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Gravitational instantons with faster than quadratic curvature decay (III)

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Abstract
This is our third paper in a series on the gravitational instantons. In this paper, we classify ALG and ALH gravitational instantons. In ALG case, we extend Tian–Yau–Hein's construction slightly and show that it is the only ALG gravitational instanton. In ALH case, we prove a Torelli-type theorem.

Contents

1 Introduction

There are lots of different definitions of gravitational instantons. The broader definition of gravitational instanton means a four dimensional Einstein manifold. In this paper, as in our previous work [12,13], we choose the following narrower definition: A noncompact complete hyperkähler manifold M of real dimension 4 is called a gravitational instanton if the curvature at x satisfies $|\mathrm{Rm}(x)| = O(r(x)^{-2-\tau})$, where $r(x)$

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denotes the distance to a fixed point in M, τ is any small positive number. It is worthwhile to notice that in real dimension 4, the hyperkähler condition hol $\subset$ Sp(1) is equivalent to the Calabi–Yau condition hol $\subset$ SU(2).

It is quite easy to prove the following theorem:

Theorem 1.1 *For gravitational instanton M, the following conditions are equivalent:*

(1) *M is flat;*
(2) *M has trivial holonomy;*
(3) *M splits as $\mathbb{R}^{4-k} \times \mathbb{T}^k$, $k = 0, 1, 2, 3$.*

It will be proved in Sect. 3. For simplicity, in this paper, we will exclude the flat gravitational instantons.

In our first paper [12], according to different kinds of asymptotic geometries, we classified gravitational instantons into the following categories: ALE (Asymptotically Locally Euclidean), ALF-A_k, ALF-D_k (Asymptotically Locally Flat), ALG and ALH ("G" and "H" are the letters after "E" and "F"). Its unique tangent cone at infinity is $\mathbb{C}^2/\Gamma$, $\mathbb{R}^3$, $\mathbb{R}^3/\mathbb{Z}_2$, a flat cone with cone angle $2\pi\beta$ or $\mathbb{R}^+$, respectively.

In the ALE case, after Bando–Kasue–Nakajima's work [4] about the improvement of asymptotic rate, Kronheimer [32,33] proved that any ALE gravitational instanton must be diffeomorphic to the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma}$ of the quotient singularity $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of SU(2). Moreover, the Torelli theorem holds for ALE gravitational instantons.

$H_2(\widetilde{\mathbb{C}^2/\Gamma}, \mathbb{Z})$ is generated by holomorphic curves with self intersection number -2. Let k be the number of generators. Then, their intersection patterns can be classified into $A_k (k \geq 1)$, $D_k (k \geq 4)$, $E_k (k = 6, 7, 8)$ Dynkin diagrams. They correspond to different types of Γ.

Later, Minerbe [38] proved that the multi-Taub-NUT metric is the only ALF-A_k gravitational instanton. When $k = 0$, it is called the Taub-NUT metric. The Taub-NUT metric is diffeomorphic to $\mathbb{C}^2$. When $k \geq 1$, the ALF-A_k gravitational instanton is diffeomorphic to the ALE-A_k gravitational instanton.

In ALF-D_k case, Biquard and Minerbe [7] proved that k must be nonnegative. Ivanov and Roček [27] conjectured a formula using generalized Legendre transform developed by Lindström and Roček [36]. This conjecture was proved by Cherkis and Kapustin [16]. A more explicit formula was computed by Cherkis and Hitchin [15]. In our second paper [13], we proved that it is the only possible ALF-D_k gravitational instanton. When $k = 0$, it is called the Atiyah-Hitchin metric [3]. When $k = 2$, it is called the Page-Hitchin metric [25,45]. When $k \geq 4$, the ALF-D_k gravitational instanton is diffeomorphic to the ALE-D_k gravitational instanton. As a corollary of the classification result in ALF case, the Torelli theorem holds for ALF gravitational instantons [13].

In the ALG and ALH cases, we [12] proved a compactification result and thus confirmed a conjecture of Yau [56]:

Theorem 1.2 [12] *For any ALG or ALH gravitational instanton M, there exists a compact elliptic surface $\bar{M}$ with a meromorphic function $z : \bar{M} \to \mathbb{CP}^1$ whose generic fiber has genus 1. The fiber $D = \{z = \infty\}$ is regular if M is ALH, while it*

is either regular or of type I_0^*, II, II*, III, III*, IV, IV* if M is ALG. There exists an (a_1, a_2, a_3) in $\mathbb{S}^2$ such that when we use $a_1 I + a_2 J + a_3 K$ as the complex structure, M is biholomorphic to $\bar{M}\backslash D$.

Remark The type of D is related to the tangent cone at infinity of M. See the table in Definition 2.3.

In this paper, we will start from an improvement of the above theorem:

Theorem 1.3 *The $(\bar{M}, z)$ in the above theorem must be a rational elliptic surface (see Definition 2.7). Moreover, in the ALG case, D can not be regular.*

In [54], Tian–Yau proved that there exists a non-compact complete Calabi–Yau metric if there exists a good background metric. Moreover, they found two ways of constructing good background metrics to produce complete Calabi–Yau metrics with volume growth rate 1 or $\frac{2n}{n+1}$ on the complement of anti-canonical divisors to n-dimensional manifolds.

Besides Tian–Yau's Kähler geometry method, hyperKähler 4-manifolds can also be constructed using gauge theory. In [17], Cherkis and Kapustin predicted complete hyperkähler metrics on the moduli space of periodic monopoles, which is a rational elliptic surface minus a fiber of type I_0^*, I_1^*, I_2^*, I_3^*, I_4^*. In [6], Biquard and Boalch constructed complete hyperkähler manifolds using wild non-abelian Hodge theory on curves.[1]

In [7], Biquard and Minerbe constructed ALF-$D_k (k \geq 4)$, ALG and ALH gravitational instantons on the minimal resolutions of the quotient of Taub-NUT metric by the binary dihedral group, $(\mathbb{R}^2 \times \mathbb{T}^2)/\mathbb{Z}_k (k = 2, 6, 4, 3)$ or $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$, respectively.

In [24], Hein chose many background metrics, used Tian–Yau's result [54] to get more complete hyperkähler manifolds on the complement of a fiber D (the anti-canonical divisor in this case) to a rational elliptic surface $\bar{M}$. Moreover, Hein [24] proved the asymptotic geometry of such construction. When D is regular, the metric is ALH. When D has type I_0^*, II, II*, III, III*, IV, or IV*, the metric is ALG. When D has type I_b^*, $b = 1, 2, \ldots, 4$, the volume growth, injectivity radius decay, curvature decay rates and tangent cone at infinity are r^2, $(\log r)^{-1/2}$, $r^{-2}(\log r)^{-1}$ and $\mathbb{R}^2/\mathbb{Z}_2$.

[1] We are grateful for the anonymous referee who kindly pointed out to us that *"Biquard and Boalch [6] proved that the many moduli spaces of solutions to Hitchin's equations on a non-compact curve (with wild boundary conditions) form complete hyperkähler manifolds. In one complex structure in the hyperkähler family they are moduli spaces of meromorphic connections and in another complex structure they are moduli spaces of meromorphic Higgs bundles (and so are algebraic integrable systems). The examples of complex dimension two are thus the complement of the anticanonical divisor D to the rational elliptic surface $\bar{M}$ (the meromorphic Hitchin systems of complex dimension two). Moreover, there is a nice relationship between the type of D and Dynkin diagrams extending that of Kronheimer commonly used in the ALE case:*

I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	II	II*	III	III*	IV	IV*
D_4	D_3	D_2	D_1	D_0	E_8	A_0	E_7	A_1	E_6	A_2

See [8] for the relationship between Biquard–Boalch metric and Painlevé equations. See [48] for the relationship between Painlevé equations and rational elliptic surfaces following the works in [41–44]. The readers may also read [9] and the references therein for the survey about Biquard–Boalch metric. See also [55] as well as the talk 'Some geometry of irregular connections on curves' given by Boalch at the workshop on 'Gauge Theory and Representation Theory' at Institute for Advanced Study in 2007."

It was called ALG by some authors but a more appropriate terminology would be ALG* because the curvature decay is too slow and the fibers have no limit at infinity. When D has type I_b, $b = 1, 2, \ldots, 9$, the volume growth, injectivity radius decay, curvature decay rates and tangent cone at infinity are $r^{4/3}$, $r^{-1/3}$, r^{-2} and $[0, \infty)$. An appropriate terminology for this kind of asymptotic geometry would be ALH*.

The background metrics constructed in [54] provide ALH and ALH* examples using their existence result. In [20,21], Cherkis-Kapustin's prediction [17] was partially verified. It was proved that the moduli space of periodic monopoles is a non-empty hyperkähler manifold. However, it is still unknown whether this metric is complete or whether it is an elliptic surface. It is conjectured to be ALG or ALG* but this conjecture has not been proved yet. The Biquard–Boalch's construction [6] is also conjectured to be ALG or ALG*, but this conjecture has not been proved, too.

In this paper, we will slightly modify Tian–Yau–Hein's construction [24,54] and then prove that any ALG gravitational instanton must be obtained by this modified construction:

Theorem 1.4 (1) *Let $(\bar{M}, z)$ be a rational elliptic surface with $D = \{z = \infty\}$ of type I_0^*, II, II*, III, III*, IV, or IV*. Let $\omega^+ = \omega^2 + i\omega^3$ be a rational 2-form on $\bar{M}$ with $[D] = \{\omega^+ = \infty\}$. For any Kähler form ω on $\bar{M}$, there exists a real smooth polynomial growth function ϕ on $M = \bar{M}\backslash D$ such that $(M, \omega^1 = \omega + i\partial\bar{\partial}\phi, \omega^2, \omega^3)$ is an ALG gravitational instanton.*

(2) *The form $\omega + i\partial\bar{\partial}\phi$ in the first part is uniquely determined by its asymptotic geometry.*

(3) *Given any ALG gravitational instanton, after a hyperkähler rotation, we can assume that $a_1 I + a_2 J + a_3 K$ in Theorem 1.2 is in fact I. Then $\omega^+ = \omega^2 + i\omega^3$ is a rational 2-form on $\bar{M}$ with $[D] = \{\omega^+ = \infty\}$. There exist a Kähler form ω on $\bar{M}$ and a real smooth polynomial growth function ϕ on $M = \bar{M}\backslash D$ such that $\omega^1 = \omega + i\partial\bar{\partial}\phi$. When D is of type II*, III*, or IV*, we may need a new choice of $\bar{M}$ to achieve this.*

In the ALH case, as a corollary of Theorems 1.2 and 1.3, any ALH gravitational instantons are diffeomorphic to each other. In particular, they are diffeomorphic to the minimal resolution of $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$ by [7]. The torus $\mathbb{T}^3 = \mathbb{R}^3/\Lambda$ is determined by the lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3$. It is easy to see that $H_2(\widetilde{(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2}, \mathbb{R}) = \mathbb{R}^{11}$ is generated by three faces F_{jk} spanned by v_j and v_k and eight rational curves Σ_j coming from the resolution of eight orbifold points in $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$. Using those notations, we will prove the following classification result of ALH gravitational instantons:

Theorem 1.5 (Torelli theorem for ALH gravitational instantons) *Let M be the smooth 4-manifold which underlies the minimal resolution of $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$. Let $[\alpha^1], [\alpha^2], [\alpha^3] \in H^2(M, \mathbb{R})$ be three cohomology classes which satisfy the non-degeneracy conditions:*

(1) *The integrals f_{ijk} of α^i on the three faces F_{jk} satisfy*

$$\begin{vmatrix} f_{123} & f_{131} & f_{112} \\ f_{223} & f_{231} & f_{212} \\ f_{323} & f_{331} & f_{312} \end{vmatrix} > 0;$$

(2) *For each* $[\Sigma] \in H_2(M, \mathbb{Z})$ *with* $[\Sigma]^2 = -2$, *there exists* $i \in \{1, 2, 3\}$ *with* $[\alpha^i][\Sigma] \neq 0$.
Then there exists on M *an ALH hyperkähler structure such that* Φ *in Definition* 2.6 *can be chosen to be the identity map and the cohomology classes of the Kähler forms* $[\omega^i]$ *are the given* $[\alpha^i]$. *It is unique up to tri-holomorphic isometries which induce identity on* $H_2(M, \mathbb{Z})$.
Moreover, any ALH gravitational instanton must be constructed by this way.

Remark Recently, Haskins, Hein and Nordström [23] classified asymptotically cylindrical Calabi–Yau manifolds of complex dimension at least 3. In dimension 2, their analytic existence theorem (Theorem 4.1 of [23]) still holds. However, when $\mathbb{T}^3$ does not split isometrically as $\mathbb{S}^1 \times \mathbb{T}^2$, their geometric existence theorem (Theorem D of [23]) fails due to the lack of background Kähler form in the cohomology class.

Remark In [24], Hein proved that the space of ALH gravitational instantons module isometries is 30 dimensional. After adding 3 parameters of hyperkähler rotations, the space of ALH gravitational instantons module tri-holomorphic isometries which induce identity on $H_2(M, \mathbb{Z})$ is 33 dimensional. Our Theorem 1.5 is consistence with Hein's computation. Remark that by Theorem 1.10 of [24], the number of parameters of Tian–Yau's ALH construction is smaller than the dimension of moduli space of ALH gravitational instantons. So Theorem 1.5 indeed provides ALH metrics which can not be constructed by Tian–Yau's ALH method.

