ON BERTRAND'S AND RODRIGUEZ VILLEGAS' HIGHER-DIMENSIONAL LEHMER CONJECTURE

WITH AN APPENDIX BY FERNANDO RODRIGUEZ VILLEGAS

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ABSTRACT. Let L be a number field and let $E \subset \mathcal{O}_L^*$ be any subgroup of the units of L. If $\mathrm{rank}_{\mathbb{Z}}(E) = 1$, Lehmer's conjecture predicts that the height of any non-torsion element of E is bounded below by an absolute positive constant. If $\mathrm{rank}_{\mathbb{Z}}(E) = \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_L^*)$, Zimmert proved a lower bound on the regulator of E which grows exponentially with $[L : \mathbb{Q}]$. By sharpening a 1997 conjecture of Daniel Bertrand's, Fernando Rodriguez Villegas "interpolated" between these two extremes of rank with a new higher-dimensional version of Lehmer's conjecture. Here we prove a high-rank case of the Bertrand-Rodriguez Villegas conjecture. Namely, it holds if L contains a subfield K for which $[L : K] \gg [K : \mathbb{Q}]$ and E contains the kernel of the norm map from $\mathcal{O}_L^*$ to $\mathcal{O}_K^*$.

1. INTRODUCTION

If $P \in \mathbb{Z}[x]$ is a polynomial of degree n with leading coefficient $a_0 \neq 0$ and roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, its Mahler measure is defined as

$$M(P) := |a_0| \prod_{|\alpha_\nu|>1} |\alpha_\nu|.$$

In 1933 D. H. Lehmer [15] published an innocent-sounding question:

Is there a $P \in \mathbb{Z}[x]$ with Mahler measure satisfying $1 < M(P) < M(P_L) = 1.176\cdots$, where $P_L(x) := x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$?

Lehmer's question still stands unanswered. The reader is referred to [23] and [26, §3.6] for surveys of many interesting partial solutions to this problem.

Using Jensen's formula, Mahler gave the alternate expression

$$m(P) := \log M(P) = \int_0^1 \log |P(\mathrm{e}^{2\pi i t})| \, dt. \tag{1}$$

As $M(P_1 P_2) = M(P_1)M(P_2)$ and $M(P) \geq |a_0| \geq 1$, in studying $P \in \mathbb{Z}[x]$ with $M(P) < 2$ we may assume that P is irreducible and $a_0 = \pm 1$. Moreover, since it follows from (1) that the reciprocal polynomial $P^*(x) := x^n P(1/x)$ satisfies

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$M(P^*) = M(P)$, we find that $M(P) < 2$ implies that P is the minimal polynomial of an algebraic unit ε. Thus, Lehmer's problem is really about the size of a unit.

This point of view lead Bertrand in 1997 to propose a higher-dimensional version of Lehmer's question involving several units [5]. Suppose $\varepsilon_1, \dots, \varepsilon_k$ are independent units in some number field L, let $\mathcal{A}_L$ denote the set of Archimedean places of L and define the logarithmic embedding of the units $\mathrm{LOG}\colon \mathcal{O}_L^* \to \mathbb{R}^{\mathcal{A}_L}$ into a Euclidean space as usual by

$$\big(\mathrm{LOG}(\varepsilon)\big)_v := e_v \log|\varepsilon|_v, \qquad\qquad e_v := \begin{cases} 1 & \text{if } v \text{ is real,} \\ 2 & \text{if } v \text{ is complex} \end{cases}, \tag{2}$$

where $|\ |_v$ is the absolute value associated to $v \in \mathcal{A}_L$ extending the usual absolute value on $\mathbb{Q}$. Bertrand asked if for each integer $j \geq 2$ there is a universal constant $c_j > 0$ such that the j-dimensional co-volume V_j of the lattice generated by $\mathrm{LOG}(\varepsilon_1), \dots, \mathrm{LOG}(\varepsilon_j)$ satisfies $V_j \geq c_j$. He only posed this question for $j \geq 2$, since for $j = 1$ it was known that the right measure of size is $m(P) = \frac{1}{2}\|\mathrm{LOG}(\varepsilon)\|_1$, i.e., one should use an L^1-norm instead of length if $j = 1$ (see §7.3). Bertrand's conjecture was soon solved in the affirmative by Amoroso and David for $j \geq 3$ [2].

A few years ago Rodriguez Villegas proposed a version of Bertrand's conjecture which has a much sharper dependence on the rank j. For $j = 1$ Rodriguez Villegas' conjecture is equivalent to Lehmer's, while for j maximal, i.e., $j = \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_L^*)$, it is equivalent to Zimmert's 1981 theorem stating that the regulator of a number field grows at least exponentially with the degree of the number field [28]. More precisely, Rodriguez Villegas conjectured a strong lower bound on the natural L^1-norm of any non-trivial element ω of the j-th exterior power of the units of a number field.[1]

To define this L^1-norm, start with the orthonormal basis $\{\delta^v\}_{v\in\mathcal{A}_L}$ on $\mathbb{R}^{\mathcal{A}_L}$,

$$\delta^v_w := \begin{cases} 1 & \text{if } \ w = v, \\ 0 & \text{if } \ w \neq v. \end{cases} \tag{3}$$

This gives rise to the orthonormal basis $\{\delta^I\}_{I\in\mathcal{A}_L^{[j]}}$ of $\bigwedge^j \mathbb{R}^{\mathcal{A}_L}$, where $\mathcal{A}_L^{[j]}$ denotes the set of subsets $I = \{v_{i_1}, \dots, v_{i_j}\}$ of $\mathcal{A}_L$ having cardinality j and $\delta^I := \delta^{v_{i_1}} \wedge \cdots \wedge \delta^{v_{i_j}}$.

