TOWARD CRITICALITY OF THE NAVIER-STOKES REGULARITY PROBLEM

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Abstract. The goal of this review paper is to summarize the current state-of-the-art of a regularity theory of the 3D Navier-Stokes (NS) equations based on the concept of 'sparseness at scale' of the regions of intense fluid activity that has recently lead to the first mathematical evidence of the *asymptotically critical nature* of the NS regularity problem, as well as to briefly discuss several possible avenues of arriving at stronger and/or more classical manifestations of criticality.

1. Reducing the 'scaling gap' and asymptotic criticality

Recall that 3D Navier-Stokes equations (NSE) – describing a flow of 3D incompressible, viscous Newtonian fluid – read

$$u_t + (u \cdot \nabla)u = -\nabla p + \Delta u,$$

complemented with the incompressibility condition $\operatorname{div} u = 0$. Here, a vector-valued function u is the velocity of the fluid and a scalar-valued function p is the pressure; the viscosity is set to 1 (without loss of generality) and the external force to zero (for simplicity). In what follows, the spatial domain will be the whole space.

Denoting the vorticity of the fluid by ω ($\omega = \operatorname{curl} u$), the vorticity-velocity formulation of the 3D NSE is then

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \Delta\omega;$$

the LHS is the transport of the vorticity by the velocity, the first term on the RHS is the vortex-stretching term, and the second one the diffusion–these are the three principal mechanical drivers of the system.

The NS system features a unique scaling invariance. Let $\lambda > 0$ be a scaling parameter; it is transparent that if a pair $u = u(x,t),\ p = p(x,t)$ is a solution to the NSE, then the pair

$$u_\lambda(x,t) = \lambda\, u(\lambda x, \lambda^2 t),\ p_\lambda(x,t) = \lambda^2\, p(\lambda x, \lambda^2 t)$$

is a solution to the NSE as well (corresponding to the rescaled initial condition, and over the rescaled interval of time).

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Super-criticality of the NS regularity problem is manifested by the actuality that all the regularity criteria–the mathematical conditions preventing possible formation of a singularity–are at best scaling-invariant (with respect to the above scaling; this is precisely the scaling level at which the nonlinearity and the diffusion balance out), while all the corresponding *a priori* bounds had been on the scaling level of the bounded kinetic energy, $u \in L^\infty(0,T;L^2)$, i.e., there has been a 'scaling gap'.

An emblematic example is furnished by the classical Ladyzhenskaya-Prodi-Serrin regularity criterion, $u \in L^p(0,T;L^q)$,

$$\frac{3}{q} + \frac{2}{p} = 1$$

vs. the corresponding *a priori* bound $u \in L^p(0,T;L^q)$,

$$\frac{3}{q} + \frac{2}{p} = \frac{3}{2}$$

(for a suitable range of the parameters).

In order to escape the above scaling entrapment, one may wish to visit the world of turbulence phenomenology and search for the source(s) of scaling anomalies and scaling corrections therein. Considering the vorticity description of the flow, a frontrunner appears to be coherent vortex structures (e.g., vortex sheets and vortex tubes/filaments) [JWSR93, S81, SJO91, VM94], and more broadly, spatial intermittency of the regions of intense fluid activity. This was one of the principal motives for designing a mathematical framework based on the 'scale of sparseness' of the vorticity super-level sets as an environment amenable to mathematical analysis of the phenomenon of spatial intermittency ([Gr13]). Several key definitions are listed below.

Let S be an open subset of $\mathbb{R}^3$ and μ_d the d-dimensional Lebesgue measure.

Definition 1.1. For a spatial point x_0 and $\delta \in (0,1)$, an open set S is 1D δ-sparse around x_0 at scale r if there exists a unit vector ν such that

$$\frac{\mu_1\left(S \cap (x_0 - r\nu, x_0 + r\nu)\right)}{2r} \le \delta \ .$$

The volumetric version is the following.

