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This empirical paper explores students' conceptions of transformation as substitution equivalence by linking it to their definitions of substitution and equivalence. This work draws on Sfard's (1995) framework to conceptualize conceptions of substitution equivalence and its components, equivalence and substitution, each on a spectrum from computational to structural. We provide examples of student work to illustrate how students' understandings of substitution, equivalence, and substitution equivalence as an approach to justifying transformation may relate to one another.

Keywords: Equivalence, substitution, substitution equivalence, transformation, definitions.

Transformation has been framed as a core mathematical activity (Kieran, 2004), and all mathematical calculation can be viewed as a process of transformation. Researchers such as Kirschner & Awtry (2004) have found that students' thinking about transforming symbols tends to be rooted in visual patterns of symbols rather than a deeper understanding of mathematical structures. Since algebraic transformation is often taught procedurally, there is a need to frame these manipulations in a structural way. By exploring the core mathematical ideas that justify why particular transformations are mathematically valid, we conceptualize transformation as a process of replacing one symbolic object with an equivalent one, and name this process substitution equivalence (Wladis et al., 2020). This includes the process of identifying sub-objects and replacing them with equivalent ones in order to generate a new equivalent object. This identification process is non-trivial for many students, and we hypothesize that thinking around substitution equivalence may be intimately connected to many of the struggles that students have with transforming symbolic mathematics in various contexts, yet this idea has rarely been explored in research. Here we present a model of students' thinking around substitution equivalence and illustrate potential affordances it might have in analyzing student work.

Conceptual framework and Prior Research

Substitution equivalence as a lens for mathematical transformation

In this paper, we focus on students' thinking around *substitution equivalence*, or the notion that two expressions, equations, or other mathematical objects are equivalent if one can be generated from the other through a sequence of substitutions carried out using standard interpretations of syntactic structure and mathematically valid uses of mathematical properties (Wladis et al., 2020).

Definition of substitution

In order to see how all mathematical activity could be viewed through the lens of substitution equivalence, we define *substitution* more broadly than has been done explicitly in much existing research and curricula. Jones and colleagues (2012) describe substitution as "the replacement of one representation with another" (p.167). Our definition builds on this idea by requiring the equivalence of mathematical objects being replaced and extending to other mathematical objects as well, such as equations. For us, substitution is the process of replacing *any* mathematical object (or any unified subpart of an object) with any *equivalent* object, regardless of complexity. This includes not only the replacement of x in $2x^2 - 2x + 1$ with -3, but also the replacement of $x^2 - 6x = 1$ with the equivalent equation $x^2 - 6x - 1 = 0$ during solving.

Definition of equivalence

Substitution equivalence is dependent upon an underlying equivalence relation. This may be a context-specific definition of equivalence (e.g., insertion equivalence in Zwetzschler and Prediger, 2013), or a more generalized concept of equivalence (e.g., an equivalence relation). Indeed, any definition of equivalence that satisfies equivalence relation criteria could be used.

Definition of substitution equivalence

Despite the importance of substitution equivalence to algebraic justification, little research has focused explicitly on substitution equivalence (see e.g., Pinkernell et al., 2017). A search in ERIC (the education research database maintained by the US Institute of Education Sciences, https://eric.ed.gov/) yields no results for *substitution equivalence*, *substitutional equivalence* or *substitution property of equality*. Other researchers have explored the "substitution principle", which refers to the structural sameness preserved when replacing a variable with a compound term, or vice versa (Musgrave et al., 2015). While this is related to our definition, as both rely on the substitution property of equality, this is not how we use this term. We define the domain of substitution equivalence as composed of two main ideas:

1. The general notion of substitution equivalence: A student understands that we can replace an object with any other equivalent object when problem-solving.
2. The notion that substitution of unified sub-objects preserves equivalence: A student understands that objects can be broken into unified sub-objects, and that replacing any unified sub-object with an equivalent unified sub-object produces an object that is equivalent to the original one (as long as substitution leaves the rest of the structure of that object unchanged).

