

CONGRUENT NUMBER TRIANGLES WITH THE SAME HYPOTENUSE

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ABSTRACT. In this article, we discuss whether a single congruent number t can have two (or more) distinct corresponding triangles with the same hypotenuse. We describe and carry out computational experimentation providing evidence that this does not occur.

1. INTRODUCTION AND MOTIVATION

A *congruent* number is a rational number that appears as the area of a rational right triangle. The *congruent number classification problem* is the problem of determining whether a given number t is congruent. By scaling the triangle, one may consider only the classification of squarefree integers t ; we do this throughout the article. The study of congruent numbers is ancient and has a long history; see the survey article of Conrad [Con08] for more.

We consider whether a single congruent number t can have two distinct rational right triangles with the same hypotenuse. This question was first motivated by an observation in [HKLDW18].

In that paper, Hulse, Kuan, Walker, and the author consider a convolution sum whose main term exists only when t is congruent. Let $\tau(n)$ denote the square indicator function, taking 1 if n is a square and 0 otherwise. The primary theorem of [HKLDW18] is that

$$\sum_{n \leq X} \sum_{m \leq X} \tau(m+n) \tau(m) \tau(m-n) \tau(nt) = C_t \sqrt{X} + O_t((\log X)^{r/2}),$$

where r is the rank of the elliptic curve $E_t : Y^2 = X^3 - t^2 X$ and

$$C_t = \sum_{h \in \mathcal{H}(t)} \frac{1}{h}$$

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is the (convergent) sum over the set of hypotenuses $\mathcal{H}(t)$ of dissimilar primitive right integer triangles with area equal to tu^2 (i.e. the squarefree part of the area is t).

Although there is implicit dependence on t in the error term and poorly understood dependence on t in the main term, heuristically one should expect that for large X , the main term “quickly” dominates.

In [HKLDW18], it is also shown that the set of hypotenuses $\mathcal{H}(t)$ is only logarithmically dense in $\mathbb{Z}_{\geq 1}$. Thus heuristically the coefficient C_t of the main term is well-estimated by its first term m/h , where h is the smallest hypotenuse in $\mathcal{H}(t)$ and m is the multiplicity of this hypotenuse. Further, the main term should heuristically grow larger than the error term quickly once $X > h/m$.

The question of the size of the smallest hypotenuse in $\mathcal{H}(t)$ is closely related to the least height of a non-torsion point on E_t (we make this more precise in §3), but the multiplicity m is more mysterious.

In this article, we describe numerical experimentation suggesting that the multiplicity is always exactly 1. We also propose that this is always the case as a conjecture.

Conjecture. *There do not exist two dissimilar primitive right triangles with the same hypotenuse and whose squarefree parts of the areas are equal to each other.*

In §2, we describe why this problem might be hard to fully resolve. Hassett describes this more completely in the Appendix.

We then describe and carry out numerical experimentation using the free and open source math software SageMath [Sag20]. Our experimentation comes in two forms. Firstly, we do a deep investigation for those congruent numbers below 1000. For each congruent number, we generate many different triangles by generating rational points $E_t(\mathbb{Q})$ and the corresponding right triangles.

Secondly, we do a broad investigation for several congruent numbers corresponding to curves E_t of high rank. These curves will have more rational points up to a bounding height, heuristically leading to more triangles with hypotenuses up to some bound. In this investigation, we make heavy use of [WDE⁺14], which gives several congruent number curves E_t of rank 6 and 7. In total, we investigate 1513 curves of ranks 6 and 7.

We describe these approaches and the results in §3 and in §4, respectively.

2. ALGEBRAIC FORMULATION

One can parametrize right triangles with two variables (s, t) such that the sides are given by $(s^2 - t^2, 2st, s^2 + t^2)$. We follow the convention that the hypotenuse is written last in any triple (a, b, c) giving a right triangle, and that $s > t$. In this correspondence, primitive integer right triangles correspond to relatively prime integers s and t .

Thus finding two right triangles with the same hypotenuse and whose areas are the same up to multiplication by squares can be reformulated as finding integers (s, t, S, T) such that

$$s^2 + t^2 = S^2 + T^2$$

and where

$$\frac{st(s^2 - t^2)}{ST(S^2 - T^2)} \text{ is a square.}$$

This last equation can be rewritten as

$$u^2 st(s^2 - t^2) = v^2 ST(S^2 - T^2)$$

for some positive integers u, v .