Definition 1.2. For a spatial point x_0 and $\delta \in (0,1)$, an open set S is 3D δ-sparse around x_0 at scale r if

$$\frac{\mu_3\left(S \cap B_r(x_0)\right)}{\mu_3(B_r(x_0))} \le \delta \ .$$

(It is straightforward to check that for any S, 3-dimensional δ-sparseness at scale r implies 1D $(\delta)^{\frac{1}{3}}$-sparseness at scale r around any spatial point x_0;

however the converse is false, i.e. local-1D sparseness is in general a weaker condition.)

The main mathematical scheme in this approach is as follows. Local-in-time analytic smoothing (in the spatial variables [FT89, GrKu98])–measured in L^∞–represents a very strong manifestation of the diffusion. This provides a suitable environment for the application of the harmonic measure majorization principle (the log-convexity of the modulus of an analytic function), cf. Proposition 1.5. Shortly, if the regions of the intense vorticity are 'sparse enough', the associated harmonic measure will be 'small enough' to prevent any possible growth of the L^∞-norm (the weight associated to the region producing the maximum will be small enough to counter any further growth) and–consequently–any possible formation of singularities.

Figure 1 illustrates various vorticity super-level sets in the case of the Kida-vortex initialized flow. The Kida flow features the maximal number of symmetries on a periodic cube (in addition to periodicity, there are bilateral symmetry, rotational symmetry and permutational symmetry of the velocity components), and was introduced as a model turbulent flow amenable to reduced computational complexity due to the symmetries [K85, KM87]. It was also considered as a computational laboratory for the study of possible formation of singularities in Euler and NS flows [BP94]. Here, it is transparent that the super-level sets of the positive and the negative parts of the vorticity components are much sparser than the composite.

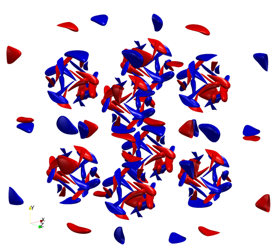
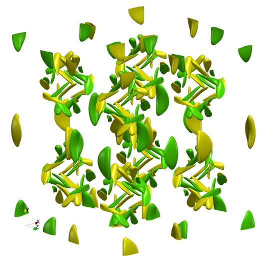
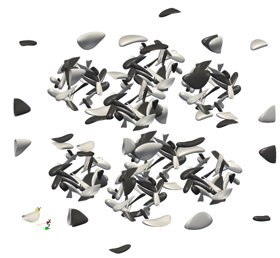
Henceforth, denote the positive and the negative parts of the vectorial components of f by $f_i^\pm$, and compute the norm of a vector as the maximum norm, $v = (a, b, c)$ as $|v| = \max\{|a|, |b|, |c|\}$. A family of uniformly-local functional classes defined below is central to the theory.

Definition 1.3 ([BrFaGr18])**.** For a positive exponent α, and a selection of parameters λ in $(0,1)$, δ in $(0,1)$ and $c_0 > 1$, the class of functions $Z_\alpha(\lambda, \delta; c_0)$ consists of bounded, continuous functions $f : \mathbb{R}^3 \to \mathbb{R}^3$ subjected to the following uniformly-local condition. For x_0 in $\mathbb{R}^3$, select the/a component $f_i^\pm$ such that $f_i^\pm(x_0) = |f(x_0)|$ (the maximum norm at the point x_0), and require that the set

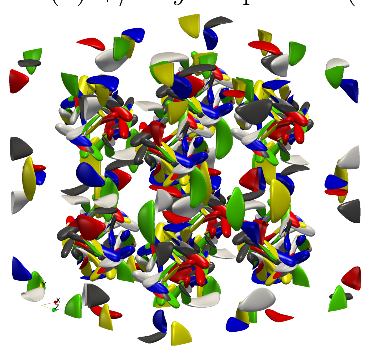
$$S_i^\pm(x_0) := \left\{ x \in \mathbb{R}^3 : \, f_i^\pm(x) > \lambda \|f\|_\infty \right\}$$

be 3D δ-sparse around x_0 at scale $c \, \frac{1}{\|f\|_\infty^\alpha}$, for some c, $\frac{1}{c_0} \leq c \leq c_0$. Enforce this for all x_0 in $\mathbb{R}^3$. Here, α is the scaling parameter, c_0 is the size-parameter, and λ and δ are the (interdependent) 'tuning parameters'.