It is interesting to compare Theorem 1.5 with the Torelli theorem for ALE gravitational instantons [32,33], ALF gravitational instantons [13] as well as K3 surfaces, which was proved by Burns–Rapoport [10], Todorov [52], Looijenga–Peters [35] and Siu [51]. It was reformulated by Besse in Section 12.K of [5]. Anderson [2] also proved a version of Torelli theorem for K3 surfaces which allows orbifold singularities.

Theorem 1.6 ([5], Torelli theorem for K3 surfaces) *Let* M *be the smooth 4-manifold which underlies the minimal resolution of* $\mathbb{T}^4/\mathbb{Z}_2$. *Let* Ω *be the space of three cohomology classes* $[\alpha^1], [\alpha^2], [\alpha^3] \in H^2(M, \mathbb{R})$ *which satisfy the following conditions:*

(1) (*Integrability*)

$$\int_M \alpha^i \wedge \alpha^j = 2\delta_{ij} V.$$

(2) (*Nondegeneracy*) *For any* $[\Sigma] \in H_2(M, \mathbb{Z})$ *with* $[\Sigma]^2 = -2$, *there exists* $i \in \{1, 2, 3\}$ *with* $[\alpha^i][\Sigma] \neq 0$.
Ω *has two components* Ω^+ *and* Ω^-. *For any* $([\alpha^1], [\alpha^2], [\alpha^3]) \in \Omega^+$, *there exists on* M *a hyperkähler structure for which the cohomology classes of the Kähler forms* $[\omega^i]$ *are the given* $[\alpha^i]$. *It is unique up to tri-holomorphic isometries which induce identity on* $H_2(M, \mathbb{Z})$.
Moreover, any hyperkähler structure on K3 surface must be constructed by this way.

One may ask whether Torelli theorem holds for ALG gravitational instantons. This is not a easy problem. First of all, one may ask whether all ALG gravitational instantons are diffeomorphic to each other if they have same type of D. It involves deep understandings of the moduli space of rational elliptic surfaces and is hard partially because it may be GIT unstable [39]. Even if it is true, the uniqueness part fails at least when D is of type II*, III*, or IV*.

Theorem 1.7 *When D is of type II*, III*, or IV*, there exist two different ALG gravitational instantons with same $[\omega^i]$.*

Roughly speaking, the failure of the uniqueness part of ALG Torelli theorem in such case is due to the existence of an exact anti-self-dual form with decay rate less than 1. So in order to describe the moduli space of ALG gravitational instantons, one may need additional parameters corresponding to such exact anti-self-dual forms. However, such modified version is still very hard at current stage. Since the uniqueness part of ALH Torelli Theorem used the gluing of ALH gravitational instantons into K3 surfaces, one may need to do the similar thing in ALG case which is much harder then the ALH case. As for the existence part, it seems that our proof of ALH Torelli theorem is helpless for ALG manifolds because the total energy $\int_M (\mathrm{tr}_{\omega_i}\omega_0 - 2)\frac{\omega_i^2}{2}$ used in Theorem 7.2 is infinity if the decay rate of $\omega_i - \omega_0$ is 2. However, recall that the ALF Torelli theorem was proved using gauge theory. So it might be very helpful if one can prove that the Biquard–Boalch metric [6] or probably its generalization is ALG and is general enough so that any classes of $[\omega^i]$ as well as the additional parameters corresponding to the exact anti-self-dual forms can be achieved.

In Sect. 3, we will study the topology of ALG and ALH gravitational instantons. In Sect. 4, we will prove Theorem 1.4. In Sect. 5, we will prove Theorem 1.7 and the theorem that the gluing of any ALH gravitational instanton with itself is a K3 surface. In Sect. 6, we will use the gluing construction in Sect. 5 and the Torelli theorem for K3 surfaces to prove the uniqueness part of Theorem 1.5. In Sect. 7, we will prove the existence part of Theorem 1.5.

2 Definitions

Definition 2.1 (*Hyperkähler manifold*) A manifold (M, g, I, J, K) is called a hyperkähler manifold if I, J, K are three parallel complex structures on M which are compatible with g and satisfy the quaternion relationships. Any map between two hyperkähler manifolds is called tri-holomorphic if it preserves I,J and K.

It is well known that I, J, K are all Kähler structures. They determine three Kähler forms $\omega^1, \omega^2, \omega^3$. The I-holomorphic 2-form $\omega^+ = \omega^2 + i\omega^3$ is called the holomorphic symplectic form. Conversely, it is well known that three closed forms ω^i satisfying $\omega^i \wedge \omega^j = 2\delta_{ij} V$ for some nowhere vanishing 4-form V determine a hyperkähler structure on M. In fact, given three such 2-forms, we can call the linear span of them the "self-dual" space. The orthogonal complement of the "self-dual" space under wedge product is called the "anti-self-dual" space. These two spaces determine a star operator. It is well known that the star operator determines a conformal class

of metrics. The conformal factor can be determined by requiring V to be the volume form. Using this metric and the three forms ω^i, we can determine three almost complex structures I, J and K. It is easy to see that $IJ = K$ or $IJ = -K$. In the K3 case, the former happens on Ω^+ (the latter happens on Ω^-). In noncompact case, the former always happens if it happens on the end. By Lemma 6.8 of [26], I, J, K are parallel.

Definition 2.2 (*Hyperkähler rotation*) For any matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in \mathrm{SO}(3),$$

$(M, g, a_1 I + a_2 J + a_3 K, b_1 I + b_2 J + b_3 K, c_1 I + c_2 J + c_3 K)$ defines another hyperkähler structure on M. It is called the hyperkähler rotation of (M, g, I, J, K). After hyperkähler rotation, we can with out loss of generality, assume that the complex structure $a_1 I + a_2 J + a_3 K$ in Theorem 1.2 is actually I.

Definition 2.3 (*ALG model*) Suppose $\beta \in (0, 1]$ and $\tau \in \mathbb{H} = \{\tau | \mathrm{Im}\tau > 0\}$ are parameters in the following table:

D	Regular	I_0^*	II	II*	III	III*	IV	IV*
β	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$
τ	$\in \mathbb{H}$	$\in \mathbb{H}$	$e^{2\pi i/3}$	$e^{2\pi i/3}$	i	i	$e^{2\pi i/3}$	$e^{2\pi i/3}$

Suppose $l > 0$ is any scaling parameter. Let E be the manifold obtained by identifying (u, v) with $(e^{2\pi i\beta}u, e^{-2\pi i\beta}v)$ in the space

$$\{(u, v) | \arg u \in [0, 2\pi\beta], |u| \geq R\} \subset (\mathbb{C} - B_R) \times \mathbb{C}/(\mathbb{Z}l \oplus \mathbb{Z}\tau l).$$

Then there is a flat hyperkähler metric h on E such that $\omega^1 = \frac{i}{2}(du \wedge d\bar{u} + dv \wedge d\bar{v})$ and $\omega^+ = \omega^2 + i\omega^3 = du \wedge dv$. It is called the standard ALG model.

Definition 2.4 (*ALG*) (M, g) is called ALG of order δ if there exist a bounded domain $K \subset M$, and a diffeomorphism $\Phi : E \to M \backslash K$ such that

$$|\nabla^m(\Phi^* g - h)| = O(|u|^{-m-\delta})$$

for some $\delta > 0$ and any $m \geq 0$.

Definition 2.5 (*ALH model*) Let (E, h) be flat product of $[R, \infty) \times \mathbb{T}^3$. Let r be the coordinate of $[R, \infty)$. Let $(\theta^1, \theta^2, \theta^3)$ be the coordinates of $\mathbb{T}^3 = \mathbb{R}^3/\Lambda$. Then there exists a hyperkähler structure on it defined by

$$\mathrm{d}r = I^* \mathrm{d}\theta^1 = J^* \mathrm{d}\theta^2 = K^* \mathrm{d}\theta^3.$$

It is called the standard ALH model.

Definition 2.6 (*ALH*) (M, g) is called ALH of order τ if there exist a bounded domain $K \subset M$, and a diffeomorphism $\Phi : E \to M \backslash K$ such that

$$|\nabla^m(\Phi^* g - h)| = O(e^{-\tau r}), |\nabla^m(\Phi^* I - I)| = O(e^{-\tau r}),$$
$$|\nabla^m(\Phi^* J - J)| = O(e^{-\tau r}), |\nabla^m(\Phi^* K - K)| = O(e^{-\tau r})$$

for some $\tau > 0$ and any $m \geq 0$.

Remark We [12,13] proved that the order of any ALH gravitational instanton is at least $\lambda_1 = 2\pi \min_{\lambda \in \Lambda^* \backslash \{0\}} |\lambda|$, where $\Lambda^* = \{\lambda \in \mathbb{R}^3 | < \lambda, \theta > \in \mathbb{Z}, \forall \theta \in \Lambda\}$. Later, in Sects. 5, 6, 7, we will choose a positive number $\delta < \lambda_1/100$ and use $e^{\delta r}$ as the weight function.

Definition 2.7 (*Rational elliptic surface*) Let F, G be two linearly independent cubic homogenous polynomials on $\mathbb{CP}^2$. $\{F = 0\}$ and $\{G = 0\}$ intersect at 9 points with multiplicity. Let $\bar{M}$ be the blow up of $\mathbb{CP}^2$ on these 9 points, if needed repeatedly. Then $z = F/G$ is a well-defined meromorphic function on $\bar{M}$ whose generic fiber has genus 1. $(\bar{M}, z)$ is called the rational elliptic surface. It is well known that it has a global section σ corresponding to any exceptional curve in the blowing up construction.

Definition 2.8 (*Holomorphic radius*) The largest γ satisfying the following conditions is called the $C^{1,\alpha}$-holomorphic radius at p:

There exist holomorphic coordinates z^j on $B(p, \gamma)$ such that

$$\frac{1}{2}\delta_{i\bar{j}} \leq g_{i\bar{j}} \leq 2\delta_{i\bar{j}}$$

and

$$\gamma^{1+\alpha} ||g_{i\bar{j}}||_{C^{1,\alpha}} \leq 1.$$

Definition 2.9 χ is a smooth cut-off function from $(-\infty, +\infty)$ to $[0, 1]$ such that $\chi \equiv 1$ on $(-\infty, -\frac{1}{2}]$, $\chi \equiv 0$ on $[\frac{1}{2}, \infty)$ and $-2 < \chi' \leq 0$.

Definition 2.10 In the ALG case, by Theorem 1.2, (after hyperkähler rotation) there exists an I-holomorphic function z on M asymptotic to $u^{1/\beta}$. The letter z in this paper will always mean this function. r will always mean the function $|z|^\beta(1 - \chi(|z|^\beta - 1)) + \chi(|z|^\beta - 1)$. In the ALH case, the function $r(1 - \chi(r - R - 1))$ can be well defined on M. It is still denoted by r.

3 The topology of ALG and ALH gravitational instantons

In this section, we will study the topology of ALG and ALH gravitational instantons.

We start from the study of flat gravitational instantons:

Theorem 3.1 *For gravitational instanton M, the following conditions are equivalent:*

(1) *M is flat;*
(2) *M has trivial holonomy;*
(3) *M splits as $\mathbb{R}^{4-k} \times \mathbb{T}^k$, $k = 0, 1, 2, 3$.*

Proof By the arguments in the proof of Theorem 3.4 in our first paper [12], it is easy to see that any flat gravitational instanton M must have trivial holonomy. It is well known that M is isometric to the Euclidean space quotient by covering transforms. However, since the holonomy is trivial, any covering transform must be a pure translation. Therefore, M is isometric to the product of the Euclidean space with a flat torus. Conversely, it is trivial that (2) or (3) implies (1). □

Therefore, as mentioned in the introduction, we will assume that any gravitational instanton is non-flat.

Theorem 3.2 *The first betti number of any ALG or ALH gravitational instanton must be 0. Moreover, in the ALG case, D can not be regular.*

Proof In the ALH case, Melrose's theory [37] works. In particular, the first cohomology group $H^1(M, \mathbb{R})$ is a subspace of the space of bounded harmonic 1-forms [37]. By Weitzenböck formula, any bounded $(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)$-harmonic 1-form ϕ is also $\nabla^*\nabla$-harmonic. By Melrose's theory, $\nabla\phi$ decays exponentially. After integration by parts,

$$\int_M |\nabla\phi|^2 \chi(r - R) \leq \int_M |\nabla\phi||\nabla\chi(r - R)| \to 0,$$

as $R \to \infty$. Therefore, ϕ is a parallel 1-form. If it is nonzero, the holonomy group must be trivial since the action of $\mathrm{Sp}(1)$ is free on $\mathbb{R}^4\backslash\{0\}$. It contradicts the non-flat assumption.

In the ALG case, (after hyperkähler rotation) if D is regular, i.e. if $\beta = 1$, the I-holomorphic function z on M is asymptotic to the function u on E. $\nabla \mathrm{d}u = 0$ on E, so when we go through the construction of z in our first paper [12], it is easy to see that $|\nabla \mathrm{d}z| = O(r^{-1-\epsilon})$ for any small enough ϵ. So

$$\int_M |\nabla \mathrm{d}z|^2 \chi\left(\frac{r}{2R} - 1\right) \leq \int_M |\nabla \mathrm{d}z||\nabla\chi\left(\frac{r}{2R} - 1\right)| \to 0,$$

as $R \to \infty$. As before, it contradicts the non-flat assumption.

