PATHOLOGICAL EXAMPLES OF STRUCTURES WITH O-MINIMAL OPEN CORE

ALEXI BLOCK GORMAN, ERIN CAULFIELD, AND PHILIPP HIERONYMI

Abstract. This paper answers several open questions around structures with o-minimal open core. We construct an expansion of an o-minimal structure $\mathcal{R}$ by a unary predicate such that its open core is a proper o-minimal expansion of $\mathcal{R}$. We give an example of a structure that has an o-minimal open core and the exchange property, yet defines a function whose graph is dense. Finally, we produce an example of a structure that has an o-minimal open core and definable Skolem functions, but is not o-minimal.

1. Introduction

Introduced by Miller and Speissegger [5] the notion of an open core has become a mainstay of the model-theoretic study of ordered structures. However, there are still many rather basic questions, in particular about structures with o-minimal open cores, that have remained unanswered. In this paper, we are able to settle some of the questions raised by Dolich, Miller and Steinhorn [2, 3].

Throughout this paper, $\mathcal{R}$ denotes a fixed, but arbitrary expansion of a dense linear order $(R, <)$ without endpoints. We now recall several definitions from the aforementioned papers. We denote by $\mathcal{R}^\circ$ the structure $(R, (U))$, where U ranges over the open sets of all arities definable in $\mathcal{R}$, and call this structure **the open core of** $\mathcal{R}$.

Given two structures $\mathcal{S}_1$ and $\mathcal{S}_2$ with the same universe S, we say $\mathcal{S}_1$ and $\mathcal{S}_2$ are **interdefinable** (short: $\mathcal{S}_1 =_{\mathrm{df}} \mathcal{S}_2$) if $\mathcal{S}_1$ and $\mathcal{S}_2$ define the same sets. For a given theory T extending the theory of dense linear orders, we say that a theory T' is an **open core of** T if for every $\mathcal{M} \models T$ there exists $\mathcal{M}' \models T'$ such that $\mathcal{M}^\circ =_{\mathrm{df}} \mathcal{M}'$.

Question 1 ([2, p. 1408])**.** *If $S \subseteq R$ and $(\mathcal{R}, S)^\circ$ is o-minimal, is $(\mathcal{R}, S)^\circ =_{\mathrm{df}} \mathcal{R}^\circ$?*

We give a negative answer to this question by constructing an expansion of the real field by a single unary predicate whose open core is o-minimal, but defines an irrational power function. It is clear from the construction in Section 2 that there are similar examples of expansions of the real ordered additive group that do not define an ordered field.

We say $\mathcal{R}$ is **definably complete** (short: $\mathcal{R} \models \mathrm{DC}$) if every definable unary set has both a supremum and an infimum in $R \cup \{\pm\infty\}$. We denote by $\mathrm{dcl}_{\mathcal{R}}$ the definable closure operator in $\mathcal{R}$, and often drop the subscript $\mathcal{R}$. We say that $\mathcal{R}$ has the **exchange property** (short: $\mathcal{R} \models \mathrm{EP}$) if $b \in \mathrm{dcl}(S \cup \{a\})$ for all $S \subseteq R$ and $a, b \in R$ such that $a \in \mathrm{dcl}(S \cup \{b\}) \setminus \mathrm{dcl}(S)$. For a theory T, we say T has the exchange property (short: $T \models \mathrm{EP}$) if every model of T has the exchange property. We write

$\mathcal{R} \models$ NIP if its theory does not have the independence property as introduced by Shelah [6]. We refer the reader to Simon [8] for a modern treatment of NIP and related model-theoretic tameness notions.

Question 2 ([2, p. 1409]). *If $\mathcal{R} \models \mathrm{DC} + \mathrm{EP} + \mathrm{NIP}$ and expands an ordered group, is $\mathcal{R}$ o-minimal?*

By [2, p. 1374] we know that $\mathcal{R}$ has o-minimal open core if $\mathcal{R} \models \mathrm{DC} + \mathrm{EP}$ and expands an ordered group. Thus Question 2 asks whether there is a combinatorical model-theoretic tameness condition that can be added to force o-minimality. Again, we give a negative answer to this question. We construct a counterexample as follows: Let $\mathbb{Q}(t)$ be the field of rational functions in a single variable t. We consider an expansion $\mathcal{R}_t$ of the ordered real additive group $(\mathbb{R}, <, +, 0, 1)$ into a $\mathbb{Q}(t)$-vector space such that for all $c \in \mathbb{Q}(t) \setminus \mathbb{Q}$, the graph of the function $x \mapsto cx$ is dense in $\mathbb{R}^2$. We show that $\mathcal{R}_t \models \mathrm{DC} + \mathrm{EP} + \mathrm{NIP}$, but is not o-minimal.

The structure $\mathcal{R}_t$ has infinite dp-rank. By Simon [7], if $\mathcal{R} \models$ DC, expands an ordered group and has dp-rank 1, then $\mathcal{R}$ is o-minimal. However, we do not know whether $\mathcal{R}$ is o-minimal if $\mathcal{R} \models \mathrm{DC} + \mathrm{EC}$, expands an ordered group and has finite dp-rank.

In addition to showing that $\mathcal{R}_t \models \mathrm{DC} + \mathrm{EP} + \mathrm{NIP}$, we prove that its open core is interdefinable with its o-minimal reduct $(\mathbb{R}, <, +, 0, 1)$. Since the graph of $x \mapsto cx$ is dense for $c \in \mathbb{Q}(t) \setminus \mathbb{Q}$, the theory of $\mathcal{R}_t$ provides a negative answer to the following question.

Question 3 ([3, p. 705]). *Let T be a complete o-minimal extension of the theory of densely ordered groups. If $\widetilde{T}$ is any theory (in any language) having T as an open core, and some model of $\widetilde{T}$ defines a somewhere dense graph, must* EP *fail for $\widetilde{T}$?*

Our counterexample $\mathcal{R}_t$ does not expand a field and we don't know whether Question 2 (or Question 3) has a positive answer if we require $\mathcal{R}$ (or T) to expand an ordered field.

We say that $\mathcal{R}$ has **definable Skolem functions** if for every definable set $A \subseteq R^m \times R^n$ there is a definable function $f : R^m \to R^n$ such that $(a, f(a)) \in A$ whenever $a \in A$ and there exists $b \in \mathbb{R}^n$ with $(a, b) \in A$. Every o-minimal expansion of an ordered group with a distinguished positive element has definable Skolem functions, but all documented examples of non-o-minimal structures with o-minimal core do not.

Question 4 ([2, p. 1409]). *If $\mathcal{R}$ has definable Skolem functions and $\mathcal{R}^\circ$ is o-minimal, is $\mathcal{R}$ o-minimal?*

The answer is again negative. We say $\mathcal{R}$ satisfies **uniform finitness** (short: $\mathcal{R} \models$ UF) if for every $m, n \in \mathbb{N}$ and every $A \subseteq R^{m+n}$ definable in $\mathcal{R}$ there exists $N \in \mathbb{N}$ such that for every $a \in R^m$ the set $\{b \in R^n \ : \ (a, b) \in A\}$ is either infinite or contains at most N elements. By [2, Theorem A], if $\mathcal{R} \models \mathrm{DC} + \mathrm{UF}$ and expands an ordered group, then $\mathcal{R}^\circ$ is o-minimal. Using a construction due to Winkler [9] and following a strategy of Kruckman and Ramsey [4], we establish that if $\mathcal{R} \models \mathrm{DC} + \mathrm{UF}$, then $\mathcal{R}$ has an expansion $\mathcal{S}$ such that $\mathcal{S}$ has definable Skolem functions and satisfies

$\mathcal{S}^\circ =_{\mathrm{df}} \mathcal{R}^\circ$. Thus if $\mathcal{R}$ also expands an ordered group, then $\mathcal{R}^\circ$ is o-minimal and so is $\mathcal{S}^\circ$.