It is instructive to compare the Z_α-classes, based on the scale of sparseness of the super-level sets, with the weak Lebesgue spaces, L^p_w, based on the rate of decay of the volume of the super-level sets (the rate of decay of the distribution function of the function in view). Recall that $f \in L^p_w$ if there

(A) $+/-$ x-components (B) $+/-$ y-components (C) $+/-$ z-components



(D) vorticity magnitude

FIGURE 1. Various super-level sets from the Kida simulation [Raf19]

exists $c_p > 0$ such that

$$\mu_3\Big(\{x \in \mathbb{R}^3 : |f(x)| > \gamma\}\Big) \le \frac{c_p^p}{\gamma^p} \text{ for any } \gamma > 0.$$

It is transparent that $f \in L^p_w$ implies $f \in Z_{\frac{p}{3}}$. In addition, in the geometrically worst case scenario (for sparseness), the super-level set clumped into a single ball, being in $Z_{\frac{p}{3}}$ is consistent with being in L^p_w. However, in general, $f \in Z_{\frac{p}{3}}$ gives no information on the rate of decay of the distribution function of f.

In ([BrFaGr18]), the authors applied the Z_α framework to the vorticity formulation of the 3D NSE–this lead to the first reduction of the scaling gap since 1960s. The following table summarizes the results and provides a comparison with the classical pointwise-in-time setting of $L^\infty(0,T;L^p)$.

All the regularity classes in the table are scaling invariant. The *a priori* bounds in the first two rows are at the scaling level of the energy class while–in contrast–the *a priori* bound in the last row–*near a possible singular time*–breaks away from the energy-level scaling.

Looking ahead, let us note that there is a simple geometric scenario in which–within the Z_α framework–the scaling gap closes completely and the NS regularity problem becomes critical. Suppose that the the vorticity

Regularity class	A priori bound	Energy-level class
$u \in L^\infty(0,T;L^3)$	$u \in L^\infty(0,T;L^2)$	$u \in L^\infty(0,T;L^2)$
$\omega \in L^\infty(0,T;L^{\frac{3}{2}})$	$\omega \in L^\infty(0,T;L^1)$	$\omega \in L^\infty(0,T;L^1)$
$\omega(\tau) \in Z_{\frac{1}{2}}$ for a suitable $\tau < T$, small size-parameter	$\omega(\tau) \in Z_{\frac{2}{5}}$ on $(0,T)$, whenever $\|\omega(\tau)\|_\infty$ is sufficiently large, the size-parameter uniform in time	$\omega(\tau) \in Z_{\frac{1}{3}}$ on $(0,T)$, the size parameter uniform in time

super-level sets comprise an ensemble of $O(1)$-long vortex filaments (not 'tightly packed') and assorted geometric structures at the scales smaller than the scale signifying the diameter of the filaments. Then the *a priori* bound $\omega \in L^\infty(0,T;L^1)$ ([Co90]) and Chebyshev's inequality imply that the solution in view is in $Z_{\frac{1}{2}}$, i.e., one arrives at the criticality.

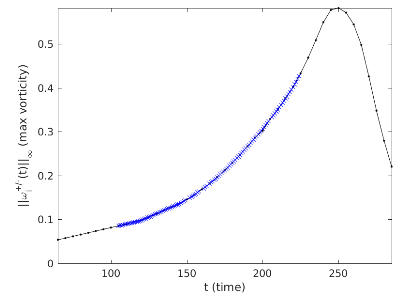
One might wonder what happens in the 'non-filamentary' geometric scenarios, e.g., in the flow initiated at the Kida-vortex. A comprehensive study of the scale of sparseness in this–plausibly 'Z_α-unfriendly'–scenario was performed in collaboration with a computational group at Aarhus University [Raf19] revealing that–over a time interval leading to the burst of the vorticity magnitude–the solution stabilized in $Z_{\frac{1}{2}+\epsilon}$ (see Figure 2). This indicated that a further progress in closing the scaling gap within the Z_α framework might be possible.