These two equations define a surface

$$X \subset \mathbb{P}_{[s,t,S,T]}^3 \times \mathbb{P}_{[u,v]}^1.$$

If we could understand all points on this surface, we could almost certainly resolve the conjecture. Unfortunately, this understanding appears to be beyond current algebraic techniques. This is considered by Hassett in the Appendix, where it is shown that the surface X admits a resolution of singularities that is of general type and simply connected (see Proposition A).

Thus we expect that a complete resolution of the conjecture is not within reach.

3. DESCRIPTION OF METHODOLOGY

With a purely theoretical resolution likely out of reach, we turn now to computational experimentation. There is a well-known correspondence between right triangles (a, b, c) with $ab/2 = t$ (which we may assume without loss of generality is a squarefree integer) and $\mathbb{Q}$ -rational points on the elliptic curve $E_t : Y^2 = X^3 - t^2X$ where $Y \neq 0$. The inverse maps of this correspondence are given by

$$\begin{aligned} (a, b, c) &\mapsto \left(\frac{tb}{c-a}, \frac{2t^2}{c-a} \right) = (X, Y), \\ (X, Y) &\mapsto \left(\frac{X^2 - t^2}{Y}, \frac{2tX}{Y}, \frac{X^2 + t^2}{Y} \right) = (a, b, c). \end{aligned}$$

(See [Kob93] for a historical overview and description of the relationship between congruent numbers and elliptic curves).

3.1. Enumeration through elliptic curves. This provides an explicit and computable way of enumerating all rational right triangles with area t : find generators for $E_t(\mathbb{Q})$ and enumerate their linear combinations. This method of enumeration is of course vastly superior to naive enumeration and is at the core of the organization of the numerical experimentation.

In principle, it is necessary to also include the torsion subgroup of $E_t(\mathbb{Q})$. However for these curves, the torsion subgroup is completely understood and can be ignored. In particular, the torsion subgroup is isomorphic to

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by two points T_1 and T_2 . If the triangle (a, b, c) corresponds to the point $P \in E_t(\mathbb{Q})$, the eight equivalent triangles $(\pm a, \pm b, \pm c)$ correspond to the points $\pm P + \epsilon_1 T_1 + \epsilon_2 T_2$, where $\epsilon_i \in \{0, 1\}$. Thus we omit no triangles (up to similarity) by omitting consideration of torsion points.

3.2. Hypotenuses, and heights, grow exponentially. Given an elliptic curve with known rank and generators, the primary obstruction to collecting numerical evidence is the size of the triangles and hypotenuses. There is a general theory that the heights of points on elliptic curves grow very rapidly under repeated addition. In Corollary 1 of [HKLDW18], it is shown that

$$|\{h \in \mathcal{H}(t) : h \leq X/2t\}| = O_t((\log X)^{r/2}),$$

where r is the rank of the corresponding elliptic curve E_t . In particular, letting $H(P)$ denote the naive height of the X -coordinate of P , it is shown that the hypotenuse h corresponding to a point P on E_t satisfies $h \geq H(P)/2t$, and thus hypotenuses grow at least as quickly as the heights.

Example 1. The elliptic curve $E_6 : Y^2 = X^3 - 36X$ has rank 1 and the free part of $E_6(\mathbb{Q})$ is generated by $g = (-3, 9)$. This point corresponds to the primitive right triangle $(3, 4, 5)$. The hypotenuse is 3 bits long. The point $2 * g = (25/4, -35/8)$ corresponds to the primitive right triangle $(49, 1200, 1201)$, whose hypotenuse is 11 bits long. The point $4 * g$ has hypotenuse 2094350404801, which is 29 bits long. The point $400 * g$ (the largest multiple we considered on this curve) corresponds to a right triangle whose hypotenuse requires 410426 bits to store.

As heights grow exponentially, we see that hypotenuses grow exponentially. If we assume that the hypotenuses behave like exponentially growing random variables, then heuristically we would expect that it is very unlikely that two points of large height correspond to the same hypotenuse. Informally, we should expect that if a pair of points yields two triangles with hypotenuse that are close in size, then both points are probably of small height.

For curves of rank 1, it is impractical to compute more than a few hundred triangles — they simply grow too quickly. For curves of higher rank, however, it is possible to compute millions of triangles before the individual sides of the triangles become prohibitively large.