Acknowledgements. A.B.G. was supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE – 1746047. P.H. was partially supported by NSF grant DMS-1654725. We thank Pantelis Eleftheriou for helpful conversations on the topic of this paper, in particular for bringing Question 1 to the attention of P.H. at a conference at the Bedlewo Conference Center. We thank Chris Miller for helpful comments on a draft of this paper.

Notation. We will use m, n for natural numbers and κ for a cardinal. Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Let $\mathcal{M} \models T$. We follow the usual convention to denote the universe of $\mathcal{M}$ by M. In this situation, $\mathcal{L}$-definable means $\mathcal{L}$-definable with parameters. Let b be a tuple of elements in M, and let $A \subseteq M$. We write $\mathrm{tp}_{\mathcal{L}}(b|A)$ for the $\mathcal{L}$-type of b over A. If $\mathcal{N}$ is another model of T and h is an embedding of a substructure of $\mathcal{M}$ containing A into $\mathcal{N}$, then $h\,\mathrm{tp}(b|A)$ is the type containing all formulas of the form $\varphi(x, h(a))$ where $\varphi(x, a) \in \mathrm{tp}(b|A)$.

2. Question 1

Let $\overline{\mathbb{R}}$ be the real ordered field $(\mathbb{R}, <, +, \cdot)$ and let $\mathbb{R}_{\exp}$ be the expansion of the real field by the exponential function exp. Let $I \subseteq \mathbb{R}$ be a dense $\mathrm{dcl}_{\mathbb{R}_{\exp}}$-independent set. Let $\tau \in I$ be such that $\tau > 1$. Set

$$J := \bigcup_{a \in I \setminus \{\tau\}} \{|a|, |a|^\tau, |a| + |a|^\tau\}.$$

By [3, 2.25] the open core of $(\mathbb{R}_{\exp}, I)^\circ$ is interdefinable with $\mathbb{R}_{\exp}$. Since $(\overline{\mathbb{R}}, J)$ is a reduct of $(\mathbb{R}_{\exp}, I)$, we have that $(\overline{\mathbb{R}}, J)^\circ$ is a reduct of $(\mathbb{R}_{\exp}, I)^\circ$. As the latter structure is o-minimal, we have that $(\overline{\mathbb{R}}, J)^\circ$ is o-minimal as well. In order to show that Question 1 has a negative answer, it is left to show that $(\overline{\mathbb{R}}, J)^\circ$ defines a set not definable in $\overline{\mathbb{R}}$. Since $\overline{\mathbb{R}}$ only defines raising to rational powers, it suffices to prove the definability of $x \mapsto x^\tau$ on an unbounded interval.

Lemma 2.1. *Let $u_1, u_2, u_3 \in J$ such that $1 < u_1 < u_2$ and $u_1 + u_2 = u_3$. Then there is $a \in I \setminus \{\tau\}$ such that $u_1 = |a|$ and $u_2 = |a|^\tau$.*

Proof. For $a \in I \setminus \{\tau\}$ observe that $|a|$, $|a|^\tau$ and $|a| + |a|^\tau$ are interdefinable in $\mathbb{R}_{\exp}$ over τ. Since $u_1 + u_2 = u_3$, we have u_1, u_2, u_3 are $\mathrm{dcl}_{\mathbb{R}_{\exp}}$-dependent. Because I is $\mathrm{dcl}_{\mathbb{R}_{exp}}$-independent, there are $a \in I \setminus \{\tau\}$ and $i, j \in \{1, 2, 3\}$ such that

$$u_i, u_j \in \{|a|, |a|^\tau, |a| + |a|^\tau\}.$$

Let $\ell \in \{1, 2, 3\}$ such that $\ell \neq i$ and $\ell \neq j$. Note u_ℓ is $\mathrm{dcl}_{\mathbb{R}_{\exp}}$-dependent over u_i and u_j. Thus $u_\ell \in \{|a|, |a|^\tau, |a| + |a|^\tau\}$. Since $|a| > 0$, we obtain from $u_1 + u_2 = u_3$ that

$$u_1, u_2 \in \{|a|, |a|^\tau\}.$$

Since $1 < u_1 < u_2$ and $\tau > 0$, we have that $u_1 = |a|$ and $u_2 = |a|^\tau$. □

Proposition 2.2. *The graph of $x \mapsto x^\tau$ on $\mathbb{R}_{\geq 1}$ is definable in $(\overline{\mathbb{R}}, J)^\circ$.*

Proof. By Lemma 2.1 the structure $(\overline{\mathbb{R}}, J)$ defines

$$\{(|a|, |a|^{\tau}) \ : |a| > 1,\ a \in I \setminus \{\tau\}\}.$$

Since I is dense in $\mathbb{R}$, the closure of this set is the graph of $x \mapsto x^{\tau}$ on $\mathbb{R}_{\geq 1}$, and hence definable in $(\overline{\mathbb{R}}, J)^{\circ}$. □

We conclude that $(\overline{\mathbb{R}}, J)^{\circ}$ is a proper expansion of $\overline{\mathbb{R}}$.

3. Questions 2 and 3

In this section we give negative answers to Questions 2 and 3. Let $\mathbb{Q}(t)$ be the field of rational functions in the variable t. We expand $(\mathbb{R}, <, +, 0, 1)$ to a $\mathbb{Q}(t)$-vector space such that for each non-constant $q(t) \in \mathbb{Q}(t)$ the graph of multiplication by $q(t)$ is dense.

We now construct such a $\mathbb{Q}(t)$-vector space structure on $\mathbb{R}$. Let $\mathbf{1}$ be the multiplicative identity of $\mathbb{Q}(t)$. We fix a dense basis $\mathcal{B}$ of $\mathbb{R}$ as a $\mathbb{Q}$-vector space, and a basis I of $\mathbb{Q}(t)$ as a $\mathbb{Q}$-vector space such that $\mathbf{1} \in I$. We choose a sequence of functions $\{\widetilde{f_\gamma} : I \to \mathcal{B}\}_{\gamma \in 2^{\aleph_0}}$ such that

$$\mathcal{B} = \bigcup_{\gamma \in 2^{\aleph_0}} \widetilde{f_\gamma}(I)$$

and for all $\gamma \in 2^{\aleph_0}$:

- $\widetilde{f_\gamma}$ is injective,
- for all $\eta \in 2^{\aleph_0}$ with $\eta \neq \gamma$, $\widetilde{f_\eta}(I) \cap \widetilde{f_\gamma}(I) = \emptyset$,
- for all open intervals $J_1, \dots, J_m \subseteq \mathbb{R}$ open intervals and all pairwise distinct $p_1(t), \dots, p_m(t) \in I$ there exists $\gamma \in 2^{\aleph_0}$ such that
$$\widetilde{f_\gamma}(p_1(t)) \in J_1, \dots, \widetilde{f_\gamma}(p_m(t)) \in J_m.$$

Since the order topology on the real line has a countable base, it is easy to see that such a sequence of functions exists. For each $\gamma \in 2^{\aleph_0}$, $\widetilde{f_\gamma}$ is defined on the basis I of $\mathbb{Q}(t)$. Therefore, we can extend each $\widetilde{f_\gamma} : I \to \mathcal{B}$ to a $\mathbb{Q}$-linear map $f_\gamma : \mathbb{Q}(t) \to \mathbb{R}$.