[1] We are grateful to F. Rodriguez Villegas for allowing his conjecture to appear in print for the first time as an appendix to this paper. In fact, in 2002 Rodriguez Villegas wrote up a weaker (unpublished) version of his conjecture without knowing that it followed from Bertrand's. Around 2015 we began work on the B-RV conjecture, still embarrassingly ignorant of Bertrand's and Amoroso-David's work on the subject. After posting an earlier arXiv version of this paper, containing the 2002 write-up by Rodriguez Villegas, our attention was fortunately called to earlier work.

For the reader who compares our version here with the appendix, we note that although Rodrigez Villegas phrases the L^1-norm in terms of Archimedean embeddings rather than places, his L^1-norm coincides with ours in (4) below because of the factor of 2 at complex places in (2). However, using places gives a larger L^2-norm if the field is not totally real, and so is better for our purposes.

The L^1-norm on $\bigwedge^j \mathbb{R}^{\mathcal{A}_L}$ is defined with respect to this basis. Namely,

$$\|\omega\|_1 := \sum_{I \in \mathcal{A}_L^{[j]}} |c_I| \qquad \Big(\omega = \textstyle\sum_{I \in \mathcal{A}_L^{[j]}} c_I \delta^I\Big). \tag{4}$$

Let $\bigwedge^j \mathrm{LOG}(\mathcal{O}_L^*)$ denote the j^{th} exterior power of the lattice $\mathrm{LOG}(\mathcal{O}_L^*) \subset \mathbb{R}^{\mathcal{A}_L}$.

B-RV Conjecture. *(Bertrand-Rodriguez Villegas) There exist two absolute constants $c_0 > 0$ and $c_1 > 1$ such that for any number field L and any $j \in \mathbb{N}$,*

$$\|\omega\|_1 \geq c_0 c_1^j \qquad \text{for all nonzero} \qquad \omega \in \bigwedge^j \mathrm{LOG}(\mathcal{O}_L^*) \subset \bigwedge^j \mathbb{R}^{\mathcal{A}_L}. \tag{5}$$

Aside from Zimmert's theorem on the regulator [28] and the known cases of Lehmer's conjecture [23], the cleanest result in favor of the B-RV conjecture is

$$\|\mathrm{LOG}(\varepsilon_1) \wedge \cdots \wedge \mathrm{LOG}(\varepsilon_j)\|_1 > 0.001 \cdot 1.4^j, \tag{6}$$

proved for all j, but only for totally real fields L. This follows from work of Schinzel [18] and Pohst [17] dating back to the 1970's. Indeed, Schinzel showed in 1973 (and Pohst independently in 1978) for L totally real that for any unit $\varepsilon \in \mathcal{O}_L^*,\ \varepsilon \neq \pm 1$,

$$\|\mathrm{LOG}(\varepsilon)\|_2 := \Big(\sum_{v \in \mathcal{A}_L} \big(e_v \log|\varepsilon|_v\big)^2\Big)^{\frac{1}{2}} \geq \sqrt{[L:\mathbb{Q}]}\,\log\big((1+\sqrt{5})/2\big).$$

Using estimates of Hermite's constant, Pohst deduced good lower bounds for the regulator of a totally real field. The same calculations show that the j-dimensional co-volume μ of the lattice spanned by $\mathrm{LOG}(\varepsilon_1), \ldots, \mathrm{LOG}(\varepsilon_j)$ satisfies [12, p. 293]

$$\mu > \frac{([L:\mathbb{Q}]/j)^{j/2} 1.406^j}{(j+2)\sqrt{j}} \qquad (1 \leq j < [L:\mathbb{Q}]). \tag{7}$$

Since

$$\|\mathrm{LOG}(\varepsilon_1) \wedge \cdots \wedge \mathrm{LOG}(\varepsilon_j)\|_1 \geq \|\mathrm{LOG}(\varepsilon_1) \wedge \cdots \wedge \mathrm{LOG}(\varepsilon_j)\|_2 = \mu,$$

a short numerical computation with (7) yields (6).

As far as we know, the only proved cases of the B-RV conjecture involve "pure wedges" of the form $\omega = \mathrm{LOG}(\varepsilon_1) \wedge \cdots \wedge \mathrm{LOG}(\varepsilon_j)$, where the ε_i are independent elements of $\mathcal{O}_L^*$. If $j = r_L := \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_L^*)$ or $j = 1$, every element of $\bigwedge^j$ is (trivially) a pure wedge, but this also holds if $j = r_L - 1$ (see Lemma 28 below). In particular, if L is a totally real field of degree n over $\mathbb{Q}$, then $\|\omega\|_1 > 0.001 \cdot 1.4^{n-2}$ for all $\omega \in \bigwedge^{n-2} \mathrm{LOG}(\mathcal{O}_L^*)$. In general, however, the B-RV conjecture makes a stronger prediction than simply a lower bound on the L^1-norm of pure wedges.

