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Bounding Zolotarev Numbers Using Faber Rational Functions

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Abstract
By closely following a construction by Ganelius, we use Faber rational functions to derive tight explicit bounds on Zolotarev numbers. We use our results to bound the singular values of matrices, including complex-valued Cauchy matrices and Vandermonde matrices with nodes inside the unit disk. We construct Faber rational functions using doubly connected conformal maps and use their zeros and poles to supply shift parameters in the alternating direction implicit method.

Keywords Faber rational functions · Zolotarev · Conformal maps · Singular values · Rational approximation

Mathematics Subject Classification 26C15 · 30C20

1 Introduction

The Zolotarev number from rational approximation theory is given by [35]:

$$Z_n(E, F) = \inf_{s_n \in \mathcal{R}_{n,n}} \frac{\sup_{z \in E} |s_n(z)|}{\inf_{z \in F} |s_n(z)|}, \qquad n \geq 0, \tag{1.1}$$

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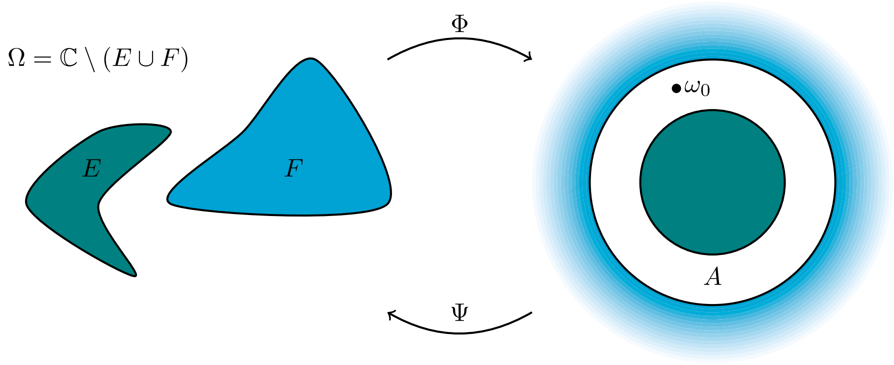

Fig. 1 We mainly focus on the situation when E and F are disjoint and compact sets in the complex plane. Here, $\Phi : \Omega \to A$ is the conformal map that transplants Ω onto an annulus $A = \{z \in \mathbb{C} : 1 < |z| < h\}$ with $h = \exp(1/\mathrm{cap}(E, F))$. The location $\omega_0 \in \mathbb{C}$ is the pole of the inverse map $\Psi = \Phi^{-1}$

where $\mathcal{R}_{n,n}$ denotes the set of rational functions of type (n, n) and $E, F \subset \mathbb{C}$ are disjoint sets in the complex plane. Due to the infimum over $\mathcal{R}_{n,n}$ in (1.1), we know that $Z_n(E, F) \leq \sup_{z\in E} |s_n(z)|/\inf_{z\in F} |s_n(z)|$ for any $s_n \in \mathcal{R}_{n,n}$. In this paper, we closely follow a construction by Ganelius [12–14] to derive Faber rational functions and use them to derive explicit upper bounds on $Z_n(E, F)$ when E and F are such that $\mathbb{C}\setminus F$ is open and simply connected and E is a compact, simply connected subset of $\mathbb{C}\setminus F$. The Faber rational functions are rational analogues of the Faber polynomials [10, 22]. A formal definition and a list of relevant properties can be found in [14, Sec. 3]. Throughout this paper, we assume that the boundaries of E and F are rectifiable Jordan curves. To be concrete, the main situation we focus on is when:

(A1) E and F are disjoint, simply connected, compact sets (see Fig. 1).

In Sect. 5, we discuss two other types of sets E and F: (A2) $\mathbb{C}\setminus F$ is a bounded domain containing E (see Fig. 7) and (A3) F is an unbounded domain and E is a compact domain contained in $\mathbb{C}\setminus F$ (see Fig. 8).

The Zolotarev number, $Z_n(E, F)$, has applications for explicitly bounding the singular values of matrices [6], solving Sylvester matrix equations [23], the computation of the singular value decomposition of a matrix [25], and the solution of generalized eigenproblems [19]. It is often important to have a tight explicit bound as well as the zeros and poles of a rational function that attains the bound. Explicit and tight bounds on $Z_n(E, F)$ are available in the literature when (i) E and F are disjoint intervals [6, Sec. 3.2] and (ii) E and F are disjoint disks [32]. Faber rationals offer a general approach for obtaining explicit bounds.

It is immediate that $Z_0(E, F) = 1$ and $Z_{n+1}(E, F) \leq Z_n(E, F)$ for $n \geq 0$. As a general rule, the number $Z_n(E, F) \to 0$ rapidly as $n \to \infty$ if E and F are disjoint, compact, and well-separated. More precisely, for disjoint sets E and F, a lower bound on $Z_n(E, F)$ as well as its asymptotic behavior is known [15]:

$$Z_n(E,F) \geq h^{-n}, \qquad \lim_{n\to\infty} (Z_n(E,F))^{1/n} = h^{-1}, \qquad h = \exp\left(\frac{1}{\mathrm{cap}(E,F)}\right), \tag{1.2}$$

where $\mathrm{cap}(E, F)$ is the condenser capacity of a condenser with plates E and F [29, Thm. VIII. 3.5]. Our goal is to derive explicit upper bounds on Zolotarev numbers of the form:

$$Z_n(E,F) \leq K_{E,F} h^{-n}, \qquad n \geq 0,$$

where $K_{E,F}$ is a constant that depends on the geometry of E and F. When E and F are disjoint disks, it is known that $K_{E,F}$ can be taken to be 1 [32] (see Sect. 2) and when E and F are disjoint real intervals, $K_{E,F}$ can be taken to be 4 [6]. To the authors' knowledge, the best previous explicit upper bound for sets satisfying one of (A1)–(A3) is $Z_n(E, F) \leq 4000n^2h^{-n}$ [12].

1.1 The Total Rotation of a Domain

Our upper bound on $Z_n(E, F)$ involves the so-called total rotation of the domains E and F [11, 28].

Definition 1.1 Let $E \subset \mathbb{C}$ be a simply connected domain with a rectifiable Jordan curve boundary. The total rotation of E is defined as:

$$\mathrm{Rot}(E) = \frac{1}{2\pi} \int_{\partial E} |\mathrm{d}\theta(s)|\,, \tag{1.3}$$

where $\theta(s)$ for $s \in (0, 1)$ is the angle of the boundary tangent of E (which exists for almost every $s \in (0, 1)$).

If $\theta(s)$ can be extended to a function of bounded variation to $s \in [0, 1]$, then $\mathrm{Rot}(E) < \infty$. For any simply connected domain, we note that $\mathrm{Rot}(E) \geq 1$. When E is a polygon, $2\pi\mathrm{Rot}(E)$ equals the sum of the absolute values of E's exterior angles. Moreover, when E is a convex domain, $\mathrm{Rot}(E) = 1$ [2, p. 6].

1.2 Main Theorem

We are now ready to state our main theorem.

Theorem 1.2 (Main Theorem) *Let $E, F \subset \mathbb{C}$ be disjoint, simply connected, compact sets with rectifiable Jordan boundaries. Then, for $h = \exp(1/\mathrm{cap}(E, F))$, we have*

$$Z_n(E,F) \leq (2\mathrm{Rot}(E)+2)(2\mathrm{Rot}(F)+2)h^{-n} + \mathcal{O}(h^{-2n}), \qquad \textit{as } n \to \infty,$$

where $\mathrm{Rot}(E)$ *and* $\mathrm{Rot}(F)$ *are the total rotation of the boundaries of the domains E and F, respectively. If, in addition, E and F are convex sets, then we simply have $Z_n(E, F) \leq 16h^{-n} + \mathcal{O}(h^{-2n})$ as $n \to \infty$.*

The rational function that we use to derive our upper bound in (1.4) below is the so-called Faber rational function associated with the sets E and F (see (3.8)). Theorem 1.2 shows that the lower bound on $Z_n(E, F)$ in (1.2) is sharp up to a constant. In particular, for disjoint, simply connected, compact sets $E, F \subset \mathbb{C}$ with rectifiable Jordan boundaries, we have

$$1 \leq \limsup_{n\to\infty} \frac{Z_n(E, F)}{h^{-n}} \leq (2\mathrm{Rot}(E) + 2)(2\mathrm{Rot}(F) + 2).$$

We do not believe that the upper bound of $(2\mathrm{Rot}(E) + 2)(2\mathrm{Rot}(F) + 2)$ is sharp and highlight in our derivation where there might be possible improvements.

The actual upper bound that we derive is an inelegant expression that is nevertheless explicit and computable. With the same assumptions as in Theorem 1.2, we have

$$Z_n(E, F) \leq \left(\frac{\frac{M_n(E,F)M_n(F,E)}{1-h^{-2n}} + \frac{32nM_n(E,F)h^{-n}}{(1-C_nh^{-n})^2(1+C_nh^{-n})}}{\max\left\{0, 1 - \frac{M_n(E,F)M_n(F,E)}{1-h^{-2n}}h^{-n} - \frac{M_n(E,F)}{1-C_nh^{-n}}h^{-n} - h^{-2n}\right\}} \right) h^{-n} \tag{1.4}$$

for any $n > N_0$. Here, $N_0 = \max\{1 + 1/(h - 1), \log(x_1)/\log(h)\}$, $M_n(E, F) = 2\mathrm{Rot}(E)+2h^{-n}\mathrm{Rot}(F)+2h^{-n}+2$, $C_n = 1+M_n(E, F)/(1-h^{-2n})$. The expression in the denominator is positive for $n > \log(x_1)/\log(h)$, where x_1 is the largest real root of a seventh-degree polynomial with coefficients depending on $\mathrm{Rot}(E)$ and $\mathrm{Rot}(F)$; in the case where E and F are convex, $x_1 \approx 29.9$ (see Lemma A.1).