Lemma 3.1. *Let $a \in \mathbb{R}$. Then there are unique $\gamma_1, \dots, \gamma_n \in 2^{\aleph_0}$ and $p_1(t), \dots, p_n(t) \in \mathbb{Q}(t)$ such that*

$$a = f_{\gamma_1}(p_1(t)) + \dots + f_{\gamma_n}(p_n(t)).$$

Proof. Since $\mathcal{B}$ is a basis of $\mathbb{R}$ as a $\mathbb{Q}$-vector space, there are unique $b_1, \dots, b_n \in \mathcal{B}$ and $u_1, \dots, u_n \in \mathbb{Q}$ such that $a = u_1 b_1 + \dots + u_n b_n$. By the above construction, there are unique $\gamma_1, \dots, \gamma_n \in 2^{\aleph_0}$ and $q_1(t), \dots, q_n(t) \in I$ such that $b_i = f_{\gamma_i}(q_i(t))$ for $i = 1, \dots, n$. Then by $\mathbb{Q}$-linearity of the f_{γ_i}'s

$$\begin{aligned} a &= u_1 b_1 + \dots + u_n b_n \\ &= u_1 f_{\gamma_1}(q_1(t)) + \dots + u_n f_{\gamma_n}(q_n(t)) \\ &= f_{\gamma_1}(u_1 q_1(t)) + \dots + f_{\gamma_n}(u_n q_n(t)). \end{aligned}$$

Set $p_i := u_i q_i(t)$ for $i = 1, \dots, n$. □

We now introduce a $\mathbb{Q}$-linear map $\lambda : \mathbb{Q}(t) \times \mathbb{R} \to \mathbb{R}$. Let $q(t) \in \mathbb{Q}(t)$ and $a \in \mathbb{R}$. By Lemma 3.1 there are unique $\gamma_1, \dots, \gamma_n \in 2^{\aleph_0}$ and $p_1(t), \dots, p_n(t) \in \mathbb{Q}(t)$ such that

$$a = f_{\gamma_1}(p_1(t)) + \dots + f_{\gamma_n}(p_n(t)).$$

We define

$$\lambda(q(t), a) := f_{\gamma_1}(q(t) \cdot p_1(t)) + \cdots + f_{\gamma_n}(q(t) \cdot p_n(t)).$$

By Lemma 3.1, the function λ is well-defined. For $q(t) \in \mathbb{Q}(t)$, we write $\lambda_{q(t)}$ for the map taking $a \in \mathbb{R}$ to $\lambda(q(t), a)$.

Proposition 3.2. *The additive group $(\mathbb{R}, +)$ with λ as scalar multiplication is an $\mathbb{Q}(t)$-vector space.*

Proof. We only verify the following vector spaces axioms: for all $a \in \mathbb{R}$ and for all $q_1(t), q_2(t) \in \mathbb{Q}(t)$.

$$\lambda_{q_1(t)\cdot q_2(t)}(a) = \lambda_{q_1(t)}\big(\lambda_{q_2(t)}(a)\big).$$

The other axioms can be checked using similar arguments and we leave this to the reader.

Let $a \in \mathbb{R}$ and let $q_1(t), q_2(t) \in \mathbb{Q}(t)$. By Lemma 3.1 there are unique $\gamma_1, \ldots, \gamma_n \in 2^{\aleph_0}$ and $p_1(t), \ldots, p_n(t) \in \mathbb{Q}(t)$ such that

$$a = f_{\gamma_1}(p_1(t)) + \cdots + f_{\gamma_n}(p_n(t)).$$

We obtain

$$\begin{aligned}
\lambda_{q_1(t)}(\lambda_{q_2(t)}(a)) &= \lambda_{q_1(t)}\big(\lambda_{q_2(t)}\big(\sum_{i=1}^{n} f_{\gamma_i}(p_i(t))\big)\big) \\
&= \lambda_{q_1(t)}\big(\sum_{i=1}^{n} f_{\gamma_i}(q_2(t) \cdot p_i(t))\big) \\
&= \sum_{i=1}^{n} f_{\gamma_i}(q_1(t) \cdot (q_2(t) \cdot p_i(t))) \\
&= \lambda_{q_1(t)\cdot q_2(t)}\big(\sum_{i=1}^{n} f_{\gamma_i}(p_i(t))\big) \\
&= \lambda_{q_1(t)\cdot q_2(t)}(a).
\end{aligned}$$

$\square$

Let $\mathcal{L}$ be the language of $(\mathbb{R}, <, +, 0, 1)$, and let T be its theory; that is the theory of ordered divisible abelian groups with a distinguished positive element. It is well-known that T has quantifier-elimination and is o-minimal. We will use various consequences of this fact throughout this section. Most noteworthy: when $\mathcal{M} \models T$, $X \subseteq M^n$ is $\mathcal{L}$-definable over A and there is $b = (b_1, \ldots, b_n) \in X$ such that $b_1, \ldots, b_n$ are $\mathbb{Q}$-linearly independent over A, then X has interior.

Let $\mathcal{R}_t = (\mathbb{R}, <, +, 0, 1, (\lambda_{q(t)})_{q(t)\in\mathbb{Q}(t)})$ be the expansion of $(\mathbb{R}, <, +, 0, 1)$ by function symbols for $\lambda_{q(t)}$ where $q(t) \in \mathbb{Q}(t)$. We denote the language of $\mathcal{R}_t$ by $\mathcal{L}_t$.

3.1. **Density.** Let $p = (p_1(t), \ldots, p_n(t)) \in \mathbb{Q}(t)^n$. Let $\lambda_p : \mathbb{R} \to \mathbb{R}^n$ denote the function from $\mathbb{R}$ that maps a to $(\lambda_{p_1(t)}(a), \ldots \lambda_{p_n(t)}(a))$. The main goal of this subsection is to show the density of the image of λ_p when the coordinates of p are $\mathbb{Q}$-linearly independent.

Lemma 3.3. *Let $p = (p_1(t), \ldots, p_n(t)) \in I$ be such that $p_i(t) \neq p_j(t)$ for $i \neq j$. Then the image of λ_p is dense in $\mathbb{R}^n$.*

Proof. Let $J_1, \dots, J_n$ be open intervals in $\mathbb{R}$. Since $p_1(t), \dots, p_n(t)$ are distinct elements of I, there exists $\gamma \in 2^{\aleph_0}$ such that

$$f_\gamma(p_1(t)) \in J_1, \dots, f_\gamma(p_n(t)) \in J_n.$$

For each $i \in \{1, \dots, n\}$, we have

$$\lambda_{p_i(t)}(f_\gamma(\mathbf{1})) = f_\gamma(p_i(t))$$

by definition of $\lambda_{p_i(t)}$. Therefore,

$$\left(\lambda_{p_1(t)}(f_\gamma(\mathbf{1})), \dots, \lambda_{p_n(t)}(f_\gamma(\mathbf{1}))\right) \in J_1 \times J_2 \times \dots \times J_n.$$

□

Proposition 3.4. *Let $q = (q_1(t), \dots, q_m(t)) \in \mathbb{Q}(t)^m$ be such that $q_1(t), \dots, q_m(t)$ are $\mathbb{Q}$-linearly independent. Then the image of λ_q is dense in $\mathbb{R}^n$.*

Proof. Let $n \in \mathbb{N}$, let $p_1(t), \dots, p_n(t) \in I$ be distinct non-constant, and let $A = (u_{i,j})_{i=1,\dots,m,j=0,\dots,n}$ be an $m \times (n+1)$ matrix with rational entries such that

$$\begin{aligned} q_1(t) &= u_{1,0}\mathbf{1} + u_{1,1}p_1(t) + \dots + u_{1,n}p_n(t) \\ q_2(t) &= u_{2,0}\mathbf{1} + u_{2,1}p_1(t) + \dots + u_{2,n}p_n(t) \\ &\vdots \\ q_m(t) &= u_{m,0}\mathbf{1} + u_{m,1}p_1(t) + \dots + u_{m,n}p_n(t). \end{aligned}$$

By definition of $\lambda_{\mathbf{1}}, \lambda_{p_1(t)}, \dots, \lambda_{p_n(t)}$, we have for each $i \in \{1, \dots, m\}$

$$\lambda_{q_i(t)}(x) = u_{i,0}\lambda_{\mathbf{1}}(x) + u_{i,1}\lambda_{p_1(t)}(x) + \dots + u_{i,n}\lambda_{p_n(t)}(x).$$

Therefore,

$$A\lambda_{(\mathbf{1},p_1(t),\dots,p_n(t))} = \lambda_{(q_1(t),\dots,q_m(t))}.$$

Since $q_1(t), \dots, q_m(t)$ are $\mathbb{Q}$-linearly independent, the matrix A has rank m. Thus multiplication by A is a surjective map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Since matrix multiplication is continuous and continuous surjections preserve density, the image of $A\lambda_{(\mathbf{1},p_1(t),\dots,p_n(t))}$ is dense in $\mathbb{R}^m$ by Lemma 3.3. □

3.2. **Axiomatization and QE.** In this subsection, we will find an axiomatization of $\mathcal{R}_t$ and show that this theory has quantifier elimination. Indeed, we will prove that the following subtheory of the $\mathcal{L}_t$-theory of $\mathcal{R}_t$ already has quantifier-elimination.