Therefore $\beta < 1$. Inspired by Lemma 6.11 of [37] ,we define

$$f(r) = \left(\frac{1}{r} + \frac{1}{4R}\chi\left(\frac{2R}{r} - \frac{3}{2}\right)\right)^{-1}.$$

Then f is increasing. $f(r) = r$ when $r \leq R$ and $\lim_{r\to\infty} f(r) = 4R$. Let $u = re^{i\theta}$. The map $F(re^{i\theta}, v) = (f(r)e^{i\theta}, v)$ on M is homotopic to the identity. Therefore, any smooth closed 1-form ϕ is cohomologous to $F^*\phi$. It is easy to see that $F^*\phi = O(r)$.

By Theorem 4.12 of [12], for any small positive ϵ, there exists a smooth 1-form ψ such that

$$\int_M |\psi|^2 r^{-8-\epsilon} + |\nabla\psi|^2 r^{-6-\epsilon} + |\nabla^2\psi|^2 r^{-4-\epsilon} < \infty$$

and

$$F^*\phi = \mathrm{d}\mathrm{d}^*\psi + \mathrm{d}^*\mathrm{d}\psi.$$

Since $F^*\phi$ is closed, it is easy to see that $\mathrm{d}^*\mathrm{d}\psi$ is a closed harmonic 1-form.

Similar to Theorem 4.6 of [13], the leading term of $\mathrm{d}^*\mathrm{d}\psi$ can be written as $au^\delta \mathrm{d}u + b\bar{u}^\delta \mathrm{d}\bar{u}$ for some $\delta \le 1$. To make it well-defined, $(\delta+1)\beta$ must be an integer. The first available choice is $\delta = 1/\beta - 1$ if $\beta \ge 1/2$ or $\delta = -1$ if $\beta < 1/2$. In the first case, $\mathrm{d}^*\mathrm{d}\psi - a\beta \mathrm{d}z - b\beta \mathrm{d}\bar{z}$ is a much smaller closed harmonic 1-form on M. Its order is also at most r^{-1}. However, by maximal principle and the Ricci flatness, any decaying harmonic 1-form on M must be 0. In conclusion, ϕ must be exact. In other words, $H^1(M) = 0$ for any ALG gravitational instanton. □

Theorem 3.3 *$\bar{M}$ is a rational elliptic surface.*

Proof Choose a tubular neighborhood T of D. Then $\bar{M} = M \cup T$. The Mayer-Vietoris sequence is

$$H^0(M) \oplus H^0(T) \to H^0(M \cap T) \to H^1(\bar{M}) \to H^1(M) \oplus H^1(T).$$

The first map is surjective, so the second map is 0. So the third map is injective. Notice that $H^1(T) = H^1(D) = 0$ because D is of type I_0^*, II, II*, III, III*, IV, or IV*. $H^1(M)$ also vanishes by Theorem 3.2. So $H^1(\bar{M}) = 0$.

A careful examination of our construction of $\bar{M}$ in [12] and Kodaira's paper [28,29] yields that ω^+ can be extended to a meromorphic 2-form on $\bar{M}$ with a pole D. In other words, D is the anti-canonical divisor of $\bar{M}$. Since D is homologous to another fiber of z, the self intersection number of D is 0. In other words $c_1^2(\bar{M}) = c_1^2(-K) = [D]^2 = 0$. It is also very easy to see that $H^0(\bar{M}, mK) = 0$ for any $m > 0$. In particular the geometric genus $p_g = \dim H^0(\bar{M}, K) = 0$.

By Kodaira's classification of complex surfaces [30], since the first betti number of $\bar{M}$ is even and $p_g = 0$, $\bar{M}$ must be algebraic. By Castelnuovo theorem, $\bar{M}$ must be rational because $H^{0,1}(\bar{M}) = H^0(\bar{M}, 2K) = 0$. By Kodaira's Equation 13 in [30], $c_1^2 + \dim H^{0,1} + b^- = 10p_g + 9$. So $b^- = 9$. By Theorem 3 of [30], $b^+ = 1 + 2p_g = 1$. Therefore the second betti number b_2 of $\bar{M}$ equals to 10. It is standard [40] to prove that $(\bar{M}, z)$ is a rational elliptic surface defined in Definition 2.7. □

Theorem 3.4 *Given any ALH gravitational instantons M, there exists a diffeomorphism from the minimal resolution of $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$ to M whose restriction on $[R, \infty) \times \mathbb{T}^3$ is Φ in Definition 2.6.*

Proof The divisor D is smooth by our construction of the compactification. So for any small enough deformation in the coefficients of F and G, the diffeomorphism

type of $M = \bar{M}\backslash D$ is invariant. For generic choice of coefficients of F and G, $\{G = 0\}$ is smooth and $\{F = 0\}$ intersects $\{G = 0\}$ in distinct points. Since the non-generic parameters have real codimension 2, generic points can be connected by paths inside the set of generic points. Therefore, it is easy to see that any ALH gravitational instantons are diffeomorphic to each other. In particular, they are diffeomorphic to the specific example of Biquard and Minerbe [7] on the minimal resolution of $(\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}_2$. □

4 Classification of ALG gravitational instantons

In this section, we will slightly modify Hein's result in [24] to get Theorem 1.4.

Let ω be any Kähler form on $\bar{M}$. Let a be the area of each regular fiber with respect to ω. Recall that for any section σ' of z on $\Delta^* = \{|z|^\beta \geq R\}$, Hein [24] wrote down some explicit formula of the semi-flat Calabi–Yau metric $\omega_{\mathrm{sf},a}[\sigma']$ on $M|_{\Delta^*}$ whose area of each regular fiber is also a:

Definition 4.1 [24] Using σ' as the zero section, $M|_{\Delta^*}$ is locally biholomorphic to

$$M|_U = (U \times \mathbb{C})/(z, v) \sim (z, v + m\tau_1(z) + n\tau_2(z))$$

for some holomorphic functions τ_1 and τ_2. So locally, $\omega^+ = g(z)\mathrm{d}z \wedge \mathrm{d}v$ for some holomorphic function $g : U \to \mathbb{C}$. Then locally

$$\omega_{\mathrm{sf},a}[\sigma'] = i|g|^2\frac{\mathrm{Im}(\bar{\tau}_1\tau_2)}{a}\mathrm{d}z \wedge \mathrm{d}\bar{z} + \frac{i}{2}\frac{a}{\mathrm{Im}(\bar{\tau}_1\tau_2)}(\mathrm{d}v - \Gamma\mathrm{d}z)(\mathrm{d}\bar{v} - \bar{\Gamma}\mathrm{d}\bar{z}),$$

where

$$\Gamma(z, v) = \frac{1}{\mathrm{Im}(\bar{\tau}_1\tau_2)}(\mathrm{Im}(\bar{\tau}_1 v)\frac{\mathrm{d}\tau_2}{\mathrm{d}z} - \mathrm{Im}(\bar{\tau}_2 v)\frac{\mathrm{d}\tau_1}{\mathrm{d}z}).$$

It is easy to check that $\omega_{\mathrm{sf},a}[\sigma']$ is actually a globally well-defined form.

After that, the following theorem is essential:

Theorem 4.2 *There exist a real smooth polynomial growth function ϕ_1 on $M|_{\Delta^*}$ and a polynomial growth holomorphic section σ' of z over Δ^* such that $\omega_{\mathrm{sf},a}[\sigma'] = \omega + i\partial\bar{\partial}\phi_1$.*

Remark Compared to Hein's Claim 1 in page 382 of [24], the key improvement in our paper is that both σ' and ϕ_1 grow at most polynomially.

Proof As Hein did in [24], there exists a real 1-form ζ on $M|_{\Delta^*}$ such that $\mathrm{d}\zeta = \omega_{\mathrm{sf},a}[\sigma] - \omega$. Choose the map F as in Theorem 3.2. By Cartan's formula, the homotopy between F and the identity map implies that $F^*\omega - \omega = \mathrm{d}\zeta_1$ and $F^*\omega_{\mathrm{sf},a}[\sigma] - \omega_{\mathrm{sf},a}[\sigma] = \mathrm{d}\zeta_2$ for some real polynomial growth 1-forms ζ_1 and ζ_2. However, $\mathrm{d}F^*\zeta =$

$F^*\omega_{\mathrm{sf},a}[\sigma] - F^*\omega$ for some polynomial growth 1-form $F^*\zeta$. In conclusion, we can without loss of generality assume that ζ grows polynomially.

Using σ as the zero section, any section σ' of z can be written as $v = \sigma'(z)$ in local coordinates. Hein calculated that there exists a real 1-form $\tilde{\zeta}$ such that $\omega_{\mathrm{sf},a}[\sigma'] - \omega_{\mathrm{sf},a}[\sigma] = \mathrm{d}\tilde{\zeta}$ and the (0,1)-part $\tilde{\xi}$ of $\tilde{\zeta}$ can be written as

$$\tilde{\xi} = -\frac{i}{2}\frac{a}{\mathrm{Im}(\bar{\tau}_1\tau_2)}\left[\sigma'(z)(\mathrm{d}\bar{v} - \bar{\Gamma}(z,v)\mathrm{d}\bar{z}) - \frac{1}{2}\bar{\Gamma}(z,\sigma'(z))\mathrm{d}\bar{z}\right].$$

Choose σ' so that $\frac{i}{2}\frac{a}{\mathrm{Im}(\bar{\tau}_1\tau_2)}\sigma'$ equals to the average of the coefficient of $\mathrm{d}\bar{v}$ term of the (0,1)-part ξ of ζ on each fiber. Then σ' and $\tilde{\xi}$ grow polynomially. Moreover, the average of the coefficient of $\mathrm{d}\bar{v}$ term of $\xi + \tilde{\xi}$ on each fiber vanishes. So on each fiber, $\xi + \tilde{\xi}$ can be written as $i\bar{\partial}\phi_2$ by solving the $\bar{\partial}$-equation on each fiber. It is easy to see that ϕ_2 also grows polynomially. So the (0,1)-form $\xi + \tilde{\xi} - i\bar{\partial}\phi_2$ can be written as $f(z,v)\mathrm{d}\bar{z}$. However, it is $\bar{\partial}$-closed, so $f(z,v) = f(z)$. By solving the $\bar{\partial}$-equation on Δ^*, $\xi + \tilde{\xi} - i\bar{\partial}\phi_2 = i\bar{\partial}\phi_3$ for some polynomial growth function $\phi_3(z)$. In conclusion

$$\omega_{\mathrm{sf},a}[\sigma'] - \omega = \mathrm{d}[i\bar{\partial}(\phi_2 + \phi_3) - i\partial(\bar{\phi}_2 + \bar{\phi}_3)] = i\partial\bar{\partial}(\phi_2 + \bar{\phi}_2 + \phi_3 + \bar{\phi}_3).$$

□

Theorem 4.3 *There exists a real smooth polynomial growth function ϕ_4 such that $\omega + i\partial\bar{\partial}\phi_4$ is ALG and*

$$(\omega + i\partial\bar{\partial}\phi_4)^2 = \frac{1}{2}\omega^+ \wedge \bar{\omega}^+$$

Remark Compared to Hein's Theorem 1.3 of [24], the key improvements in our paper are that ϕ_4 grows polynomially and that we obtain $\frac{1}{2}\omega^+ \wedge \bar{\omega}^+$ instead of $\frac{\alpha}{2}\omega^+ \wedge \bar{\omega}^+$ for large enough α.

Proof To achieve this, we still introduce a real positive bump function b on $\mathbb{C}$ supported in $\{R \le |z|^\beta \le 4R\}$ such that $b = 1$ on $\{2R \le |z|^\beta \le 3R\}$. The involution with the Green function provides a real at most polynomial growth function ϕ_5 on $\mathbb{C}$ such that $i\partial\bar{\partial}\phi_5 = ib(z)\mathrm{d}z \wedge \mathrm{d}\bar{z}$.

Now let us look at the form $\omega + i\partial\bar{\partial}((1 - \chi(\frac{r}{R} - \frac{5}{2}))\phi_1)$. It equals to ω when $r \le 2R$ and $\omega_{\mathrm{sf},a}[\sigma']$ when $r \ge 3R$. On the part $2R \le r \le 3R$, this form may not be positive. However, as Hein did in Claim 3 of [24],

$$\omega_t = \omega + i\partial\bar{\partial}\left(\left(1 - \chi\left(\frac{r}{R} - \frac{5}{2}\right)\right)\phi_1 + t\phi_5\right)$$

is positive for large enough t.