Note that as $n \to \infty$, the denominator in (1.4) takes the limiting value of 1 . In particular, the bound in (1.4) has the correct geometric decay to zero as $n \to \infty$. If the bound in (1.4) turns out to > 1 (which it may for small n), then one is welcome to take $Z_n(E, F) \leq 1$ instead. Figure 2 illustrates how the bounds behave for convex sets with varying values of h.

1.3 Conformal Mapping of Doubly Connected Sets

The construction of Faber rationals requires conformal maps of doubly connected sets. A domain $\Omega \subset \mathbb{C}$ is said to be doubly connected if between any two points in Ω there are two distinct paths, i.e., two paths that cannot be smoothly deformed into each other. Any doubly connected domain, except for regions conformally equivalent to a punctured disk and a punctured plane, is conformally equivalent to $A = \{z \in \mathbb{C} : 1 < |z| < h\}$ for some $h > 1$ [9, Ch.1, sec.7]. When $E, F \subset \mathbb{C}$ are as in Theorem 1.2, $\Omega = \mathbb{C}\backslash(E \cup F)$ is doubly connected and can be conformally mapped to an annulus, i.e.,

$$\Phi : \Omega \to A, \qquad A = \{z \in \mathbb{C} : 1 < |z| < h\}. \tag{1.5}$$

Since conformal maps preserve the logarithmic capacity of two plate condensers and the capacity of A is $1/\log(h)$ [20], the outer radius in (1.5) is $h = \exp(1/\mathrm{cap}(E, F))$ (see (1.2)). If E and F are disjoint polygons, then the conformal map, Φ, can be

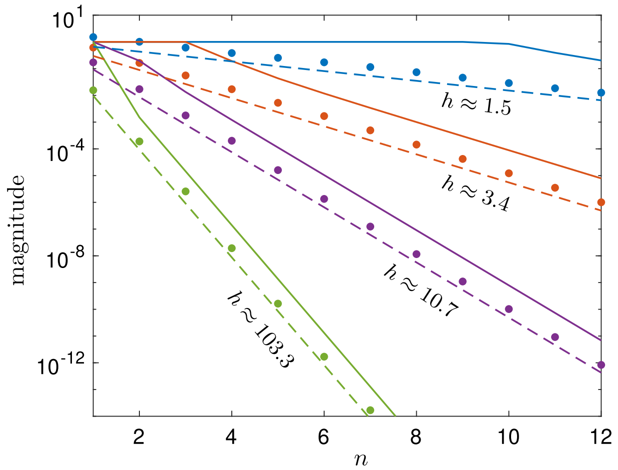

Fig. 2 Bounds on $Z_n(E_\alpha, -E_\alpha)$ with $E_\alpha = \{z \in \mathbb{C} : Re(z) \in [-.4 - \alpha, .4 - \alpha], \ Im(z) \in [-.6, .6]\}$, where $\alpha = .45$ (blue), .6 (orange), 1 (purple), 3 (green). As α grows, $h = \exp(-1/\text{cap}(E_\alpha, -E_\alpha))$ grows, and $Z_n(E_\alpha, -E_\alpha)$ decays more rapidly. The solid lines are the bounds from Theorem 1.2, combined with the trivial bound $Z_n(E, F) \leq 1$. The dotted lines are the lower bounds of $Z_n(E_\alpha, -E_\alpha) \geq h^{-n}$ in (1.2). The dots are computed by first constructing the Faber rational function $r_n(z)$ associated with $(E_\alpha, -E_\alpha)$ and then computing the $\max_{z\in E_\alpha} |r_n(z)|/ \min_{z\in -E_\alpha} |r_n(z)|$ (Color figure online)

constructed as a doubly connected Schwarz–Christoffel mapping [21]. The inverse conformal map is denoted by $\Psi = \Phi^{-1} : A \to \Omega$.

1.4 Paper Summary

This paper is structured as follows: In Sect. 2, we briefly describe the simplest case when the conformal map Φ is a Möbius transform. In Sect. 3, we describe the general construction of a Faber rational function associated with sets E and F satisfying the assumptions in Theorem 1.2. In Sect. 4, we bound Faber rational functions to obtain the explicit upper bound on $Z_n(E, F)$ given in (1.4). We extend our results to cases (A2) and (A3) in Sect. 5. In Sect. 6, we provide numerical details, and we provide some examples of applications in Sect. 7.

2 When Φ is a Möbius Transformation

Suppose that $E, F \subset \mathbb{C}$ are such that there exists a Möbius transform $\boldsymbol{\Phi} : \Omega \to A$ in (1.5). Since $\boldsymbol{\Phi}(z) = (az + b)/(cz + d)$ with $ad - bc \neq 0$, we find that Φ is a type $(1, 1)$ rational function. One can immediately verify that $\Phi^n \in \mathcal{R}_{n,n}$, $|\Phi(z)| \leq 1$ for $z \in E$, and $|\Phi(z)| \geq h$ for $z \in F$. This means that $Z_n(E, F) = h^{-n}$ as

$$h^{-n} \leq Z_n(E, F) \leq \frac{\sup_{z\in E} |\Phi^n(z)|}{\inf_{z\in F} |\Phi^n(z)|} \leq h^{-n},$$

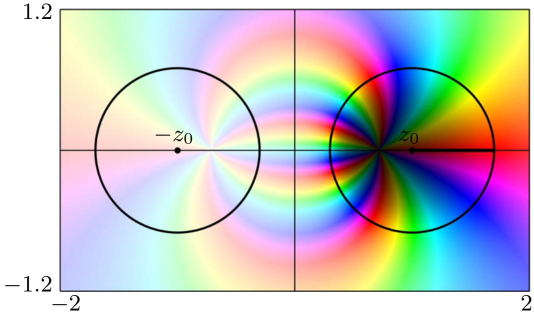

Fig. 3 A phase plot of the type (5, 5) Faber rational function on two disjoint disks, $E = \{z \in \mathbb{C} : |z-1| \leq .7\}$ and $F = -E$. Since there is a Möbius transformation from $\mathbb{C}\backslash(E \cup F)$ to an annulus, $Z_n(E, F)$ is known explicitly

where the lower bound is from (1.2). Moreover, the rational function that attains the value of $Z_n(E, F)$ is known because it is simply given by Φ^n. (An alternative proof of the optimality of Φ^n for $Z_n(E, F)$ is the near-circularity criterion [32].)

When E and F are disjoint disks, there is a Möbius transform that maps Ω to an annlulus [32]. For example, suppose that $E = \{z \in \mathbb{C} : |z - z_0| \leq \eta_0\}$ and $F = -E$ with $0 < \eta_0 < z_0$ and $z_0, \eta_0 \in \mathbb{R}$. Then, the Möbius transform

$$\Phi(z) = \frac{z_0 + \eta_0 + c}{z_0 + \eta_0 - c}\frac{z - c}{z + c}, \qquad c = \sqrt{z_0^2 - \eta_0^2}$$

maps $\Omega = \mathbb{C}\backslash(E \cup F)$ onto the annulus $A = \{z \in \mathbb{C} : 1 < |z| < h\}$ with $h = (z_0 + c)/(z_0 - c)$. Therefore, we know that $Z_n(E, F) = (z_0 - c)^n/(z_0 + c)^n$, and this value is attained by the rational function $r_n(z) = \Phi^n(z)$ (see Fig. 3).

3 Constructing Faber Rational Functions

When Φ in (1.5) is not a Möbius transform, we find that $\Phi^n \notin \mathcal{R}_{n,n}$. Therefore, Φ^n is not immediately useful for bounding $Z_n(E, F)$. However, we still expect Φ^n to be $\mathcal{O}(h^n)$ near F and $\mathcal{O}(1)$ near E. Thus, the idea is to construct a rational function from Φ^n by "filtering" Φ^n using the Faber operator associated with $\Psi = \Phi^{-1}$ [2] (see Fig. 4). The rational function obtained from Φ^k after applying this "filtering" process is called the Faber rational associated with E and F. The Faber operator was first introduced as a means for constructing polynomial approximations known as Faber polynomials [10, 22].

We now describe how one constructs a Faber rational, which closely follows the procedure in [12]. There are two main steps: (1) constructing a function, $R_n(z)$, defined on $\mathbb{C}\backslash F$ with precisely n zeros, and (2) constructing a rational function, $r_n(z)$, of type (n, n). Both steps are accomplished by taking Cauchy integrals along the boundaries of E and F.

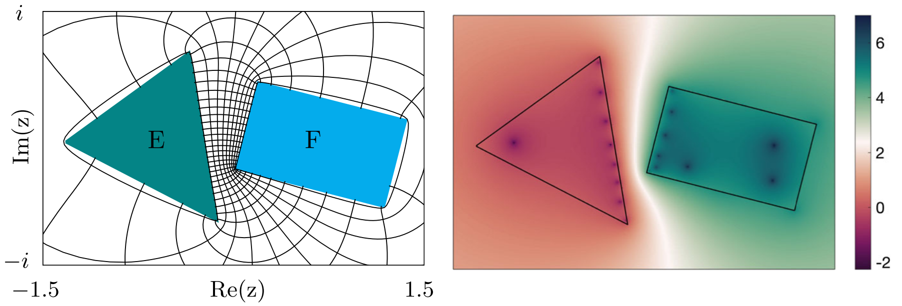

Fig. 4 Left: A plot of the conformal map $\Psi = \Phi^{-1}$, where Φ maps $\mathcal{C}\backslash E \cup F$ to the annulus $A = \{z \in \mathbb{C} : 1 < |z| < h\}$. Right: The magnitude of the degree $(9, 9)$ Faber rational function is plotted on a logarithmic scale. As n increases, the Faber rational function grows increasingly larger on F and smaller on E, making it useful for bounding the Zolotarev number $Z_n(E, F)$

3.1 Step 1: Constructing a function $\mathbf{R_n(z)}$ with n Zeros Near E

Let $\gamma : [0, 1] \to \Omega$ be a positively oriented parameterization of the boundary E. We can define the following "filtered" function inside E:

$$R_n(z) := \frac{1}{2\pi i} \int_\gamma \frac{\Phi^n(\zeta)\mathrm{d}\zeta}{\zeta - z}, \qquad z \in E. \tag{3.1}$$

The holomorphic function $R_n(z)$ is initially defined inside E. If it is possible to extend Φ homomorphically to the whole interior of E, then $R_n(z) = \Phi^n(z)$ there, but this occurs only in exceptional cases. Intuitively one might expect $R_n(z)$ to behave like a degree n polynomial whose zeros are all in E. This is because $R_n(z)$ has boundary values on ∂E close to those of Φ^n, which are the same as the function z^n on the boundary of the unit circle. We show this using Villat's strategy for solving the Dirichlet problem in an annulus [1, Section 56].