Definition 3.5. *Let T_t be the $\mathcal{L}_t$-theory extending T by axiom schemata stating that for every model $\mathcal{M} = (M, <, +, 0, 1, (\lambda_{q(t)})_{q(t)\in\mathbb{Q}(t)}) \models T_t$*

- (T1) *$(M, +, 0, (\lambda_{q(t)})_{q(t)\in\mathbb{Q}(t)})$ is a $\mathbb{Q}(t)$-vector space.*
- (T2) *If $q_1(t), \dots, q_m(t) \in \mathbb{Q}(t)$ are $\mathbb{Q}$-linearly independent, then the image of the $\lambda_{(q_1(t),\dots,q_m(t))}$ is dense in M^m.*

By Proposition 3.2 and Proposition 3.4 we know that $\mathcal{R}_t \models T_t$. Let $\mathcal{M} \models T_t$. We observe that by (T1) the $\mathcal{L}_t$-substructures of $\mathcal{M}$ are precisely the $\mathbb{Q}(t)$-linear subspaces of $\mathcal{M}$ containing 1.

Lemma 3.6. *Let $\mathcal{M} \models T_t$ and let $\mathcal{A}$ be an $\mathcal{L}_t$-substructure of $\mathcal{M}$. Let $b \in M \setminus A$ and let $p_1(t), \dots, p_n(t) \in \mathbb{Q}(t)$ be $\mathbb{Q}$-linearly independent. Then $\lambda_{p_1(t)}(b), \dots, \lambda_{p_n(t)}(b)$ are $\mathbb{Q}$-linearly independent over A.*

Proof. Since $\mathcal{A}$ is a $\mathbb{Q}(t)$-linear subspace of $\mathcal{M}$, we know that $\lambda_{q(t)}(b) \notin A$ for all non-zero $q(t) \in \mathbb{Q}(t)$. Let $u_1, \ldots, u_n \in \mathbb{Q}$. Since $\mathcal{M}$ is a $\mathbb{Q}(t)$-vector space,

$$u_1\lambda_{p_1(t)}(b) + \cdots + u_n\lambda_{p_n(t)}(b) = \lambda_{u_1p_1(t)+\cdots+u_np_n(t)}(b).$$

Because $p_1(t), \ldots, p_n(t) \in \mathbb{Q}(t)$ are $\mathbb{Q}$-linearly independent and $b \notin A$, we get that

$$u_1\lambda_{p_1(t)}(b) + \cdots + u_n\lambda_{p_n(t)}(b) \in A \Rightarrow u_1 = \cdots = u_n = 0.$$

Thus $\lambda_{p_1(t)}(b), \ldots, \lambda_{p_n(t)}(b)$ are $\mathbb{Q}$-linearly independent of A. $\square$

Proposition 3.7. *The theory T_t has quantifier elimination.*

Proof. Let $\mathcal{M}, \mathcal{N} \models T_t$ be such that $\mathcal{N}$ is $|\mathcal{M}|^+$-saturated. Let $\mathcal{A} \subseteq \mathcal{M}$ be a substructure that embeds into $\mathcal{N}$ via the embedding $h : \mathcal{A} \hookrightarrow \mathcal{N}$. Let $b \in M \setminus A$. To prove quantifier elimination of T_t, it is enough to show that the embedding h extends to an embedding of the $\mathcal{L}_t$-substructure generated by $\mathcal{A}$ and b.

Consider the tuple $(\lambda_{p(t)}(b))_{p(t)\in I}$. We first find $c \in N$ such that

$$h\,\mathrm{tp}_{\mathcal{L}}\big((\lambda_{p(t)}(b))_{i\in I}|A\big) = tp_{\mathcal{L}}\big((\lambda_{p(t)}(c))_{i\in I}|h(A)\big) \tag{1}$$

By saturation of $\mathcal{N}$ it is enough to show that for every

- pairwise distinct $p_1(t), \ldots, p_n(t) \in I$,
- $\mathcal{L}$-formula $\psi(x, y)$ and $a \in A^{|y|}$ such that

$$\mathcal{M} \models \psi\big(\lambda_{p_1(t)}(b), \ldots, \lambda_{p_n(t)}(b), a\big),$$

there is $c \in N \setminus h(A)$ such that

$$\mathcal{N} \models \psi\big(\lambda_{p_1(t)}(c), \ldots, \lambda_{p_n(t)}(c), h(a)\big).$$

Because I is a $\mathbb{Q}$-linear basis of $\mathbb{Q}(t)$, the sequence $(\lambda_{p(t)}(b))_{i\in I}$ is $\mathbb{Q}$-linear independent over A by Lemma 3.6. Thus the set

$$\{d \in N^n \ : \ \mathcal{N} \models \psi(d, h(a))\}$$

has interior. The existence of c now follows from (T2) and saturation of $\mathcal{N}$.

Let $c \in N$ be such that (1) holds. Let $\mathcal{X}$ be the $\mathbb{Q}$-linear subspace of $\mathcal{M}$ generated by $(\lambda_{p(t)}(b))_{i\in I}$ and $\mathcal{A}$. Let $\mathcal{Y}$ be the $\mathbb{Q}$-linear subspace of $\mathcal{N}$ generated by $(\lambda_{p(t)}(c))_{i\in I}$ and $h(\mathcal{A})$. Observe that $\mathcal{X}$ is the $\mathbb{Q}(t)$-subspace of $\mathcal{M}$ generated by b and $\mathcal{A}$, and $\mathcal{Y}$ is the $\mathbb{Q}(t)$-subspace of $\mathcal{N}$ generated by c and $h(\mathcal{A})$. Hence $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{L}_t$-structures of $\mathcal{M}$ and $\mathcal{N}$ respectively. Since c satisfies (1), there is an $\mathcal{L}$-isomorphism $h' : \mathcal{X} \to \mathcal{C}$ extending h and mapping $\lambda_{p(t)}(b)$ to $\lambda_{p(t)}(c)$ for each $p(t) \in I$. It follows easily that this h' is $\mathbb{Q}(t)$-linear and hence an $\mathcal{L}_t$-isomorphism extending h. $\square$

Corollary 3.8. *Let $\mathcal{M}, \mathcal{N} \models T_t$, let $\mathcal{A} \subseteq \mathcal{M}$ be an $\mathcal{L}_t$-substructure such that $h : A \hookrightarrow \mathcal{N}$ is an $\mathcal{L}_t$-embedding. Let $b \in M \setminus A$ and $c \in N \setminus h(A)$ such that*

$$h\,\mathrm{tp}_{\mathcal{L}}(\big(\lambda_{p(t)}(b)\big)_{p(t)\in I}|A) = \mathrm{tp}_{\mathcal{L}}(\big(\lambda_{p(t)}(c)\big)_{p(t)\in I}|h(A)).$$

Then $\mathrm{tp}_{\mathcal{L}_t}(b|A) = \mathrm{tp}_{\mathcal{L}_t}(c|h(A))$.