Another known case of the B-RV conjecture occurs when

$$E = E(L/K) := \big\{\varepsilon \in \mathcal{O}_L^* \,\big|\, \mathrm{Norm}_{L/K}(\varepsilon) \text{ is a root of unity}\big\} \tag{8}$$

is the group of relative units associated to an extension L/K. Skoruppa and Friedman [13] proved in 1999 that inequality (5) in the B-RV conjecture holds for pure wedges if $[L:K] \geq n_0$ for some absolute constant n_0.[2]

[2] The inequality proved in [13] is for the relative regulator $\mathrm{Reg}(L/K)$ rather than for the co-volume μ of the relative units. This suffices since $\mu = \mathrm{Reg}(L/K) \prod_{w \in \mathcal{A}_K} \sqrt{r_w} \geq \mathrm{Reg}(L/K)$, where

To prove their result, Skoruppa and Friedman defined a Θ-type series Θ_E associated to any subgroup $E \subset \mathcal{O}_L^*$ of arbitrary rank and used it to produce a complicated inequality for the co-volume $\mu(E)$ associated to the lattice $\mathrm{LOG}(E)$. In the case of $E = E(L/K)$ they obtained the desired inequality using the saddle-point method to estimate the terms in the series Θ_E as $[L : K] \to \infty$. Although the saddle-point method in one variable is a standard tool, the difficulty in the asymptotic estimates in [13, §5] was that the estimates needed to depend only on $[L : K]$.

The results cited so far all pre-date the B-RV conjecture and essentially dealt with regulators or Lehmer's conjecture. Motivated by the B-RV conjecture, Sundstrom [24] [25] dealt in his 2016 thesis with a new kind of subgroup of the units. Namely, suppose L contains two distinct real quadratic subfields K_1, K_2, and let

$$E := E(L/K_1) \cap E(L/K_2).$$

Here $E(L/K) \subsetneq E$, where $K := K_1K_2$ is the compositum of the K_i. The series Θ_E is still defined and yields an inequality for the co-volume $\mu(E)$ associated to the lattice $\mathrm{LOG}(E)$, but to estimate the terms in the inequality Sundstrom had to apply the saddle-point method to a triple integral. Keeping all estimates uniform in this case proved considerably harder than in the one-variable case treated in [13]. In the end, Sundstrom was able to verify the B-RV conjecture in this case for pure wedges. More precisely, he proved the existence of absolute constants N_0, $c_0 > 0$ and $c_1 > 1$ such that $\mu(E_{K_1,K_2}) \geq c_0 c_1^j$, where $[L : \mathbb{Q}] \geq N_0$ and $j := \mathrm{rank}_{\mathbb{Z}}(E_{K_1,K_2}) = \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_L^*) - 2$.

We prove the following generalization of Sundstrom's result.

Theorem 1. *Suppose $E \subset \mathcal{O}_L^*$ is such that $E(L/K) \subset E$ for some subfield $K \subset L$, where $E(L/K)$ are the relative units defined in (8). Let $\varepsilon_1, ..., \varepsilon_j$ be independent elements of E, where $j := \mathrm{rank}_{\mathbb{Z}}(E)$, and let $k := 1 + \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_L^*/E)$. Then*

$$\|\varepsilon_1 \wedge \cdots \wedge \varepsilon_j\|_1 \geq \|\varepsilon_1 \wedge \cdots \wedge \varepsilon_j\|_2 \geq 1.1^j,$$

provided that

$$[L : K] \geq N_0 := \max\bigl(100k^6(\#\mathcal{A}_K)^{3/2}, 20000k^2\tbinom{\#\mathcal{A}_K}{k}, 1000(\#\mathcal{A}_K)^2\bigr).$$

Thus the B-RV conjecture (5) holds for $\omega := \varepsilon_1 \wedge \cdots \wedge \varepsilon_j$ when $[L : K] \gg [K : \mathbb{Q}]$.

Here $\#$ denotes cardinality and $\binom{a}{b} := \frac{a!}{b!(a-b)!}$. The hypothesis $E(L/K) \subset E$ implies that $k \leq \#\mathcal{A}_K$, so that N_0 above could be replaced by a coarser bound involving only $\#\mathcal{A}_K$ or $[K : \mathbb{Q}]$.

r_w is the number of places of L above w. The proof of this relation between the co-volume and the relative regulator mimics the determinant manipulations in the case $K = \mathbb{Q}$ [8, p. 115]. We note that J. Sundstrom, in the appendix to his doctoral thesis [24], corrected an error in Skoruppa and Friedman's proof. Namely, in the bound on what is called J_1 in the proof of Lemma 5.5 of [13], the real part of the error term ρ in the exponential was neglected. This did not affect the proof of their Main Theorem, but it did affect the numerical constants claimed in Theorem 4.1 and its corollaries. However, if we are willing to settle for $n_0 = 400$, the proof in [13] will easily do after correcting the constants. By improving the asymptotic estimates in [13] and using extensive computer calculations, Sundstrom was able to prove the estimate in Theorem 4.1 of [13], with the constants as given in [13]. In particular, $n_0 = 40$ is allowed.

Motivated by a first version of this paper posted on arxiv.org, Amoroso and David [3] made a comprehensive study of the B-RV conjecture and suggested even stronger forms [3, Conjs. 1.3 and 1.9]. They also gave a different proof of an improved version of Theorem 1 and proved interesting new cases of the B-RV conjecture [3, Thms. 1.4–1.7].