3.3. Two strategies of experimentation. We investigate two potential sources for a counterexample to the conjecture: when all numbers are small and coincidental collisions are most likely, and when there are many triangles up to a certain size due to high rank. Thus we organize our experimentation into two strategies: one for elliptic curves E_t with small t , and one for high rank elliptic curves.

First, we study the curves E_t for all squarefree $t \leq 1000$. Each of these curves has rank 0, 1, or 2. In total, this includes 361 curves corresponding

to congruent numbers. These are the curves with the smallest t and with the simplest triangles.

As these curves have low rank, we can carry out “deep” investigation on each curve. More specifically, if $\{g_i\}$ denote generators for the free part of $E_t(\mathbb{Q})$, then we can study triangles coming from $\sum a_i g_i$ for all coefficients $\{a_i\}$ in a large box $|a_i| \leq K$ for a constant $K = K(t)$. For rank 1 curves, we choose $K(t) \geq 300$, and for rank 2 curves we choose $K(t) \geq 75$. As the points P and $-P$ generate the same triangle, this translates to considering at least 300 triangles from each rank 1 curve and at least $(151^2 - 1)/2 = 11400$ triangles from each rank 2 curve.

Second, we study curves E_t of known high rank. To get these curves, we use the large-scale project detailed in [WDE⁺14], which builds off of earlier work of Rogers [Rog00, Rog04]. In this project, Watkins et al. investigated ranks of congruent number curves. They identify 1486 curves E_t of rank at least 6 and 27 curves of rank 7, giving 1513 curves in total of high rank. We examine rational points on each of these 1513 curves.

We note that no curves of higher rank are known, and it is conjectured that ranks in the family of congruent number curves are bounded [Hon61]. In §11 of [WDE⁺14], it is further demonstrated that a heuristic of Granville’s suggests that the maximum rank of a congruent number elliptic curve is 7.

As noted in §3.2, we should heuristically expect that it is more likely for points of small height to give two triangles with the same hypotenuse. For this reason, we carry out a “wide but shallow” investigation of these curves. Again letting $\{g_i\}$ denote a set of generators for the free part of $E_t(\mathbb{Q})$, we study triangles coming from points $\sum a_i g_i$, where all coefficients $\{a_i\}$ are in the small box $|a_i| \leq 4$. This gives 267520 triangles from each rank 6 curve and 2391484 triangles from each rank 7 curve.

Remark 2. It is likely that using the box $|a_i| \leq 2$ would be sufficient. In §4, we observe that $|a_i| \leq 2$ finds all closest pairs of hypotenuses on the curves investigated. We use the larger box to conduct a more robust investigation. Computing and manipulating these points is by far the most time intensive portion of this experiment. With the larger coefficient boxes $|a_i| \leq 4$, we spend approximately 32 minutes on average for each rank 6 curve, and approximately 5 hours on average for each rank 7 curve. In total, we used approximately 940 CPU hours on this part of the experimentation.

3.4. Computing rank and generators. In order to generate triangles for a congruent number t , we require that we can compute generators for (the free part of) $E_t(\mathbb{Q})$. We perform computations for two different sets of curves: those curves E_t with $t < 1000$ and for several chosen curves of higher rank.

For the 1513 curves of higher rank, we use ranks and generators available as an electronic supplement to [WDE⁺14].¹ We note that not all of these curves have explicitly known rank: all have rank at least 6, and 27 have rank 7, but Watkins et al. couldn't verify the ranks of each curve. We do not try to complete this verification here, and instead use the 6 known generators for each curve of presumed rank 6.

Curves corresponding to $t < 1000$ can be handled within SageMath [Sag20]. Directly using SageMath's `EllipticCurve.rank` and `EllipticCurve.gens` works for all but 84 cases. The smallest problematic example is 113. We note that the curve E_{113} is contained within the L-function and Modular Form Database (LMFDB) [LMF19]. However, it is also possible to directly use Tunnell's criterion [Tun83] to quickly confirm that 113 is not congruent. Applying Tunnell's criterion on the remaining 83 cases to remove non-congruent numbers reduces this to 54 cases. The smallest remaining example is 157.

The number 157 is congruent and is somewhat famous for its late classification by Zagier [Zag89]; the simplest rational triangle with area 157 is extremely complicated.

The techniques SageMath uses to compute the rank and generators for elliptic curves are based on John Cremona's MWRANK [CMP⁺19]). SageMath includes a copy of MWRANK and a partial interface to MWRANK's functionality. Calling MWRANK directly and allowing up to 2000 seconds of computation time per curve provides generators and ranks for an additional 20 cases (including 157). This leaves 34 congruent numbers up to 1000, the smallest of which is 277.