One of the follow-up avenues taken was to study the *higher-order spatial fluctuations* of the velocity field featuring more pronounced spatial intermittency and–consequently–harboring a chance for more pronounced deviation from 'the scaling'. Considering the sequence of functional classes $Z^{(k)}_{\alpha_k}$ defined by the following rule,

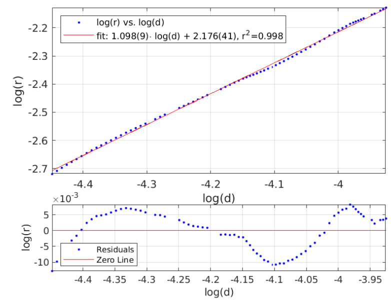
$$u \in Z^{(k)}_{\alpha_k} \quad \text{if} \quad D^{(k)}u \in Z_{\alpha_k},$$

for a suitable decreasing sequence $\{\alpha_k\}$, turned out to be the most profitable route (this could be viewed as constructing a 'Sobolev scale' based on Z_α).

Two central results in this setting ([GrXu19ac]) are delineated in the first two columns of the table below (the third column is included for comparison). The regularity class result holds for any $k^* \geq k^{**}(\|u_0\|_2, \|u_0\|_\infty)$.



(A) formation of the peak of the vorticity magnitude in the Kida-vortex simulation



(B) log-log plot of the scale of sparseness r realized in the flow *vs* the theoretical dissipation scale d over the time slices marked in (A)

FIGURE 2. Detecting the onset of turbulent dissipation in the Kida-vortex simulation within the Z_α framework [Raf19]

Regularity class	A priori bound	Energy-level class
$u(\tau) \in \bigcap_{k\geq k^*} Z^{(k)}_{\frac{1}{k+1}}$ on a suitable $(T^*-\epsilon, T^*)$, small size-parameters, uniform in time	$u(\tau) \in \bigcap_{k\geq 0} Z^{(k)}_{\frac{1}{k+\frac{3}{2}}}$ on a suitable $(T^*-\epsilon, T^*)$, the size-parameters uniform in time	$u(\tau) \in \bigcap_{k\geq 0} Z^{(k)}_{\frac{1}{\frac{3}{2}(k+1)}}$ on $(0, T^*)$, the size-parameters uniform in time

The level-k scales of sparseness realizing the above functional classes are as follows.

Regularity class-scale	A priori bound-scale	Energy-level class-scale
$\frac{1}{C_1(k)} \frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{k+1}}}$	$C_2(\|u_0\|_2) \frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{k+\frac{3}{2}}}}$	$C_3(u_0) \frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{\frac{3}{2}(k+1)}}}$

As we are (at this point) primarily interested in the 'scaling distance' between various scales, comparing the scaling of *dynamic quantities* in the table above is key. This is summarized in the table below.

Regularity class-scale	A priori bound-scale	Energy-level class-scale
$\frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{k+1}}} \approx r$	$\frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{k+\frac{3}{2}}}} \approx r^{\frac{k+1}{k+\frac{3}{2}}}$	$\frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{\frac{3}{2}(k+1)}}} \approx r^{\frac{2}{3}}$

Since

$$r^{\frac{k+1}{k+\frac{3}{2}}} \to r, \quad k \to \infty$$

we arrive at the *asymptotic criticality*, closing (asymptotically) the scaling gap in the NS regularity problem.

Remark 1.4. Asymptotic criticality demonstrated above can be interpreted as an 'asymptotic cascade' near a possible singular time. Schematically, the scale of sparseness can be thought of as a size of the largest eddy. At level-0, the *a priori* upper bound on this scale is at the energy-level and abyss away from the lower bound sufficient for the regularity (representing the dissipation scale within this framework). As k increases, the gap between the bounds on these two dynamic quantities (the gap between the scaling signature of the largest eddy and the dissipation scale) decreases, vanishing as k goes to infinity.