The second notion leads us to another core definition: We use the term *subexpression* (or sub-object, more generally) to denote a substring of an expression (or other object) that can be treated as a unified object without changing the syntactic meaning of the original expression (or object). For example, $a - b$ is a subexpression of $a - b - c$, but $b - c$ is not (because putting brackets around $b - c$ would change the syntactic meaning of the whole expression). This different from, but related to, what Kieran (1989) refers to as surface structure (identifying the syntactic meaning of a symbolic algebraic representation) and what Malle (1993) refers to as *Termstrukturen* ("expression structuring") (identifying all algebraic expressions with the same syntactic meaning).

In this paper, we use Sfard's (1995) work to frame our thinking about student conceptions, where student thinking about a concept may be operational (as a process, often of computation) or structural (as an abstract object in and of itself). We conceptualize students' definitions on a continuum that can be primarily structural, primarily operational, or somewhere in between. A student may vary along the spectrum flexibly, but the ability to think structurally, at least some of the time, is necessary in order to progress to some higher order processes (Sfard, 1995).

Aside from Wladis et al. (2020), we have found little (if any) work on student conceptions of substitution equivalence, although there has been substantial work around equality. One common strand focuses on conceptions of the equal sign, where students see the symbol either operationally (as a ‘do something symbol), or relationally (as a relationship between two entities) (Knuth et al., 2006). In terms of substitution, relatively little work has been done, although some research has explored student notions of substitution equivalence in the context of arithmetic (Jones et al., 2012).

Model of operational and structural view of substitution equivalence

Wladis et al (2020) described key features of students’ thinking around substitution equivalence on a spectrum from structural to operational approaches. This paper aims to take this further by drawing on empirical data to explicitly describe how students’ conceptions of substitution equivalence may be dependent upon their definitions of substitution and equivalence (see Figure 1).

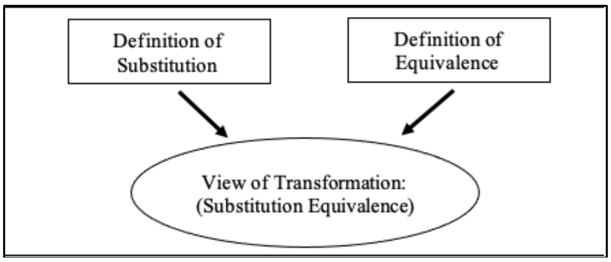


Figure 1: Model of student thinking about substitution equivalence

Method

This work draws on data collected from multiple classes across six years at an urban community college in the US, including cognitive interviews and open-ended questionnaires. Open-ended questionnaires and cognitive interviews were distributed to participants in courses from elementary algebra through linear algebra within a larger data collection process in efforts to develop an algebra concept inventory. These data were analyzed using grounded theory (Strauss & Corbin, 1990) to generate and refine models of students’ conveyed meanings to explain their written and spoken work. Categories developed during analysis were heavily influenced by the work of Sfard (1995), and existing literature about students’ understanding of the equals sign (e.g., Knuth et al., 2006). An initial coding scheme was developed by a single coder, and then in subsequent rounds, multiple coders revised the scheme until consensus was reached; coders included mathematicians, mathematics education researchers, and elementary algebra instructors.

