Fewest Moves Inequality Constraint Puzzles[1]

Robert M. Alaniz[2], Tim Gomez[3], Robert Schweller[2], Tim Wylie[2]

Abstract

Inequality constraint games are normal play impartial games where users place available numbers, while respecting existing inequalities on the board. While the 1D and 2D 1-player versions are known to be in P, other versions have yet to be explored. Here, we show that the 2D 1-player fewest moves version is NP-complete, i.e., given an $n \times n$ board (with $\lceil n^2/2 \rceil$ spaces), and an integer $k > 0$, can you place k unique numbers from $\{1, \ldots, \lceil n^2/2 \rceil\}$ such that there are no more available moves? This is true whether the inequalities are strict or not, and with only one direction of inequalities on all left/right comparisons (only $<$'s or $>$'s).

1 Introduction

An inequality constraint puzzle is a game where 1-player plays on an $n \times n$ checkered board, placing $\lceil n^2/2 \rceil$ numbers in spots where inequalities are satisfied. The numbers may only be used once (Fig. 1a). Although it could be more general, for any north/south comparison, we use north on left, south on right.

Problem 1.1 (Inequality Constraint Game)**.** Given an $n \times n$ checkerboard with $\lceil \frac{n^2}{2} \rceil$ spaces and inequalities (inequality constraint board), and a $k \in \mathbb{N}$, can k numbers be placed from universe U?

There are many variations to the 1-player game, such as non-unique numbers, a limited universe (only 0 or 1), pattern objectives (four consecutive numbers or four 0's in a row), specific inequality patterns, more placement constraints (a row must sum to less than a value), etc. There are also the obvious extensions to 2-player normal-play impartial (or partizan) combinatorial games.

The 1D 1-player version is simple (see [4]). The 2D 1-player version was proven to be in P if all numbers must be placed, regardless of the inequalities used [3]. A similar generalized version has been explored where 4-sided tiles are used with a different number on each side [1]. If the tiles have the same number on all sides, the models are similar, but slightly different (in ICG, an inequality used to compare a north/south pair is also used to compare a west/east pair). There is also considerable combinatorial work counting the permutations of numbers given a sequence of inequalities (Up-Down Sequences) [2].

2 Unique Fewest Moves 1-player ICG is NP-complete

We show that, if a number may only be placed once, deciding whether k numbers can be placed is NP-complete. We use a reduction from a restricted version of Set Element-Label Cover, which we show is hard by a reduction from Vertex Cover in cubic graphs (VC3) [5] following a similar reduction as [6].[4] Given space constraints, the ideas are outlined without the formal proof.

Problem 2.1 (Unique Fewest Moves Inequality Constraint (k-UIC))**.** Given an Inequality Constraint board B with strict inequalities, a universe of numbers $U = \{1, \ldots, n\}$, and an integer $k > 0$. Can k unique numbers from U be placed on B such that no more moves are possible?

Problem 2.2 (k-Set Element-Label Cover (SELC))**.** Given sets $S = \{s_1, \ldots, s_m\}$ over some universe $U = \{x_1, \ldots, x_n\}$ and $k \in \mathbb{N}$. A Set Element-Label Cover is an injective mapping $M : S' \to U'$ where $S' \subseteq S$ and $U' \subseteq U$, and $M(s_i) \in s_i$. Does there exist a maximal partial cover such that $|S'| \leq k$?

Theorem 2.3 (SELC is NP-complete)**.** *Set Element-Label Cover is NP-complete even with sets of size 2 and when an element may occur in at most 4 elements.*

Proof. For clarity, we reduce to a graph variant of Set Element-Label Cover where, for an undirected simple graph $G = (V, E)$, let $S = E$ and $U = V$ such that each edge is defined as $\{v_i, v_j\}$ where $v_i, v_j \in V$ and $i \neq j$. This is essentially uniquely labeling edges with an incident vertex, and the labels can not be used more than once. Using only k edges for the bipartite labeling graph (Examples in Fig. 1) is equivalent to finding a minimal maximum matching.