The main mathematical device facilitating utilization of sparseness in the estimates is the harmonic measure maximum principle for subharmonic functions (cf. [Ahl10] and [Ran]). Here is a version suitable for our purposes.

Proposition 1.5 ([Ran])**.** *Let Ω be an open, connected set in $\mathbb{C}$ such that its boundary has nonzero Hausdorff dimension, and let K be a Borel subset of the boundary. Suppose that u is a subharmonic function on Ω satisfying*

$$u(z) \le M \ , \quad \textit{for } z \in \Omega$$
$$\limsup_{z\to\zeta} u(z) \le m \ , \quad \textit{for } \zeta \in K.$$

Then

$$u(z) \le m\, h(z,\Omega,K) + M(1 - h(z,\Omega,K)) \ , \quad \textit{for } z \in \Omega.$$

(Here, $h(z,\Omega,K)$ denotes the harmonic measure of K with respect to Ω, evaluated at z.)

Note that this majorization principle can be applied to the positive and the negative parts of the vectorial components of $D^{(k)}u$ since any solution originating from L^∞-data is spatially analytic (with a uniform control on the L^∞-norm over the complexified domain) at least locally in time.

The main idea in this type of a no-blow-up argument ([Gr13]) is that if the super-level sets–cut at a fraction of the maximum–are sparse enough, the associated harmonic measure will be small enough for the majorization principle to produce a favorable bound. Applying this argument–locally-in-time–near a possible singular time leads to 'self-improving bounds' on the L^∞-norm of the solution in view, contradicting the finite time blow-up.

Consequently–in this approach–there are two main tasks to complete. One is deriving sharp lower bounds on the radius of spatial analyticity measured in L^∞, and the other one demonstrating *a priori* sparseness of the super-level sets near a possible singular time. In the previous works ([Gr13, BrFaGr18]) these two tasks could be completed independently, the reason being that the analyticity estimates needed could be obtained via the existing state-of-the-art. However, in the case of the time-evolution of $D^{(k)}u$ this is no longer the case as the asymmetric terms present a serious obstacle on the road to the optimal dissipation scale/analyticity radius. This necessitated developing a novel technique–utilizing the sparseness intrinsically–based on the local-in-time dynamics of 'chains of derivatives',

$$\left\{c_{i,k}\, \|D^{(i)}u(t)\|_\infty^{\frac{k+1}{i+1}}\right\}_{0\le\kappa_1\le i\le\kappa_2\le k}$$

where $c_{i,k}$ are suitable combinatorial coefficients, and $\kappa_2-\kappa_1$ the length of the chain. Shortly, the nonlinear effect–at high enough (with respect to the size of the initial data) k-levels–is weakened through the interplay between the enhanced sparseness and the local-in-time monotonicity properties of chains of derivatives ('ascendind' *vs.* 'descending').

Lastly, one should remark that the argument leading to the *a priori* bounds on the scale of sparseness of the super-level sets of $D^{(k)}u$ eventually boils down to the *a priori* boundedness of the kinetic energy (Leray bound [Le34]) which–in turn–follows from the (only known) fundamental cancellation enjoyed by the NS nonlinearity,

$$\int (u\cdot\nabla)u\cdot u\,dx = 0.$$

Hence, the framework in hand seems to provide the first instance in the mathematical theory of the 3D NSE in which the base energy bound is capable of improving the bounds on 'regularity norms' (in this case, on $\|D^{(k)}u\|_\infty$).

2. Toward criticality

At this point, a question arises of whether the current methodology could be refined/repurposed in order to arrive at stronger and/or more classical displays of criticality. Two natural avenues to pursue are criticality with respect to the strength of the diffusion and criticality with respect to the strength of the nonlinearity. In what follows, some thoughts on what might be possible are presented.

2.1. **Criticality with respect to the diffusion.** Arguably the most classical setting here is the one of the 3D hyper-dissipative NSE,

$$u_t + (u \cdot \nabla)u = -\nabla p - (-\Delta)^\beta u, \qquad \operatorname{div} u = 0, \tag{2.1}$$

for some $\beta > 1$. For simplicity, we will continue to consider the spatial domain to be the whole space $\mathbb{R}^3$; then, $(-\Delta)^\beta$ is a Fourier multiplier with the symbol $|\xi|^{2\beta}$.