Lemma 3.1 *Let $E, F \in \mathbb{C}$ be sets satisfying the assumptions in Theorem 1.2. For $n \geq 1$, the function in (3.1) satisfies*

$$\sup_{z\in E} |R_n(z)| \leq \frac{M_n(E, F)}{1 - h^{-2n}}, \qquad M_n(E, F) = 2(\mathrm{Rot}(E) + h^{-n}\mathrm{Rot}(F) + 1 + h^{-n}),$$

where $\mathrm{Rot}(E)$ *and* $\mathrm{Rot}(F)$ *are defined in* (1.3)*, and h is defined in* (1.2).

Proof Let $\Psi : A \to \Omega$ be the inverse conformal map to Φ, which is meromorphic inside A with a simple pole at $\omega_0 = \Phi(\infty) \in A$ (see Fig. 1). For any $z \in E$, we can use a change of variables to write

$$R_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{\Phi^n(\zeta)\mathrm{d}\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{|\omega|=\rho} \frac{\omega^n \Psi'(\omega)\mathrm{d}\omega}{\Psi(\omega) - z}, \qquad 1 \leq \rho \leq h,$$

One can take $\rho = 1$ and $\rho = h$ since the integrand extends continuously to the boundary by Caratheodory's theorem [18, Thm. 13.2.3]. We note that the logarithmic derivative $\frac{\mathrm{d}}{\mathrm{d}\omega}\log(\Psi(\omega)-z)$ has a simple pole at ω_0 with residue -1. If we set $G_z(\omega)=\omega\Psi'(\omega)/(\Psi(\omega)-z)$, then $G_z(\omega)$ can be written as the sum of a term of the form $(\omega-\omega_0)^{-1}$ and a doubly infinite convergent Laurent series.

Setting

$$G_z(\omega)=\frac{-\omega_0}{\omega-\omega_0}+\sum_{k=-\infty}^{\infty}a_k(z)\omega^k,$$

we have

$$a_k(z)=\frac{1}{2\pi i}\int_{|\omega|=\rho}\frac{1}{\omega^{k+1}}\left(G_z(\omega)+\frac{\omega_0}{\omega-\omega_0}\right)\mathrm{d}\omega$$

for any $1<|\rho|<h$ with $|\rho|\neq|\omega_0|$.

Now, we observe that

$$\frac{1}{2\pi i}\int_{|\omega|=1}\frac{1}{\omega^{k+1}}\frac{\omega_0}{\omega-\omega_0}\mathrm{d}\omega=\begin{cases}-\omega_0^{-k}, & k\geq 0,\\ 0, & k<0,\end{cases}$$

and

$$\frac{1}{2\pi i}\int_{|\omega|=h}\frac{1}{\omega^{k+1}}\frac{\omega_0}{\omega-\omega_0}\mathrm{d}\omega=\begin{cases}\omega_0^{-k}, & k\geq 0,\\ 2\omega_0^{-k}, & k<0,\end{cases}$$

because when integrating over the outer boundary of the annulus there is a nonzero residue at 0 for $k<0$ and a residue at ω_0 for all k.

Now consider $n>0$ and compare $a_{-n}(z)$ to $\overline{a_n(z)}$:

$$\begin{aligned}a_{-n}(z)+\overline{a_n(z)}&=\frac{1}{\pi}\int_0^{2\pi}\operatorname{Re}\left(G_z(e^{i\theta})\right)e^{in\theta}\mathrm{d}\theta-\overline{\omega_0}^{-n},\\ a_{-n}(z)h^{-n}+\overline{a_n(z)}h^n&=\frac{1}{\pi}\int_0^{2\pi}\operatorname{Re}\left(G_z(he^{i\theta})\right)e^{in\theta}\mathrm{d}\theta+2h^{-n}\omega_0^n+h^n\overline{\omega_0}^{-n}.\end{aligned}$$

Since $R_n(z)=a_{-n}(z)$, we find that for $z\in E$ we have

$$\begin{aligned}R_n(z)=&\frac{1}{\pi(1-h^{-2n})}\int_0^{2\pi}e^{in\theta}\left(\operatorname{Re}(G_z(e^{i\theta}))-h^{-n}\operatorname{Re}(G_z(he^{i\theta}))\right)\mathrm{d}\theta\\ &-\frac{(2h^{-2n}\omega_0^n+2\overline{\omega_0}^{-n})}{1-h^{-2n}}.\end{aligned}$$

The geometric significance of this integrand is revealed by the identity:

$$\operatorname{Re}\left(G_z(e^{i\theta})\right)=\operatorname{Im}\left(iG_z(e^{i\theta})\right)=\frac{\mathrm{d}}{\mathrm{d}\theta}\operatorname{Im}\left(\log(\Psi(e^{i\theta})-z)\right)=\frac{\mathrm{d}}{\mathrm{d}\theta}\operatorname{Arg}(\Psi(e^{i\theta})-z).$$

Therefore, we find that

$$\left|\int_0^{2\pi} e^{in\theta}\mathrm{Re}\left(G_z(e^{i\theta})\right)\mathrm{d}\theta\right| \le \int_0^{2\pi}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Arg}\left(\Psi(e^{i\theta})-z\right)\right|\mathrm{d}\theta \le 2\pi\mathrm{Rot}(E), \tag{3.2}$$

where the last inequality follows since the total variation in argument around a closed curve as measured from a point not on the curve is bounded by total rotation [11, (1.6.14)]. Similarly, the same integral as (3.2) with integrand $e^{in\theta}\mathrm{Re}(G_z(he^{i\theta}))$ is bounded by $2\pi\mathrm{Rot}(F)$. The upper bound on $\sup_{z\in E}|R_n(z)|$ follows by noting that $1<|\omega_0|<h$.

Now we address the contribution of the pole term

$$p_n = -2\frac{h^{-2n}\omega_0^n + \overline{\omega_0}^{-n}}{1-h^{-2n}}.$$

Note that since $1<|\omega_0|<h$, we have that

$$|h^{-2n}\omega_0^n| < h^{-n}, \qquad |\overline{\omega_0}^{-n}| < 1.$$

The $\overline{\omega_0}^{-n}$ term gives the larger contribution to $R_n(z)$ and could be bounded by a quantity decaying exponentially if $|\Phi(\infty)|$ is bounded away from 1. For the statement of the lemma, we take the simplest bound $|p_n| \le (2(1+h^{-n}))/(1-h^{-2n})$. □

Lemma 3.1 simplifies when E and F are, in addition, convex sets because we have $\mathrm{Rot}(E)=\mathrm{Rot}(F)=1$. We obtain

$$\sup_{z\in E}|R_n(z)| \le \frac{4(1+h^{-n})}{1-h^{-2n}}, \quad n\ge 0. \tag{3.3}$$

Previously, it was shown by Ganelius that $\sup_{z\in E}|R_n(z)| \le 4e^2 n$ [12]. For most practical n and h, the bound in Lemma 3.1 is sharper than the bound in [12]. For example, for convex sets E, F, the bound in (3.3) is an improvement over $4e^2n$ for all $n\ge 1$ if $h > e^2/(e^2-1) \approx 1.157$. Similarly, when $h>1.072$, the bound is an improvement for $n\ge 2$, and for any $n\ge 3$ when $h>1.047$.

There are opportunities to improve the bound in Lemma 3.1 as (3.2) can be weak, especially when $h\approx 1$. The bound in (3.2) ignores potential cancellation in the integral $\int e^{in\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Arg}(\Psi - z)\mathrm{d}\theta$. However, as the point z approaches the boundary of E, the function $\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Arg}(\Psi - z)$ tends to a delta function centered at the value of θ corresponding to the limit on the boundary, which is π if the boundary point is smooth. For this reason, we suspect that one can improve the bound in Lemma 3.1 by a factor of about 2.

By analytic continuation, the definition of R_n can now be extended to $\Omega = \mathbb{C}\backslash(E\cup F)$. Fix $z\in\Omega$. First, we continuously deform the contour γ to a contour γ' that is contained in Ω and encircles z. By continuously deforming the contour γ' back to γ

plus a path traversed in both directions extending to an arbitrarily small circle around z, we find that

$$R_n(z) = \frac{1}{2\pi i}\int_{\gamma'} \frac{\Phi^n(\zeta)\mathrm{d}\zeta}{\zeta - z} = \Phi^n(z) + \frac{1}{2\pi i}\int_{\gamma} \frac{\Phi^n(\zeta)\mathrm{d}\zeta}{\zeta - z}, \qquad z \in \Omega.$$

Here, the term $\Phi^n(z)$ appears because it is the average value of the Cauchy integral over an arbitrarily small circle around z. Since $|\Phi^n(z)| < h^n$ for $z \in \Omega$, we find that R_n is a bounded function in Ω.