Given a cubic VC3 graph $G = (V, E)$, create a graph $G' = (V', E')$ where every $v_i \in V$ is replaced with the 3-clique $u_i, m_i, p_i \in V'$ and edges $\{u_i, m_i\}, \{m_i, p_i\}, \{p_i, u_i\} \in E'$ as a vertex gadget (Fig. 1b). WLOG, for an edge $\{v_i, v_j\} \in E$, we add one vertex $z_{ij} \in V'$ and edges $\{X_i, Y_j\}, \{X_i, z_{ij}\}, \{z_{ij}, Y_j\} \in E'$ where X, Y are one of the u, m, p vertices from v_i, v_j, respectively (Fig. 1b). G' has max degree 4 and $|V'| = 3|V| + |E|$ and $|E'| = 3|V| + 3|E|$.

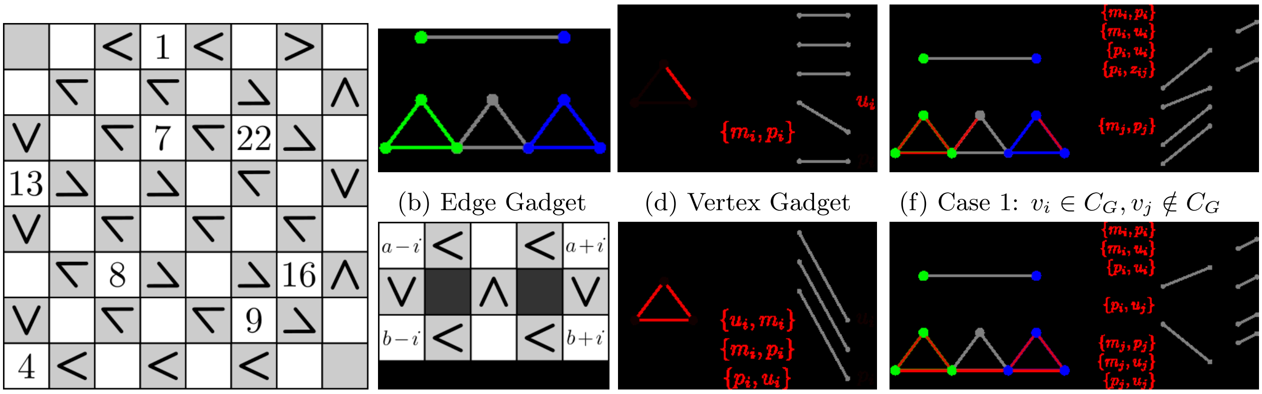
We show a bound on the minimum number of labels assigned with no other labeling possible. Let $\beta_{G'}$ be the size of a minimum labeling on G', and C_G be a minimum vertex cover of G. We assert that $\beta_{G'} = 2|V| + 2|E| - 2|C_G|$, and thus, G has a vertex cover of size k if and only if G' has a minimal

[1]This research was supported in part by National Science Foundation Grant CCF-1817602.

[2]{robert.alaniz01,robert.schweller,timothy.wylie}@utrgv.edu,
Department of Computer Science, University of Texas Rio Grande Valley, Edinburg, TX, USA.

[3]tagomez7@mit.edu, Department of Electrical Engineering and Computer Science,
Massachusetts Institute of Technology, Boston, MA, USA.

[4]VC3 usually refers to Vertex Cover on graphs of max degree 3, but we use it as cubic graphs for convenience.



(a) Example game (b) Edge Gadget (c) UIC Mapping (d) Vertex Gadget (e) Vertex Gadget (f) Case 1: $v_i \in C_G, v_j \notin C_G$ (g) Case 2: $v_i, v_j \in C_G$

Figure 1: (a) Example game with 32 blanks, thus the pieces are $\{1, \ldots, 32\}$. Given the board with 8 numbers placed, can you fill in 10 numbers s.t. there are no more valid moves? (b) Edge gadget connected to two vertex gadgets. (c) k-UIC gadget for two numbers a, b with $a > b$. Each a, b can be in up to 4 gadgets- each using a different i ($1 \leq i \leq 4$). (d-e) Vertex gadget and the mapping in SELC based on whether v is part of the vertex cover. x, y, z are vertices of adjacent vertex gadgets. (f-g) Edge gadget labeling based on whether one or both vertices are in the vertex cover. Red indicates an unlabeled edge.

maximal labeling of size $2|V| + 2|E| - 2k$ ($|V| + |E| + 2k$ unlabeled edges). The basic idea is, if a vertex $v_i \notin C_G$, then the edges in the vertex gadget can be labeled by the nodes u_i, m_i, or p_i. If $v_i \in C_G$, then u_i, m_i, and p_i are used to label edges to neighboring nodes. Note that a minimum labeling is not unique.

