing above.

Since $0<\varepsilon<1$, first we choose $\lambda = 2(1+\max_M|\eta|_{\omega_0})$. Then we fix λ, and choose δ to be $2n\delta\lambda = \alpha$. We need to make sure the coefficient in front of $tr_\varphi g$ to be positive. This can be achieved by choosing θ to be sufficiently small. Indeed, with above choice of δ and λ, we may calculate:

$$
\begin{aligned}
&\delta\eta_{p_0}(\lambda-\varepsilon-\max_M|\eta|_{\omega_0}) - \frac{4\theta\eta_{p_0}}{d_0^2(1-\theta)} - \frac{4\theta^2}{d_0^2(1-\theta)^2} \\
&\geq \frac{1}{2}\delta(1-\theta)\lambda - \frac{4\theta\eta_{p_0}}{d_0^2(1-\theta)} - \frac{4\theta^2}{d_0^2(1-\theta)^2} \geq \frac{(1-\theta)\alpha}{4n} - \frac{4\theta}{d_0^2(1-\theta)} - \frac{4\theta^2}{d_0^2(1-\theta)}.
\end{aligned} \tag{5.14}
$$

Hence if we choose θ small enough, above ≥ 0. After we made all the choices of δ, λ, θ, we obtain from (5.13) that

$$
\Delta_\varphi\big(e^{\delta(F+\varepsilon\psi-\lambda\varphi)}\eta_{p_0}\big) \geq \delta\eta_{p_0}e^{\delta(F+\varepsilon\psi-\lambda\varphi)}(f-\lambda n+\varepsilon n A_\Phi^{-\frac{1}{n}}\Phi^{\frac{1}{n}}(F)). \tag{5.15}
$$

Denote $u = e^{\delta(F+\varepsilon\psi-\lambda\varphi)}$. Now we are ready to apply Alexandrov estimate in $B_{d_0}(p_0)$:

$$
\begin{aligned}
\sup_{B_{d_0}(p_0)} u\eta_{p_0} \leq & \sup_{\partial B_{d_0}(p_0)} u\eta_{p_0} \\
& + C_n d_0 \Big(\int_{B_{d_0}(p_0)} \frac{u^{2n}\big((f - \lambda n + \varepsilon n A_\Phi^{-\frac{1}{n}} \Phi^{\frac{1}{n}}(F))^-\big)^{2n}}{e^{-2F}} dvol_g \Big)^{\frac{1}{2n}}.
\end{aligned}
\tag{5.16}
$$

We want to claim the integral appearing on the right hand side is bounded. Indeed, the function been integrated is nonzero only if

$$
f - \lambda n + \varepsilon n A_\Phi^{-\frac{1}{n}} \Phi^{\frac{1}{n}}(F) < 0.
$$

By the coercivity of Φ, this will imply an upper bound for F, say $F \leq C_{5.3}$, where the constant $C_{5.3}$ depends on ε, the choice of Φ, the integral bound A_Φ, and the background metric g. With this observation, we see

$$
\begin{aligned}
& \int_{B_{d_0}(p_0)} \frac{u^{2n}\big((f - \lambda n + \varepsilon n A_\Phi^{-\frac{1}{n}} \Phi^{\frac{1}{n}}(F))^-\big)^{2n}}{e^{-2F}} dvol_g \\
& \leq \int_{B_{d_0}(p_0)\cap\{F\leq C_{5.3}\}} e^{2n\delta(F+\varepsilon\psi-\lambda\varphi)} e^{2F}(|f| + \lambda n)^{2n} dvol_g \\
& \leq (||f||_0 + \lambda n)^{2n} e^{(2n\delta+2)C_{5.3}} \int_{B_{d_0}(p_0)} e^{2n\delta\varepsilon\psi - 2n\delta\lambda\varphi} dvol_g.
\end{aligned}
\tag{5.17}
$$

But recall $\psi \leq 0$, and $2n\delta\lambda = \alpha$, we know

$$
\int_{B_{d_0}(p_0)} e^{2n\delta\varepsilon\psi - 2n\delta\lambda\varphi} dvol_g \leq \int_{B_{d_0}(p_0)} e^{-\alpha\varphi} dvol_g \leq C_{5.4}. \tag{5.18}
$$

Denote $I = (||f||_0 + \lambda n)^{2n} e^{(2n\delta+2)C_{5.3}} \int_{B_{d_0}(p_0)} e^{-\alpha\varphi} dvol_g$. Now we go back to (5.16) and obtain:

$$
u(p_0) = \sup_M u \leq (1-\theta) \sup_M u + C_n d_0 I^{\frac{1}{2n}}. \tag{5.19}
$$