Aside from proving Theorem 1, our aim here is to lay the ground work for an approach to proving the B-RV conjecture for any high-rank subgroup $E \subset \mathcal{O}_L^*$, as we now explain. Our starting point is Skoruppa and Friedman's inequality, valid for any $t > 0$ and any subgroup $E \subset \mathcal{O}_L^*$,

$$\frac{\mu_{E_{\mathbb{R}}}(E_{\mathbb{R}}/E)}{|E_{\mathrm{tor}}|} \geq \sum_{\substack{a \in \mathcal{O}_L/E \\ a \neq 0}} \int_{x \in E_{\mathbb{R}}} \Big(\frac{2t\|ax\|^2}{[L:\mathbb{Q}]} - 1 \Big) \mathrm{e}^{-t\,\|ax\|^2}\, d\mu(x), \tag{9}$$

where $\mathcal{O}_L$ denotes the algebraic integers of L, $E_{\mathbb{R}} \cong E \otimes_{\mathbb{Z}} \mathbb{R}$, which acts on $\mathbb{R}^{\mathcal{A}_L}$ since E does, $\|ax\|^2 := \sum_{v \in \mathcal{A}_L} e_v |a|_v^2 x_v^2$, and μ is a suitable Haar measure on $E_{\mathbb{R}}$.

To make use of (9) one tries to prove that for a well chosen t the term $a = 1 \in \mathcal{O}_L$ produces a contribution growing exponentially with $[L : \mathbb{Q}]$, while the terms for other a are at least non-negative. The integral in (9) is not very useful, for this purpose, because it seems to depend on $\#\mathcal{A}_L$ variables, namely on each of the absolute values $|a|_v$. In fact, it depends only on $k = \#\mathcal{A}_L - \mathrm{rank}_{\mathbb{Z}}(E)$ variables, as integrating over $E_{\mathbb{R}}$ removes $\mathrm{rank}_{\mathbb{Z}}(E)$ of them. Hence our first task is to write the integral in (9) as a k-dimensional inverse Mellin transform. This we do in §3, as summarized in Corollary 6.

As in [13] and [25], the next step is to apply the saddle-point method to the k-dimensional complex contour integral obtained in §2. To do this we need a saddle point. In the case of [13] one could easily write down a formula for the saddle point in terms of the logarithmic derivative of the classical Γ-function. In [25] the equations for the critical point were explicit enough that monotonicity arguments proved the existence of the saddle point. In our case the equations are too complicated to analyse directly. Instead, in §4 we obtain the existence and uniqueness of the saddle point by re-interpreting it as the value of the Legendre transform of a convex function on $\mathbb{R}^k$, closely related to $\log \Gamma$. Since the saddle point $\sigma \in \mathbb{R}^k$ is far from explicit, in §5 we prove useful inequalities which depend only on its first coordinate σ_1, which we can control by choosing t in (9) appropriately.

In §6 we show that the contribution from the saddle point actually dominates the integral when $\#\mathcal{A}_L$ is large enough compared with the dimension k of the contour integral. It is only here that we need the hypothesis that that the relative units $E(L/K) \subset E$.

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2. Overview of the proof

To clarify the proof of Theorem 1 we decribe its main steps in some detail here. We begin by recalling in §3.1 how theta functions lead to the basic inequality (9).

The terms on the right of (9) are indexed by $a \in (\mathcal{O}_L - \{0\})/E$. We show in Corollary 6 of §3.2 how the term associated to a can be written as an inverse Mellin transform of a function of $s = (s_1, \ldots, s_k) \in \mathbb{C}^k$ of the form $\exp\big(\alpha(s) - n\sum_{j=1}^{k} y_j s_j\big)$, in which $\alpha(s)$ is a sum of logarithms of the values of the Γ-function at linear forms in s and $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ depends on a.

The saddle point method for estimating this k-dimensional integral consists of first determining a critical point σ of the integrand, moving the integration contour so that it passes through σ, and then attempting to show that the dominant term arises from a neighborhood of σ. We show in Lemma 8 of §4 that for each a there is a unique critical point $\sigma = \sigma(y) \in \mathbb{R}^k$ associated to the value of y arising from a. The inverse Mellin transform associated to a then takes the form

$$\int_{T\in\mathbb{R}^k} \mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\, dT =: \int_{\mathbb{R}^k} \mathcal{G}(T)\, dT.$$

To control this integral, in §5 we prove some inequalities concerning σ and $\alpha(\sigma)$.

In §6 we define the Gaussian $\mathcal{H}(T)$ approximating $\mathcal{G}(T)$ in a suitable bounded neighborhood Δ of $T = 0$. The saddle point method then leads to estimating the integrals $I_1, I_2, -I_3$ and I_4 on the right side of

$$\int_{\mathbb{R}^k} \mathcal{G}(T)dT = \int_{\mathbb{R}^k} \mathcal{H}(T)dT + \int_{\mathbb{R}^k-\Delta} \mathcal{G}(T)dT - \int_{\mathbb{R}^k-\Delta} \mathcal{H}(T)dT + \int_{\Delta} \big(\mathcal{G}(T) - \mathcal{H}(T)\big)dT \tag{10}$$

The first integral $I_1 = \int_{\mathbb{R}^k} \mathcal{H}(T)dT$ is readily computed; see Corollary 16 of §6.1. To show it is the main term by our method requires the assumption that E is (M,Ω)-dispersed in the sense of Definition 13 for suitable values of M and Ω. This assumption is implied by the hypotheses of Theorem 1. It amounts to requiring that there be a partition $\mathcal{M}$ of the archimedean places $\mathcal{A}_L$ of L into large subsets of approximately the same size m such that the orthogonal complement of the image of E under the log map is spanned by vectors whose components are constant over the places in each subset. This assumption leads to the equality

$$\alpha(s) := \sum_{w\in\mathcal{M}} m_w \alpha_{\kappa_w}\big(S_w(s)\big) \tag{11}$$

in equation (60) in which S_w is a linear form depending on E, α_{κ_w} is a linear combination of $\log(\Gamma(z))$ and $\log(\Gamma(z+\frac{1}{2})$, and the number m_w of elements in the subset w of the partition $\mathcal{M}$ is approximately $m := \min_w\{m_w\}$.