Since the fundamental work of J.L. Lions (cf. [Li59, Li69]), it is known that the system (2.1) is regular globally-in-time for any $\beta \geq \frac{5}{4}$. Tao (relatively) recently extended this result to the case of the diffusion realized by a Fourier multiplier logarithmically-weaker than the critical exponent $\beta = \frac{5}{4}$ [Tao09]. Modulo the logarithmic correction(s), the interval of super-criticality remains $\beta \in (1, \frac{5}{4})$.

It is worth noting that the current state-of-the-art essentially does not differentiate between the NS dissipation ($\beta = 1$) and the super-critical hyper-dissipation $\beta \in (1, \frac{5}{4})$. Two examples, from two different sub-realms are summarized below.

In the context of the search for possible finite-time singularities in the initially smooth flows, Tao considered a modified 3D NS model in which the NS nonlinearity is replaced by an 'averaged nonlinearity' satisfying the same scaling properties and featuring (essentially) the same degree of smoothness as the original NS nonlinearity, as well as the same fundamental cancellation leading to the *a priori* boundedness of the energy. The nonlinearity is designed in a way that–when considering a class of carefully constructed Schwarz initial data–the diffusion is not given a chance to catch up with the nonlinearity, leading to a finite time blow-up. This blow-up is super-critical, and–as remarked in [Tao16]–the construction can be adopted to any super-critical hyper-dissipative NSE.

In the context of demonstrating non-uniqueness of finite-energy weak solutions to dissipative fluid models, building on the fundamental work of Buckmaster and Vicol [BV], a proof of a more refined result–non-uniqueness of finite-energy, integrable-vorticity, smooth outside of the time-singular set of the box-counting dimension less than one-weak solutions–for any dissipation exponent $\beta \in [1, \frac{5}{4})$ is given in [BCV18].

In contrast, in light of the asymptotic criticality result for $\beta = 1$ ([GrXu19ac]), and the fact that the local-in-time analyticity estimates in L^∞ (a central player in the theory) are directly related to the strength of the diffusion, global-in-time regularity results for $\beta \in (1, \frac{5}{4})$ seem within reach.

As remarked in the previous section, the main technical barrier to achieving the optimal (lower bound on the) radius of spatial analyticity when working with $D^{(k)}u$ are the asymmetric terms. Nevertheless, after an elaborate dynamical argument based on local-in-time monotonicity properties of chains of derivatives ([GrXu19ac]), one does end up–for k large enough (depending on the size of the initial data)–with essentially the same regularity class as what would follow if the local-in-time complexified iteration scheme, starting from $D^{(k)}u_0 \in L^\infty$, would have the (optimal, corresponding to the symmetric terms) life-span comparable to

$$\frac{1}{c_1(k)} \frac{1}{\|D^{(k)}u_0\|_\infty^{\frac{2}{k+1}}}$$

(which would–in turn–yield to the analyticity radius of

$$\frac{1}{C_1(k)} \frac{1}{\|D^{(k)}u_0\|_\infty^{\frac{1}{k+1}}}$$

near the end-point of the interval).

That was for $\beta = 1$. A simple dimensional analysis reveals that the corresponding (optimal, corresponding to the symmetric terms) bounds on the life-span and the analyticity radius in the case of the super-critical hyper-dissipation $\beta \in (1, \frac{5}{4})$ would be

$$\frac{1}{c_1(k,\beta)} \frac{1}{\|D^{(k)}u_0\|_\infty^{\frac{2\beta}{(2\beta-1)(k+1)}}} \quad \text{and} \quad \frac{1}{C_1(k,\beta)} \frac{1}{\|D^{(k)}u_0\|_\infty^{\frac{1}{(2\beta-1)(k+1)}}},$$

respectively. If this was indeed achievable, global-in-time regularity for any super-critical hyper-dissipative NS would follow in a straightforward fashion.