To achieve the integrability condition $\int_M(\omega_1^2 - \frac{1}{2}\omega^+ \wedge \bar{\omega}^+) = 0$, we start from choosing large enough R and t such that $\omega_{\mathrm{sf},a}[\sigma']$ is close enough to the standard ALG

model $\frac{i}{2}\frac{a}{\mathrm{Im}\tau}(\mathrm{d}z^\beta\wedge\mathrm{d}\bar{z}^\beta+\mathrm{d}v\wedge\mathrm{d}\bar{v})$ and $\int_M(\omega_t^2-\frac{1}{2}\omega^+\wedge\bar{\omega}^+)>0$. Then we consider

$$\omega_{s,t}=\omega_t-\frac{i}{4}\frac{a}{\mathrm{Im}\tau}\left(1-\chi\left(\frac{r}{R}-6\right)\right)\chi\left(\frac{r}{R}-s\right)\beta^2|z|^{2\beta-2}\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

It is easy to see that for any $s\geq 5$, $\omega_{s,t}$ must be positive. What is more, since $\int_M(\omega_{s,t}^2-\frac{1}{2}\omega^+\wedge\bar{\omega}^+)$ decreases to negative infinity when s goes to infinity, by intermediate value theorem, there exists s such that the integrability condition is achieved. By the work of Tian–Yau [54], there exists a real smooth bounded function ϕ_6 such that $(\omega_{s,t}+i\partial\bar{\partial}\phi_6)^2=\frac{1}{2}\omega^+\wedge\bar{\omega}^+$. By Proposition 2.9 of [24], the solution $\omega_{s,t}+i\partial\bar{\partial}\phi_6$ is actually ALG. □

Thus, the first part of Theorem 1.4 has been proved. The second part is quite simple:

Theorem 4.4 *Suppose there exist two ALG metrics $\omega_j=\omega+i\partial\bar{\partial}\phi_j$, $j=7,8$, satisfying $\omega_7^2=\omega_8^2=\frac{1}{2}\omega^+\wedge\bar{\omega}^+$, $|\nabla^m(\omega_7-\omega_8)|=O(r^{-m-\delta})$ and $|\phi_j|=O(r^N)$ for all $j=7,8$, $m\geq 0$ and some $\delta, N>0$. Then $\omega_7=\omega_8$.*

Proof It is easy to see that $\tilde{\omega}=\omega+i\partial\bar{\partial}(\phi_7+\phi_8)/2$ also defines a Kähler metric which is asymptotic to the standard ALG model. Since $\tilde{\omega}\wedge i\partial\bar{\partial}(\phi_7-\phi_8)=0$, $\phi_7-\phi_8$ is harmonic with respect to $\tilde{\omega}$. By Theorem 4.13 of [12], it is asymptotic to $a_nz^n+b_n\bar{z}^n$ for some constants a_n and b_n. The difference has smaller order. Repeat the procedure until the order is smaller than 0. The maximal principle implies that the difference is 0. In other words,

$$\phi_7-\phi_8=\sum_{k=0}^{n}a_kz^k+b_k\bar{z}^k.$$

□

Theorem 4.5 *The third part of Theorem 1.4 holds.*

Proof Let $(M,\omega_{\mathrm{ALG}},\omega^2,\omega^3)$ be the ALG gravitational instanton in the third part of Theorem 1.4. Let a be the area of each regular fiber with respect to ω_{ALG}. Now pick a Kähler metric ω_0 on $\bar{M}$ whose area of each regular fiber is a. Then $\omega_{\mathrm{sf},a}[\sigma']=\omega_0+i\partial\bar{\partial}\phi_9$ for some holomorphic section σ' on $\{|z|^\beta\geq R\}$ and some real function ϕ_9. It is easy to see that Theorem 4.2 also holds for ω_{ALG}, i.e. $\omega_{\mathrm{sf},a}[\sigma'']=\omega_{\mathrm{ALG}}+i\partial\bar{\partial}\phi_{10}$ for some holomorphic section σ'' on $\{|z|^\beta\geq R\}$.

When D is of type I_0^*, II, III, or IV, i.e. $\beta\leq 1/2$, our goal is to show that the action $T(z,v)=(z,v+\sigma''(z)-\sigma'(z))$ as well as its inverse can be extended across D. If it is true, then

$$\omega_{\mathrm{ALG}}=\omega_{\mathrm{sf},a}[\sigma'']-i\partial\bar{\partial}\phi_{10}=(T^{-1})^*\omega_{\mathrm{sf},a}[\sigma']-i\partial\bar{\partial}\phi_{10}=(T^{-1})^*\omega_0-i\partial\bar{\partial}\phi_{11}.$$

So $\omega=\omega_{\mathrm{ALG}}+i\partial\bar{\partial}((1-\chi(\frac{r}{R}-\frac{5}{2}))\phi_{11}+t\phi_5)$ will be the required Kähler form on $\bar{M}$ for large enough t.

To understand the structure near D, we start from the elliptic surface over $\Delta = \{|\tilde{u}| \leq R^{-1}\}$ constructed by $(\Delta \times \mathbb{C})/(\tilde{u}, v) \sim (\tilde{u}, v + m\tau_1(\tilde{u}) + n\tau_2(\tilde{u}))$. Take the quotient by $(\tilde{u}, v) \sim (e^{2\pi i\beta}\tilde{u}, e^{2\pi i\beta}v)$, then there are several orbifold points in the central fiber. As Kodaira did in [28,29], those orbifold points can be resolved by replacing the neighborhoods by the non-singular models N_{+m} constructed in page 583 of [28,29]. Then blow down exceptional curves if they exist. $M|_{\Delta^*} \cup D$ is biholomorphic to such model by the relationship $z = u^{1/\beta} = \tilde{z}^{-1} = \tilde{u}^{-1/\beta}$.

In those coordinates, if T is given by $T(\tilde{u}, v) = (\tilde{u}, v + f(\tilde{u})\tilde{u})$, then by the proof of Theorem 4.2, $\frac{i}{2}\frac{af(\tilde{u})\tilde{u}}{\mathrm{Im}(\bar{\tau}_1\tau_2)}$ will be the average of the coefficient of $\mathrm{d}\bar{v}$ term of the (0,1)-part ξ of ζ on each fiber, where ζ is a real smooth polynomial growth 1-form satisfying $\mathrm{d}\zeta = \omega_{\mathrm{sf},a}[\sigma'] - \omega_{\mathrm{ALG}}$. By Main Theorem 1 of [13], the difference between the two ALG metrics ω_{ALG} and $\omega_{\mathrm{sf},a}[\sigma']$ is bounded by $|u|^{-2}$. By Theorem 4.4 of [13], $\zeta = (\mathrm{d}^*\mathrm{d} + \mathrm{dd}^*)\psi_1$ and $\omega_{\mathrm{sf},a}[\sigma'] - \omega_{\mathrm{ALG}} = (\mathrm{d}^*\mathrm{d} + \mathrm{dd}^*)\psi_2$ for some ψ_1 and ψ_2 on $M|_{\Delta^*}$. So $(\mathrm{d}^*\mathrm{d} + \mathrm{dd}^*)(\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2) = 0$. Therefore, the leading term of $\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2$ must be the linear combinations of $u^\delta \mathrm{d}u$, $\bar{u}^\delta \mathrm{d}u$, $u^\delta \mathrm{d}v$, $\bar{u}^\delta \mathrm{d}v$ and their conjugates. However, $\mathrm{d}(\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2) = \mathrm{d}^*\mathrm{d}\psi_2$ has small order. So if δ is large, the leading term of $\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2$ must be the linear combinations of $u^\delta \mathrm{d}u$ and its conjugate. However, such kind of term can be written as the linear combinations of $\mathrm{d}z^m$ and its conjugate for some integer m. We can then subtract the leading term from $\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2$ and repeat the process. Finally, it is easy to see that $f(\tilde{z}) = f(\tilde{u})$ is bounded by $|\tilde{z}|^{-\epsilon}$ for any small positive ϵ. By removal of singularity theorem of holomorphic functions on the punctured disc Δ^*, $f(\tilde{z})$ can be extended to a holomorphic function on Δ.

Therefore, the induced map of T on the resolution is holomorphic outside the central curves in N_{+m} and continuous across those curves. By removal of singularity theorem, it can be extended holomorphically. Then, in the blow down procedure, the induced map is holomorphic outside the blow down of the exceptional curves. By Hartog's theorem, it can be extended on $\bar{M}|_\Delta$. Similarly, T^{-1} can also be extended.

When D is of type II*, III*, IV*, i.e. $1/2 < \beta < 1$, the arguments above fail because the meromorphic function f may have a pole at $\{\tilde{z} = 0\}$ corresponding to the term $u^{1/\beta-2}\mathrm{d}u \wedge \mathrm{d}\bar{v}$ in the difference $\omega_{\mathrm{sf},a}[\sigma'] - \omega_{\mathrm{ALG}}$. However, recall that in [12], we used the section σ as zero section to compactify M into $\bar{M}$. If we use the section $\sigma + \sigma'' - \sigma'$ instead, then we may get a different $\bar{M}$. For this new choice of $\bar{M}$, the form $\omega_{\mathrm{sf},a}[\sigma''] + i\partial\bar{\partial}\phi_{12} = \omega_{\mathrm{ALG}} + i\partial\bar{\partial}\phi_{13}$ is a smooth Kähler form on $\bar{M} \cap \{|\tilde{z}| \leq R^{-1/\beta}\}$ for some real smooth polynomial growth functions ϕ_{12}, ϕ_{13} on $M \cap \{|\tilde{z}| \leq R^{-1/\beta}\}$. So $\omega = \omega_{\mathrm{ALG}} + i\partial\bar{\partial}((1 - \chi(\frac{r}{R} - \frac{5}{2}))\phi_{13} + t\phi_5)$ will be the required Kähler form on the new choice of $\bar{M}$ for large enough t. □

5 Gluing of ALH gravitational instantons

In this section, we will prove that the gluing of any ALH gravitational instanton with itself is a K3 surface. We learned this idea as well as some initial set-ups of the gluing construction [19] from the lecture of Sir Simon Donaldson in the spring of 2015 at

Stony Brook University. We will also construct a counterexample of ALG Torelli Theorem when D is of type II*, III*, or IV*.

We will use the notations of Kovalev and Singer [31] in order to apply their results. Pick two copies of M. Define $t_1 = r$ on the first copy M_1. Define $t_2 = -r$ on the second copy M_2. For any gluing parameters $(\rho, \Theta) \in [R+8, \infty) \times \mathbb{T}^3$, the gluing manifold $M_{\rho,\Theta}$ is defined by truncating the two manifolds at $t_j = \pm\rho$ and identifying the boundary points $(\rho, \theta) \in M_1$ with the points $(-\rho, \Theta - \theta) \in M_2$. On $M_{\rho,\Theta}$, the function t is defined by $t = t_1 - \rho = t_2 + \rho$. The picture can be found in page 10 of [31].

Our metric on $M_{\rho,\Theta}$ is slightly different from [31]. In fact, there are three Kähler forms ω^1, ω^2 and ω^3 on M. The closed forms

$$\omega^i - \omega^i_{\text{flat}} = a^i_j(r,\theta)\mathrm{d}r \wedge \mathrm{d}\theta^j + b^i_{jk}(r,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k$$

are very small on $\{\rho - 1 \le r \le \rho + 1\}$ by Definition 2.6. Now define

$$\phi^i = \left[\int_\rho^r a^i_j(s,\theta)\mathrm{d}s\right]\mathrm{d}\theta^j.$$

Then

$$\begin{aligned}
\mathrm{d}\phi^i &= a^i_j(r,\theta)\mathrm{d}r \wedge \mathrm{d}\theta^j + \left[\int_\rho^r \frac{\partial}{\partial\theta^k} a^i_j(s,\theta)\mathrm{d}s\right]\mathrm{d}\theta^k \wedge \mathrm{d}\theta^j \\
&= a^i_j(r,\theta)\mathrm{d}r \wedge \mathrm{d}\theta^j + \left[\int_\rho^r \frac{\partial}{\partial r} b^i_{jk}(s,\theta)\mathrm{d}s\right]\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k \\
&= a^i_j(r,\theta)\mathrm{d}r \wedge \mathrm{d}\theta^j + b^i_{jk}(r,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k - b^i_{jk}(\rho,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k.
\end{aligned}$$

Therefore

$$\omega^i - \omega^i_{\text{flat}} = \mathrm{d}\phi^i + b^i_{jk}(\rho,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k$$

are cohomologous to the forms $b^i_{jk}(\rho,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k$ on $\mathbb{T}^3$.

Notice that any closed form on $\mathbb{T}^3$ can be cohomologous to a form with constant coefficients and any 2-form with constant coefficients is invariant under the map $\theta \to \Theta - \theta$. Therefore, when we glue the part $\{\rho - 1 \le t_1 \le \rho + 1\}$ on M_1 with the part $\{\rho - 1 \le -t_2 \le \rho + 1\}$ on M_2, the difference $\omega^i_{M_2} - \omega^i_{M_1} = \mathrm{d}\psi^i$ for some small ψ^i. Now define the forms $\omega^i_{\rho,\Theta}$ on $M_{\rho,\Theta}$ by

$$\omega^i_{\rho,\Theta} = \omega^i_{M_1} + \mathrm{d}((1-\chi(t))\psi^i) = \omega^i_{M_2} - \mathrm{d}(\chi(t)\psi^i).$$

Then $\omega^i_{\rho,\Theta}$ are three closed forms and $|\nabla^m(\omega^i_{\rho,\Theta} - \omega^i_{M_j})| = O(e^{-\lambda_1\rho})$ for all $m \ge 0$.