Since the Cauchy transform of a continuous function on a closed contour can be used to define two distinct holomorphic functions—one in the interior of the region bounded by the contour and the other on the exterior—we can write

$$\begin{aligned}
\mathcal{C}^+_{\partial E}(\Phi^n)(z) &= \frac{1}{2\pi i}\int_{\gamma} \frac{\Phi^n(\zeta)\mathrm{d}\zeta}{\zeta - z}, \qquad z \text{ inside of } \gamma,\\
\mathcal{C}^-_{\partial E}(\Phi^n)(z) &= \frac{1}{2\pi i}\int_{\gamma} \frac{\Phi^n(\zeta)\mathrm{d}\zeta}{\zeta - z}, \qquad z \text{ outside of } \gamma,
\end{aligned}$$

where the subscript indicates that the integral is taken over the boundary of E. Therefore, the function $R_n(z)$ can be expressed as

$$R_n(z) = \begin{cases} \mathcal{C}^+_{\partial E}(\Phi^n)(z), & z \in E, \\ \Phi^n(z) + \mathcal{C}^-_{\partial E}(\Phi^n)(z), & z \in \mathbb{C}\backslash E \cup F). \end{cases} \tag{3.4}$$

To further emphasize the interpretation that $R_n(z)$ is a filtered version of $\Phi^n(z)$, we show that $R_n(z)$ is relatively close to $\Phi^n(z)$ for $z \in \Omega$.

Lemma 3.2 *Let E, $F \in \mathbb{C}$ be sets satisfying the assumptions in Theorem 1.2, and let $\Omega = \mathbb{C}\backslash(E \cup F)$. Then, $R_n(z)$ in (3.4) satisfies*

$$\sup_{z\in\Omega} \left|R_n(z) - \Phi^n(z)\right| \leq 1 + \sup_{z\in E} |R_n(z)|\,.$$

Proof From the definition of $R_n(z)$ for $z \in \Omega$ (see (3.4)), we just need to bound $|\mathcal{C}^-_{\partial E}(\Phi^n)(z)|$. Note that $\mathcal{C}^-_{\partial E}(\Phi^n)(z)$ is a bounded analytic function outside of γ whose maximum modulus is attained on the curve γ. By the Sokhotski–Plemelj theorem [20, (14.1-9)], we find that $\mathcal{C}^-_{\partial E}(\Phi^n)(z_0) = \mathcal{C}^+_{\partial E}(\Phi^n)(z_0) - \Phi^n(z_0)$ for $z_0 \in \partial E$. Therefore, $|\mathcal{C}^-_{\partial E}(\Phi^n)(z)| \leq \sup_{z\in E} |R_n(z)| + 1$. □

Note that the bound in Lemma 3.2 is the worst possible based on the maximum difference attained for $z \in \partial E$. The bound can be improved for z sufficiently far from E using the decay of $\mathcal{C}^-_{\partial E}(\Phi^n)(z)$.

Lemma 3.2 allows us to show that all the zeros of R_n lie in E or within a small neighborhood of E. Rouché's theorem says that the winding numbers of Φ^n and R_n around a closed curve Γ will be equal, provided that $|\Phi^n(z) - R_n(z)| < |\Phi^n(z)|$ for

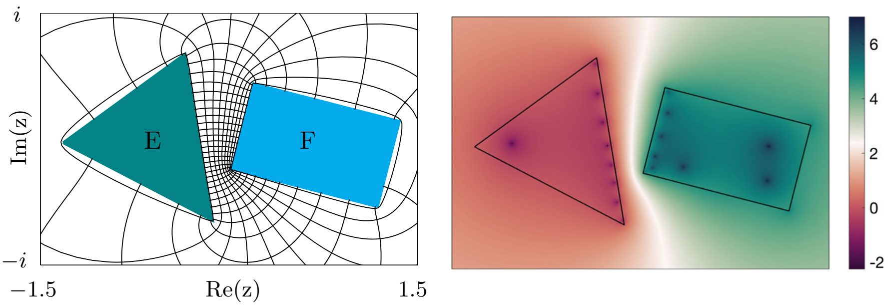

Fig. 5 Left: A plot of the conformal map $\Psi = \Phi^{-1}$, where Φ maps $\mathcal{C}\backslash E \cup F$ to the annulus $A = \{z \in \mathbb{C} : 1 < |z| < h\}$. Right: The magnitude of the type $(9, 9)$ Faber rational function is plotted on a logarithmic scale. As n increases, the Faber rational function grows increasingly larger on F and smaller on E, making it useful for bounding the Zolotarev number $Z_n(E, F)$

z on Γ [18, Theorem 5.3.1]. By Lemma 3.2, the theorem applies on any closed curve Γ in Ω winding once around E such that $1 + \sup_{z\in E} |R_n(z)| < |\Phi^n(z)|$ for z on Γ. Such a curve Γ can always be found when the bound $1+\sup_{z\in E} |R_n(z)| < h^n$, say, by taking the image of Γ to be an appropriate level set of $|\Phi^n|$. The map Φ^n has winding number precisely n around Γ by definition (though it is not defined in E) and hence so does R_n. Since R_n is analytic inside Γ, it has n zeros (counting multiplicities) inside Γ. Moreover, the same reasoning shows that R_n has no additional zeros outside of Γ in Ω (Fig. 5).

At this point, we have not attempted to address whether or under what conditions the roots of R_n actually lie in the domain E. A paper of Goodman [16] gives an example of a domain and a Faber polynomial with a zero that lies outside the convex hull of the domain. We believe that if E is a convex set, then the zeros of $R_n(z)$ lie in E, analogous to Faber polynomials [22, Thm. 2].

3.2 Step 2: Constructing a Faber Rational Function

While R_n has precisely n zeros, it is typically not a rational function. We must “filter" R_n again to obtain a rational function. Let $0 < \delta < 1$, and let $\eta : [0, 1] \to \Omega$ be a curve that is close to the boundary of F with $|\Phi(\eta(t))| \geq h - \delta$ for $t \in [0, 1]$ and winds around F once in the counterclockwise direction.[1] By Lemma 3.2, R_n is close to Φ^n on η, and $|\Phi^n|$ is close to h^n on η, so make sure that δ is sufficiently small to avoid encircling any zeros of R_n. Therefore, we can assume that $1/R_n$ is analytic on the curve η (see (3.4)). We can construct analytic functions inside and outside of η (the inside of η contains F) as

[1] The requirement that $\delta > 0$ is a technical necessity as $R_n(z)$ is defined for $z \in \mathbb{C}\backslash F$. Later, we take $\delta \to 0$ so conceptually one may prefer to think of η as a parameterization of the boundary of F.

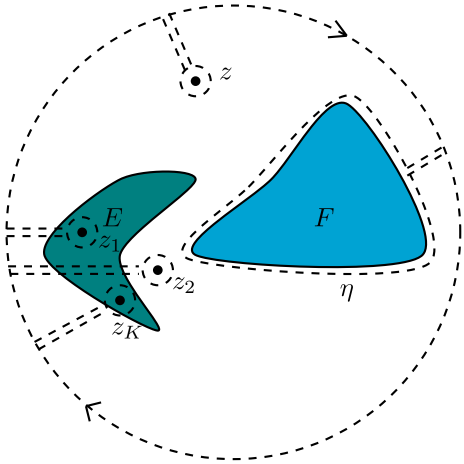

Fig. 6 The contour Γ in the proof of Lemma 3.3

$$\begin{aligned}\mathcal{C}^+_\eta(1/R_n)(z) &= \frac{1}{2\pi i}\int_\eta \frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)}, \qquad z \text{ inside of } \eta,\\ \mathcal{C}^-_\eta(1/R_n)(z) &= \frac{1}{2\pi i}\int_\eta \frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)}, \qquad z \text{ outside of } \eta.\end{aligned} \tag{3.5}$$

It is possible to give an exact expression for $\mathcal{C}^-_\eta(1/R_n)(z)$ in terms of $R_n(z)$ for z outside of η.

Lemma 3.3 *Let $E, F \in \mathbb{C}$ be sets satisfying the assumptions in Theorem 1.2 and $R_n(z)$ be defined as in (3.4). If $z_1, \ldots, z_K$ are the distinct zeros of $R_n(z)$ with multiplicities $m_1 + \cdots + m_K = n$, then for z outside of η we have*

$$\mathcal{C}^-_\eta(1/R_n)(z) = -\frac{1}{R_n(z)} + \sum_{k=1}^{K}\sum_{j=1}^{m_k} \frac{a^k_{-j}}{(z - z_k)^j} + \frac{1}{R_n(\infty)}, \tag{3.6}$$

where a^k_{-j} is the z^{-j} coefficient of the principal part of the Laurent series for $R_n(z)$ about z_k.

Proof The proof is an application of the Cauchy residue formula. We evaluate $\mathcal{C}^-_\eta(1/R_n)(z)$ on a large circle Γ of radius $1/\Delta$ oriented clockwise enclosing E, F and z, with detour paths in both directions leading to small counterclockwise circles around z, η, and each of the zeros of R_n, as well as the curve η (see Fig. 6). For an arbitrarily small $\epsilon > 0$, we have

$$\begin{aligned}-\int_\Gamma \frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)} &= \int_\eta \frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)} + \int_{|\zeta - z| = \epsilon} \frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)}\\ &\quad + \sum_{k=1}^{K} \int_{|\zeta - z_k| = \epsilon} \frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)}.\end{aligned}$$

If we perform the change-of-variables $\zeta = 1/t$ on the left-hand side, then we find that

$$-\frac{1}{2\pi i}\int_{\Gamma}\frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)} = \frac{1}{2\pi i}\int_{|t|=\Delta}\frac{\mathrm{d}t}{tR_n(\frac{1}{t})(1-zt)}$$
$$= \mathrm{Res}_{t=0}\frac{1}{tR_n(\frac{1}{t})(1-zt)} = \frac{1}{R_n(\infty)}.$$

Since $|\Phi^n(\infty)| \le h^n$, we note that, for fixed n, $|R_n(\infty)|$ is finite by Lemma 3.2. For each circle of radius ϵ around z_k for $1 \le k \le K$, we find from the residue theorem that