To understand the remaining 34 curves, we use functionality from PARI-GP [BBB⁺00], which is also packaged with SageMath. Although MWRANK struggles to exactly compute the rank and generators, it can compute that the upper bound on the rank is 1 for each of these curves. For the curve E_{277} for example, calling MWRANK through Sagemath with the command `EllipticCurve([-277*277,0]).rank_bound()` shows an upper bound of 1 for the rank.

Thus one can use the Heegner point method [Elk94] to find a generator for the rational points on these curves. The existence of such a generator confirms that the rank is exactly 1. This is implemented remarkably well in PARI. Calling `pari(EllipticCurve([-277*277,0])).ellheegner()` from SageMath quickly returns a generator for $E_{277}(\mathbb{Q})$.

Remark 3. The CAS Magma [BCP97] is also capable of generating ranks and generators for these elliptic curves. An anonymous Referee noted that `Generators(EllipticCurve([-277*277,0]))` returns a set of generators almost immediately, and functions almost as quickly for all congruent numbers up to 1000.

¹The supplement to their paper made this data available in Magma code. We've converted it to a SageMath-friendly format and made it available at <https://github.com/davidlowryduda/notebooks/blob/master/Papers/largetdata.sage>

4. EXPERIMENTAL RESULTS

We performed the strategies outlined in §3 for each squarefree congruent number t with $t \leq 1000$ and for the 1513 curves of higher rank from [WDE⁺14]. (See the author's demonstration github [LD] for a reference implementation to obtain and manipulate generators when $t \leq 1000$; for higher t , we do similar manipulations with the generators found by Watkins et al).

4.1. Results for $t \leq 1000$. Out of the 608 squarefree numbers up to 1000, we verified that 327 of them are congruent numbers and produced several triangles for each one. Of these, 274 corresponding to curves with rank 1 and 53 to curves with rank 2. We note that this is consistent with the celebrated conjecture of Goldfeld [Gol79], which implies that although there may be infinitely many such curves of rank at least 2, these should be sparse and correspond to 0% of all congruent numbers in natural density.

Among all these computed triangles, we found no counterexample to the conjecture. There were no two dissimilar right triangles with the same hypotenuse (after scaling to a primitive right triangle) corresponding to the same congruent number.

Recall that we computed at least 300 triangles from each curve of rank 1 and at least 11400 from each curve of rank 2. Per the heuristic in §3.2, we would expect that pairs of points corresponding to triangles with the close hypotenuses should come from points of small height.

This is true for all the curves tested for $t \leq 1000$. Further, the nearest pairs of hypotenuses among triangles associated to the same curve were always among the smallest hypotenuses. In terms of our generators $\{g_i\}$, the smallest pair of hypotenuses always came from triangles corresponding to the points $\sum a_i g_i$ with $|a_i| \leq 2$. (Although there are not canonical choices of generators, we note that MWRANK returns generators with small naive height [Cre97]).

Example 4. On $E_6(\mathbb{Q})$, whose free part is generated by $(-3, 9)$, the two nearest hypotenuses come from the triangles $(3, 4, 5)$ and $(49, 1200, 1201)$, which come from g and $2g$ respectively.

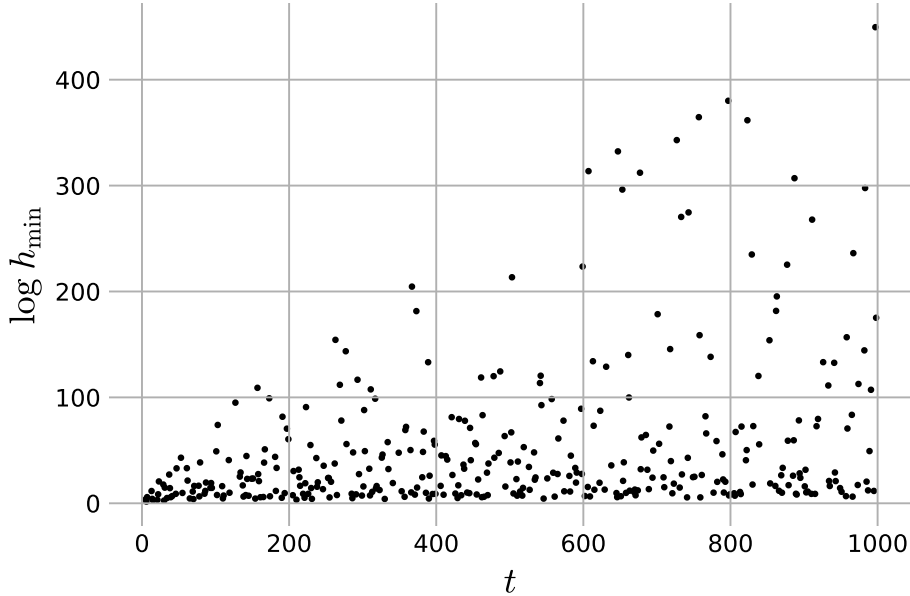
For $t = 34$, we have that $\text{rank}(E_{34}(\mathbb{Q})) = 2$. The free part of $E_{34}(\mathbb{Q})$ is generated by the points $g_1 = (-16, 120)$ and $g_2 = (-2, 48)$. The two closest hypotenuses come from the triangles $(225, 272, 353)$ and $(17, 144, 145)$, corresponding to g_1 and g_2 respectively.

