Reachability in Population Protocols is PSPACE-Complete[1]

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Abstract

Recently developed motion planning techniques lend powerful tools to formulate insights across numerous research fields. This work investigates the application of motion planning gadgets to reachability problems for the small agent-based Population Protocols model of Distributed Computing and, in turn, the nearly equivalent research fields of Vector Addition Systems [7], Petri-Nets [5], and Chemical Reaction Networks [8]. By two reductions derived from the edge encoding of graphs into the interactions of a Population Protocol, we show that general reachability is PSPACE-complete and is NP-hard in Feed-Forward versions.

1 Introduction

Population Protocols are a model of distributed computation where there is a discrete count of agents that interact pairwise and change their states according to a set of rules. This model has been shown to be able to compute exactly semi-linear sets [1]. Here, we focus on the Reachability problem and show it is PSPACE-complete. For the case of Feed-Forward Population Protocols, where there is an ordering on the rules such that agents transition in a directed way [3], we show it is NP-hard.

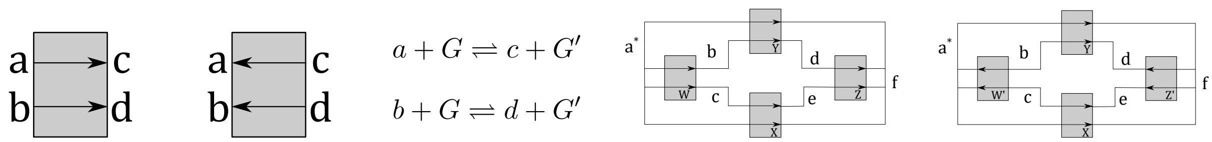
Definition 1.1 (Population Protocol)**.** A Population Protocol $P = (\Lambda, R)$, where Λ is the set of *states* and $R : \Lambda^2 \to \Lambda^2$ is a partial function mapping pairs of states to other pairs, which are called the *rules.* For convenience, the rules of R, $R(x_1, x_2) = (y_1, y_2)$, are usually denoted as $x_1 + x_2 \to y_1 + y_2$. We use $x_1 + x_2 \rightleftharpoons y_1 + y_2$ to denote that a rule is reversible. A configuration c is a multiset over the elements of Λ. Each element of c is called an agent.

Definition 1.2 (Reachability)**.** Given a Population Protocol P and two configurations c and c', does there exist a sequence of rule applications (replacing a pair of agents (a, b) within the domain of R with $R(a, b)$) to reconfigure c to c'?

Proper Chemical Reaction Networks (CRNs) can be seen as a generalization of Population Protocols where there is no bound on the size of the rules, and only that they must keep the same number of agents. Previous work on the Reachability problem in CRNs shows it is PSPACE-complete if using rules up to size 5 (up to 5 agents in a rule) [8]. Our result improves this to rules of only size 2. When considering only rules of size 1, i.e., agents that cannot interact and just change states, the reachability problem is equivalent to a directed path and thus is NL-complete.

2 Reachability is PSPACE-complete

We reduce from the reconfiguration 2-Toggle gadgets presented in [2, 4]. A motion planning system consists of a set of gadgets with labeled ports, a set of wires denoting port connections[4], and an initial signal location. Each gadget (Fig. 1a) has 2 parallel directed tunnels. A gadget's *direction* is the direction of its indicated tunnel arrows. Completing a traversal of a tunnel by a signal toggles the direction of the gadget and associated tunnels. Given an initial gadget system and target configuration, the reconfiguration problem asks if there exists a traversal of the gadgets such that the system reconfiguration meets the target configuration.



(a) State G (b) State G' (c) Rules for Gadget (d) Initial $\{a, W, Y, X, Z\}$ (e) Target $\{a, W', Y, X, Z'\}$

Figure 1: (a-b) Two states of a 2-Toggle gadget. (c) Bidirectional rules which implement a single gadget. (d) Example initial system and configuration. (e) Target system and configuration.

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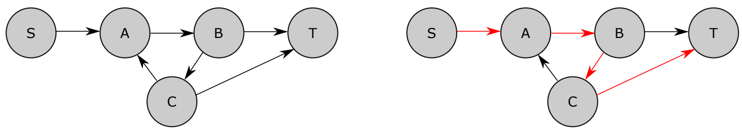
[4]In [2, 4], they define wires using a connection graph, however, the definitions are equivalent.