Of course, this is not rigorous. However, given that the same type of 'scaling numerology' –in the case of the NSE–turned out to be convertible into a rigorous argument, leading to the asymptotic criticality ([GrXu19ac]), there is a degree of optimism that the conversion in the case of the super-critical hyper-dissipative NSE might work as well.

Remark 2.1. Let us note that the methodology presented in [GrXu19ac] relies on the higher-order (differential) structure of the NS model, and in particular on the higher-order pointwise structure of the nonlinearity (reflected in the local-in-time evolution properties of chains of derivatives). As a matter of fact, when applied at the level-zero (the NSE itself), it does not yield any improvement compared to the classical theory. The first gain is seen at the level-one (or, as previously demonstrated in [BrFaGr18], in the vorticity formulation), and the scaling gap closes as the order approaches infinity. As such, it is not applicable to the class of the averaged nonlinearities introduced in [Tao16], and a possible road to the global-in-time regularity for the super-critical hyper-dissipative NSE charted here is not incompatible with the finite time super-critical blow-up for the averaged NSE constructed in [Tao16].

2.2. **Criticality with respect to the nonlinearity.** Let us consider the phenomenon of 'geometric depletion of the nonlinearity' via local coherence of the vorticity direction, a dissipation mechanism in the 3D NS system discovered by Constantin (cf. [Co94]). This approach is based on the singular integral representation of the stretching factor in the evolution of the vorticity magnitude featuring a (singular) geometric kernel regularized by local coherence of the vorticity direction. Denoting the vorticity direction vector by η, and the angle between the vectors a and b by $\varphi(a,b)$, the assumption on the local coherence is usually stated in the following form,

$$\sin\varphi\big(\eta(x,t),\eta(y,t)\big) \le c|x-y|^\alpha \tag{2.2}$$

for some $\alpha \in (0,1]$; this is assumed in the spatial regions of high vorticity, and over the interval of time $(0,T)$ where T is the first possible singular time.

The first no-blow-up result in this setting was obtained by Constantin and Fefferman in [CoFe93], stating that the Lipschitz-coherence ($\alpha = 1$) suffices to rule out a singularity. This was later relaxed to the $\frac{1}{2}$-Hölder coherence in [BdVBe02] ($\alpha = \frac{1}{2}$). The author has also contributed to this line of research, including the papers [Gr09, GrGu10], as well as a chapter in the *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids* (jointly with Beirao Da Veiga and Giga) [BdVGiGr16].

The following could be viewed as merging two mathematical approaches for the study of turbulent dissipation, based on two key signatures of the geometry of turbulent flows, spatial intermittency and local coherence of the vorticity direction (see, e.g., [JWSR93, S81, SJO91, VM94]). The foundation has already been laid in [GrXu19ac] where the results on the asymptotic criticality of the 3D NSE in the vorticity formulation were given (working with $\|D^{(k)}\omega\|_\infty$ instead of $\|D^{(k)}u\|_\infty$, and considering the initial data in $L^1 \cap L^\infty$; recall that the vorticity analogue of the Leray bound is the bound on the L^1-norm of the vorticity ([Co90])). Geometric depletion of the nonlinearity induced by a local coherence of the vorticity direction-assumption is expected to yield improved estimates on the life-span of the complexified solution and the analyticity radius (replacing the role of the hyper-dissipation in the preceding subsection).

More precisely, the asymptotic criticality of the 3D NS problem in the vorticity formulation presented in [GrXu19ac] indicates that (2.2)–featuring any Hölder exponent greater than zero–might prove to be a regularity criterion.

The ultimate goal here would be arriving at the global-in-time regularity for the 3D NSE with the coherence assumption expressed via existence of an (arbitrary) locally-uniform modulus of continuity γ,

$$\sin \varphi\big(\eta(x,t), \eta(y,t)\big) \le \gamma(|x-y|). \tag{2.3}$$

It is interesting to note that–assuming a (critical) 'Type I' blow-up–Giga and Miura obtained a no-blow-result precisely under the coherence assumption (2.3) [GiMi].

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