This adds support for our choice of box $|a_i| \leq 4$ for points on higher rank elliptic curves.

Although no pair of hypotenuses exactly matched, we can ask about how close two hypotenuses could be. Of those congruent numbers $t \leq 1000$, we determined that 5 have pairs of dissimilar triangles with hypotenuses that differ by less than t : 210, 330, 546, 609, 915. All five of these come from rank 2 curves.

The curve E_{210} has a particularly close pair of triangles whose hypotenuses differ by 8. This is the pair of triangles $(12, 35, 37)$ and $(20, 21, 29)$, which correspond to the two generators for the free part of $E_{210}(\mathbb{Q})$.

FIGURE 1. Smallest hypotenuses for congruent numbers $t \leq 1000$



Smallest hypotenuses. Further, we took this opportunity to collect data on the size of the smallest hypotenuse for each congruent number ≤ 1000 . In Figure 1, we plot squarefree congruent numbers t against the log of the smallest hypotenuse of a primitive right triangle with squarefree part of the area equal to t .

Within the plot, the first congruent number t with smallest hypotenuse greater than e^{100} is Zagier’s number, 157. The *largest* smallest hypotenuse we computed corresponds to the congruent number $t = 997$, and is approximately e^{449} .

Many smallest-hypotenuses appear to be moderate in size, but no consistent pattern emerges.

4.2. Results for curves of rank 6 and 7. We produced hundreds of thousands of triangles for each of the 1513 curves of high rank. We found no counterexample to the conjecture.

As above, we tracked closest pairs of hypotenuses for each curve. Even though we computed all rational points $\sum a_i g_i$ in the coefficient box where each coefficient satisfies the bound $|a_i| \leq 4$, we again observed that for each t , the closest pair of hypotenuses came from points in the smaller box $|a_i| \leq 2$. The heuristic from §3.2 appears to continue to hold, indicating

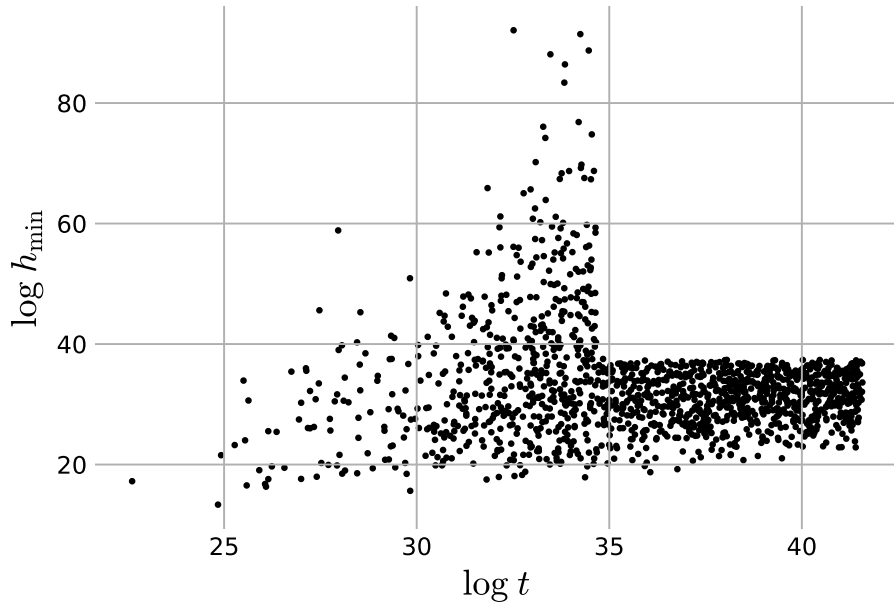
that nearest pairs of hypotenuses occur among the points of the least height on the curve.