The goal of Lemmas 17 through 25 of §6.2 is to show that (11) leads to upper bounds on the error terms $I_2, -I_3$ and I_4 on the right side of (10) that are sufficient to bound them in terms of $|I_1|$. Namely, $|I_2| + |I_3| + |I_4| \leq 0.01 I_1$, as we show in Lemma 26 for large enough m. The positivity of I_1 shown in Lemma 26 then leads to the term associated to every $a \in \mathcal{O}_K/E$ being positive. Specializing Lemma 26 to the case $a = 1$ gives a contribution that establishes the more precise version of Theorem 1 given by Theorem 27.

Lemmas 17 through 25 also show that there does exist a set of constants for which the estimates involved in the proof of Theorem 27 lead to exponential growth rather than exponential decay in the apppropriate relative regulators. As in the work of Friedman and Skoruppa, such estimates are tedious to check.

We end this section with an elementary geometric result arising in the proof that may have other applications (see Lemma 22 and Remark 23).

Lemma 2. *Define the sign-symmetrized parallelotope $P(\eta)$ associated to a set η of k vectors in $\mathbb{R}^k$ to be the set of all real linear combinations $\sum_{t\in\eta} r_t t$ with $-1 \le r_t \le 1$, and suppose we are given a subset $S \subset \mathbb{R}^k$ containing a basis of $\mathbb{R}^k$. Let η_0 be any k-element subset of S for which $P(\eta_0)$ has maximal volume. Then $S \subset P(\eta_0)$.*

3. The Θ-function

In this section we recall the series $\Theta_E(t;\mathfrak{a})$ associated to a subgroup $E \subset \mathcal{O}_L^*$ of the units and to a fractional ideal $\mathfrak{a}$ of the number field L. We also recall the inequality for the co-volume of $\mathrm{LOG}(E)$ resulting from the functional equation of Θ_E. This is all quoted from [13, §2]. Our main new task here is to express the terms in the inequality as an inverse Mellin transform.

3.1. **The basic inequality.**

Given a subgroup $E \subset \mathcal{O}_L^*$, we define $E_\mathbb{R} \subset \mathbb{R}_+^{\mathcal{A}_L}$ as the group generated by all elements of the form

$$x = (x_v)_{v\in\mathcal{A}_L} = \big(|\varepsilon|_v^\xi\big)_{v\in\mathcal{A}_L} \qquad \big(\varepsilon \in E,\ \ \xi \in \mathbb{R}\big).$$

Here $\mathbb{R}_+ := (0,\infty)$ is the multiplicative group of the positive real numbers, $\mathcal{A}_L$ denotes the set of Archimedean places of L, and $|\ |_v$ is the (un-normalized) absolute value associated to the Archimedean place $v \in \mathcal{A}_L$. Thus, for $a \in L$ we have

$$|\mathrm{Norm}_{L/\mathbb{Q}}(a)| = \prod_{v\in\mathcal{A}_L} |a|_v^{e_v}, \qquad (e_v := 1 \text{ if } v \text{ is real},\ e_v := 2 \text{ if } v \text{ is complex}). \tag{12}$$

Note that

$$\sum_{v\in\mathcal{A}_L} e_v = [L:\mathbb{Q}] =: n, \tag{13}$$

$$\prod_{v\in\mathcal{A}_L} x_v^{e_v} = 1 \qquad \big(x = (x_v)_{v\in\mathcal{A}_L} \in E_\mathbb{R}\big), \tag{14}$$

and that $\varepsilon \in E$ acts on $x = (x_v)_v \in E_\mathbb{R}$, via $(\varepsilon \cdot x)_v := |\varepsilon|_v\, x_v$.

We fix a Haar measure on $E_\mathbb{R} \subset \mathbb{R}_+^{\mathcal{A}_L}$ as follows. The standard Euclidean structure on $\mathbb{R}^{\mathcal{A}_L}$, in which the δ^v in (3) form an orthonormal basis of $\mathbb{R}^{\mathcal{A}_L}$, induces a Euclidean structure (and therefore a unique Haar measure) on any $\mathbb{R}$-subspace of $\mathbb{R}^{\mathcal{A}_L}$. We give $E_\mathbb{R}$ the Haar measure $\mu_{E_\mathbb{R}}$ that results from pulling back the Haar measure on the $\mathbb{R}$-subspace $\mathrm{LOG}(E_\mathbb{R})$ via the isomorphism LOG in (2), and let $\mu_{E_\mathbb{R}}(E_\mathbb{R}/E)$ be the measure of a fundamental domain for the action of E on $E_\mathbb{R}$.

Following [13, p. 120], for a fractional ideal $\mathfrak{a} \subset L$ and $t > 0$, we let

$$\Theta_E(t;\mathfrak{a}) := \frac{\mu_{E_\mathbb{R}}(E_\mathbb{R}/E)}{|E_{\mathrm{tor}}|} + \sum_{\substack{a\in\mathfrak{a}/E\\ a\neq 0}} \int_{x\in E_\mathbb{R}} \mathrm{e}^{-c_\mathfrak{a} t\,\|ax\|^2}\, d\mu_{E_\mathbb{R}}(x), \qquad \|ax\|^2 := \sum_{v\in\mathcal{A}_L} e_v |a|_v^2 x_v^2, \tag{15}$$

where $|E_{\mathrm{tor}}|$ is the number of roots of unity in E,

$$c_{\mathfrak{a}} := \pi\Big(\sqrt{|D_L|}\,\mathrm{Norm}_{L/\mathbb{Q}}(\mathfrak{a})\Big)^{-2/n}, \qquad D_L := \text{discriminant of } L, \quad n := [L:\mathbb{Q}].$$

Note that the integral in (15) depends only on the E-orbit of a, and hence is independent of the representative $a \in \mathfrak{a}/E$ taken for the E-orbit of a.