Here we recall that $\eta_{p_0} \equiv 1 - \theta$ on $\partial B_{d_0}(p_0)$. This implies $\sup_M u \leq \frac{C_n d_0 I^{\frac{1}{2n}}}{\theta}$. $\square$

Lemma 5.2 (Alexandroff maximum principle (c.f. [30, Lemma 9.3])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Denote $M = \sup_\Omega u - \sup_{\partial\Omega} u$. Define*

$$
\begin{aligned}
\Gamma^-(u, \Omega) = \{ & x \in \Omega : u(y) \leq u(x) + \nabla u(x) \cdot (y - x), \\
& \textit{for any } y \in \Omega \textit{ and } |\nabla u(x)| \leq \frac{M}{3diam\Omega} \}.
\end{aligned}
\tag{5.20}
$$

Then for some dimensional constant $C_d > 0$:

$$
M \leq C_d \Big(\int_{\Gamma^-(u,\Omega)} \det(-D^2 u) dx \Big)^{\frac{1}{d}}.
$$

In particular, suppose u satisfies $a_{ij}\partial_{ij} u \geq f$. Here a_{ij} satisfies the ellipticity condition $a_{ij}\xi_i\xi_j \geq 0$. Define $D^* = (\det a_{ij})^{\frac{1}{d}}$. Then the following estimate holds:

$$
M \leq C_d' \, diam\, \Omega || \frac{f^-}{D^*} ||_{L^d(\Omega)}. \tag{5.21}
$$

Here C_d' is another dimensional constant.

Remark 5.1. In this section (Proof of Theorem 5.2 and Lemma 5.1), we apply this estimate with $d = 2n$ to the operator Δ_φ. After rewriting Δ_φ in terms of real coefficients, one can find $D^* = \big(\det(g_\varphi)_{i\bar{j}}\big)^{-\frac{1}{n}} = e^{-\frac{F}{n}}\big(\det g_{i\bar{j}}\big)^{-\frac{1}{n}}$.

Finally, we want to give a proof to Theorem 5.1.

Proof. It is well known that in a given Kähler class, cscK metrics is global minimizer of the K-energy functional, by the main result of [3]. In particular, it follows that the K-energy functional of φ is apriori bounded from above. Recall the decomposition formula for K-energy functional E, proved in [12]:

$$K(\varphi) = \int_M \log\frac{\omega_\varphi^n}{\omega_0^n}\frac{\omega_\varphi^n}{n!} + J_{-Ric}(\varphi). \tag{5.22}$$

In the above, J_{-Ric} is defined in terms of its derivative, namely

$$\frac{dJ_{-Ric}}{dt} = \int_M \frac{\partial\varphi}{\partial t}(-tr_\varphi Ric + \underline{R})\frac{\omega_\varphi^n}{n!}.$$

It is well known in the literature that J_{-Ric} can be bounded in terms of C^0 norm of the potential function φ. A bound for $\int_M e^F|F|dvol_g$ follows from here.

Now we prove the second part of the theorem. First Corollary 5.2 gives a bound for F from above and Corollary 5.1 gives a bound for $||\varphi||_0$. Proposition 2.1 gives a bound for F from below. ☐

For future reference, we record Theorem 5.3, which follows from combining Proposition 4.2, Theorem 5.1 and Theorem 4.1.

Theorem 5.3. *Let φ be a smooth solution to* (1.1), (1.2) *normalized to be* $\sup_M \varphi = 0$*, then for any $p < \infty$, there exist a constant $C_{0.5}$, depending only on the background metric (M, g), $||f||_0$, $\max_M |\eta|_{\omega_0}$, p, and the upper bound of $\int_M e^F F dvol_g$ such that $||\varphi||_{W^{4,p}} \leq C_{0.5}$, $||F||_{W^{2,p}} \leq C_{0.5}$.*

Acknowledgments

In the Fall of 1997, Sir Simon invited the first named author to join him in exploring the space of Kähler potentials, and Sir Simon has been remarkably generous about sharing his time and ideas ever since. It is therefore a deep pleasure to dedicate this paper to Sir Simon Donaldson, in celebration of his 60th birthday, and in acknowledgment of the far-reaching influence of his profound mathematical ideas, which have changed the landscape of mathematics so much.

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