The Model

Table 1: Components of substitution equivalence model

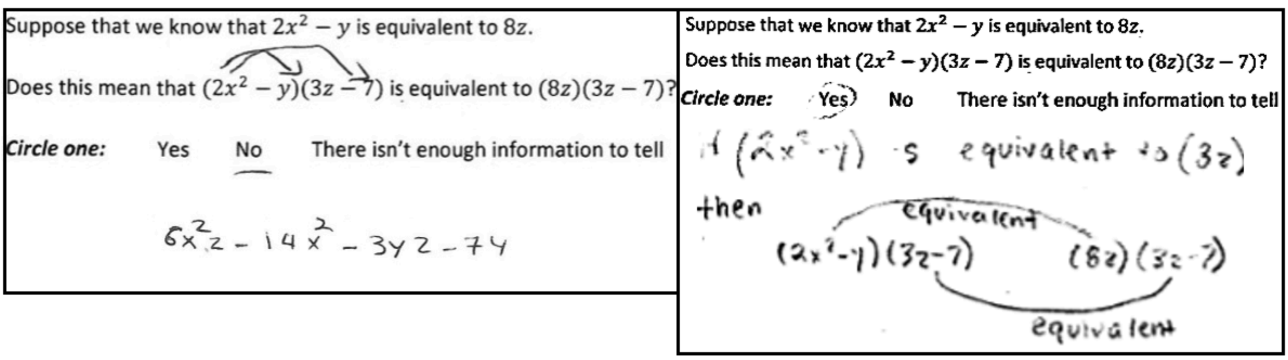
	Operational Thinking	**Structural Thinking**
View of Transformation	Students see transformations of expressions and equations (or other objects) as a process of “operating on” the original object itself. They may or may not see this as linked to any notion of equivalence.	Students see each step in a transformation as the process of replacing one object with an equivalent one through substitution, using properties and existing syntactic structure. They appear to have some notion of an equivalence class as an object (which need not be formally defined).

Definition of Equivalence	Students either ignore the notion of equivalence entirely, or appear to have only vague, ill-defined, or unstable notions of equivalence, or try to apply one definition of equivalence that works only in one context to another context.	Students have a well-defined and relatively stable definition of equivalence, and recognize that it is context-dependent. They recognize that equivalence is a fixed trait (two objects are either equivalent under a particular definition or not—they do not "become" equivalent).
Definition of Substitution	Students see substitution only as plugging a number in for a variable (and then computing the result). They see variables as representing only numbers.	Students see replacement of any object (or sub-object) with an equivalent one as substitution. They see variables as representing any valid mathematical object, including numbers or (potentially complex) expressions.

In the model in Figure 1, holding well-defined and standard definitions of both substitution and equivalence are necessary but not sufficient conditions for students to develop a view of transformation justified by substitution equivalence. A student may have trouble thinking of transformation as substitution equivalence because (a) their definitions of substitution are too narrow; (b) their definitions of equivalence are ill-defined, unstable, or mathematically invalid; (c) they do not draw on their knowledge of substitution and/or equivalence when performing transformation; or a combination of all of these. We conceptualize students' views of substitution, equivalence, and transformation as being on a continuum from operational to structural (Table 1). This model is based on the notion that the *ability* to conceptualize transformation as a process of substitution equivalence may be useful for students in developing deeper understanding of the justification behind their transformation work (and a way of checking the validity of that work).

Vignettes: A model in action

We now provide examples of students' written work from our dataset to illustrate how one might use the model we present here. These are intended to highlight the continuum of operational and structural views. To see how students' views of transformation as substitution equivalence can vary along this spectrum, we present two developmental algebra (a non-credit course with similar content to Algebra I from secondary school) students' responses about assessing whether two expressions are equivalent (Figure 2), where the first response (Figure 2a) exemplifies an operational view and the second response (Figure 2b) exemplifies a structural view.



(a) **(b)**

Figure 2: Examples of responses rooted in an operational (a) and structural view (b) of equivalence

The first student's response (Figure 2a) foregrounds computation and symbolic manipulation, so we classify it as an operational view of transformation. In cognitive interviews, students on similar problems have provided similar work and explained that they can only tell if two expressions are equivalent if they both simplify to the same final "answer". Hence we see the approach taken in

Figure 2a as indicative of having an internal computational definition of equivalence of “expressions that simplify to the same thing”. In contrast, the response in Figure 2b illustrates exactly how the two equivalent subexpressions are substituted into the larger expressions using arrows to indicate the relationship between each piece and to highlight the structure of the two expressions. This student mapped each unified subexpression in the first expression to an equivalent unified subexpression in the same place in the second, in order to illustrate why the two expressions are equivalent. Though the student didn’t use the word “substitution”, we see evidence that they were depicting a replacement or exchange of one equivalent sub-part with another.