Our starting point for proving lower bounds on co-volumes is the inequality [13, Corol. p. 121], valid for any $t>0$ and any fractional ideal $\mathfrak{a}$ of L.

$$\Theta_E(t;\mathfrak{a}) + \frac{2t\Theta_E'(t;\mathfrak{a})}{n} \geq 0 \qquad \Big(t>0,\ \ \Theta_E' := \frac{d\Theta_E}{dt}\Big). \tag{16}$$

Writing out the individual terms of (16), we have [13, p. 121, eq. (2.6)] the

Basic Inequality.

$$\frac{\mu_{E_{\mathbb{R}}}(E_{\mathbb{R}}/E)}{|E_{\mathrm{tor}}|} \geq \sum_{\substack{a\in\mathfrak{a}/E\\ a\neq 0}} \int_{x\in E_{\mathbb{R}}} \Big(\frac{2t\|ax\|^2}{n} - 1\Big) \mathrm{e}^{-t\,\|ax\|^2}\, d\mu_{E_{\mathbb{R}}}(x) \qquad (t>0). \tag{17}$$

Note that in [13] we find $tc_{\mathfrak{a}}$ instead of t in (17), but $t>0$ is arbitrary there too.

3.2. **Mellin transforms.** Our main task in this section is to re-write the r-dimensional integral in (15) as an inverse Mellin transform. For this it will prove convenient to characterize $E_{\mathbb{R}} \subset G := \mathbb{R}_+^{\mathcal{A}_L}$ not through generators, but rather through generators of the orthogonal complement in $\mathbb{R}^{\mathcal{A}_L}$ of $\mathrm{Log}(E_{\mathbb{R}})$. Here $\mathrm{Log}\colon G \to \mathbb{R}^{\mathcal{A}_L}$ is the group isomorphism defined by

$$\big(\mathrm{Log}(g)\big)_v := \log(g_v) \qquad \big(v\in\mathcal{A}_L,\ \ g=(g_v)_v \in G := \mathbb{R}_+^{\mathcal{A}_L}\big). \tag{18}$$

Note that Log is *not* the traditional logarithmic embedding LOG in (2), as we do not insert a factor of e_v in (18). Instead we endow $\mathbb{R}^{\mathcal{A}_L}$ with a new inner product

$$\langle\beta,\gamma\rangle := \sum_{v\in\mathcal{A}_L} e_v\beta_v\gamma_v \qquad \big(\beta=(\beta_v)_v,\ \gamma=(\gamma_v)_v \in \mathbb{R}^{\mathcal{A}_L}\big), \tag{19}$$

where $e_v = 1$ or 2 as in (12). Let $\big\{q_j\big\}_{j=1}^k = \big\{(q_{jv})_v\big\}_{j=1}^k$ be an $\mathbb{R}$-basis of the orthogonal complement of $\mathrm{Log}(E_{\mathbb{R}})$ in $\mathbb{R}^{\mathcal{A}_L}$ such that

$$q_{1v} := 1 \ \ (\forall v\in\mathcal{A}_L), \quad \sum_{v\in\mathcal{A}_L} e_v q_{iv} q_{jv} = 0 \ \ \big(1\leq i\neq j\leq k := 1+\mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_L^*/E)\big). \tag{20}$$

Thus, for $g=(g_v)_v \in G := \mathbb{R}_+^{\mathcal{A}_L}$,

$$g\in E_{\mathbb{R}} \qquad \Longleftrightarrow \qquad \sum_{v\in\mathcal{A}_L} e_v q_{jv}\log(g_v) = 0 \qquad (1\leq j\leq k). \tag{21}$$

Let $H := \mathbb{R}_+^k$. Define a homomorphism $\delta\colon G\to H$ by

$$\big(\delta(g)\big)_j := \prod_{v\in\mathcal{A}_L} g_v^{e_v q_{jv}} \qquad \big(1\leq j\leq k,\ \ g=(g_v)_v\in G := \mathbb{R}_+^{\mathcal{A}_L}\big), \tag{22}$$

so that by (21) we have an exact sequence

$$1 \longrightarrow E_{\mathbb{R}} \longrightarrow G \xrightarrow{\ \delta\ } H \longrightarrow 1. \tag{23}$$

Let $\sigma\colon H \to G$ be a homomorphism splitting the exact sequence (23), i.e., $\delta \circ \sigma$ is the identity map on H. Such a splitting exists because G and H are real vector spaces. Let

$$d\mu_G := \prod_{v \in \mathcal{A}_L} \frac{dg_v}{g_v}, \qquad\qquad d\mu_H := \prod_{j=1}^{k} \frac{dh_j}{h_j} \tag{24}$$

be the usual Haar measures on $G := \mathbb{R}_+^{\mathcal{A}_L}$ and $H := \mathbb{R}_+^k$.