Now we can call the linear span of $\omega^i_{\rho,\Theta}$ the "self-dual" space. The orthogonal complement of the "self-dual" space under wedge product is called the "anti-self-dual" space. These two spaces determine a star operator. It is well known that the

star operator determines a conformal class of metrics. The the conformal factor is determined by requiring the volume form to be $\frac{1}{2}(\det(\omega^i_{\rho,\Theta}\wedge\omega^j_{\rho,\Theta}))^{1/3}$. The resulting metric is called $g_{\rho,\Theta}$. It is slightly different from [31], but it satisfies all the properties needed in [31].

Now we define three operators on the space of self-dual 2-forms by

$$P_1\phi = e^{-\delta t_1}(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(e^{\delta t_1}\phi),$$
$$P_2\phi = e^{-\delta t_2}(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(e^{\delta t_2}\phi),$$

and

$$P_{\rho,\Theta}\phi = e^{-\delta t}(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(e^{\delta t}\phi),$$

where $\delta < \lambda_1/100$ is a small positive number. It is easy to prove the following theorem:

Theorem 5.1 (1) *P_1, P_2, $P_{\rho,\Theta}$ are Fredholm operators from $W^{k+2,2}$ to $W^{k,2}$ for any $k \geq 0$. In other words, the kernels are finite dimensional and the cokernels, i.e. the kernels of P_1^*, P_2^*, $P^*_{\rho,\Theta}$ are also finite dimensional. The range is the L^2-orthogonal complement of the cokernel. The operator from the L^2-orthogonal complement of the kernel to the range is an isomorphism.*

(2) $\ker P_1 = \mathrm{span}\{e^{-\delta t_1}\omega^i\}$, $\mathrm{coker}\, P_1 = \{0\}$,
$\ker P_2 = \{0\}$, $\mathrm{coker}\, P_1 = \mathrm{span}\{e^{\delta t_2}\omega^i\}$,
$\mathrm{span}\{e^{-\delta t}\omega^i_{\rho,\Theta}\} \subset \ker P_{\rho,\Theta}$, $\mathrm{span}\{e^{\delta t}\omega^i_{\rho,\Theta}\} \subset \mathrm{coker}\, P_{\rho,\Theta}$.

Proof The first part was proved by Lockhart and McOwen in [34]. As for the second part, on any Kähler manifold with Kähler form ω, define the operator L by $L\phi = \phi\wedge\omega$. Then by Kähler identities, $[L, \bar\partial] = 0$ and $[L, \bar\partial^*] = -i\partial$. Therefore

$$[L, \mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*] = 2[L, \bar\partial^*\bar\partial + \bar\partial\bar\partial^*] = 2[L, \bar\partial^*]\bar\partial + 2\bar\partial[L, \bar\partial^*] = -2i(\partial\bar\partial + \bar\partial\partial) = 0.$$

In particular for any function f,

$$(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(f\omega) = (-\Delta f)\omega,$$

where $\Delta f = -(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)f$ is the ordinary Laplacian operator on functions. On hyperkähler manifolds M_1 and M_2, there are three Kähler structures I, J and K. Therefore,

$$\sum_{i=1}^3(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(f_i\omega^i) = -\sum_{i=1}^3(\Delta f_i)\omega^i.$$

In other words, the Laplacian on the self-dual forms is exactly the Laplacian on the coefficients. On $M_{\rho,\Theta}$, even though the metric is not hyperkähler, $\omega^i_{\rho,\Theta}$ are still harmonic since they are closed self-dual forms. The second part follows directly from the two facts above. □

The most important result of [31] is the following theorem: (Proposition 4.2 in their paper)

Theorem 5.2 *There exists $\rho_* > 0$ such that for all $\rho \geq \rho_*$, the induced map $P''_{\rho,\Theta}$ from the L^2-orthogonal complement of* $\mathrm{span}\{\chi(t_1 - \rho/2)e^{-\delta t_1}\omega^i_{\rho,\Theta}\}$ *in $W^{k+2,2}$ to the L^2-orthogonal complement of* $\mathrm{span}\{\chi(-t_2 - \rho/2)e^{\delta t_2}\omega^i_{\rho,\Theta}\}$ *in $W^{k,2}$ is an isomorphism and the operator norm of $[P''_{\rho,\Theta}]^{-1}$ is bounded independent of ρ and Θ.*

It is easy to prove the following lemmas in functional analysis:

Lemma 5.3 (1) *Suppose $V = \mathrm{span}\{v_1, \ldots, v_m\}$ is a finite dimensional subspace in $W^{k,2}$ for some $k \geq 0$. Let $V^\perp$ be the L^2-orthogonal complement of V in $W^{k,2}$, then $W^{k,2} = V \oplus V^\perp$ and*

$$||f + g||_{W^{k,2}} \leq ||f||_{W^{k,2}} + ||g||_{W^{k,2}} \leq (1 + 2C_1)||f + g||_{W^{k,2}}$$

for all $f \in V$ and $g \in V^\perp$, where $C_1 = \sup_{f \in V\setminus\{0\}} \frac{||f||_{W^{k,2}}}{||f||_{L^2}}$.

(2) *Suppose $W = \mathrm{span}\{w_1, \ldots, w_m\}$ is another subspace. Let $a_{ij} = (w_i, v_j)_{L^2}$. If the matrix $A = \{a_{ij}\}$ is invertible with $A^{-1} = \{a^{ij}\}$, then the composition of the inclusion and the projection maps $P = \mathrm{Proj}_{W^\perp} \circ i : V^\perp \to W^\perp$ is an isomorphism. What is more*

$$(1 + C_2)^{-1}||Pf||_{W^{k,2}} \leq ||f||_{W^{k,2}} \leq C_3||Pf||_{W^{k,2}},$$

where

$$C_2 = \sup_{f \in W\setminus\{0\}} \frac{||f||_{W^{k,2}}}{||f||_{L^2}}, C_3 = 1 + C(m)||A^{-1}|| \max ||v_i||_{L^2} \max ||w_j||_{W^{k,2}}.$$

Proof The proof is quite obvious. The only thing to notice is that

$$P^{-1}f = f - \sum_{i,j=1}^{m} a^{ij}(f, v_i)_{L^2} w_j.$$

□

The following corollary of Theorem 5.2 and Lemma 5.3 provides the main estimate of this section:

Corollary 5.4 *There exists $\rho_* > 0$ such that for all $\rho \geq \rho_*$, the space of harmonic self-dual 2-forms on $M_{\rho,\Theta}$ equals to $\mathcal{H}^+ = \mathrm{span}\{\omega^i_{\rho,\Theta}\}$. The Laplacian operator $\Delta_{\rho,\theta} = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ from the L^2-orthogonal complement of $\mathcal{H}^+$ in $W^{k+2,2}(\Lambda^+)$ to the L^2-orthogonal complement of $\mathcal{H}^+$ in $W^{k,2}(\Lambda^+)$ is an isomorphism and the operator norm of $G_{\rho,\theta} = \Delta^{-1}_{\rho,\theta}$ is bounded by $Ce^{2\delta\rho}$ for some constant C independent of ρ and Θ.*

Proof The isomorphism map in Theorem 5.2 can be decomposed into the composition of following maps:

$$\begin{aligned}&(\chi(t_1-\rho/2)e^{-\delta t_1}\omega^i_{\rho,\Theta})^\perp \to (e^{-\delta t}\omega^i_{\rho,\Theta})^\perp \to (\ker P_{\rho,\Theta})^\perp\\ &\to (\mathrm{coker}\, P_{\rho,\Theta})^\perp \to (e^{\delta t}\omega^i_{\rho,\Theta})^\perp \to (\chi(-t_2-\rho/2)e^{\delta t_2}\omega^i_{\rho,\Theta})^\perp.\end{aligned}$$

The first and the fifth maps are isomorphisms by Lemma 5.3. Therefore, the second map must be injective and the fourth map must be surjective. In other words, $\ker P_{\rho,\Theta} = \mathrm{span}\{e^{-\delta t}\omega^i_{\rho,\Theta}\}$ and $\mathrm{coker}\, P_{\rho,\Theta} = \mathrm{span}\{e^{\delta t}\omega^i_{\rho,\Theta}\}$. So all the maps are actually isomorphisms. By Theorem 5.2 and Lemma 5.3, the operator norm of the inverse of the map $P_{\rho,\Theta} : (\ker P_{\rho,\Theta})^\perp \to (\mathrm{coker}\, P_{\rho,\Theta})^\perp$ is bounded. It is straight forward to switch this estimate into the estimate of the Laplacian operator. □

We are ready for the main theorem of this section:

Theorem 5.5 *Fix $k \geq 3$. For large enough ρ_* and any $\rho \geq \rho_*$, there exists a hyperkähler structure $\tilde{\omega}^i_{\rho,\Theta}$ on $M_{\rho,\Theta}$ such that $||\tilde{\omega}^i_{\rho,\Theta} - \omega^i_{\rho,\Theta}||_{W^{k,2}} \leq Ce^{(-\lambda_1+2\delta)\rho}$ for some constant C independent of ρ and Θ.*

Proof Fix the volume form $V = \frac{1}{2}\det(\omega^i_{\rho,\Theta}\wedge\omega^j_{\rho,\Theta})^{1/3}$ on $M_{\rho,\Theta}$. When two symmetric matrices $A = \frac{\omega^i_{\rho,\Theta}\wedge\omega^j_{\rho,\Theta}}{2V}$ and B are close enough to the identity matrix, the equation $CAC^{\mathrm{T}} = B$ has a solution $C = B^{1/2}A^{-1/2}$. Define $F^i(B)$ by $F^i(B) = C_{ij}\omega^j_{\rho,\Theta}$. Then $F^i(B)\wedge F^j(B) = 2b_{ij}V$.

Recall that the map $G_{\rho,\Theta}$ on $(\mathcal{H}^+)^\perp \subset \Lambda^+$ satisfies

$$\begin{aligned}\psi &= (\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)G_{\rho,\Theta}\psi = -(*\mathrm{d}*\mathrm{d}+\mathrm{d}*\mathrm{d}*)G_{\rho,\Theta}\psi\\ &= -(*+\mathrm{Id})\mathrm{d}*\mathrm{d}G_{\rho,\Theta}\psi = \mathrm{d}^+(-2*\mathrm{d}G_{\rho,\Theta}\psi).\end{aligned}$$

So if $\phi^i \in \Lambda^1$ satisfy the equation

$$\phi^i = -2*\mathrm{d}G_{\rho,\Theta}\mathrm{Proj}_{(\mathcal{H}^+)^\perp}F^i\left(\delta_{\alpha\beta} - \frac{\mathrm{d}^-\phi^\alpha\wedge\mathrm{d}^-\phi^\beta}{2V}\right),$$

then the closed forms $\tilde{\omega}^i_{\rho,\Theta} = \mathrm{d}\phi^i + \mathrm{Proj}_{\mathcal{H}^+}F^i(\delta_{\alpha\beta} - \frac{\mathrm{d}^-\phi^\alpha\wedge\mathrm{d}^-\phi^\beta}{2V})$ will satisfy the required equation $\tilde{\omega}^i_{\rho,\Theta}\wedge\tilde{\omega}^j_{\rho,\Theta} = 2\delta_{ij}V$.

We will solve the equation by iterations

$$\begin{aligned}\phi^i_0 &= 0,\\ \phi^i_{n+1} &= -2*\mathrm{d}G_{\rho,\Theta}\mathrm{Proj}_{(\mathcal{H}^+)^\perp}F^i\left(\delta_{\alpha\beta} - \frac{\mathrm{d}^-\phi^\alpha_n\wedge\mathrm{d}^-\phi^\beta_n}{2V}\right).\end{aligned}$$

Since $W^{k,2}$ embeds into C^0, if $||\phi_n^i||_{W^{k+1,2}} \leq e^{-\lambda_1\rho/2}$ and $\rho \geq \rho_*$, then

$$\left|\left|\frac{\omega^i_{\rho,\Theta}\wedge\omega^j_{\rho,\Theta}}{2V}-\delta_{ij}\right|\right|_{C^0}+\left|\left|\frac{\mathrm{d}^-\phi_n^i\wedge\mathrm{d}^-\phi_n^j}{2V}\right|\right|_{C^0}\leq Ce^{-\lambda_1\rho}$$

can be arbitrarily small if ρ_* is large. So

$$||\phi^i_{n+1}||_{W^{k+1,2}}\leq Ce^{2\delta\rho}\left|\left|\mathrm{Proj}_{(\mathcal{H}^+)^\perp}F^i\left(\delta_{\alpha\beta}-\frac{\mathrm{d}^-\phi_n^\alpha\wedge\mathrm{d}^-\phi_n^\beta}{2V}\right)\right|\right|_{W^{k,2}}\leq Ce^{(-\lambda_1+2\delta)\rho}.$$

As long as ρ_* is large enough, the above estimate holds by induction. It follows that

$$||\phi^i_{n+2}-\phi^i_{n+1}||_{W^{k+1,2}}\leq Ce^{(-\lambda_1+4\delta)\rho}||\phi^i_{n+1}-\phi^i_n||_{W^{k+1,2}}.$$

As long as ρ_* is large enough, $\phi^i=\lim_{n\to\infty}\phi^i_n$ will be the solution. □

Corollary 5.6 *For any ALH gravitational instanton M, $\int_M|Rm|^2=96\pi^2$.*

Proof It is easy to deduce this conclusion from the well known fact that for K3 surface $M_{\rho,\Theta}$, $\int_{M_{\rho,\Theta}}|Rm(\tilde{\omega}^i_{\rho,\Theta})|^2=8\pi^2\chi(M_{\rho,\Theta})=192\pi^2$. □

Theorem 5.7 *Suppose α^i are three 2-forms satisfying the following conditions:*

(1)

$$\int_{M_{\rho,\Theta}}\alpha^2\wedge\alpha^3=\int_{M_{\rho,\Theta}}\alpha^3\wedge\alpha^1=\int_{M_{\rho,\Theta}}\alpha^1\wedge\alpha^2=0,$$
$$\int_{M_{\rho,\Theta}}\alpha^1\wedge\alpha^1=\int_{M_{\rho,\Theta}}\alpha^2\wedge\alpha^2=\int_{M_{\rho,\Theta}}\alpha^3\wedge\alpha^3;$$

(2)

$$||\alpha^i-\tilde{\omega}^i_{\rho,\Theta}||_{L^2}\leq e^{-3\lambda_1\rho/4}.$$

Then there exists a hyperkähler structure ω^i on $M_{\rho,\Theta}$ such that $\omega^i\in[\alpha^i]$ and $||\omega^i-\tilde{\omega}^i_{\rho,\Theta}||_{W^{k,2}}\leq Ce^{2\delta\rho}||\alpha^i-\tilde{\omega}^i_{\rho,\Theta}||_{L^2}$.