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{|\zeta - z_k|=\epsilon}\frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)} = -\sum_{j=1}^{m_i}\frac{a^i_{-j}}{(z-z_i)^j}.$$

These residues and the residue at the point z are summed together to give (3.6). □

Lemma 3.3 can be combined with the Sokhotski–Plemelj Theorem [20, (14.1–9)] to find an expression for $\mathcal{C}^+_\eta(1/R_n)(z)$ in terms of $R_n(z)$. We have

$$\mathcal{C}^+_\eta(1/R_n)(z) - \mathcal{C}^-_\eta(1/R_n)(z) = \frac{1}{R_n(z)}, \qquad \text{for } z \text{ on } \eta.$$

and, by analytic continuation, we have

$$\mathcal{C}^+_\eta(1/R_n)(z) = \sum_{k=1}^{K}\sum_{j=1}^{m_k}\frac{a^k_{-j}}{(z-z_k)^j} + \frac{1}{R_n(\infty)}, \qquad z \text{ inside of } \eta. \tag{3.7}$$

We conclude that $\mathcal{C}^+_\eta(1/R_n)$ is a rational function of type (n, n). Finally, we define the Faber rational associated with E and F as

$$\frac{1}{r_n(z)} = \sum_{k=1}^{K}\sum_{j=1}^{m_k}\frac{a^k_{-j}}{(z-z_k)^j} + \frac{1}{R_n(\infty)}. \tag{3.8}$$

The expression for $r_n(z)$ in (3.8) is ideal for identifying $r_n(z)$ as a rational function of type (n, n). The relationship $1/r_n(z) = \mathcal{C}^+_\eta(1/R_n)(z)$ in (3.5) is more convenient for practical computations as it does not involve computing z_k for $1 \le k \le K$ or the Laurent coefficients $\{a_k^{-j}\}$.

4 Using the Faber Rational to Bound a Zolotarev Number

In this section, we set out to find an upper bound on the Zolotarev number $Z_n(E, F)$ by bounding the Faber rational in (3.8) associated with the sets E and F. Since $s_n \in \mathcal{R}_{n,n}$

if and only if $1/s_n \in \mathcal{R}_{n,n}$, we note that $Z_n(E, F) = Z_n(F, E)$. Given our setup, we find it more convenient to derive a bound on $Z_n(E, F)$ as follows:

$$Z_n(E, F) = Z_n(F, E) = \inf_{s_n \in \mathcal{R}_{n,n}} \frac{\sup_{z\in F} |s_n(z)|}{\inf_{z\in E} |s_n(z)|} \leq \frac{\sup_{z\in F} |1/r_n(z)|}{\inf_{z\in E} |1/r_n(z)|}, \tag{4.1}$$

where r_n is the Faber rational associated with E and F. Therefore, we seek an upper bound on $\sup_{z\in F} |1/r_n(z)|$ in Sect. 4.1 and a lower bound on $\inf_{z\in E} |1/r_n(z)|$ in Sect. 4.2.

4.1 Bounding the Faber rational on F

From (3.7) and (3.8), we know that $1/r_n(z) = \mathcal{C}_\eta^+(1/R_n)(z)$ for $z \in F$. Thus, an upper bound on $\sup_{z\in F} |1/r_n(z)|$ follows from an upper bound on $\mathcal{C}_\eta^+(1/R_n)(z)$ for $z \in F$. From simple algebra, we have

$$\begin{aligned} \mathcal{C}_\eta^+(1/R_n)(z) &= \frac{1}{2\pi i}\int_\eta \frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)} \\ &= \frac{1}{2\pi i}\int_\eta \frac{\mathrm{d}\zeta}{\Phi^n(\zeta)(\zeta - z)} + \frac{1}{2\pi i}\int_\eta \frac{\Phi^n(\zeta) - R_n(\zeta)}{R_n(\zeta)\Phi^n(\zeta)}\frac{\mathrm{d}\zeta}{\zeta - z} \\ &= \underbrace{\frac{-1}{2\pi i}\int_{|\omega|=h} \frac{1}{\omega^n}\frac{\Psi'(\omega)\mathrm{d}\omega}{\Psi(\omega) - z}}_{=I(z)} + \underbrace{\frac{1}{2\pi i}\int_{|\omega|=h} \frac{\tilde{\varepsilon}(\omega)\Psi'(\omega)\mathrm{d}\omega}{\Psi(\omega) - z}}_{=II(z)}, \end{aligned} \tag{4.2}$$

where $\tilde{\varepsilon}(\omega) = (\omega^n - R_n(\Psi(\omega)))/(R_n(\Psi(\omega))\omega^n)$. (One can take the contours over $|\omega| = h$ since the integrand extends continuously to the boundary by Caratheodory's theorem.) Here, the minus sign in the definition of I appears to respect the orientation of η with respect to the interior of η. The integral I may be bounded using the same argument as in the proof of Lemma 3.1 to obtain

$$\sup_{z\in F} |I(z)| \leq \frac{M_n(F, E)}{1 - h^{-2n}} h^{-n}, \quad M_n(F, E) = \left(2\mathrm{Rot}(F) + 2h^{-n}\mathrm{Rot}(E) + 2h^{-n} + 2\right). \tag{4.3}$$

To bound $|II(z)|$, we note that $\tilde{\varepsilon}(\omega)$ is holomorphic in the annulus A with a pole at 0. So, for any $0 < \alpha < 1 - 1/h$, we have

$$\begin{aligned} |II(z)| &= \left| \frac{1}{2\pi i}\int_{|\omega|=(1-\alpha)h} \frac{\tilde{\varepsilon}(\omega)\Psi'(\omega)\mathrm{d}\omega}{\Psi(\omega) - z} \right| \\ &\leq \sup_{|\omega|=(1-\alpha)h} |\tilde{\varepsilon}(\omega)| \frac{1}{2\pi}\int_{|\omega|=(1-\alpha)h} \left| \frac{\Psi'(\omega)\mathrm{d}\omega}{\Psi(\omega) - z} \right|. \end{aligned}$$

By the same argument as Ganelius in [12, p. 411] using the Koebe–1/4 theorem, we find that

$$\frac{1}{2\pi}\int_{|\omega|=(1-\alpha)h}\left|\frac{\Psi'(\omega)\mathrm{d}\omega}{\Psi(\omega)-z}\right| \le \frac{4(1-\alpha)h}{d}, \qquad d=\min\{\alpha h,(1-\alpha)h-1\}. \quad (4.4)$$

The bound in (4.4) simplifies to $4(1-\alpha)/\alpha$ when $\alpha < (1-1/h)/2$. For the $\tilde{\varepsilon}$ term, we have

$$\sup_{|\omega|=(1-\alpha)h}|\tilde{\varepsilon}(\omega)| \le \frac{C_n}{(1-\alpha)^n h^n((1-\alpha)^n h^n - C_n)}, \qquad C_n = 1+\sup_{z\in E}|R_n(z)| \quad (4.5)$$

as long as $((1-\alpha)^n h^n - C_n) > 0$. We now want to find $0<\alpha<(1-1/h)/2$ with $((1-\alpha)^n h^n - C_n) > 0$ to minimize the product of (4.4) and (4.5) as that will derive a reasonable bound on $|\mathcal{C}_\eta^+(1/R_n)(z)|$ for $z\in F$. We have the following result:

Lemma 4.1 *For any $n > N_0$ with $N_0 = \max\{1+1/(h-1), \log(x_1)/\log(h)\}$, we have*

$$\min_{\substack{0<\alpha<(1-1/h)/2\\((1-\alpha)^n h^n - C_n)>0}} \frac{4(1-\alpha)C_n}{\alpha(1-\alpha)^n h^n((1-\alpha)^n h^n - C_n)} \le \frac{32nh^n}{(h^n-C_n)^2(h^n+C_n)}, \quad (4.6)$$

where it is sufficient to take x_1 as given in Lemma A.1.

Proof Let $f(\alpha) = (4(1-\alpha)C_n)/(\alpha(1-\alpha)^n h^n((1-\alpha)^n h^n - C_n))$. Using calculus, we find that the minimum of (4.6) is given by a unique value $0<\alpha_*<1/(2n)$ such that

$$(1-\alpha_*)^n h^n = \frac{1-n\alpha_*}{1-2n\alpha_*}C_n \quad \Rightarrow \quad \alpha_* = \frac{h^n - \frac{1-n\alpha_*}{(1-\alpha_*)^n}C_n}{2nh^n}.$$

Since $n > 1+1/(h-1)$, we find that $\alpha_* < 1/(2n) < (1-1/h)/2$. Moreover, by using $(1-x)^n \ge 1-nx$ for $n\ge 1$ and $x\in\mathbb{R}$, we have $\alpha_*\ge\alpha_0$, where $\alpha_0 = (h^n - C_n)/(2nh^n)$. Note that we also have

$$(1-\alpha_0)^n h^n - C_n \ge (1-n\alpha_0)h^n - C_n = \left(1-\frac{h^n-C_n}{2h^n}\right)h^n - C_n = \frac{h^n-C_n}{2}.$$

The restriction $n > N_0$ guarantees that $h^n > C_n$, and therefore also that α_0 satisfies the constraints in (4.6), and we have the following upper bound on $f(\alpha_*)$:

$$\begin{aligned} f(\alpha_*) \le f\left(\frac{h^n-C_n}{2nh^n}\right) &\le \frac{4(1-\frac{h^n-C_n}{2nh^n})C_n}{\frac{h^n-C_n}{2nh^n}\frac{h^n+C_n}{2h^n}h^n(\frac{h^n+C_n}{2}-C_n)} \\ &= \frac{16((2n-1)h^n - C_n)C_n}{(h^n-C_n)^2(h^n+C_n)}, \end{aligned}$$

where the second inequality follows from $(1-\alpha_0)^n \geq 1-n\alpha_0$. The result follows as $((2n-1)h^n - C_n) < 2nh^n$. $\square$

By combining (4.2), (4.3), and Lemma 4.1, we conclude that

$$\sup_{z\in F}\left|\frac{1}{r_n(z)}\right| \leq \frac{M_n(F,E)}{1-h^{-2n}}h^{-n} + \frac{32nh^n}{(h^n-C_n)^2(h_n+C_n)}, \tag{4.7}$$

where $M_n(F,E)$ is defined in (4.3). The upper bound in (4.7) controls the numerator in (4.1).