Recall that in order to define Θ_E in (15) we fixed a Haar measure $\mu_{E_\mathbb{R}}$ on $E_\mathbb{R}$. In order to calculate Mellin transforms below, we will need to compare the Haar measure $\mu_H \times \mu_{E_\mathbb{R}}$ on $H \times E_\mathbb{R}$ with a Haar measure coming from μ_G. Namely, if $\gamma\colon E_\mathbb{R} \times H \to G$ is the isomorphism defined by the splitting σ, i.e.,

$$\gamma(x,h) := x\sigma(h), \tag{25}$$

then the measure $\mu_G \circ \gamma$ is a Haar measure on $E_\mathbb{R} \times H$. Hence

$$c\, \mu_G \circ \gamma = \mu_{E_\mathbb{R}} \times \mu_H, \tag{26}$$

where the positive constant c is evaluated in the next lemma.

Lemma 3. *Let Q be the $\#\mathcal{A}_L \times k$ matrix whose rows are indexed by $v \in \mathcal{A}_L$ and whose columns are indexed by $j = 1, \dots, k$, with entry $Q_{v,j} := q_{jv}$ in the v^{th} row and the j^{th} column, with q_{jv} as in* (20). *Then c in* (26) *is independent of the splitting σ in* (25) *and is given by $c = 2^{r_2}\sqrt{\det(Q^\intercal Q)}$, where $Q^\intercal$ is the transpose of Q and r_2 is the number of complex places of L.*

Proof. For $x = (x_v)$ and $y = (y_v) \in \mathbb{R}^{\mathcal{A}_L}$, let $x \cdot y$ be the standard dot product $x \cdot y := \sum_{v \in \mathcal{A}_L} x_v y_v$. Recall that we defined in (19) another inner product on $\mathbb{R}^{\mathcal{A}_L}$, namely $\langle x, y\rangle := \sum_v e_v x_v y_v$. To relate these products, let $T : \mathbb{R}^{\mathcal{A}_L} \to \mathbb{R}^{\mathcal{A}_L}$ be given by $\big(T(x)\big)_v := e_v x_v$. Then

$$\langle x, y\rangle = x \cdot T(y) = T(x) \cdot y. \tag{27}$$

Note that $\det(T) = 2^{r_2}$.

Let $u_1, ..., u_r$ be an orthonormal basis of $\mathrm{LOG}(E_\mathbb{R})$ (with respect to the dot product), let $C_1 := \big\{ \sum_\ell x_\ell u_\ell \big| \, 0 \le x_\ell \le 1 \big\} \subset \mathrm{LOG}(E_\mathbb{R})$ be the unit r-cube spanned by the u_ℓ, and let $B_1 := \mathrm{LOG}^{-1}(C_1)$. By the definition of the measure $\mu_{E_\mathbb{R}}$ given in the paragraph preceding (15), $\mu_{E_\mathbb{R}}(B_1) = 1$.

We define next an analogous subset $B_2 \subset H := \mathbb{R}_+^k$ with $\mu_H(B_2) = 1$. Let $F_1, \dots, F_k$ be the "standard" orthonormal basis of $\mathbb{R}_+^k$ as an $\mathbb{R}$-vector space; that is, $(F_j)_i = \mathrm{e}$ if $i = j$, and $(F_j)_i = 1$ otherwise. Let $B_2 \subset \mathbb{R}_+^k$ be the k-cube spanned by $F_1, \dots, F_k$, so that $\mu_H(B_2) = 1$.

Set $B := B_1 \times B_2 \subset E_\mathbb{R} \times H$, so that $(\mu_{E_\mathbb{R}} \times \mu_H)(B) = 1$. Thus c in (26) satisfies $c^{-1} = \mu_G\big(\gamma(B)\big)$. Now, $\gamma(x,h) := x\sigma(h)$ and μ_G is the measure on G that maps by Log to the standard Haar measure on $\mathbb{R}^{\mathcal{A}_L}$ $\big($see (18), (24) and (25)$\big)$. Hence, $c^{-1} = |\det(M)|$, where M is the $(\#\mathcal{A}_L \times \#\mathcal{A}_L)$-matrix whose first r columns are the vectors $w_\ell := \mathrm{Log}\big(\mathrm{LOG}^{-1}(u_\ell)\big) \in \mathbb{R}^{\mathcal{A}_L}$ $(1 \le \ell \le r)$. The remaining k columns of M are the vectors $\mathrm{Log}\big(\sigma(F_j)\big)$ $(1 \le j \le k)$.

Suppose $\tilde{\sigma}$ is another splitting of (23). Then $\sigma(F_j)\tilde{\sigma}(F_j)^{-1} \in E_{\mathbb{R}}$, and therefore $\mathrm{Log}\big(\sigma(F_j)\big) - \mathrm{Log}\big(\tilde{\sigma}(F_j)\big)$ lies in the span of the columns $w_1, ..., w_r$. Hence c is independent of the splitting σ, as claimed in the lemma. We are therefore free to use the splitting σ determined by

$$\big(\sigma(F_j)\big)_v := \exp(q_{jv}/d_j) \qquad \Big(v \in \mathcal{A}_L,\ 1 \le j \le k,\ d_j := \langle q_j, q_j \rangle := \sum_{\rho \in \mathcal{A}_L} e_\rho q_{j\rho}^2\Big).$$

Using (22) and the orthogonality relations (20), one checks that this is indeed a splitting of δ. With this σ, the last k columns of M are just $\mathrm{Log}\big(\sigma(F_j)\big) = d_j^{-1} q_j \in \mathbb{R}^{\mathcal{A}_L}$. As $T \circ \mathrm{Log} = \mathrm{LOG}$ and $\det(T) = 2^{r_2}$ $\big($see (27)$\big)$, we have

$$c^{-1} = |\det(M)| = 2^{-r_2} |\det(N)|,$$

where N is the $(\#\mathcal{A}_L \times \#\mathcal{A}_L)$-matrix whose columns are T applied to the columns of M, i.e., the columns of N are $u_1, ..., u_r$, followed by $d_1^{-1}T(q_1), ..., d_k^{-1}T(q_k)$.