We consider how close hypotenuses from two different triangles could be. Of all the 1513 higher rank curves investigated, 375 has two hypotenuses that differed by less than the corresponding congruent number t .

In absolute terms, the closest pair came from $t = 6611719866 \approx 6.611 \times 10^9$, which has triangles with hypotenuses 30544225 and 67119265, differing by about 3.657×10^7 . This is very large. We also noted a pair for the congruent number $t = 1902736244939034 \approx 1.9027 \times 10^{15}$, which has triangles with hypotenuses given approximately by 1.061×10^{10} and 1.109×10^{10} , differing by about 4.769×10^8 . These are also the two smallest hypotenuses of triangles for t . Although this is also very large, this is significantly smaller than both t and the smallest hypotenuse.

Both of these “near-miss” examples came from rank 6 elliptic curves.

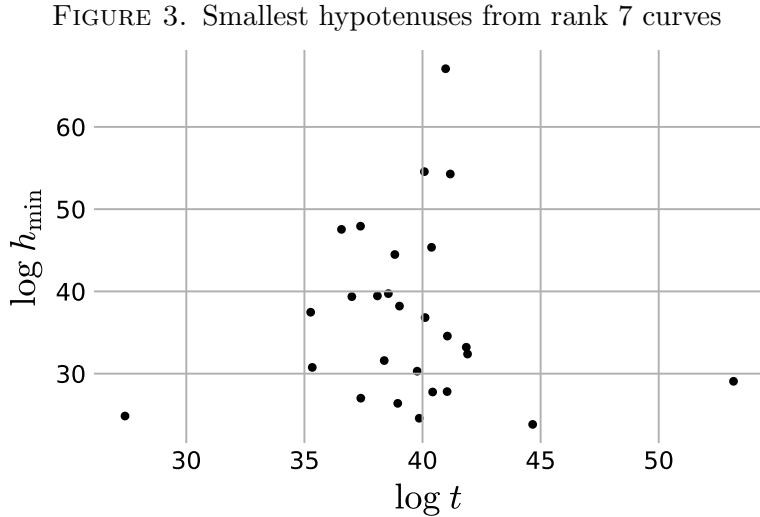
FIGURE 2. Smallest hypotenuses from (presumed) rank 6 curves



Smallest hypotenuses. We again took this opportunity to collect data on the size of the smallest hypotenuse coming from each of the 1513 curves of higher rank. As the numbers encompass a vastly larger domain, we present this in two log-log plots.

In Figure 2, we show smallest hypotenuses for each t from curves of (presumed) rank 6. We first explain the lack of points in the upper-right portion of the plot. This is caused by the methodology used to generate the 1513 curves in [WDE⁺14] — they performed an almost-exhaustive search for high

rank elliptic curves for all $t \leq 2^{50}$, while for higher t they only looked for curves with points of relatively low height. As $2^{50} \approx e^{34.5}$ and as the hypotenuse h corresponding to a point P satisfies the bound $h > H(p)/2t$, this explains the missing region of data. (Although we found it surprising that this plot reveals the choices of bounds used to find the elliptic curves so clearly).



Comparing Figure 2 with Figure 1, we note the disparity in size. Every hypotenuse coming from a higher rank elliptic curve was smaller than the smallest hypotenuse on E_{157} . On the other hand, each smallest hypotenuse was larger than e^{15} .

In Figure 3, we show smallest hypotenuses coming from curves of rank 7. As with the previous plot, there are patterns among the curves that result from the way in which the curves were initially found. In [WDE⁺14], they performed an extensive search for curves of rank 7 for $t \leq 2^{60} \approx e^{41.6}$, and for higher t they only looked for curves with rational points of low height. Although it is less obvious in this plot, we note that we should expect that there is a lot of missing data in the upper right portion of this plot.

APPENDIX A. INVARIANTS OF THE LOWRY-DUDA MODULI SPACE

by BRENDAN HASSETT

We verify that the moduli space X introduced by Lowry-Duda admits a resolution of singularities $\tilde{X} \rightarrow X$ that is of general type and simply connected. Since $\tilde{X}(\mathbb{Q}) \neq \emptyset$, existing Diophantine techniques (like Faltings' Theorem [Fal94]) shed little light on the structure of the rational points on X .