Proof. Let $\mathcal{X}$ be the $\mathbb{Q}$-linear subspace of $\mathcal{M}$ generated by $(\lambda_{p(t)}(b))_{i\in I}$ and $\mathcal{A}$, and let $\mathcal{Y}$ be the $\mathbb{Q}$-linear subspace of $\mathcal{N}$ generated by $(\lambda_{p(t)}(c))_{i\in I}$ and $h(\mathcal{A})$. It is easy to check that $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{L}_t$-substructures of $\mathcal{M}$ and $\mathcal{N}$ respectively. By our assumption on b and c, the embedding h extends on an $\mathcal{L}$-isomorphism h' between $\mathcal{X}$ and $\mathcal{Y}$ mapping $\lambda_{p(t)}(b)$ to $\lambda_{p(t)}(c)$ for each $p(t) \in I$. Since h is an $\mathcal{L}_t$-embedding, it follows easily that h' is an $\mathcal{L}_t$-isomorphism between $\mathcal{X}$ to $\mathcal{Y}$. Since T_t has quantifier elimination, we get that $\mathrm{tp}_{\mathcal{L}_t}(b|A) = \mathrm{tp}_{\mathcal{L}_t}(c|h(A))$. $\square$

Proposition 3.9. *The theory of $\mathcal{R}_t$ is axiomatized by the theory T_t in conjunction with the axiom scheme that specifies $\mathrm{tp}_{\mathcal{L}}(\big(\lambda_{p(t)}(1)\big)_{p(t)\in I})$.*

Proof. Let T_t^* be the theory described in the statement. Since $\mathcal{R}_t \models T_t$, we immediately get that $\mathcal{R}_t \models T_t^*$. It is left to show that T_t^* is complete. Let $\mathcal{M}$ and $\mathcal{N}$ be model of T_t^* of size $\kappa > \aleph_0$. By Corollary 3.8, $1_{\mathcal{M}}$ and $1_{\mathcal{N}}$ satisfy the same $\mathcal{L}_t$-type. Thus there is an $\mathcal{L}_t$-isomorphism h mapping the $\mathcal{L}_t$-substructure of $\mathcal{M}$ generated by $1_{\mathcal{M}}$ to the $\mathcal{L}_t$-substructure of $\mathcal{N}$ generated by $1_{\mathcal{N}}$. By the proof of Proposition 3.7 this isomorphism h extends to an $\mathcal{L}_t$-isomorphism between $\mathcal{M}$ and $\mathcal{N}$. $\square$

3.3. **Exchange property.** In this subsection we establish that every model of T_t has the exchange property. We will do so by showing that the definable closure in such a model is equal to the $\mathbb{Q}(t)$-linear span.

Lemma 3.10. *Let $\mathcal{M} \models T_t$ and let $\mathcal{A} \subseteq \mathcal{M}$ be an $\mathcal{L}_t$-substructure. Then $\mathcal{A}$ is definably closed.*

Proof. Without loss of generality, we can assume that $\mathcal{M}$ is $|A|^+$-saturated. Let $b \in M \setminus A$. It is enough to show that there exists $c \in M$ such that $b \neq c$ and $\mathrm{tp}_{\mathcal{L}_t}(b|A) = \mathrm{tp}_{\mathcal{L}_t}(c|A)$. By Corollary 3.8 it is sufficient to find $c \in M$ such that $b \neq c$ and

$$\mathrm{tp}_{\mathcal{L}}(\big(\lambda_{p(t)}(b)\big)_{p(t)\in I}|A) = \mathrm{tp}_{\mathcal{L}}(\big(\lambda_{p(t)}(c)\big)_{p(t)\in I}|A).$$

Let $\varphi(x, y)$ be an $\mathcal{L}$-formula, $p_1(t), \dots, p_m(t) \in I$ and $a \in A^n$ such that

$$\mathcal{M} \models \varphi(\lambda_{p_1(t)}(b), \dots, \lambda_{p_m(t)}(b), a).$$

By saturation of $\mathcal{M}$, we only need to find $c \in M$ such that $c \neq b$ and $\mathcal{M} \models \varphi(\lambda_{p_1(t)}(b), \dots, \lambda_{p_m(t)}(b), a)$. By Lemma 3.6, $\big(\lambda_{p(t)}(b)\big)_{p(t)\in I}$ is $\mathbb{Q}$-linear independent over A. Thus the set

$$X := \{d \in M^m \ : \ \mathcal{M} \models \varphi(d, a)\}$$

has interior. By axiom (T2) the intersection

$$\{(\lambda_{p_1(t)}(c), \dots, \lambda_{p_m(t)}(c)) \ : \ c \in M\} \cap X$$

is dense in X. $\square$

Corollary 3.11. *Let $\mathcal{M} \models T_t$ and let $Z \subseteq M$. Then the $\mathcal{L}_t$-definable closure of Z is the $\mathbb{Q}(t)$-subspace of $\mathcal{M}$ generated by Z.*

Proof. By Lemma 3.10 the definable closure of Z is the $\mathcal{L}_t$-substructure generated by Z. However, the latter is just the $\mathbb{Q}(t)$-subspace of $\mathcal{M}$ generated by Z. $\square$

The exchange property for T_t follows immediately from Corollary 3.11 and the classical Steinitz exchange lemma for vector spaces.

Proposition 3.12. *The theory T_t has the exchange property.*

3.4. **Open core.** Let $\mathcal{M} \models T_t$. Then by Axiom (T2) it defines functions from M to M whose graph is dense. We already know that $\mathcal{M}$ has EP by Proposition 3.12. To give a negative answer to Question 3, it is left to show that every open subset of M^n definable in $\mathcal{M}$ is already definable in the reduct $(M, <, +, 0, 1)$.

Theorem 3.13. *The theory T is an open core of T_t.*

Proof. Let $\mathcal{M} \models T_t$. We prove that every open set is $\mathcal{L}$-definable. Without loss of generality, we can assume that $\mathcal{M}$ is $\aleph_0$-saturated. Let C be a finite subset. Using Boxall and Hieronymi [1, Corollary 3.1], we will show that for every $n \in \mathbb{N}$ and every subset of M^n that is $\mathcal{L}_t$-definable over C, is also $\mathcal{L}$-definable over C. Let $n \in \mathbb{N}$ and $p_1(t), \dots, p_{n-1}(t) \in I$ be distinct and non-constant. We define

$$D := \{\big(a, \lambda_{p_1(t)}(a), \dots, \lambda_{p_{n-1}(t)}(a)\big) \ : \ a \notin \mathrm{dcl}_{\mathcal{L}_t}(C)\}.$$

From (T2) and saturation of $\mathcal{M}$, it follows easily that D is dense in M^n. Thus Condition (1) of [1, Corollary 3.1] is satisfied. Condition (3) of [1, Corollary 3.1] holds by Corollary 3.8. It is only left to establish Condition (2).

Let $b \in D$ and $a \notin \mathrm{dcl}_{\mathcal{L}_t}(C)$ be such that $b = \big(a, \lambda_{p_1(t)}(a), \dots, \lambda_{p_{n-1}(t)}(a)\big)$. Let $U \subseteq M^n$ be open and suppose that $\mathrm{tp}_{\mathcal{L}}(b|C)$ is realized in U. We need to show that $\mathrm{tp}_{\mathcal{L}}(b|C)$ is realized in $U \cap D$. By Lemma 3.6 we know that the coordinates of b are $\mathbb{Q}$-linearly independent over $\mathrm{dcl}_{\mathcal{L}_t}(C)$. Thus the set of realizations of $\mathrm{tp}_{\mathcal{L}}(b|C)$ is open, and so is its intersection with the open set U. Denote this intersection by V. By (T2) and $\aleph_0$-saturation of $\mathcal{M}$, we find $a' \notin \mathrm{dcl}_{\mathcal{L}_t}(C)$ such that $b' = \big(a', \lambda_{p_1(t)}(a'), \dots, \lambda_{p_{n-1}(t)}(a')\big)$. Now b' is the desired realization of $\mathrm{tp}_{\mathcal{L}}(b|C)$ in $U \cap D$. $\square$

By Theorem 3.13 every model of T_t has o-minimal open core and thus is definably complete.

3.5. **Neostability results.** We will now show that T_t is NIP, but not strong. We use an equivalent definition of the independence property in the theorem below, namely that in a monster model $\mathcal{M}$ of T_t there is no formula $\varphi(x, y)$ and no element $a \in M$ such that for some indiscernible sequence $(b_i)_{i<\omega}$ of tuples in $M^{|y|}$ we have

$$\mathcal{M} \models \varphi(a, b_i) \text{ if and only if } i < \omega \text{ is even.}$$

For a proof that this is equivalent to the classic definition of NIP, see [8].