Obs. 1. A minimum labeling on a vertex gadget (with no other edges) labels 2 edges. (Fig. 1d).

Obs. 2. A minimum labeling on a vertex gadget with vertices of degree 3 can label 0 edges. (Fig. 1e).

Obs. 3. For two adjacent vertices $v_i, v_j \in V$. The edge gadget between the two vertex gadgets in G' can always have one unlabeled edge.

Obs. 4. For the subgraph of two adjacent vertex gadgets connected by an edge gadget, if one is part of a vertex cover, the minimum labeling labels 4 edges (Fig. 1f).

Obs. 5. For the subgraph of two adjacent vertex gadgets connected by an edge gadget, if both are part of a vertex cover, the minimum labeling labels 2 edges (Fig. 1g).

Obs. 4-5 note how to always give a minimum labeling for the 3-clique on the edges with 2 vertex gadgets. The assertion is that any deviation from this labeling will not yield a smaller labeling (since it is not unique). Finally, the problem is in NP since k assignments can be guessed. □

Theorem 2.4. *k-UIC is NP-complete.*

Proof. Given an instance of SELC with (S, U, k), we enumerate U and let $U' = \{1, \ldots, (9|U| + 4)\}$. Further, take S' to be the pairs of enumerated values times 9, i.e., if $s_i = \{x_7, x_{11}\}$, then $s'_i = \{63, 99\}$. Now, since each $s \in S'$ is simply two numbers, say $a, b \in U'$ and WLOG, assume $a > b$. We create $|S'|$ gadgets as shown in Figure 1c. Either a or b may be placed in their space, but not both, nor any numbers besides a or b. Since the game only allows a number to be placed once, this is equivalent to picking one of the numbers as the mapping for the set (gadget). Note that number a may occur in up to 4 pairs, so the multiplication by 9 lets us use unique numbers for each gadget a is in ($a - i << a + i$ for $1 \leq i \leq 4$). Let $q = \lceil \sqrt{|U'|}\, \rceil$, g_i represent the 5×3 gadget for s_i, and X represent a 3×3 gadget used for spacing that is nonplayable (prefilled with numbers). The gadgets are laid out in a grid with the first row being $g_1, X, g_2, X, \ldots, X, g_q$, then a spacing row of X gadgets. In general, gadget row r is $g_{q(r-1)+1}, X, \ldots, X, g_{q(r-1)+q}$. This grid has size $(5q + 3(q-1)) \times (3q + 3(q-1))$, which we fill out to a square with nonplayable prefilled area. Let $T = 5q + 3(q-1)$, so we have a $T \times T$ board with $U' = \{1, \ldots, \lceil T^2/2 \rceil\}$. Finally, the problem is in NP since k numbers can be guessed with location. □

References

[1] J. Bosboom, C. Chen, L. Chung, S. Compton, M. Coulombe, E. Demaine, M. Demaine, I. Filho, D. Hendrickson, A. Hesterberg, C. Hsu, W. Hu, O. Korten, Z. Luo, and L. Zhang. Edge matching with inequalities, triangles, unknown shape, and two players. *J. Info. Processing*, 28:987–1007, 2020.

[2] L. Carlitz. Enumeration of up-down sequences. *Discrete Mathematics*, 4(3):273–286, 1973.

[3] C. Chalk and B. Wang. One-player constraint inequality is easy. In *Proc. of the* 1^{st} *Hack Research Hackathon*, HackR'18, pages 7–8, 2018.

[4] A. Levitin and M. Levitin. *Algorithmic Puzzles.* Oxford University Press, 2011.

[5] D. Maier and J. A. Storer. A note on the complexity of the superstring problem. Technical Report 233, Princeton University, 1977.

[6] M. Yannakakis and F. Gavril. Edge dominating sets in graphs. *SIAM J App Math*, 38:364–372, 1980.