Double cover realization. Let

$$X \subset \mathbb{P}_{[s,t,S,T]}^3 \times \mathbb{P}_{[u,v]}^1$$

be the surface over $\mathbb{Q}$ defined by the equations

$$\begin{aligned} s^2 + t^2 &= S^2 + T^2 \\ u^2 st(s^2 - t^2) &= v^2 ST(S^2 - T^2). \end{aligned}$$

The first equation defines a surface $Y \subset \mathbb{P}^3$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Projection onto the first factor induces a morphism $\varpi : X \rightarrow Y$ that is generically finite of degree two. The branch curve

$$B := \{st(s^2 - t^2)ST(S^2 - T^2) = 0\} \subset Y$$

is a union of eight planar sections of Y , each a smooth conic curve defined over $\mathbb{Q}$.

The morphism ϖ fails to be flat over the locus

$$Z := \{st(s^2 - t^2) = ST(S^2 - T^2) = 0\} \subset B \subset Y$$

consisting of the following 32 points:

$$\begin{aligned} [0, 1, 0, \pm 1] &= \{s = 0\} \cap \{S = 0\} \\ [0, 1, \pm 1, 0] &= \{s = 0\} \cap \{T = 0\} \\ [0, \pm\sqrt{2}, 1, 1] &= \{s = 0\} \cap \{S - T = 0\} \\ [0, \pm\sqrt{2}, 1, -1] &= \{s = 0\} \cap \{S + T = 0\} \\ [1, 0, 0, \pm 1] &= \{t = 0\} \cap \{S = 0\} \\ [1, 0, \pm 1, 0] &= \{t = 0\} \cap \{T = 0\} \\ [\pm\sqrt{2}, 0, 1, 1] &= \{t = 0\} \cap \{S - T = 0\} \\ [\pm\sqrt{2}, 0, 1, -1] &= \{t = 0\} \cap \{S + T = 0\} \\ [1, 1, 0, \pm\sqrt{2}] &= \{s - t = 0\} \cap \{S = 0\} \\ [1, 1, \pm\sqrt{2}, 0] &= \{s - t = 0\} \cap \{T = 0\} \\ [1, 1, \pm 1, \pm 1] &= \{s - t = 0\} \cap \{S - T = 0\} \\ [1, 1, \pm 1, \mp 1] &= \{s - t = 0\} \cap \{S + T = 0\} \\ [1, -1, 0, \pm\sqrt{2}] &= \{s + t = 0\} \cap \{S = 0\} \\ [1, -1, \pm\sqrt{2}, 0] &= \{s + t = 0\} \cap \{T = 0\} \\ [1, -1, \pm 1, \pm 1] &= \{s + t = 0\} \cap \{S - T = 0\} \\ [1, -1, \pm 1, \mp 1] &= \{s + t = 0\} \cap \{S + T = 0\} \end{aligned}$$

Points of Z are nodes of B . The double cover of Y branched over B has A_1 singularities over these 32 nodes which are automatically resolved in X .

The remaining points of intersection among components of B are:

$$\begin{aligned} [0, 0, 1, \pm i] &= \{s = 0\} \cap \{t = 0\} \cap \{s - t = 0\} \cap \{s + t = 0\} \\ [1, \pm i, 0, 0] &= \{S = 0\} \cap \{T = 0\} \cap \{S - T = 0\} \cap \{S + T = 0\} \end{aligned}$$

Write

$$W := \{[0, 0, 1, \pm i], [1, \pm i, 0, 0]\},$$

where B has multiplicity-four singularities. There are $\binom{8}{2} = 28$ pairs of conics altogether so all intersections are accounted for.

Resolving the bad singularities. We briefly review background on surface singularities: A double cover of a smooth surface branched along a curve has ADE singularities if and only if the curve has simple (ADE) singularities. An isolated hypersurface singularity $s \in S$ is *Du Val* if there exists a resolution $\beta : \tilde{S} \rightarrow S$ such that $K_{\tilde{S}} - \beta^*K_S$ is effective. Complex Du Val singularities coincide with the ADE surface singularities [Rei97, §4.20]. Given a projective surface S with Du Val singularities, pluricanonical differentials on S are the same as pluricanonical differentials on a resolution $\tilde{S}$. In particular, if K_S is ample then $\tilde{S}$ is of general type.