Student definitions of equivalence

To see how students’ definitions of equivalence can vary, we refer to the previous examples and consider the definitions of equivalence the students seem to be evoking. These responses exemplify operational and structural definitions of equivalence, respectively. In Figure 2a, the student attempted to simplify the expressions to determine whether they are equivalent, and then appeared to decide that the expressions were not equivalent after they could not simplify them further. This definition (“two expressions are equivalent only if they simplify to the same thing”) of equivalence appears to be computational, and their work doesn’t seem to explicitly acknowledge equivalence relationships which justify their work. Because the student abandoned the attempt after simplifying did not work, this suggests that they did not see a way to use the structure of the given expressions to determine equivalence beyond simplifying both sides to see if the results are the same. In contrast, the response in Figure 2b suggests that the student may have a structural definition of equivalence. They drew on the structure of two complex expressions to show how they map to one another in such a way that each subexpression is either the same or equivalent, and they leveraged that structure to show that the final result is equivalent. This definition of equivalence appears to be well-defined and to be identifying fixed traits of the expressions.

Student definitions of substitution

To demonstrate differences along this spectrum, we look at two students’ definitions of substitution evoked from the prompt “In math, what is substitution? (Or what does it mean to substitute?)”. One student wrote “To substitute is to replace a number with a variable”, and further provided an example “$2x + 3 = 9$; $2(3) + 3 = 9$; $x = 3$”. This response (“putting a number in for a letter”) was one of the most common given by students at all levels, from elementary algebra through linear algebra. We classify this narrow definition of substitution as operational, whereas the response “To replace one number, variable, or expression for another” (a response from another student) was classified as structural definition of substitution. This is because their definition affords a greater variety of terms to be replaced for one another, which involves conceptualizing complex subexpressions as objects.

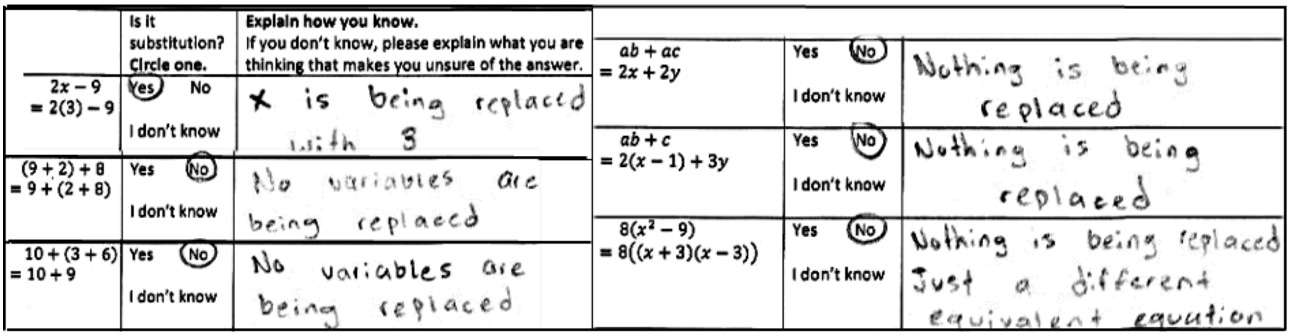
In order to see how students’ views of substitution may impact their view of transformation of expressions, we further examined their responses to a task to identify instances of substitution, and found that their responses were typically consistent with their definitions (e.g., only recognizing transformation as substitution when it involved a number being substituted in for a letter if that was their stated definition); we include one such example of this in the next section.

Using the framework to analyze student work longitudinally

In order to illustrate the potential of this model for deeper analysis, we consider examples from a single Algebra I student (whom we call Epsilon, like ε) across multiple tasks and points in time.