4.2 Bounding the Faber Rational on E

From the triangle inequality, we have

$$\inf_{z\in E}\left|\frac{1}{r_n(z)}\right| \geq \inf_{z\in E}\left|\frac{1}{R_n(z)}\right| - \sup_{z\in E}\left|\frac{1}{r_n(z)} - \frac{1}{R_n(z)}\right|,$$

where r_n is the Faber rational associated with the sets E and F and R_n is defined in (3.4). Since $R_n(z) = \mathcal{C}^+_{\partial E}(\Phi^n)(z)$ for $z \in E$, we have $1/r_n(z) = 1/R_n(z) + \mathcal{C}^-_\eta(1/R_n)(z)$ for z outside of η. Thus, we have

$$\inf_{z\in E}\left|\frac{1}{r_n(z)}\right| \geq \frac{1}{\sup_{z\in E}\left|\mathcal{C}^+_{\partial E}(\Phi^n)(z)\right|} - \sup_{z\in E}\left|\mathcal{C}^-_\eta(1/R_n)(z)\right|.$$

Lemma 3.1 provides an upper bound on the first term $\sup_{z\in E}\left|\mathcal{C}^+_{\partial E}(\Phi^n)(z)\right|$. For the second term, observe that $\mathcal{C}^-_\eta(1/R_n)$ is an analytic function outside of the contour η. Therefore, the maximum of $|\mathcal{C}^-_\eta(1/R_n)|$ is on the curve η, where we have the Sokhostski–Plemelj theorem. We find that for z outside of the contour η,

$$\begin{aligned}|\mathcal{C}^-_\eta(1/R_n)(z)| &\leq \sup_{z\in\eta}|\mathcal{C}^+_\eta(1/R_n)(z)| + \sup_{z\in\eta}|1/R_n(z)| \\ &\leq \sup_{z\in\eta}|\mathcal{C}^+_\eta(1/R_n)(z)| + \frac{1}{(h-\delta)^n - C_n},\end{aligned}$$

where the second inequality follows from $|R_n(z)| \geq |\Phi^n(z)| - |R_n(z) - \Phi^n(z)|$, the fact that $|\Phi^n(z)| \geq (h-\delta)^n$ for z on the curve η and Lemma 3.2. Because $0<\delta<1$ is arbitrarily small, we may take $|\mathcal{C}^-_\eta(1/R_n)(z)| \leq \sup_{z\in\eta}|\mathcal{C}^+_\eta(1/R_n)(z)| + 1/(h^n - C_n)$. Since E is contained in the region outside of the contour η, we conclude that

$$\inf_{z\in E}\left|\frac{1}{r_n(z)}\right| \geq \frac{1-h^{-2n}}{M_n(E,F)} - \frac{M_n(F,E)}{1-h^{-2n}}h^{-n} - \frac{1}{h^n - C_n}. \tag{4.8}$$

The lower bound in (4.8) controls the denominator in (4.1). As $n\to\infty$, this lower bound becomes $1/M_n(E,F)$.

Note that the bound in (4.8) is only effective when the degree n is sufficiently large so that the bound is positive. Fortunately the degree does not have to be too big; see Lemma A.1.

4.3 Bounding the Zolotarev Number

Putting (4.7) and (4.8) together completes the proof of (1.4) and Theorem 1.2. That is, we have

Theorem 4.2 (Main Theorem) *Let $E, F \subset \mathbb{C}$ be disjoint, simply connected, compact sets with rectifiable Jordan boundaries. Then, for $h = \exp(1/\mathrm{cap}(E, F))$ and $n > N_0$, we have*

$$Z_n(E,F) \le \left(\frac{\frac{M_n(E,F)M_n(F,E)}{1-h^{-2n}} + \frac{32nM_n(E,F)h^{-n}}{(1-C_nh^{-n})^2(1+C_nh^{-n})}}{\max\left\{0,\, 1 - \frac{M_n(E,F)M_n(F,E)}{1-h^{-2n}}h^{-n} - \frac{M_n(E,F)}{1-C_nh^{-n}}h^{-n} - h^{-2n}\right\}} \right) h^{-n},$$

where $M_n(E,F) = 2\mathrm{Rot}(E) + 2h^{-n}\mathrm{Rot}(F) + 2h^{-n} + 2$, and $C_n = 1 + M_n(E,F)/(1-h^{-2n})$. Here, $N_0 = \max\{1 + 1/(h-1), \log(x_1)/\log(h)\}$, x_1 is the largest real root of a certain seventh-degree polynomial (see Lemma A.1*), and* $\mathrm{Rot}(E)$ *and* $\mathrm{Rot}(F)$ *are the total rotation of the boundaries of the domains E and F, respectively.*

The explicit bound in Theorem 4.2 slightly simplifies when E and F are convex sets as $\mathrm{Rot}(E) = \mathrm{Rot}(F) = 1$. We find that $x_1 \approx 29.901$ and $M_n(E,F) = M_n(F,E) = 4(1+h^{-n})$. We have

$$Z_n(E,F) \le \left(\frac{\frac{16(1+h^{-n})^2}{1-h^{-2n}} + \frac{128n(1+h^{-n})h^{-n}}{(1-(1+\frac{4(1+h^{-n})}{1-h^{-2n}})h^{-n})^2(1+(1+\frac{4(1+h^{-n})}{1-h^{-2n}})h^{-n})}}{\max\left\{0,\, 1 - \frac{16(1+h^{-n})^2}{1-h^{-2n}}h^{-n} - \frac{4(1+h^{-n})}{1-(1+\frac{4(1+h^{-n})}{1-h^{-2n}})h^{-n}}h^{-n} - h^{-2n}\right\}} \right) h^{-n},$$

for any $n > \max\{1 + 1/(h-1), \log(29.901)/\log(h)\}$.

5 Other Cases

In addition to E and F both being compact sets, there are two other types of sets E and F that may be of interest to the reader.

5.1 $\mathbb{C}\backslash F$ is a bounded domain containing E

Let $E, F \subset \mathbb{C}$ be disjoint sets with rectifiable Jordan boundaries so that $\mathbb{C}\backslash F$ is an open and bounded domain containing a compact set E (see Fig. 7). Small adjustments

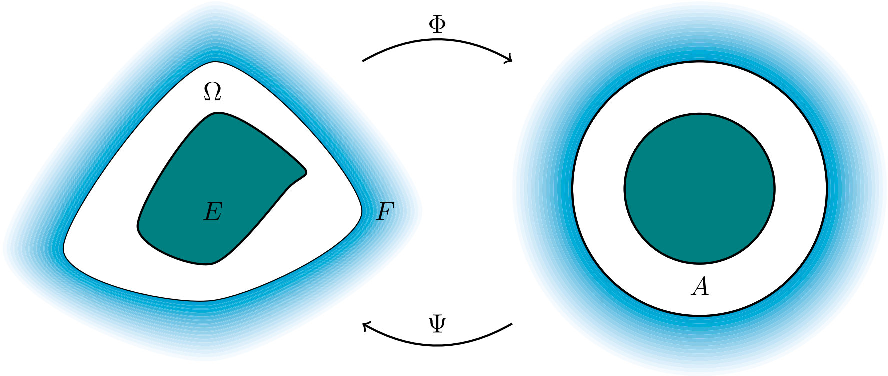

Fig. 7 Illustration of the typical setup when $\mathbb{C}\backslash F$ is a bounded domain containing a compact set E

to our arguments in this paper are needed to bound $Z_n(E, F)$ here. In particular, due to the fact that Ψ no longer has a pole in the annulus A, Lemma 3.1 becomes

$$\sup_{z\in E} |R_n(z)| \leq \frac{\tilde{M}_n(E, F)}{1 - h^{-2n}}, \qquad \tilde{M}_n(E, F) = 2\mathrm{Rot}(E) + 2\mathrm{Rot}(F)h^{-n},$$

which causes minor changes to the final bounds. Let $\tilde{C}_n = 1 + \tilde{M}_n(E, F)/(1 - h^{-2n})$. We find that

Theorem 5.1 *Let $E, F \subset \mathbb{C}$ be disjoint sets with rectifiable Jordan boundaries so that $\mathbb{C}\backslash F$ is bounded and E is a compact subset of $\mathbb{C}\backslash F$. Then, for $h = \exp(1/\mathrm{cap}(E, F))$ and $n > \tilde{N}_0$, we have*

$$Z_n(E, F) \leq \left(\frac{\frac{\tilde{M}_n(E,F)\tilde{M}_n(F,E)}{1-h^{-2n}} + \frac{32n\tilde{M}_n(E,F)h^{-n}}{(1-\tilde{C}_n h^{-n})^2(1+\tilde{C}_n h^{-n})}}{\max\left\{0, 1 - \frac{\tilde{M}_n(E,F)\tilde{M}_n(F,E)}{1-h^{-2n}}h^{-n} - \frac{\tilde{M}_n(E,F)}{1-\tilde{C}_n h^{-n}}h^{-n} - h^{-2n}\right\}} \right) h^{-n}. \tag{5.1}$$

Here, we also have $\tilde{N}_0 = \max\{1 + 1/(h - 1), \log(\tilde{x}_1)/\log(h)\}$, with $\tilde{x}_1$ defined analogously to x_1 in Lemma A.1.