Theorem 3.14. *Every completion of the theory T_t has* NIP.

Proof. We let $\mathcal{M} \models T_t$ be a monster model of T_t. We suppose for a contradiction that there is an $\mathcal{L}_t$-formula $\varphi(x, y)$ along with an element $a \in M$ and indiscernible sequence $(b_i)_{i<\omega}$ of elements in $M^{|y|}$ that witnesses IP, i.e. $\mathcal{M} \models \varphi(a, b_i)$ precisely if i is even. Let $|y| = n$ and for each $i < \omega$ we denote the j-coordinate of b_i by $b_{i,j}$. By quantifier elimination in the language $\mathcal{L}_t$, we can assume that the formula $\varphi(a, y)$ is equal to a boolean combination of formulas of the form

(1) $a - \sum_{j=1}^n \lambda_{q_j(t)}(y_j) = 0$
(2) $a - \sum_{j=1}^n \lambda_{q_j(t)}(y_j) > 0$

where $q_1, \dots, q_n(t) \in \mathbb{Q}(t)$. Since NIP is preserved under boolean combinations, we can assume that φ is of the form (1) or (2). For ease of notation, let $f(b_i) = \sum_{j=1}^n \lambda_{q_j(t)}(b_{i,j})$ for each $i < \omega$.

Suppose that φ is of form (1). We suppose without loss of generality that $a - f(b_i) = 0$ holds if and only if $i < \omega$ is odd. Then we have that $f(b_1) = f(b_3)$, but $f(b_2) \neq f(b_1)$. Thus we conclude $\mathrm{tp}_{\mathcal{L}_t}(b_1 b_2) \neq \mathrm{tp}_{\mathcal{L}_t}(b_2 b_3)$, contradicting indiscernability.

Now assume that φ is of the form (2). Without loss of generality assume that $a - f(b_i) > 0$ holds if and only if $i < \omega$ is odd. Then for all $i < \omega$, we have $a < f(b_{2i})$ and $a > f(b_{2i+1})$. However, this means that $f(b_1) < f(b_2)$ and $f(b_2) > f(b_3)$. So we again obtain $\mathrm{tp}_{\mathcal{L}_t}(b_1 b_2) \neq \mathrm{tp}_{\mathcal{L}_t}(b_2 b_3)$, contradicting indiscernability.

□

Thus $\mathcal{R}_t \models \mathrm{DC} + \mathrm{EP} + \mathrm{NIP}$, but $\mathcal{R}_t$ is not o-minimal. This gives a negative answer to Question 2.

Proposition 3.15. *No completion of the theory T_t is strong.*

Proof. Fix a family $(q_j(t))_{j \in \mathbb{N}}$ of distinct elements of I. Consider the family of $\mathcal{L}_t$-formulas given by $(\lambda_{q_j(t)}(x) \in (a_k, b_k))_{j,k \in \mathbb{N}}$ such that

- $(a_k, b_k) \cap (a_\ell, b_\ell) = \emptyset$, for all $\ell \neq k \in \mathbb{N}$, and
- the tuples $(a_k, b_k)_{k \in \mathbb{N}}$ form an indiscernible sequence.

In the array that corresponds to varying $j \in \mathbb{N}$ along the rows and the $k \in \mathbb{N}$ along the columns, it is easy to see that formulas in the same row are pairwise inconsistent. However, for every path $(\lambda_{q_{\gamma(k)}(t)}(x) \in (a_{\gamma(k)}, b_{\gamma(k)}))_{k \in \mathbb{N}}$, every finite subset of these formulas are consistent by our axiom scheme (T2). So by compactness, every path through the entire array is consistent. □

4. Question 4

Let T be a theory extending the theory of dense linear orders without endpoints in an language $\mathcal{L}$. We write $T \models \mathrm{UF}$ if every model of T satisfies UF. The main goal of this section is to establish the following theorem.

Theorem 4.1. *Suppose that $T \models \mathrm{DC} + \mathrm{UF}$. Let T' be an open core of T. There is a theory T^{∞}_{Sk} extending T such that T^{∞}_{Sk} has definable Skolem functions and T' is an open core of T^{∞}_{Sk}.*

This immediately gives a negative answer to Question 4, as there are many documented examples of a theory T with $T \models \mathrm{DC} + \mathrm{UF}$ and o-mininal open core that is not o-minimal itself. To prove Theorem 4.1 we follow a strategy of Kruckman and Ramsey [4] and rely on a construction due to Winkler [9] allowing us to successively add definable Skolem functions to the language $\mathcal{L}$ of a given theory T while preserving uniform finiteness. As explained below, this construction preserves the open definable sets by a result from [1]. We begin by recalling notations and results from [9].

4.1. **Skolem expansions.** Let $\mathcal{L}$ be a language and $\Theta = \{\theta_t(x, y) : t < |\mathcal{L}|\}$ be an enumeration of all $\mathcal{L}$-formulas $\varphi(x, y)$ where the variable y has length 1. Define $\mathcal{L}_{Sk}$ to be $\mathcal{L} \cup \{f_t : t < |\mathcal{L}|\}$, where the arity of f_t is the length of the tuple x appearing in $\theta_t(x, y)$.

The **Skolem expansion** T_+ **of** T is the $\mathcal{L}_{Sk}$-theory

$$T_+ = T \cup \{\forall x \exists y\, (\theta_t(x, y) \to \theta_t(x, f_t(x))) \ : \ t < |\mathcal{L}|\}.$$

We refer to the f_t's as Skolem functions.

From here on we assume that T has quantifier elimination in the language $\mathcal{L}$ and assume that for each $\mathcal{L}$-definable function f there is an $\mathcal{L}$-term t such that $T \models \forall x\, f(x) = t(x)$. Let $\mathcal{M}_+ \models T_+$, and denote its reduct to $\mathcal{L}$ by $\mathcal{M}$. For $A \subseteq M$ we denote by $\langle A \rangle_{\mathrm{Sk}}$ the $\mathcal{L}_{\mathrm{Sk}}$-substructure generated by A.

Following [9], we say an $\mathcal{L}_{\mathrm{Sk}}$-formula $\chi(x_1, \ldots, x_n)$ is a **uniform configuration** if it is a conjunction of equalities of the form $f_t(x_{i_1}, \ldots, x_{i_m}) = x_{i_0}$ involving Skolem functions. We need the following result about uniform configurations from [9].

Fact 4.2 ([9, p. 448])**.** *Let $\chi(x)$ be a uniform configuration. Then there exists an $\mathcal{L}$-formula $\chi'(x)$ such that for all $\mathcal{A} \models T_+$ and $a \in A^{|x|}$ the following are equivalent:*

- $\mathcal{A} \models \chi'(a)$.
- *The result of altering the Skolem structure of $\mathcal{A}$ precisely so that $\mathcal{A} \models \chi(a)$ is again a model of T_+.*

In the case of Fact 4.2, we say that $\chi'(x)$ **codes the eligibility of the configuration** $\chi(x)$.