Proof Using $\tilde{\omega}^i_{\rho,\Theta}$ as the background hyperkähler structure, we can choose harmonic representatives β^i from the cohomology classes $[\alpha^i]$. Therefore,

$$C^{-1}||\beta^i-\tilde{\omega}^i_{\rho,\Theta}||_{W^{k,2}}\leq||\beta^i-\tilde{\omega}^i_{\rho,\Theta}||_{L^2}\leq||\alpha^i-\tilde{\omega}^i_{\rho,\Theta}||_{L^2}\leq e^{-3\lambda_1\rho/4}.$$

After replacing $\omega^i_{\rho,\Theta}$ by β^i in the proof of Theorem 5.5, we can find a hyperkähler structure ω^i_0 such that $\mathrm{span}\{[\omega^i_0]\}=\mathrm{span}\{[\alpha^i]\}$. By the condition (1) of α^i, the hyperkähler structure ω^i can be chosen to be some rescaling and hyperkähler rotation of ω^i_0. □

Now we will use the method in this section to construct a counterexample of Torelli Theorem in ALG case:

Theorem 5.8 *When D is of type II*, III*, or IV*, there exist two different ALG gravitational instantons with same $[\omega^i]$.*

Proof In Example 3.1 of [24], Hein explains how the pairs (IV, IV*) occur in rational elliptic surfaces $\bar{M}$ birational to $(\mathbb{P}^1 \times \mathbb{T}^2)/\Gamma$ with $\Gamma = \mathbb{Z}_3$. Let D be the fiber of type IV*. Then, the construction in [24] provides an ALG gravitational instanton ω^i on $M = \bar{M} \backslash D$. Moreover, the asymptotic rate is $2 + \frac{1}{\beta}$. In particular, $|\mathrm{Rm}| = O(r^{-\frac{1}{\beta}-4})$. There is a similar example when D is of type II* or III*.

By Theorem 4.12 of [12], there exists a harmonic (0,1) form h on M asymptotic to $\frac{1}{\frac{1}{\beta}-1} u^{1/\beta-1} \mathrm{d}\bar{v}$. So $\mathrm{d}(\mathrm{Re}h)$ is an exact harmonic form asymptotic to $\mathrm{Re}(u^{1/\beta-2}\mathrm{d}u \wedge \mathrm{d}\bar{v})$. Moreover, it is anti-self-dual because the coefficients of its self-dual part are decaying harmonic and thus 0.

Let us use the notations in [12]. For example,

$$||\phi||_{H^2_\delta} = \sqrt{\int_M |\phi|^2 r^\delta \mathrm{dVol} + \int_M |\nabla\phi|^2 r^{\delta+2} \mathrm{dVol} + \int_M |\nabla^2\phi|^2 r^{\delta+4} \mathrm{dVol}}.$$

Then by Theorem 4.12 of [12], for $k \geq 5$ and small positive ϵ, there exists a map $G : H^k_{6-\frac{4}{\beta}-\epsilon}(\Lambda^+) \to H^{k+2}_{2-\frac{4}{\beta}-\epsilon}(\Lambda^+)$ such that $\psi = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)G\psi$. We still define $F^i : \Gamma(\mathbb{R}^{3\times 3}) \to \Lambda^+$ as in Theorem 5.5 so that $F^i(B) \wedge F^j(B) = 2b_{ij}V$. Then we do the iteration

$$\phi_0^1 = \phi_0^3 = 0, \phi_0^2 = t\mathrm{Re}h,$$
$$\phi^i_{n+1} = -2 * \mathrm{d}G\left(F^i\left(\delta_{\alpha\beta} - \frac{\mathrm{d}^-\phi_n^\alpha \wedge \mathrm{d}^-\phi_n^\beta}{2V}\right) - \omega^i\right) + \delta_{i2}t\mathrm{Re}h$$

When t is small enough, $(\phi_n^1, \phi_n^2 - t\mathrm{Re}h, \phi_n^3) \to (\phi^1, \phi^2 - t\mathrm{Re}h, \phi^3) \in H^{k+1}_{4-\frac{4}{\beta}-\epsilon}$. Then $\omega^i + \mathrm{d}\phi^i$ will be an ALG gravitational instanton. By direct computation, the curvature of the metric corresponding to $(J, \omega^2 + t\mathrm{d}(\mathrm{Re}h))$ is proportional to $r^{\frac{1}{\beta}-4}$. It is also true for the metric corresponding to $\omega^i + \mathrm{d}\phi^i$ because their difference is in $H^k_{6-\frac{4}{\beta}-\epsilon}$. In particular, the metric corresponding to $\omega^i + \mathrm{d}\phi^i$ is not isometric to the metric corresponding to ω^i. □

6 Uniqueness of ALH gravitational instantons

In this section, we will prove the uniqueness part of Theorem 1.5. We start from the understanding of the cross section:

Theorem 6.1 *The integrals of ω^i on the three faces determine the torus $\mathbb{T}^3$.*

Proof On the flat model, recall that

$$\mathrm{d}r = I^*\mathrm{d}\theta^1 = J^*\mathrm{d}\theta^2 = K^*\mathrm{d}\theta^3.$$

So

$$\begin{aligned}\omega^1 &= \mathrm{d}r \wedge \mathrm{d}\theta^1 + \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3,\\ \omega^2 &= \mathrm{d}r \wedge \mathrm{d}\theta^2 + \mathrm{d}\theta^3 \wedge \mathrm{d}\theta^1,\\ \omega^3 &= \mathrm{d}r \wedge \mathrm{d}\theta^3 + \mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2.\end{aligned}$$

The torus $\mathbb{T}^3 = \mathbb{R}^3/\Lambda$ is determined by the lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3$. Let v_{ij} be the $\frac{\partial}{\partial\theta^j}$ components of v_i. Then

$$\begin{pmatrix} f_{123} & f_{131} & f_{112}\\ f_{223} & f_{231} & f_{212}\\ f_{323} & f_{331} & f_{312}\end{pmatrix}$$

equals to

$$\begin{pmatrix} v_{22}v_{33} - v_{23}v_{32} & v_{32}v_{13} - v_{33}v_{12} & v_{12}v_{23} - v_{13}v_{22}\\ v_{23}v_{31} - v_{21}v_{33} & v_{33}v_{11} - v_{31}v_{13} & v_{13}v_{21} - v_{11}v_{23}\\ v_{21}v_{32} - v_{22}v_{31} & v_{31}v_{12} - v_{32}v_{11} & v_{11}v_{22} - v_{12}v_{21}\end{pmatrix}.$$

It is exactly the adjunct matrix $\mathrm{adj}(A)$ of

$$A = \begin{pmatrix} v_{11} & v_{12} & v_{13}\\ v_{21} & v_{22} & v_{23}\\ v_{31} & v_{32} & v_{33}\end{pmatrix}.$$

Since $\mathrm{adj}(A)A = \det(A)I$, it is easy to see that $\det(\mathrm{adj}(A)) = (\det(A))^2$. Thus, $A = (\det(\mathrm{adj}(A)))^{-1/2}\mathrm{adj}(\mathrm{adj}(A))$ is determined by $\mathrm{adj}(A)$. On M, the hyperkähler structure is asymptotic to the flat model. So we can get the same conclusion. □

Theorem 6.2 *ALH gravitational instantons are uniquely determined by their three Kähler classes* $[\omega^i]$ *up to tri-holomorphic isometry which induces identity on* $H_2(M, \mathbb{Z})$.

Proof Suppose two ALH hyperkähler structures $\omega^{k,i}, k = 1, 2$ on M satisfy $[\omega^{1,i}] = [\omega^{2,i}]$. By the results in the previous section, we have two families of K3 surfaces $(M_{\rho_k,\Theta_k}, \tilde{\omega}^{k,i}_{\rho_k,\Theta_k})$. To understand the relationship between M and M_{ρ_k,Θ_k}, let us start from the flat orbifold $(\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}_2$. Take two copies of it. On the first copy, define $t_1 = r$. On the second copy, define $t_2 = -r$. Now we glue them by truncating the two manifolds at $t_j = \pm\rho$ and identifying the boundary points (ρ, θ) with the points $(-\rho, \Theta - \theta)$. An alternating way to describe the gluing is to start from $[0, 2\rho]\times\mathbb{T}^3$, and then identify $(0, \theta)$ with $(0, -\theta)$ and identify $(2\rho, \theta)$ with $(2\rho, 2\Theta - \theta)$. Let

$\mathbb{T}^4 = (\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}(4\rho, 2\Theta)$. Then it is easy to see that the gluing is actually the orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

The resolution of this picture provides the topological picture of the construction of M_{ρ_k,Θ_k}. The second homology group $H_2(M_{\rho_k,\Theta_k}, \mathbb{R}) = \mathbb{R}^{22}$ is generated by 16 curves Σ_α corresponding to 16 orbifold points, 3 faces $F_{\alpha\beta}$ spanned by v_α and v_β and 3 faces F_α spanned by $(4\rho, 2\Theta)$ and $(0, v_\alpha)$. Any hyperkähler structure ω^i on the K3 surface determines 48 integrals $c_{i\alpha}$ on Σ_α, 9 integrals $f_{i\alpha\beta}$ on $F_{\alpha\beta}$ and 9 integrals $f_{i\alpha}$ on F_α. The integrability condition $\int_M \omega^i \wedge \omega^j = 2\delta_{ij} V$ is equivalent to

$$-\frac{1}{2}\sum_{\alpha=1}^{16} c_{i\alpha}c_{j\alpha} + \frac{1}{2}\sum_{(\alpha,\beta,\gamma)=(1,2,3),(2,3,1),(3,1,2)} f_{i\alpha}f_{j\beta\gamma} + f_{j\alpha}f_{i\beta\gamma} = 2\delta_{ij}V.$$

If $c_{i\alpha}$, $f_{i\alpha\beta}$ are given, it is a rank 5 linear system in 9 variables $f_{i\alpha}$.

By the construction of $\tilde{\omega}^{k,i}_{\rho_k,\Theta_k}$ on M_{ρ_k,Θ_k}, the differences $c^{1,\rho_1,\Theta_1}_{i\alpha} - c^{2,\rho_2,\Theta_2}_{i\alpha}$ and $f^{1,\rho_1,\Theta_1}_{i\alpha\beta} - f^{2,\rho_2,\Theta_2}_{i\alpha\beta}$ are all bounded by $Ce^{(-\lambda_1+2\delta)\rho_1} + Ce^{(-\lambda_1+2\delta)\rho_2}$ for large enough ρ_k. However $f^{1,\rho_1,\Theta_1}_{i\alpha} - f^{2,\rho_2,\Theta_2}_{i\alpha}$ may be very large. Fortunately, we are free to change the 8 parameters ρ_k, Θ_k. When ρ_k and Θ_k are changed by adding $\delta\rho_k$ and $\delta\Theta_k$, the integrals $f^{k,\rho_k,\Theta_k}_{i\alpha}$ are changed by adding the almost linear terms $L(\delta\rho_k, \delta\Theta_k) + O(e^{(-\lambda_1+3\delta)\rho_1} + e^{(-\lambda_1+3\delta)\rho_2})$, where

$$L(\delta\rho_k, \delta\Theta_k) = 4\delta\rho_k\left(v_\alpha, \frac{\partial}{\partial\theta^i}\right) + 2\left(\delta\Theta_k \times v_\alpha, \frac{\partial}{\partial\theta^i}\right)$$

are determined by the cross section $\mathbb{R}^3/(\mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3)$, $\delta\rho_k$ and $\delta\Theta_k$. The image of L is exactly the linear space

$$\left\{(f_{i\alpha})|\exists C \text{s.t.} \sum_{(\alpha,\beta,\gamma)=(1,2,3),(2,3,1),(3,1,2)} f_{i\alpha}f_{j\beta\gamma} + f_{j\alpha}f_{i\beta\gamma} = 2\delta_{ij}C\right\},$$

where $f_{i\alpha\beta} = (v_\alpha \times v_\beta, \frac{\partial}{\partial\theta^i})$. Therefore, after increasing the gluing parameters ρ_1 or ρ_2 and changing the parameters Θ_k,

$$\min_{\phi\in\left[\tilde{\omega}^{1,i}_{\rho_1,\Theta_1} - \tilde{\omega}^{2,i}_{\rho_2,\Theta_2}\right]} ||\phi||_{L^2} \le Ce^{(-\lambda_1+4\delta)\rho_1} + Ce^{(-\lambda_1+4\delta)\rho_2}.$$

By Theorem 5.7, there exists a hyperkähler structure ω^i on M_{ρ_2,Θ_2} such that $[\omega^i] = [\tilde{\omega}^{1,i}_{\rho_1,\Theta_1}]$ and $||\omega^i - \tilde{\omega}^{2,i}_{\rho_2,\Theta_2}||_{W^{k,2}} \le Ce^{(-\lambda_1+6\delta)\rho_1} + Ce^{(-\lambda_1+6\delta)\rho_2}$. By Theorem 1.6, ω^i and $\tilde{\omega}^{1,i}_{\rho_1,\Theta_1}$ are tri-holomorphically isometric to each other. Moreover the isometry induces identity on $H_2(M_{\rho_k,\Theta_k}, \mathbb{Z})$. Notice that the long neck regions are almost flat but by Corollary 5.6, the compact parts are not flat. So all the isometrics must map compact parts to compact parts. In particular, we can apply the Arzela–Ascoli theorem

and the diagonal argument to get a limiting tri-holomorphic isometry on the original manifold M which induces identity on $H_2(M,\mathbb{Z})$ when ρ_k go to infinity. □

Theorem 6.3 *The Kähler classes* $[\omega^i]$ *satisfy the two conditions in Theorem* 1.5.