When the denominator in (5.1) is zero, then one can take the trivial bound of $Z_n(E, F) \leq 1$ instead. For large n, we conclude that

$$1 \leq \limsup_{n\to\infty} \frac{Z_n(E, F)}{h^{-n}} \leq 4\mathrm{Rot}(E)\mathrm{Rot}(F).$$

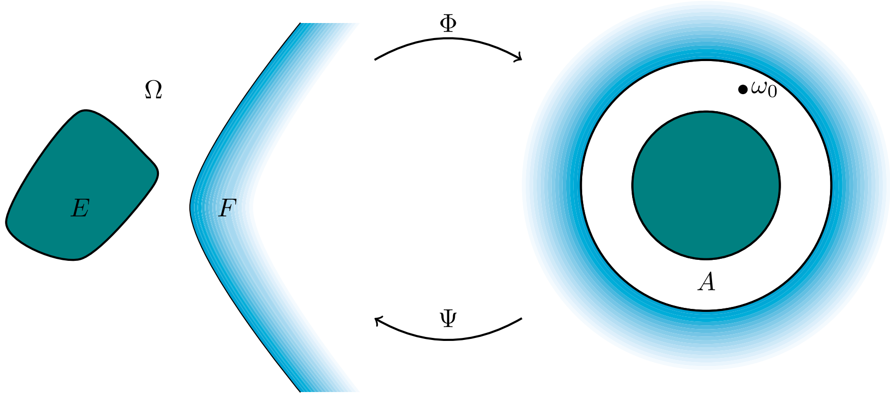

Fig. 8 Illustration of the typical setup when F is an unbounded domain and E is a compact domain contained in $\mathbb{C}\backslash F$. The location $\omega_0 \in \mathbb{C}$ is the pole of the inverse map $\Psi = \Phi^{-1}$

5.2 F is an unbounded domain and E is a compact domain contained in $\mathbb{C}\backslash$F

Let $E, F \subset \mathbb{C}$ be disjoint sets with rectifiable Jordan boundaries, where F is an unbounded domain and E is a compact domain contained in $\mathbb{C}\backslash F$ (see Fig. 8). In this situation, our bound on $Z_n(E, F)$ is the same as that found in Theorem 1.2 and (1.4).

6 Numerical Methods

In this section, we briefly describe the algorithms we use to evaluate Faber rational functions, as well compute $h = \exp(1/\text{cap}(E, F))$. We also discuss a method for finding the poles and zeros of r_n.

6.1 Evaluating $\mathbf{r_n}$

To evaluate $r_n(z)$, we use the integral formulations for $1/r_n(z)$ developed in Sect. 3.2. It is acceptable for numerical purposes to choose the contour η in Lemma 3.3 as ∂F. Taking this liberty, we have from the lemma that

$$1/r_n(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\partial F} \dfrac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)}, & z \in F, \\ -\dfrac{1}{R_n(z)} + \dfrac{1}{2\pi i} \displaystyle\int_{\partial F} \dfrac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)}, & z \in \mathbb{C}\backslash F, \end{cases} \tag{6.1}$$

where the first integral is understood in the principal value sense for $z \in \partial F$,[2] and R_n is defined in (3.4).

[2] We avoid sampling directly on ∂F in our applications, and so omit discussion on the numerical computation of principle value integrals.

The integrals in (3.4) and (6.1) can be computed using a quadrature rule. These computations can become numerically unstable when z is close to the contour of the integral being evaluated. To alleviate this issue, we apply a variant of the barycentric interpolation formula [7]. For $z \in F$, this takes the following form:

$$\frac{1}{r_n(z)} = \frac{\displaystyle\int_{\partial F} \frac{\mathrm{d}\zeta}{R_n(\zeta)(\zeta - z)}}{\displaystyle\int_{\partial F} \frac{\mathrm{d}\zeta}{\zeta - z}} \approx \frac{\sum_{j=1}^{N_Q} \frac{w_j}{R_n(x_j)(x_j - z)}}{\sum_{j=1}^{N_Q} \frac{w_j}{x_j - z}},$$

where $\{(w_j, x_j)\}_{j=1}^{N_Q}$ are an appropriate set of quadrature weights and nodes. A similar procedure is used when evaluating $R_n(z)$ for $z \in E$ near ∂E. Once $f_z = 1/r_n(z)$ is computed, we set $r_n(z) = 1/f_z$. After one can evaluate r_n on $E \cup F$, r_n can be represented as a rational function via the AAA algorithm [26], which makes further evaluations more efficient.

6.2 Computing the Conformal Map

Evaluating $R_n(z)$ requires the conformal map $\Phi : \Omega \to A$ (see Sect. 1.3). We construct Φ using the method in [33]. In this approach, Φ is computed via the Green's function, $u(z)$, associated with the Laplacian operator with zero Dirichlet boundary conditions on Ω (see [30, p. 253], [33, Sec. 4]). To solve for u, boundary data are used to find the least squares fit to the coefficients of an approximate rational expansion of u. This is especially effective for resolving singularities in corners of the domain because the poles of the expansion are chosen to be exponentially clustered near the singular points [17]. The modulus h is treated as an additional unknown in the least squares system of equations, and it is recovered along with u.

This method is versatile and can be used when E and F are polygons, as well as when their boundaries are either analytic curves or piecewise continuous analytic curves. It can be adapted for use in the case from Sect. 5.1 where F is unbounded.

6.3 The Poles and Zeros of r_n

To compute the poles and zeros of r_n, we first construct a representation of r_n in barycentric form via the AAA algorithm [26]. This construction is computationally expensive because r_n must be sufficiently sampled on the sets E and F. The poles and zeros are then computed by solving an $(n+2) \times (n+2)$ generalized eigenvalue problem. To improve the accuracy of the computation, we apply AAA twice: first to r_n on E to compute the zeros and then again to r_n on F to compute the poles. For an application involving poles and zeros, see Sect. 7.2.

7 Applications

We give two examples from numerical linear algebra where our results can be applied. In the first, we bound the singular values of Cauchy and Vandermonde matrices. In the second example, we treat r_n as a proxy to the true infimal rational function that attains $Z_n(E, F)$. We show that the poles and zeros of r_n are near-optimal parameters in the alternating direction implicit (ADI) method.

7.1 Bounding Singular Values of Matrices with Low Displacement Rank

A matrix $X \in \mathbb{C}^{m\times p}$ is said to have displacement rank ν if there are $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{p\times p}$ so that $\mathrm{rank}(AX - XB) \le \nu$. When A and B are normal matrices with spectra $\lambda(A) \subset E, \lambda(B) \subset F$, the normalized singular values of X are bounded above in terms of Zolotarev numbers [6, Thm. 2.1]. Specifically,

$$\sigma_{j\nu+1}(X) \le Z_j(E, F)\|X\|_2, \quad 0 \le j \le \lfloor (N-1)/\nu \rfloor, \quad N = \min(m, p), \quad (7.1)$$

where $\sigma_1(X) \ge \cdots \ge \sigma_N(X)$ are the nonzero singular values of X. Pairing this observation with Theorems 1.2, 4.2 and 5.1 gives bounds on $\sigma_{j\nu+1}(X)$ whenever E and F are as in the theorems. We illustrate the point with two examples.

7.1.1 Complex-Valued Cauchy Matrices

Let C be a Cauchy matrix in $\mathbb{C}^{m\times p}$, with entries given by

$$C_{jk} = 1/(x_j - y_k), \quad \underline{x} = \{x_j\}_{j=1}^m \subset E, \quad \underline{y} = \{y_k\}_{k=1}^p \subset F,$$

where E and F are as in Theorem 1.2 and the sets $\underline{x}$, $\underline{y}$ are each collections of distinct points. Since $\mathrm{rank}(D_{\underline{x}}C - CD_{\underline{y}}) \le 1$, where $D_{\underline{x}} = \mathrm{diag}(x_1, \ldots, x_m)$, it immediately follows from (7.1) and Theorem 1.2 that for $0 \le j \le N-1$,

$$\sigma_{j+1}(C) \le K_{E,F} h^{-j}\|C\|_2, \qquad h = \exp(1/\mathrm{cap}(E, F)).$$

Here, $K_{E,F}$ is given in (1.4). To implement the bound, we compute h using the method in Sect. 6.2. A comparison of the bounds to computed singular values is shown in Fig. 9.

7.1.2 Vandermonde Matrices with Nodes Inside the Unit Circle

Let V_α be an $m \times p$ Vandermonde matrix with entries $(V_\alpha)_{jk} = \alpha_j^{(k-1)}$, where the nodes $\alpha = \{\alpha_j\}_{j=1}^m$ are distinct points in $\mathbb{C}$. The singular values of V_α are known to decay rapidly when each α_j is real [6], and there are multiple results on the (extremal) singular values of V_α when all $|\alpha_j| = 1$ [4, 24]. Less is known about singular value decay when $|\alpha_j| < 1$, despite the fact that this assumption is encountered in several applications [5, 8, 27]. We give the following lemma:

Lemma 7.1 *Let $V_\alpha \in \mathbb{C}^{m\times p}$, $N = \min(m, p)$, have a set of distinct nodes contained in the disk $E := \{|z - z_0| < \eta_0\}$, $z_0 \neq 0$, where E is in the open unit disk. Then, the following bound holds for $0 \le j \le N-1$:*

$$\sigma_{j+1}(V_\alpha) \le h^{-j}\|V_\alpha\|_2,$$

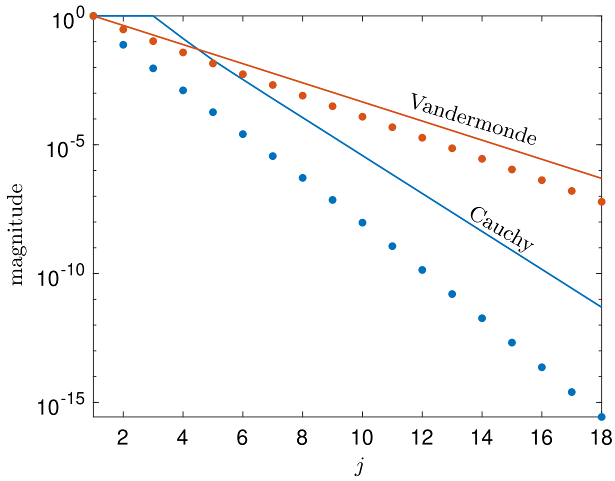

Fig. 9 The first 18 normalized singular values of a Cauchy matrix (blue dots) and Vandermonde matrix (red dots) are plotted against the singular value index j on a logarithmic scale. The Cauchy matrix is given by $C_{jk} = 1/(x_j - y_k)$, $1 \leq j, k \leq 100$, where for all (j, k), $x_j \in E_C := \{z \in \mathbb{C} : .3 \leq \mathrm{Re}(z) \leq 1.3, |\mathrm{Im}(z)| \leq .5\}$ and $y_k \in -E_C$. The nodes of the Vandermonde matrix $V \in \mathbb{C}^{100\times 80}$ all lie in $E_V = \{z \in \mathbb{C} : |z - (2+i)/10| < .4\}$. The solid lines show bounds on $\sigma_j(C)/\sigma_1(C)$ (blue) and $\sigma_j(V)/\sigma_1(V)$ (red) obtained via Theorem 1.2 and Lemma 7.1, respectively (Color figure online)

where

$$h = \left|\frac{z_0 - |z_0|\beta(z_0 + \eta_0)}{|z_0|(z_0 + \eta_0) - \beta z_0}\right|, \quad \beta = \frac{1}{2|z_0|}\left(1 + c - \sqrt{(c+1)^2 - 4|z_0|^2}\right), \quad c = |z_0|^2 - \eta_0^2.$$

Proof We observe that $\mathrm{rank}(D_\alpha V - VQ) = 1$, where $Q = \left[\begin{smallmatrix} 0 & 1 \\ I_{n-1} & 0 \end{smallmatrix}\right]$ is the circulant shift matrix. The eigenvalues $\lambda(Q)$ are the pth roots of unity. We choose F as the set exterior to the open unit disk and note that $\lambda(Q) \subset F$. We map $\Omega := C\backslash E \cup F$ to the annulus $A := \{z \in \mathbb{C} : 1 < |z| < h\}$ with the following Möbius transformation:

$$T(z) := \frac{h(|z_0|z - z_0\beta)}{z_0 - |z_0|\beta z},$$

where h, β, and c are as in the theorem. Since T maps $\Omega \to A$ conformally and T is rational, $r_j = T^j$ is the rational function that attains $Z_j(E, F)$, and $Z_j(E, F) = h^{-j}$ (see Sect. 2). Applying (7.1) completes the proof. $\square$

The bounds from Lemma 7.1 are shown in Fig. 9 along with computed singular values. We remark that if E is centered on the origin, then $h = 1/\eta_0$. For more general choices of E, an argument similar to the proof of Lemma 7.1 can be applied using the bound on $Z_j(E, F)$ from Theorem 4.2.

7.2 ADI Shift Parameters

The ADI method is an iterative method used to solve the Sylvester matrix equation

$$AX - XB = M, \qquad X, M \in \mathbb{C}^{m\times p}. \tag{7.2}$$

For an overview with applications, see [31]. One iteration of ADI consists of the following two steps:

1. Solve for $X^{(j+1/2)}$, where

$$\left(A - \tau_{j+1}I\right) X^{(j+1/2)} = X^{(j)} \left(B - \tau_{j+1}I\right) + M.$$

2. Solve for $X^{(j+1)}$, where

$$X^{(j+1)} \left(B - \kappa_{j+1}I\right) = \left(A - \kappa_{j+1}I\right) X^{(j+1/2)} - M.$$

The numbers (κ_j, τ_j) are referred to as shift parameters, and an initial guess $X^{(0)} = 0$ is used to begin the iterations. After k iterations, an approximate solution $X^{(k)}$ is constructed. Suppose that A and B are normal matrices with spectra $\lambda(A) \subset E$, $\lambda(B) \subset F$. Then, the ADI error is bounded by a rational function determined by the shift parameters [23]:

$$\|X - X^{(k)}\|_2 \leq \frac{\sup_{z\in E} |s_k(z)|}{\inf_{z\in F} |s_k(z)|} \|X\|_2, \quad s_k(z) = \prod_{j=1}^{k} \frac{(z-\kappa_j)}{(z-\tau_j)}, \quad k \geq 1. \tag{7.3}$$

The bound in (7.3) is minimized by selecting as shift parameters the poles and zeros of the rational function that attains $Z_k(E, F)$ in (1.1). The solution to (1.1) is generally unknown, but when E and F are as in Theorems 1.2,4.2 and 5.1, we choose s_k as the Faber rational function r_k. We refer to the poles and zeros of r_k as Faber shifts. The bounds on $Z_k(E, F)$ in Theorems 1.2,4.2 and 5.1 also bound the expression involving s_k in (7.3). Since the bounds decay with k at essentially the same rate as $Z_k(E, F)$, the Faber shifts are nearly optimal shift parameters.

We do not claim to have an efficient method for computing Faber shifts; the approach in Sect. 6.3 is impractical for applications. For convex E, F, we observe that ADI with shifts derived from other so-called asymptotically optimal rational functions [32], i.e., rationals s_n with the property that

$$\lim_{n\to\infty} \left(\frac{\sup_{z\in E} |s_n(z)|}{\inf_{z\in F} |s_n(z)|}\right)^{1/n} = h^{-1}, \qquad h = \exp\left(\frac{1}{\mathrm{cap}(E, F)}\right),$$

often performs comparably to ADI with Faber shifts (see Fig. 10). This includes the generalized Fejér points [34], which can be computed with the inverse conformal map Ψ from Sect. 1.3, and the generalized Leja points, which are computed recursively by a greedy process [3, 32].

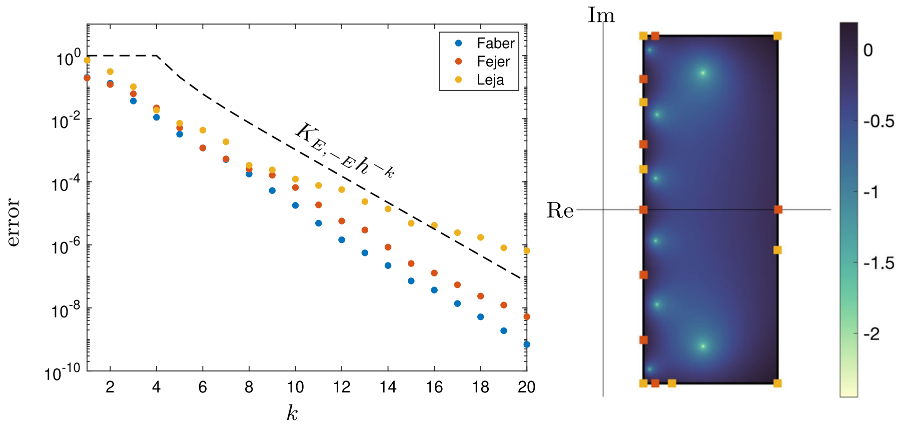

Fig. 10 Left: The computed ADI error $\|X - X^{(k)}\|_2/\|X\|_2$ is plotted against the indices k on a logarithmic scale, where ADI is applied using Faber shifts (blue), generalized Fejér points (red), and generalized Leja points (yellow). The bound on the error for ADI with Faber shifts is shown as a dotted line. Here, X satisfies (7.2), with $m = p = 100$, $\lambda(A) \in E$, $\lambda(B) \in -E$, where $E = \{z \in \mathbb{C} : .3 \le \mathrm{Re}(z) \le 1.3, \ -1.3 \le \mathrm{Im}(z) \le 1.3\}$. Right: The magnitude of the Faber rational r_8 is plotted on a logarithmic scale over E. Generalized Fejér points (red squares) and generalized Leja points (yellow squares) associated with $(E, -E)$ are plotted. These are selected as the κ_j parameters for ADI, and due to the symmetry of the domain, we choose $\tau_j = -\overline{\kappa}_j$. The Faber shifts are formed by using the zeros of r_8 as κ_j parameters, and the poles of r_8 (not depicted) as τ_j parameters (Color figure online)

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Appendix A: Ensuring n is Large Enough for a Valid Bound

For the bound in (1.4) to be effective, n must be taken large enough so that both $h^n > C_n$, and the lower bound in (4.8) must also be greater than 0, or

$$\frac{1 - h^{-2n}}{M_n(E, F)} - \frac{M_n(F, E)}{1 - h^{-2n}} h^{-n} - \frac{1}{h^n - C_n} \ge 0 \tag{A.1}$$

so that the denominator in (1.4) is positive. This latter condition subsumes the first, and so a minimal effective value of n may be given as follows.

Lemma A.1 *For any integer $n > \log(x_1)/\log(h)$, we have that the expression in (4.8) is greater than 0, where x_1 is the largest positive real root of the seventh-degree polynomial*

$$
\begin{aligned}
&x^7 - (4ef + 8e + 4f + 9)x^6 + (8e^2 f + 4e^2 + 20ef + 12e - 4f^2 - 3)x^5 \\
&+ (8e^3 + 28e^2 + 16ef^2 + 32ef + 56e + 20f^2 + 40f + 43)x^4 \\
&+ (16e^2 f + 20e^2 + 32ef + 40e + 8f^3 + 28f^2 + 56f + 43)x^3 \\
&+ (-4e^2 + 8ef^2 + 20ef + 4f^2 + 12f - 3)x^2 \\
&- (4ef + 4e + 8f + 9)x + 1,
\end{aligned}
$$

where we have abbreviated $\mathrm{Rot}(E)$ *and* $\mathrm{Rot}(F)$ *as* e *and* f*, respectively.*

In the case where E and F are convex, the polynomial above reduces to

$$x^7 - 25x^6 + 37x^5 + 243x^4 + 243x^3 + 37x^2 - 25x + 1, \tag{A.2}$$

whose rightmost root satisfies $29.9 < x_1 < 29.901$.

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