Substitution

We first consider Epsilon's definition of substitution, who gave the response that classified as an operational definition of substitution in the prior section. This correlates with the extent to which they identify different computations as substitution in the following work (Figure 3).



	Is it substitution? Circle one.	Explain how you know. If you don't know, please explain what you are thinking that makes you unsure of the answer.
$2x - 9 = 2(3) - 9$	(Yes) No I don't know	x is being replaced with 3
$(9 + 2) + 8 = 9 + (2 + 8)$	Yes (No) I don't know	No variables are being replaced
$10 + (3 + 6) = 10 + 9$	Yes (No) I don't know	No variables are being replaced
$ab + ac = 2x + 2y$	Yes (No) I don't know	Nothing is being replaced
$ab + c = 2(x - 1) + 3y$	Yes (No) I don't know	Nothing is being replaced
$8(x^2 - 9) = 8((x + 3)(x - 3))$	Yes (No) I don't know	Nothing is being replaced Just a different equivalent equation

Figure 3: Epsilon's interpretations of substitution in specific contexts

We can see in Figure 3 that Epsilon rarely identified computation as substitution when it was more complex or generalized. They noticed, for example, that the expressions in the last question in Figure 3 are equivalent, but they did not see replacement of the subexpression $x^2 - 9$ with $(x + 3)(x - 3)$ as an instance of substitution ("nothing is being replaced"), which is consistent with the more limited operational definition of substitution that they provided in the previous section.

Equivalence

Now we consider Epsilon's definition of equivalent expressions. When given the prompt "How could you check whether two mathematical expressions are equivalent? (An expression is a mathematical phrase that does not contain an equals or inequality sign)", Epsilon wrote "If they both have the same correct answer". Epsilon provided a seemingly correct (if perhaps incomplete or ill-defined) definition of equivalent expressions. We cannot be sure whether they understand that expressions must have the same value for every possible combination of variable values or that this applies to algebraic and not just arithmetic expressions, and the word "answer" is ill-defined; however, their definition is in line with the standard definition used in algebra, and they correctly identified that the algebraic expressions in the last question in Figure 3 were equivalent (as well as in other questions not shown here). Their definition also appears to be operational, as it is rooted in computations with expressions.

Substitution equivalence

Now we consider the extent to which Epsilon recognized instances of substitution equivalence in certain algebra examples. Epsilon was given the following two questions: 1.) "Suppose we know that $2x^2 - y$ is equivalent to $8z$. Does that mean that $(2x^2 - y)(3z - 7)$ is equivalent to $(8z)(3x - 7)$?" and "Suppose we know that $3a + b$ is equivalent to $42a$. Does that mean that $7a - 5 + (3a + b) + b^2 - 3a^2$ is equivalent to $7a - 5 + 42a + b^2 - 3a^2$?". Epsilon did not recognize either example as substitution equivalence, given that his response was "I don't know". To the first prompt, they wrote

"$24z^2 - 64z$" seemingly to simplify the expression "$(8z)(3x - 7)$" (in line with their definition of equivalence in the prior section), but this did not help them to identify whether the two expressions are equivalent. They did not appear to draw on the given fact that $2x^2 - y$ is equivalent to $8z$ when attempting to determine if the two larger expressions are equivalent. They provided no additional inscriptions in response to the second question.

This suggests that they may not have a notion of substitution equivalence or may be unable to draw on it in this problem context. Epsilon's operational approach when attempting to determine if the two expressions are equivalent suggests that their operational conception of equivalence may be limiting their ability to recognize and use substitution equivalence when performing mathematical transformations. Another potential barrier to Epsilon developing a robust notion of substitution equivalence and linking this to their transformation work may be their narrow notion of substitution itself. They likely did not recognize the transformations in the questions presented here as substitution just like they did not recognize most of the transformations in Figure 3 as substitution.