Proof The first condition is a trivial consequence of $\det(\mathrm{adj}(A)) = (\det(A))^2$ and $\det(A) \neq 0$ in the proof of Theorem 6.1. As for the second condition, any ALH gravitational instanton M can be glued with itself to obtain a K3 surface. By Theorem 5.7, we can modify the hyperkähler metric on the K3 surface so that the integrals of ω^i on the 11 cycles are unchanged in the gluing process. For any $[\Sigma] \in H_2(M,\mathbb{Z})$ such that $[\Sigma]^2 = -2$, we can find a corresponding element in the second homology group of the K3 surface. By Theorem 1.6, there exists i such that $[\omega^i][\Sigma] \neq 0$. Since the integrals of ω^i on the K3 surface are the same as the integrals on M, the second condition must be satisfied. □

7 Existence of ALH gravitational instantons

In this section, we will use the continuity method to prove the existence part of Theorem 1.5. Given any three classes $[\alpha_1^i]$ satisfying two conditions in Theorem 1.5, the cross section $\mathbb{T}^3$ is determined by Theorem 6.1. By the work of Biquard and Minerbe [7], there exists an ALH hyperkähler structure ω_0^i on $\widetilde{(\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}_2}$. Now we are going to connect $[\alpha_1^i]$ with $[\alpha_0^i] = [\omega_0^i]$. We require that along the path, the cross section $\mathbb{T}^3$, i.e. the integrals on the faces F_{jk} are invariant.

We already know that for any $k = 0, 1$, any $[\Sigma] \in H_2(M,\mathbb{Z})$ with $[\Sigma]^2 = -2$, there exists $i \in \{1, 2, 3\}$ with $[\alpha_k^i][\Sigma] \neq 0$. After a hyperkähler rotation, we can assume that $[\alpha_k^i][\Sigma] \neq 0$ for any $k = 0, 1$, any $i = 1, 2, 3$ and any $[\Sigma] \in H_2(M,\mathbb{Z})$ with $[\Sigma]^2 = -2$.

Now we can connect $[\alpha_0^i]$ with $[\alpha_1^i]$ by several pieces of segments. Along each segment, two of $[\alpha^i]$ are fixed while the remaining one is varying. We require that the actions of the two fixed $[\alpha^i]$ on any $[\Sigma] \in H_2(M,\mathbb{Z})$ with $[\Sigma]^2 = -2$ are nonzero. Therefore along the path, the two conditions of Theorem 1.5 are always satisfied.

So we only need to consider each segment. Without loss of generality, we can assume that there is only one segment and $[\alpha^2]$, $[\alpha^3]$ are fixed along the segment. Actually, we can assume that I, ω^2 and ω^3 are invariant along the continuity path. Only $[\alpha^1]$, i.e. the I- Kähler class is varying. We denote the original $\omega_0^1 \in [\alpha_0^1]$ by ω_0. We will use it as the background metric.

By Proposition 6.16 of [37], for ω_0, the second cohomology group $H^2(M,\mathbb{R})$ is naturally isomorphic to the space of bounded harmonic forms which are asymptotic to the linear combinations of $\mathrm{d}\theta^2\wedge\mathrm{d}\theta^3$, $\mathrm{d}\theta^3\wedge\mathrm{d}\theta^1$ and $\mathrm{d}\theta^1\wedge\mathrm{d}\theta^2$. We only care about the forms whose integrals on F_{jk} are 0. Such kind of forms must decay exponentially. By the calculation in Theorem 5.1 and the maximal principle, the self-dual part of any decaying harmonic form vanishes. It is well known that any anti-self-dual form must be (1,1).

Thus, we can add linear combinations of those exponential decay anti-self-dual harmonic forms to change the Kähler class. However, the integrability condition

$\int_M((\alpha^1)^2-(\omega^2)^2)=0$ may not be satisfied. Fortunately, there is an exponential decay exact form $\mathrm{d}((1-\chi(r-R-2))I^*\mathrm{d}r)$ on M. Moreover, it is (1,1) since in local coordinates

$$\begin{aligned}\mathrm{d}((1-\chi(r-R-2))I^*\mathrm{d}r)&=\mathrm{d}\left((1-\chi(r-R-2))\left(ir_j\mathrm{d}z^j-ir_{\bar{j}}\mathrm{d}\bar{z}^j\right)\right)\\&=-2i(1-\chi(r-R-2))r_{j\bar{k}}\mathrm{d}z^j\wedge\mathrm{d}\bar{z}^k+2i\chi'(r-R-2)r_jr_{\bar{k}}\mathrm{d}z^j\wedge\mathrm{d}\bar{z}^k.\end{aligned}$$

If we add this term with α^1, then

$$\int_M(\alpha^1+a\mathrm{d}((1-\chi(r-R-2))I^*\mathrm{d}r))^2-(\alpha^1)^2=2a\lim_{R\to\infty}\int_{r=R}I^*\mathrm{d}r\wedge\alpha^1.$$

The integral $\int_{r=R}I^*\mathrm{d}r\wedge\alpha^1$ on M converges to the term $\int_{\mathbb{T}^3}-\mathrm{d}\theta^1\wedge\mathrm{d}\theta^2\wedge\mathrm{d}\theta^3$ on the flat model, which is non-zero. So we can choose a suitable a to achieve the integrability condition. We call the resulting (1,1) form α_t. It satisfies the following conditions:

(1) For any $m\geq 0$, $||e^{\lambda_1 r}\nabla^m_{\omega_0}(\alpha_t-\alpha_T)||_{C^0}$ converges to 0 when t goes to T. In particular, $||e^{\lambda_1 r}\nabla^m_{\omega_0}(\alpha_t-\omega_0)||_{C^0}$ is uniformly bounded.
(2) $\int_M(\alpha_t^2-\omega_0^2)=0$.

Remark α_t is positive in far enough region. However, it may not be positive in the compact part. That is the reason why the geometric existence part of [23] fails.

Now define I as the set

$$\left\{t\in[0,1]|\exists\phi_t s.t.\forall m\geq 0,|\nabla^m_{\omega_0}\phi_t|=O(e^{-\lambda_1 r}),\omega_t=\alpha_t+i\partial\bar{\partial}\phi_t>0,\omega_t^2=\omega_0^2\right\}.$$

It is trivial that $0\in I$.

Theorem 7.1 *I is open*

Proof Suppose $T\in I$, then as long as t is close enough to T, $\alpha_t+i\partial\bar{\partial}\phi_T$ is positive. It satisfies the integrability condition

$$\int_M((\alpha_t+i\partial\bar{\partial}\phi_T)^2-\omega_0^2)=\int_M((\alpha_t+i\partial\bar{\partial}\phi_T)^2-\alpha_t^2)+\int_M(\alpha_t^2-\omega_0^2)=0.$$

By Theorem 4.1 of [23], $(\alpha_t+i\partial\bar{\partial}\phi_T+i\partial\bar{\partial}\phi)^2=\omega_0^2$ has a solution ϕ. So $t\in I$ with $\phi_t=\phi_T+\phi$. □

Now we are going to show that I is closed. Assume that $\{t_i\}\in I$ converge to T. To make the notation simpler, we will use α_i, ω_i and ϕ_i to denote α_{t_i}, ω_{t_i} and ϕ_{t_i}.

We start from an estimate:

Theorem 7.2

$$\int_M(\mathrm{tr}_{\omega_0}\omega_i-2)\frac{\omega_0^2}{2}\leq C.$$

Moreover

$$\int_M (\mathrm{tr}_{\omega_j}\omega_i - 2)\frac{\omega_j^2}{2} \to 0$$

as $i, j \to \infty$.

Proof

$$\int_M (\mathrm{tr}_{\omega_0}\omega_i - 2)\frac{\omega_0^2}{2} = \int_M \omega_0 \wedge \omega_i - \omega_0^2 = \int_M \omega_0 \wedge (\alpha_i - \omega_0) \leq C.$$

Moreover

$$\int_M (\mathrm{tr}_{\omega_j}\omega_i - 2)\frac{\omega_j^2}{2} = \int_M \omega_j \wedge \omega_i - \omega_j^2 = \int_M \alpha_j \wedge (\alpha_i - \alpha_j) \to 0$$

as $i, j \to \infty$. □

Remark By mean inequality, both $\mathrm{tr}_{\omega_0}\omega_i - 2$ and $\mathrm{tr}_{\omega_j}\omega_i - 2$ are non-negative since $\omega_0^2 = \omega_i^2 = \omega_j^2$.

Theorem 7.3 *Let* U_N *be the sets* $\{N \leq r \leq N+1\}$ *in the sense of* ω_0. *Then for all large enough* N, *there exist subsets* $V_{Ni} \subset U_N$ *such that the volume* $\mathrm{Vol}(V_{Ni}) \geq \mathrm{Vol}(U_N)/2 \geq C$ *and for any* $y_1, y_2 \in V_{Ni}$, $d_{\omega_i}(y_1, y_2) \leq C_1$.

Proof It was proved by Demailly, Peternell and Schneider as Lemma 1.3 of [18] from the bound in Theorem 7.2. □

By the volume comparison theorem on Ricci flat manifolds, if we pick any point $p_{Ni} \in V_{Ni}$, then the volume of radius R ball centered at p_{Ni} in the sense of ω_i has a uniform lower bound depending on R.

Theorem 7.4 *For any fixed number* R, *the* ω_i*-curvature in* $B_{\omega_i}(p_{Ni}, R)$ *is uniformly bounded. Moreover, the* ω_i*-holomorphic radius in* $B_{\omega_i}(p_{Ni}, R)$ *is uniformly bounded below.*

Proof Suppose on the contrary, the ω_i-curvature goes to infinity. Then we can rescale the metric so that the largest curvature equals to 1. By Theorem 4.7 of [11], the volume lower bound and the curvature bound imply the lower bound on the injectivity radius. Then, by Lemma 4.3 of [46], the holomorphic radius has a lower bound. By Page 483 of [1], the bound on the L^2-norm of curvature, the lower bound on the volume and the harmonic radius imply that the rescaled metric converges to an Einstein ALE space M_∞. Replacing the harmonic radius by the holomorphic radius, we can show that M_∞ is actually Kähler. Moreover, before taking limit, the manifold has a parallel holomorphic symplectic form $\omega^2 + i\omega^3$. Thus, on M_∞, there exists a parallel holomorphic symplectic form, too. In other words, M_∞ is actually hyperkähler.

By Bando–Kasue–Nakajima [4] and Kronheimer [32,33], the (non-flat) ALE-gravitational instanton M_∞ contains a curve Σ_∞ with self intersection number -2. Before rescaling, the integrals of ω_i, ω^2 and ω^3 on Σ_i converge to 0.

Recall that $H_2(M,\mathbb{R})$ is generated by 8 curves Σ_α and 3 faces F_{23}, F_{31}, F_{12}. In fact, similar to Section B of [50], any element in $H_2(M,\mathbb{Z})$ can be represented by half integral linear combinations of Σ_α and $F_{\alpha\beta}$. Let

$$[\Sigma_i] = \frac{1}{2}\sum (m_{i\alpha}[\Sigma_\alpha] + m_{i\alpha\beta}[F_{\alpha\beta}]).$$

Then $[\Sigma_i]^2 = -2 = \frac{-2}{4}\sum m_{i\alpha}^2$. So there are only finitely many possibilities of $m_{i\alpha}$. By condition (1) of Theorem 1.5, the actions of $\lim_{i\to\infty}[\omega_i]$, $[\omega^2]$ and $[\omega^3]$ on $F_{\alpha\beta}$ are linearly independent. Since the integrals of ω_i, ω^2 and ω^3 on Σ_i converge to 0, we know that $m_{i\alpha\beta}$ also has a uniform bound. In other words, the homology class $[\Sigma_i]$ only has finitely many possibilities. Taking a subsequence where the homology class of Σ_i are same, we obtain a contradiction to the condition (2) of Theorem 1.5.