Lemma 4.3. *Let $t_1(x), \ldots, t_n(x)$ be $\mathcal{L}_{\mathrm{Sk}}$-terms such that for every $i \leq n$ there is an $\mathcal{L}_{\mathrm{Sk}}$-function symbol f_i with*

$$t_i(x) = f_i(x, t_1(x), \ldots, t_{i-1}(x)).$$

Then there is an $\mathcal{L}$-formula $\varphi(x, y)$ and a uniform configuration $\chi(x, y)$ such that

$$T_+ \models \forall x \forall y \Big(\big(\varphi(x, y) \wedge \chi(x, y)\big) \leftrightarrow \big(\bigwedge_{i=1}^{n} y_i = f_i(x, y_1, \ldots, y_{i-1})\big)\Big).$$

Proof. Let $J \subseteq \{1, \ldots, n\}$ be the set of all i such that $f_i \in \mathcal{L}_{\mathrm{Sk}} \setminus \mathcal{L}$. Let $\chi(x, y)$, where $y = (y_1, \ldots, y_n)$, be the uniform configuration given by

$$\bigwedge_{i \in J} f_i(x, y_1, \ldots, y_{i-1}) = y_i$$

and let $\varphi(x, y)$ be the $\mathcal{L}$-formula given by

$$\bigwedge_{i \in \{1,\ldots,n\} \setminus J} f_i(x, y_1, \ldots, y_{i-1}) = y_i.$$

It is easy check this pair of formulas has the desired property. ☐

One of the main results in [9] is that if $T \models \mathrm{UF}$, then the Skolem expansion has a model companion. Indeed, more is true:

Fact 4.4 ([9, Theorem 2, Corollary 3])**.** *Let $T \models$ UF. Then the Skolem expansion T_+ has a model companion T_{Sk} that satisfies* UF.

From here on we assume that $T \models$ UF. We have the following axiomatization of the model companion of the Skolem expansion.

Fact 4.5 ([9, p. 447])**.** *The theory T_{Sk} is axiomatized as the expansion of T_+ by the set Φ of all sentences of the form $\forall x_1 \ldots \forall x_k \psi(x)$, where $x = (x_1, \ldots, x_n)$ and*

(i) $\psi(x) = \exists^\infty x_{k+1} \ldots x_n \varphi(x) \wedge \chi'(x) \to \exists x_{k+1}, \ldots x_n \varphi(x) \wedge \chi(x)$,
(ii) $\varphi(x)$ *is a quantifier free $\mathcal{L}$-formula,*

(iii) $\chi(x)$ *is a uniform configuration,*
(iv) $\chi'(x)$ *codes the eligibility of the configuration* $\chi(x)$.

Let $\mathcal{M}_{\mathrm{Sk}}$ be an $|\mathcal{L}|^+$ -saturated model of T_{Sk} with underlying set M, and denote its reduct to $\mathcal{L}$ by $\mathcal{M}$. We need following easy corollary of the axiomatization of T_{Sk}.

Fact 4.6. *Let* $\mathcal{I}$ *be the set of partial* $\mathcal{L}$*-elementary maps* $\iota : X \to Y$ *between* $\mathcal{M}_{\mathrm{Sk}}$ *and itself such that*

- ι *is a partial* $\mathcal{L}_{\mathrm{Sk}}$*-isomorphisms and*
- $X = \langle X \rangle_{\mathrm{Sk}}$ *and* $Y = \langle Y \rangle_{\mathrm{Sk}}$.

Then $\mathcal{I}$ *is a back-and-forth system.*

Proof. Let $\iota : X \to Y$ in $\mathcal{I}$. Let $a \in M \setminus X$. By symmetry, it is enough to find $a' \in M$ such that there exists $\iota' \in \mathcal{I}$ extending ι such that $\iota(a) = a'$. By saturation of $\mathcal{M}_{\mathrm{Sk}}$, we just need to find $a' \in M$ such that for all $\mathcal{L}_{\mathrm{Sk}}(X)$-terms $t(x) = (t_1(x), \dots, t_n(x))$

$$\mathrm{tp}_{\mathcal{L}}(a', t(a')|Y) = \iota\, \mathrm{tp}_{\mathcal{L}}(a, t(a)|X).$$

Without loss of generality, we can assume that there is $c \in X^m$ such that for every $i \in \{1, \dots, n\}$ there is a function symbol $f_i \in \mathcal{L}_{\mathrm{Sk}}$ with

$$t_i(x) = f_i(x, t_1(x), \dots, t_{i-1}(x), c).$$

Let $\varphi(x, y_1, \dots, y_n, z)$ be the $\mathcal{L}$-formula and $\chi(x, y_1, \dots, y_n, z)$ be the uniform configuration given by Lemma 4.3. Let the $\mathcal{L}$-formula $\chi'(x, y_1, \dots, y_n, z)$ code the eligibility of $\chi(x, y_1, \dots, y_n, z)$. For ease of notation, set $y := (y_1, \dots, y_n)$.

Consider an $\mathcal{L}$-formula $\psi(x, y, z)$ and $c' \in X^{|c'|}$ such that $\psi(x, y, c') \in \mathrm{tp}_{\mathcal{L}}(a, t(a)|X)$. Extending c, we can assume that $c = c'$. By saturation of $\mathcal{M}_{\mathrm{Sk}}$ it suffices to find $a' \in M$ such that $\mathcal{M}_{\mathrm{Sk}} \models \psi(a', t(a'), \iota(c))$. Since $\mathcal{M} \models \psi(a, t(a), c) \wedge \varphi(a, t(a), c) \wedge \chi'(a, t(a), c)$ and $a \notin X$, we have that

$$\mathcal{M} \models \exists^\infty xy\ \psi(x, y, c) \wedge \varphi(x, y, c) \wedge \chi'(x, y, c).$$

Since ι is $\mathcal{L}$-elementary,

$$\mathcal{M} \models \exists^\infty xy\ \psi(x, y, \iota(c)) \wedge \varphi(x, y, \iota(c)) \wedge \chi'(x, y, \iota(c))).$$

Thus from the axiomatization of T_{Sk} we know that there is $(a', a'_1, \dots, a'_n) \in M^{1+n}$ such that

$$\mathcal{M} \models \psi(a', a'_1, \dots, a'_n, \iota(c)) \wedge \varphi(a', a'_1, \dots, a'_n, \iota(c)) \wedge \chi(a', a'_1, \dots, a'_n, \iota(c)).$$

By our choice of φ and χ, we have that $a'_i = t_i(a')$ for each i. Thus $\mathcal{M} \models \psi(a', t(a'), c)$. ☐

We now collect the following easy corollary of Fact 4.6.

Fact 4.7. *Let* $a, a' \in M^n$ *and let* $\sigma : M \to M$ *be an* $\mathcal{L}_{\mathrm{Sk}}$*-automorphism fixing* C *such that* $\sigma(a) = a'$ *and for all* $\mathcal{L}_{\mathrm{Sk}}$*-terms* $t(x) = (t_1(x), \dots, t_n(x))$

$$\mathrm{tp}_{\mathcal{L}}(t(a)|C) = \mathrm{tp}_{\mathcal{L}}(t(a')|C).$$

Then $\mathrm{tp}_{\mathcal{L}_{\mathrm{Sk}}}(a|C) = \mathrm{tp}_{\mathcal{L}_{\mathrm{Sk}}}(a'|C)$.

4.2. **No new definable open sets in T_{Sk}.** Let $\mathcal{M}_{\mathrm{Sk}}$ be an $|\mathcal{L}_{\mathrm{Sk}}|^+$-saturated model of T_{Sk} with underlying set M, and denote its reduct to $\mathcal{L}$ by $\mathcal{M}$. Fix a subset $C \subseteq M$ of cardinality at most $|\mathcal{L}_{\mathrm{Sk}}|$.