Potential impacts of instruction

Epsilon was part of a cohort that took part in a semester-long classroom intervention in which students were taught broader structural definitions of substitution, equivalence, and how to view transformation as substitution equivalence explicitly (as well as other concepts). After the intervention, Epsilon was not able to identify substitution equivalence in all cases, but they were able to recognize it in cases similar to those in the prior section. When given the prompt "Suppose that $3x = 2y + 1$. Does that mean that $5x^2 - (3x) + 7 = 5x^2 - (2y + 1) + 7$?", Epsilon wrote "Yes because $3x = 2y + 1$; its plugged in correctly". From this response, we see how they were able to recognize a complex equation as an equivalence relationship between two structurally identical expressions where one equivalent subexpression could be conceptualized as having been substituted for another. Epsilon's use of the words "plugged in" is a common phrase often used by students to indicate substitution. We do note, however, that this language still suggests a computational approach. However, Epsilon is drawing on structural features of equivalent algebraic expressions through the lens of substitution equivalence, even if their approach is still partially operational. We have insufficient space to discuss the intervention at length here—we simply include this short example to demonstrate that more structural and well-defined definitions of substitution, equivalence, and substitution equivalence approaches to transformation can all be learned by students, even those in developmental mathematics courses, when students are given the right supports.

Conclusion

We have presented a model that describes how students' definitions of substitution and equivalence may related to their ability to justify transformation through the lens of substitution equivalence. Using students' work, we have illustrated some of the affordances of this lens. We have demonstrated that students may struggle with substitution equivalence for different reasons, which may then require different instructional approaches. For example, if a student's definition of equivalence is ill-defined, it may be important to find ways for them to improve their personal definition; whereas if a student has broad and well-defined definitions of substitution and equivalence, a more effective approach may be to help them to see connections between this existing knowledge and their computational work when performing transformations. These are very different approaches to solving what might on the surface look like similar errors, but which stem from very different underlying patterns of how students think about mathematics. Thus, we hope that this model may aid us to better tailor instruction to respond to how to students think, and to better think about how definitions of substitution and equivalence are presented in instruction. We have also shown through one student's work that, with the right kind of instructional approaches, students can learn to think about transformation through a substitution equivalence lens. Further research is needed to investigate which ways of thinking may be most productive for students in different contexts.

Acknowledgments

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References

Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of experimental child psychology*, *113*(1), 166–176. https://doi.org/10.1016/j.jecp.2012.05.003

Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of the Teaching and Learning of Algebra. The 12th ICMI Study* (pp. 21–33). Springer. https://doi.org/10.1007/1-4020-8131-6_2

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education*, *35*(4), 224–257. https://doi.org/10.2307/30034809

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*(4), 297–312. https://doi.org/10.2307/30034852

Malle, G. (1993). *Didaktische Probleme der elementaren Algebra*. Springer. Wiesbaden, Germany.

Musgrave, S., Hatfield, N., & Thompson, P. (2015). Teachers' meanings for the substitution principle. In *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 801-808). Pittsburgh, USA: RUME.

Pinkernell, G., Düsi, C., & Vogel, M. (2017, February). Aspects of proficiency in elementary algebra. In *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME 10)*. Dublin, Ireland: ERME.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior*, *14*(1),15–39. http://doi.org/10.1016/0732-3123(95)90022-5

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research*. Newbury Park: Sage publications.

Wladis, C., Offenholley, K., Beiting-Parish, M., Griffith, S., Jaffe, E., Thakkar, N., & Dawes, D. (2020). A Proposed Framework of Student Thinking around Substitution Equivalence: Structural versus Operational Views. In *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 762–768). Boston, USA: RUME.

Zwetzschler, L., & Prediger, S. (2013). Conceptual challenges for understanding the equivalence of expressions–a case study. In *Proceedings of the Eighth Congress of European Research in Mathematics Education (CERME 8)* (pp. 558–567). Antalya, Turkey: ERME.