We have obtained a bound on the curvature. Theorem 4.7 of [11] and Lemma 4.3 of [46] now provide a lower bound on the holomorphic radius. □

Let D_0 be the upper bound on the diameter of U_N with respect to ω_0. We are interested in the function $e(t_i) = \mathrm{tr}_{\omega_i}\omega_0 = \mathrm{tr}_{\omega_0}\omega_i$ on $B_{\omega_i}(p_{Ni}, 10D_0)$. We start from a theorem:

Theorem 7.5 *There exists a constant C_2 such that if*

$$\gamma^{-2}\int_{B_{\omega_i}(p,\gamma)} e(t_i) \le C_2$$

for some ball $B_{\omega_i}(p,\gamma) \subset B_{\omega_i}(p_{Ni}, 10D_0)$, then

$$\sup_{\sigma\in[0,\frac{2}{3}\gamma]} \sigma^2 \sup_{B_{\omega_i}(p,\frac{2}{3}\gamma-\sigma)} e(t_i) \le \frac{1}{C_2}\gamma^{-2}\int_{B_{\omega_i}(p,\gamma)} e(t_i) \le 1.$$

Proof It is well known that in the $C^{1,\alpha}$-holomorphic radius, there are higher derivative bounds automatically. Thus, the constant in Proposition 2.1 of [46] is uniform. (There are several errors in [46]. After correcting them, we can only get "$\frac{2}{3}\gamma$" instead of "γ" in the statement) □

Theorem 7.6 *For each i, there exists a set $A_i \subset \{0,1,2,\ldots\}$ such that for all $N \notin A_i$, $\sup_{B_{\omega_i}(p_{Ni},4D_0)} e(t_i) \le 2.5$ and the number of elements in A_i is bounded.*

Proof For any point $p \in M$, the number of N such that $p \in B_{\omega_i}(p_{Ni}, 10D_0)$ is bounded. Actually, by Theorem 7.3, for all such N, the set V_{Ni} is contained in $B_{\omega_i}(p, 10D_0+C_1)$. The volume of $B_{\omega_i}(p, 10D_0+C_1)$ has an upper bound by volume comparison, while the volumes of the disjoint sets V_{Ni} have a lower bound.

Therefore, by Theorem 7.2

$$\sum_N \int_{B_{\omega_i}(p_{Ni},10D_0)} (e(t_i)-2) \le C.$$

Let γ be a constant smaller than D_0 such that $\gamma^2 \le \frac{C_2}{2\pi^2}$. Then by the volume comparison theorem, the volume of $B(R)$ on a Ricci flat 4-manifold is bounded by $\frac{\pi^2}{2}R^4$. So

$$\int_{B_{\omega_i}(p,\gamma)} 2 = 2\mathrm{Vol}(B_{\omega_i}(p,\gamma)) \le \frac{C_2}{2}\gamma^2.$$

Therefore, as long as

$$\int_{B_{\omega_i}(p_{Ni},10D_0)} (e(t_i)-2) \le \frac{C_2}{2}\gamma^2,$$

we can get

$$\int_{B_{\omega_i}(p,\gamma)} e(t_i) \le C_2\gamma^2$$

for all $p \in B_{\omega_i}(p_{Ni}, 8D_0)$. So $e(t_i) = \mathrm{tr}_{\omega_i}\omega_0 \le \frac{9}{4}\gamma^{-2}$ in $B_{\omega_i}(p_{Ni}, 8D_0)$ by Theorem 7.5. In particular, $B_{\omega_i}(p_{Ni}, 8D_0) \subset B_{\omega_0}(p_{Ni}, 18\gamma^{-2}D_0)$.

For any $\epsilon \le \frac{C_2}{2}\gamma^2$, let $A_{i,\epsilon}$ denote the set of N such that the integral $\int_{B_{\omega_i}(p_{Ni},10D_0)}(e(t_i)-2) > \epsilon$ or $\sup_{B_{\omega_0}(p_{Ni},18\gamma^{-2}D_0)}|Rm(\omega_0)| > \epsilon$. Then the number of elements in $A_{i,\epsilon}$ is bounded. For all $N \notin A_{i,\epsilon}$, it is well known [46] that

$$-\Delta_{\omega_i} e(t_i) \le C|Rm(\omega_0)|e(t_i)^2 \le C|Rm(\omega_0)|$$

in $B_{\omega_i}(p_{Ni}, 8D_0)$. So both $\sup_{B_{\omega_i}(p_{Ni},8D_0)} -\Delta_{\omega_i} e(t_i)$ and $\int_{B_{\omega_i}(p_{Ni},8D_0)}(e(t_i)-2)$ are bounded by $C\epsilon$. By Theorem 9.20 of [22], $\sup_{B_{\omega_i}(p_{Ni},4D_0)}(e(t_i)-2) \le C\epsilon$ for all $N \notin A_{i,\epsilon}$. After a suitable choice of ϵ, we can make it smaller than $1/2$. □

Lemma 7.7 *There exists a constant R such that $M \subset \cup_{N\notin A_i}\overline{B_{\omega_i}(p_{Ni}, R)}$.*

Proof For all $N \notin A_i$, $\frac{1}{2}\omega_i \le \omega_0 \le 2\omega_i$ in $B_{\omega_i}(p_{Ni}, 4D_0)$ by Theorem 7.6. In particular,

$$U_N \subset \overline{B_{\omega_0}(p_{Ni}, D_0)} \subset B_{\omega_i}(p_{Ni}, 4D_0).$$

Let $U_{A_i} = \cup_{N\in A_i} U_N$. Suppose $R = \sup_{q\in U_{A_i}} \inf_{N\notin A_i} d_{\omega_i}(p_{Ni}, q)$ is achieved by q_i and N_i. Then we will use the argument similar to Theorem 3.1 of [47] and Theorem 3.1 of [53] proved by applying Theorem I.4.1 of [49]. Actually, if $R > 10D_0$, by Theorem 7.6, it is easy to see that $B_{\omega_i}(q_i, R-10D_0) \subset U_{A_i}$ and $U_{N_i} \subset B_{\omega_i}(q_i, R+$

$10D_0)\backslash B_{\omega_i}(q_i, R - 10D_0)$. Since the volume of U_{A_i} is bounded from above and the volume of U_{N_i} is bounded from below, it is easy to get a bound on R by the Bishop-Gromov volume comparison theorem. □

Theorem 7.8 *$e(t_i) = \mathrm{tr}_{\omega_0}\omega_i = \mathrm{tr}_{\omega_i}\omega_0$ is uniformly bounded on M.*

Proof By Theorem 7.4 and Lemma 7.7, the ω_i-holomorphic radius is bounded from below. So the constant in Theorem 7.5 is uniform if we replace ω_0 by ω_j in the statement of Theorem 7.5. By Theorem 7.2, $\int_M (\mathrm{tr}_{\omega_j}\omega_i - 2)\frac{\omega_j^2}{2} \to 0$ as $i, j \to \infty$. So for large enough i and j, $\mathrm{tr}_{\omega_j}\omega_i$ is uniformly bounded on M. Fix j and let i go to infinity. Since $C_j^{-1}\omega_0 \le \omega_j \le C_j\omega_0$, the bound on $\mathrm{tr}_{\omega_j}\omega_i$ automatically implies a bound on $\mathrm{tr}_{\omega_0}\omega_i$. □

Now we are ready to use the arguments in [23] to prove Theorem 1.5. Let N be a large constant such that when $r \ge N$, $\frac{1}{2}\omega_0 \le \alpha_i \le 2\omega_0$. We start from a theorem which can be easily deduced from Proposition 4.21 of [23]:

Theorem 7.9 *Let $w = \frac{e^{-\delta r}}{\int_M e^{-\delta r}}$ be a weight function. Define the weighted norm $||u||_{L^p_w}$ by $||u||_{L^p_w} = (\int |u|^p w)^{1/p}$, then for all $u \in C_0^\infty$,*

$$||u||_{L^4_w(\{r\ge N\})} \le C||\nabla u||_{L^2(\{r\ge N\})} + C||u||_{L^4(\{N\le r\le N+1\})}.$$

It is easy to see that for all $1 \le p \le q \le \infty$, $||u||_{L^p_w} \le ||u||_{L^q_w}$ by Hölder's inequality.

Theorem 7.10 *I is closed.*

Proof Let $\phi_{ai} = \frac{\int_M \phi_i e^{-2\delta r}}{\int_M e^{-2\delta r}}$ be the weighted average of ϕ_i. By the standard Lockhart-McOwen theory [34], since constant is the only harmonic function less than $e^{\delta r}$, we can obtain a bound on $||e^{-\delta r}(\phi_i - \phi_{ai})||_{W^{2,2}}$ from the L^2 bound of $e^{-\delta r}\Delta_{\omega_0}\phi_i = e^{-\delta r}(\mathrm{tr}_{\omega_0}\omega_i - \mathrm{tr}_{\omega_0}\alpha_i)$.

Let $u_i = \phi_i - \phi_{ai}$. We already obtain a bound on $||u_i||_{W^{2,2}(\{r\le N+4\})}$ and $||\Delta_{\omega_0}u_i||_{L^\infty(M)}$. So $||u_i||_{W^{2,p}(\{r\le N+3\})}$ is bounded for any $p \in (1, \infty)$ by Theorem 9.11 of [22].

The $C^{2,\alpha}$-estimate for real Monge–Ampère equation was done by Evans–Krylov–Trudinger. See Section 17.4 of [22] for details. Now we are in the complex case. However, the arguments in Section 17.4 of [22] still work. An alternative way to achieve the bound on $[\partial\bar\partial u_i]_{C^\alpha(\{r\le N+2\})}$ for all $0 < \alpha < 1$ was done by Theorem 1.5 of [14] using the rescaling argument. Now it is standard to get a C^∞ bound of u_i on $\{r \le N + 1\}$ through Schauder estimates.

As in [23],

$$\int_{r\ge N} |\nabla |u_i|^{\frac{p}{2}}|^2\alpha_i^2 \le \frac{p^2}{p-1}\left[\int_{r\ge N} u_i|u_i|^{p-2}(\omega_i^2 - \alpha_i^2) - \frac{1}{2}\int_{r=N} u_i|u_i|^{p-2}\mathrm{d}^c u_i \wedge (\omega_i + \alpha_i)\right].$$

Therefore, for $p \geq 2$,

$$\int_{r\geq N} |\nabla |u_i|^{\frac{p}{2}}|^2 \leq Cp\left(\int_{r\geq N} |u_i|^{p-1}w + C_3^{p-1}\right) \leq Cp\left(C_3\int_{r\geq N} |u_i|^{p-1}w + C_3^p\right),$$

where C_3 is a bound on $\sup_{\{r\leq N+1\}} |u_i|$. By Young's inequality,

$$\int_{r\geq N} |\nabla |u_i|^{\frac{p}{2}}|^2 \leq Cp^2\left(||u_i||^p_{L_w^{p-1}(\{r\geq N\})} + C_3^p\right) \leq Cp^2\left(||u_i||^p_{L_w^p(\{r\geq N\})} + C_3^p\right).$$

Apply Theorem 7.9 to $|u_i|^{p/2}$. Then

$$||u_i||^{2p}_{L_w^{2p}(\{r\geq N\})} \leq C_4 p^4(||u_i||^{2p}_{L_w^p(\{r\geq N\})} + C_3^{2p}).$$

We already know that $||u_i||_{L_w^2(\{r\geq N\})} \leq C_5$. That is our starting point. We are going to obtain a bound on $||u_i||_{L^\infty(\{r\geq N\})} = \lim_{j\to\infty} ||u_i||_{L_w^{2^j}(\{r\geq N\})}$ by Moser iteration.

(1) If $||u_i||_{L_w^{2^j}(\{r\geq N\})} \leq C_3$ for all $j \geq 1$, then $||u_i||_{L^\infty(\{r\geq N\})} \leq C_3$.

(2) If $||u_i||_{L_w^{2^j}(\{r\geq N\})} \leq C_3$ for all $1 \leq j \leq k$ but $||u_i||_{L_w^{2^j}(\{r\geq N\})} > C_3$ for all $j > k$, then

$$||u_i||_{L^\infty(\{r\geq N\})} \leq (2C_4)^{\sum_{j=k}^\infty 2^{-j-1}} 2^{\sum_{j=k}^\infty 2^{-j+1}j} C_3 \leq C$$

(3) If $||u_i||_{L_w^{2^j}(\{r\geq N\})} > C_3$ for all $j \geq 1$, then

$$||u_i||_{L^\infty(\{r\geq N\})} \leq (2C_4)^{\sum_{j=1}^\infty 2^{-j-1}} 2^{\sum_{j=1}^\infty 2^{-j+1}j} C_5 \leq C$$

The L^∞ bound on $u_i = \phi_i - \phi_{ai}$ implies a bound on ϕ_{ai} since ϕ_i decay exponentially. Therefore, we actually have a L^∞ bound on ϕ_i. Then we can obtain a global C^∞ bound as before. Finally, we can go through the Step 3 and Step 4 in [23] to get the C^∞ bound on $e^{\lambda_1 r}\phi_i$. We are done by taking the limit of some subsequence of $\{\phi_i\}$. □

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