Theorem 4.8. *Let $C = \langle C \rangle_{\mathrm{Sk}}$. Then every open set definable over C in $\mathcal{M}_{\mathrm{Sk}}$ is definable in $\mathcal{M}$.*

Proof. By [1, Therorem 2.2] it is enough to show that for every $a \in M^n$ for which the set of realisations of $\mathrm{tp}_{\mathcal{L}}(a|C)$ is dense in an open set, the set of realisations of $\mathrm{tp}_{\mathcal{L}_{\mathrm{Sk}}}(a|C)$ is dense in the set of realizations of $\mathrm{tp}_{\mathcal{L}}(a|C)$. Let $U \subseteq M^n$ be an open definable set such that the set of realizations of $\mathrm{tp}_{\mathcal{L}}(a|C)$ intersected with U is dense in U. It is left to show that there is $a' \in U$ such that $a' \models \mathrm{tp}_{\mathcal{L}_{\mathrm{Sk}}}(a|C)$. By Fact 4.7 and saturation of $\mathcal{M}_{\mathrm{Sk}}$, it is enough to find for

- every tuple $t = (t_1, \dots, t_m) : M^n \to M^m$ of $\mathcal{L}_{\mathrm{Sk}}(C)$-terms and
- every $\mathcal{L}(C)$-definable set $X \subseteq M^{n+m}$ with $(a, t(a)) \in X$

an $a' \in U$ such that $(a', t(a')) \in X$. Fix t and X. After increasing m, we can assume that there is $c \in C^\ell$ such that X is $\mathcal{L}(c)$-definable and for every $i \leq m$

$$t_i(x) = f_i(x, t_1(x), \dots, t_{i-1}(x), c)$$

where f_i is a function symbol in $\mathcal{L}_{\mathrm{Sk}}$. Let $\varphi(x, y_1, \dots, y_n, c)$ be the $\mathcal{L}$-formula and $\chi(x, y_1, \dots, y_n, c)$ be the uniform configuration given by Lemma 4.3. Set $y = (y_1, \dots, y_n)$. Let the $\mathcal{L}$-formula $\chi'(x, y, c)$ code the eligibility of $\chi(x, y, c)$.

We now prove the existence of a'. Let d_0 be a realization of $\mathrm{tp}_{\mathcal{L}}(a|C)$ in U. Let $d_1, \dots, d_m \in M^m$ be such that $(d_0, d_1, \dots, d_m) \in X$ and

$$\mathcal{M} \models (\varphi \wedge \chi')(d_0, d_1, \dots, d_m, c).$$

Since there are infinitely many realizations of $\mathrm{tp}_{\mathcal{L}}(a|C)$ in U, there are infinitely many $e \in M^{n+m}$ such that $e \in X \cap U$ and $\mathcal{M} \models (\varphi \wedge \chi')(e, c)$. Thus by Fact 4.5, there is $e = (e_0, e_1, \dots, e_m) \in M^{n+m}$ such that

$$(e_0, e_1 \dots, e_m) \in X \cap U \text{ and } \mathcal{M}_{\mathrm{Sk}} \models \chi(e_0, e_1 \dots, e_m, c).$$

Thus $(e_1, \dots, e_m) = t(e_0)$ and we can set $a' = e_0$. □

Corollary 4.9. *Let T' be an open core of T. Then T' is an open core of T_{Sk}.*

Proof. Let $\mathcal{L}'$ be the language of T'. Without loss of generality, we can assume that $\mathcal{L}' \cap \mathcal{L}_{\mathrm{Sk}} = \emptyset$. Let $\mathcal{L}^*$ be the union of $\mathcal{L}'$ and $\mathcal{L}_{\mathrm{Sk}}$. Let $\mathcal{M}_{\mathrm{Sk}} \models T_{\mathrm{Sk}}$. Since T' is an open core of T, we can expand $\mathcal{M}_{\mathrm{Sk}}$ to a model $\mathcal{M}^*$ of the $\mathcal{L}^*$-theory $T' \cup T_{\mathrm{Sk}}$. Let $X \subseteq M^n$ be an open set given by

$$X := \{a \in M^n \ : \ \mathcal{M}_{\mathrm{Sk}} \models \varphi(a, c)\},$$

where φ is an $\mathcal{L}_{\mathrm{Sk}}$-formula with parameters $c \in M^m$. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}^*$ that is $|\mathcal{L}|^+$-saturated. Set $Y := \{a \in N^n \ : \ \mathcal{N} \models \varphi(a, c)\}$. Since X is open, so is Y. By Theorem 4.8 there is an $\mathcal{L}'$-formula $\psi(x, y)$ such that there is $d \in M^\ell$ with $Y = \{a \in N^n \ : \ \mathcal{N} \models \psi(a, d)\}$. Since $\mathcal{M}^* \preceq \mathcal{N}$, there is $d' \in M^\ell$ such that $X = \{a \in M^n \ : \ \mathcal{M}^* \models \psi(a, d')\}$. Thus X is $\mathcal{L}'$-definable. □

Proof of Theorem 4.1. We are now able to complete the proof of Theorem 4.1 using the same argument as in [4, Corollary 4.9]. Suppose $T \models \mathrm{DC} + \mathrm{UF}$ and let T' be an open core of T. Set T_0 be the Morleyization of T in a language $\mathcal{L}_0$. For every $n > 0$, we will now construct a language $\mathcal{L}_n$ and an $\mathcal{L}_n$-theory T_n such that

(1) T_n has quantifier-elimination,
(2) $T_n \models \mathrm{UF}$, and
(3) T' is an open core of T_n.

Let $n \geq 0$, and suppose we already constructed a language $\mathcal{L}_n$ and an $\mathcal{L}_n$-theory T_n with the properties (1)-(3). Let Φ be the set of $\mathcal{L}_n$-formulas $\varphi(x, y)$ such that $|y| = 1$ and

$$T_n \models \forall x \exists! y \; \varphi(x, y),$$

For each $\varphi(x, y) \in \Phi$ we introduce a new function symbol f_φ of arity $|x|$. Let $\widetilde{\mathcal{L}}$ be the union of the $\mathcal{L}_n$ and $\{f_\varphi \; : \; \varphi \in \Phi\}$. Let $\widetilde{T}$ be the union of T_n with the set of all $\widetilde{\mathcal{L}}$-sentence of the form

$$\forall x \forall y (f_\varphi(x) = y) \leftrightarrow \varphi(x, y),$$

where $\varphi \in \Phi$. Since $\widetilde{T}$ is an expansion of T_n by definitions, it is easy to check that $\widetilde{T}$ satisfies (1)-(3). Now consider the model companion $(\widetilde{T})_{\mathrm{Sk}}$ of the Skolem expansion $(\widetilde{T})_+$. Let T_{n+1} be the Morleyization of $(\widetilde{T})_{\mathrm{Sk}}$ in an expanded language $\mathcal{L}_{n+1}$. We know $T_{n+1} \models \mathrm{UF}$ by Fact 4.4. By Corollary 4.9, the theory T' is an open core of T_{n+1}.

Now set $T^\infty_{\mathrm{Sk}} := \bigcup_{i \in \mathbb{N}} T_n$. From the construction, it follows immediately that T' is an open core of T^∞_{Sk} and that T^∞_{Sk} has definable Skolem functions. □

References

[1] Gareth Boxall and Philipp Hieronymi, Expansions which introduce no new open sets. *J. Symb. Log.*, 77(1):111-121, (2012).

[2] Alfred Dolich, Chris Miller, and Charles Steinhorn, Structures having o-minimal open core. *Trans. Am. Math. Soc.*, 362(3):1371–1411, (2010).

[3] Alfred Dolich, Chris Miller, and Charles Steinhorn, Expansions of o-minimal structures by dense independent sets. *Ann. Pure Appl. Logic*, 167(8):684–706, (2016).

[4] Alex Kruckman and Nicholas Ramsey, Generic expansion and Skolemization in NSOP1 theories *Ann. Pure Appl. Logic.*, 169(8): 755-774, (2018).

[5] Chris Miller and Patrick Speissegger, Expansions of the real line by open sets: o-minimality and open cores. *Fund. Math.*,162(3):193–208 (1999).

[6] Saharon Shelah. Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory, *Ann. Pure Appl. Logic*, 3 (3): 271-362 (1971)

[7] Pierre Simon, On dp-minimal structures *J. Symb. Logic*, 76(2): 448–460 (2011)

[8] Pierre Simon, A guide to NIP theories. *Lect. Notes Log.*, vol. 44, Cambridge University Press, 2015.

[9] Peter Winkler, Model completeness and skolem expansions. in *Model Theory and Algebra*, Springer (pp. 408–463), 1975.

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801

E-mail address: atb2@illinois.edu

Department of Mathematics & Statistics, McMaster University, Hamilton Hall, 1280 Main Street West, Hamilton, Ontario, Canada, L8S 4K1

E-mail address: caulfiee@mcmaster.ca

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801

E-mail address: phierony@illinois.edu

URL: http://faculty.math.illinois.edu/~phierony