Returning to X , we seek to resolve the singularities that are not Du Val, which are associated with the multiplicity four singularities of B along W . We observe that

$$W \subset \{s^2 + t^2 = S^2 + T^2 = 0\},$$

i.e., the vertices of a cycle of four rational curves contained in Y , whose complement is a torus. The blow-up

$$\tilde{Y} := \text{Bl}_W(Y) \rightarrow Y$$

is therefore toric with boundary consisting of an octagon of $\mathbb{P}^1$'s with self-intersections alternating between -1 and -2 . The induced blow-up

$$\beta : \tilde{X} \rightarrow X$$

resolves the singularities of X .

Canonical class of X . First, we choose a basis for the Picard group of $\tilde{Y}$: Let f_1 and f_2 be the rulings of $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $E_1, \dots, E_4$ the exceptional divisors over W so

$$\text{Pic}(\tilde{Y}) = \langle f_1, f_2, E_1, E_2, E_3, E_4 \rangle.$$

We order so that the boundary octagon has sides with classes

$$E_1, f_1 - E_1 - E_2, E_2, f_2 - E_2 - E_3, E_3, f_1 - E_3 - E_4, E_4, f_2 - E_4 - E_1.$$

The canonical class

$$K_{\tilde{Y}} \equiv -2f_1 - 2f_2 + E_1 + E_2 + E_3 + E_4.$$

The proper transform $\tilde{B}$ of B – noting the multiplicity four singularities along W – has class

$$8(f_1 + f_2) - 4(E_1 + E_2 + E_3 + E_4);$$

$\tilde{B}$ is disjoint from the four (-2) curves in the octagon. Blowing down these four curves induces a birational morphism

$$\tilde{Y} \rightarrow \Sigma$$

to a quartic del Pezzo surface with four A_1 singularities. This morphism is induced by the linear series $|2f_1 + 2f_2 - E_1 - E_2 - E_3 - E_4|$ corresponding to the quadratic equations for W modulo the defining equation for Y .

We refer the reader to [BHPVdV04, IV.22] for a discussion of the invariants of double covers of surfaces branched along curves with simple singularities.

The canonical class of $\tilde{X}$ is obtained by pulling back

$$K_{\tilde{Y}} + \frac{1}{2}\tilde{B} \equiv 2(f_1 + f_2) - (E_1 + \dots + E_4)$$

through the induced morphism $\tilde{X} \rightarrow \tilde{Y}$. The image of $\tilde{Y}$ and $\tilde{X}$ under this linear series equals Σ . We summarize this in the following proposition.

Proposition A. *The surface $\tilde{X}$ is of general type with $K_{\tilde{X}}^2 = 8$, and the linear series $|K_{\tilde{X}}|$ induces a morphism*

$$\tilde{X} \rightarrow \Sigma$$

that is generically finite of degree two.

In particular, we find

$$\dim \Gamma(K_{\tilde{X}}) = \dim \Gamma(\tilde{Y}, K_{\tilde{Y}} + \frac{1}{2}\tilde{B}) = 5.$$

$\tilde{X}$ is simply connected. To compute the other invariants of $\tilde{X}$, we use observations following immediately from simultaneous resolution of ADE singularities [Bri68]: $\tilde{X}$ is deformation equivalent to a double cover of $\tilde{Y}$ branched along a *generic* member of

$$|\tilde{B}| = |8(f_1 + f_2) - 4(E_1 + E_2 + E_3 + E_4)|,$$

disjoint from the (-2) -curves. Smoothing the A_1 singularities of Σ and noting that the branch divisor lies in $|-4K_{\Sigma}|$, we find that $\tilde{X}$ is also deformation equivalent to a double cover of a quartic del Pezzo surface Σ' branched over a divisor $A \in |-4K_{\Sigma'}|$. Such surfaces are simply connected by the Lefschetz hyperplane theorem, hence

$$\pi_1(\tilde{X}) = \{1\}.$$

The remaining invariants. The branch curve A satisfies

$$\deg K_A = 48$$

by adjunction on Σ' . Thus we conclude

$$\chi(\tilde{X}) = 2(\chi(\Sigma') - \chi(A)) + \chi(A) = 2(8 + 48) - 48 = 64,$$

$b_2(\tilde{X}) = 62$, and $h^1(\Omega_{\tilde{X}}^1) = 52$.

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