(a) Graph (b) Hamilton Path

$$\begin{array}{lllll} \{S_0^* & A & B & C & T\} \\ \{S^v & A_1^* & B & C & T\} \\ \{S^v & A^v & B_2^* & C & T\} \\ \{S^v & A^v & B^v & C_3^* & T\} \\ \{S^v & A^v & B^v & C^v & T^4\} \end{array}$$

(c) Configurations

Figure 2: Our starting configuration $c = \{S_0^*, A, B, C, T\}$. Our goal configuration is $c' = \{S^v, A^v, B^v, C^v, T^4\}$. In order to reach the target, each vertex must be changed to the visited state and the T must be the last vertex.

We divide our states into two sets. The first set contains the *location* states, denoting each of the wires where the signal might be (states $\{a, b, c, d, f\}$ in the example). Reachable configurations always contain exactly one agent in a single location state that represent the signal's location in the system. The second set contains the *gadget* states that correspond to each gadget of the system and its current direction (states $\{W, W', X, X', Y, Y', Z, Z'\}$ in the example). Each reachable configuration has one agent for each gadget to represent that gadget's current direction.

The rule set contains rules to facilitate traversal of the gadget system. Each rule takes in one agent state and one gadget state if the agent can traverse the gadget in its current state. The location state changes to the location prescribed by the relevant gadget after the traversal, and the gadget state switches to its toggled version. We note these rules are reversible, meaning for every rule the opposite rule also exists. The provided Population Protocol reaches the target agent configuration if and only if the given toggle system is reconfigurable into the destination system, yielding the following theorem:

Theorem 2.1. *The Reachability Problem in Population Protocols is PSPACE-complete.*

3 Reachability in Feed-Forward Systems is NP-hard

Feed-Forward Population Protocols permit an ordering on the rules such that a state occurring in the output of the i^{th} rule cannot be used in the input for the j^{th} rule for $i > j$. We show NP-hardness for such systems by a reduction from the Hamiltonian Path problem with vertices of in and out degree of at most 2 [6]. For each vertex X in the graph $G = (V, E)$, we include $2 + |V|$ states: an initial state X, a visited state X^v, and $|V|$ signal states X_i^*. We encode the edges of the graph in the rules as follows,

$$\text{Rules } R = \left\{ \begin{array}{c|c|c} S_i^* + A \to S^v + A_{i+1}^* & B_i^* + C \to B^v + C_{i+1}^* & B_i^* + T \to B^v + T_{i+1}^* \\ A_i^* + B \to A^v + B_{i+1}^* & C_i^* + A \to C^v + A_{i+1}^* & C_i^* + T \to C^v + T_{i+1}^* \end{array} \right\}$$

An example reduction is shown in Figure 2. Given this reduction, we see that any Hamiltonian path of graph G has a corresponding sequence of rules that end with every vertex, other than T, represented with the *visited* state, and T represented with the signal state matching the count of the vertices. Conversely, the only way to reach such a configuration corresponds directly to a Hamiltonian walk of G from S to T, yielding the following result:

Theorem 3.1. *The Reachability Problem in Feed-Forward Population Protocols is NP-hard, even if each state appears on the left side of at most two rules, and on the right side of at most two rules.*

References

[1] D. Angluin, J. Aspnes, D. Eisenstat, and E. Ruppert. The computational power of population protocols. *Distributed Computing*, 2007.

[2] J. Ani, E. D. Demaine, Y. Diomidov, D. Hendrickson, and J. Lynch. Traversability, reconfiguration, and reachability in the gadget framework. In *Intl. Conf. and Work. on Algorithms and Comp.*, 2022.

[3] H.-L. Chen, D. Doty, and D. Soloveichik. Rate-independent computation in continuous chemical reaction networks. In *The 5th conference on Innovations in theoretical computer science*, 2014.

[4] E. D. Demaine, I. Grosof, J. Lynch, and M. Rudoy. Computational complexity of motion planning of a robot through simple gadgets. In *9th Intl. Conf. on Fun with Algorithms (FUN 2018)*, 2018.

[5] J. Leroux. The reachability problem for petri nets is not primitive recursive. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*, 2022.

[6] J. Plesník. The np-completeness of the hamiltonian cycle problem in planar diagraphs with degree bound two. *Information Processing Letters*, 1979.

[7] S. Schmitz. The complexity of reachability in vector addition systems. *ACM SigLog News*, 2016.

[8] C. Thachuk and A. Condon. Space and energy efficient computation with dna strand displacement systems. In *International Workshop on DNA-Based Computers*, 2012.