To prove the lemma we must show that $|\det(N)|^{-1} = \sqrt{\det(Q^\intercal Q)}$. We calculate $|\det(N)|$ as $|\det(N)| = |\det(R^\intercal N)|/\sqrt{\det(R^\intercal R)}$, where R is the $(\#\mathcal{A}_L \times \#\mathcal{A}_L)$-matrix whose columns are $u_1, ..., u_r$, followed by $q_1, ..., q_k$ (i.e., Q). Using the orthonormality of the u_ℓ's (with respect to the dot product), we see that $R^\intercal R$ can be divided into four blocks, the upper left one being the $r \times r$ identity matrix $I_{r\times r}$. Below it, $R^\intercal R$ has a $k \times r$ block with entries

$$q_j \cdot u_\ell = q_j \cdot T\big(\mathrm{Log}(\mathrm{LOG}^{-1}(u_\ell))\big) = \big\langle q_j, \mathrm{Log}\big(\mathrm{LOG}^{-1}(u_\ell)\big)\big\rangle = 0,$$

where we used (27) and the definition of the q_j's as a basis of the orthogonal complement of $\mathrm{Log}(E_{\mathbb{R}}) \subset \mathbb{R}^{\mathcal{A}_L}$ $\big($with respect to $\langle\ \rangle$, see (21)$\big)$. Since the bottom right $k \times k$ block of $R^\intercal R$ is $Q^\intercal Q$, we find that $R^\intercal R = \begin{pmatrix} I_{r\times r} & 0_{r\times k} \\ 0_{k\times r} & Q^\intercal Q \end{pmatrix}$. Thus, $\det\big(R^\intercal R\big) = \sqrt{\det(Q^\intercal Q)}$. A similar calculation shows $R^\intercal N = \begin{pmatrix} I_{r\times r} & *_{r\times k} \\ 0_{k\times r} & I_{k\times k} \end{pmatrix}$, whence $\det(R^\intercal N) = 1$. $\square$

In order to study the Θ-series (15), we need to consider integrals of the form

$$\int_{x\in E_{\mathbb{R}}} \mathrm{e}^{-\|gx\|^2}\, d\mu_{E_{\mathbb{R}}}(x) \qquad \big(\|gx\|^2 := \sum_{v\in\mathcal{A}_L} e_v g_v^2 x_v^2\big), \tag{28}$$

for $g = (g_v)_v \in G := \mathbb{R}_+^{\mathcal{A}_L}$. For $h = (h_1, \ldots, h_k) \in H := \mathbb{R}_+^k$, define ψ by substituting $g = \sigma(h)$ above:

$$\psi(h) := \int_{x\in E_{\mathbb{R}}} \mathrm{e}^{-\|\sigma(h)x\|^2}\, d\mu_{E_{\mathbb{R}}}(x). \tag{29}$$

Note that the integral (28) depends only on g modulo $E_{\mathbb{R}}$, so the function ψ is independent of the choice of σ splitting the exact sequence (23). The fact that (28) depends only on g modulo $E_{\mathbb{R}}$ also shows that

$$\int_{x\in E_{\mathbb{R}}} \mathrm{e}^{-\|gx\|^2}\, d\mu_{E_{\mathbb{R}}}(x) = \int_{x\in E_{\mathbb{R}}} \mathrm{e}^{-\|\sigma(\delta(g))x\|^2}\, d\mu_{E_{\mathbb{R}}}(x) = \psi\big(\delta(g)\big), \tag{30}$$

so we will concentrate on ψ, a function of only k variables.

Define a linear map $S\colon \mathbb{C}^k \to \mathbb{C}^{\mathcal{A}_L}$ by $S(s) = Qs$, where $s \in \mathbb{C}^k$ and Q is the matrix whose j^{th} column is $q_j \in \mathbb{R}^{\mathcal{A}_L} \subset \mathbb{C}^{\mathcal{A}_L}$, as in Lemma 3. Also define maps $S_v\colon \mathbb{C}^k \to \mathbb{C}$ for each $v \in \mathcal{A}_L$ by $S_v(s) = \big(S(s)\big)_v$. That is,

$$S(s) = \sum_{j=1}^{k} s_j q_j, \qquad S_v(s) = \sum_{j=1}^{k} q_{jv} s_j \qquad \big(s = (s_1, ..., s_k)\big). \tag{31}$$

Note that S is injective since the $q_j \in \mathbb{R}^{\mathcal{A}_L}$ are linearly independent.

Our first aim is to calculate the (k-dimensional) Mellin transform

$$(M\psi)(s) := \int_H \psi(h)\, h^s \, d\mu_H(h) := \int_{h_1=0}^{\infty} \cdots \int_{h_k=0}^{\infty} \psi(h)\, h_1^{s_1} \cdots h_k^{s_k} \frac{dh_1}{h_1} \cdots \frac{dh_k}{h_k}, \tag{32}$$

where $\mathrm{Re}(s) := \big(\mathrm{Re}(s_1), \ldots, \mathrm{Re}(s_k)\big) \in \mathcal{D}$, with

$$\mathcal{D} := \Big\{\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^k \Big|\ S_v(\sigma) > 0\ \ \forall v \in \mathcal{A}_L\Big\}. \tag{33}$$

As $q_{1v} := 1$ for all $v \in \mathcal{A}_L$ $\big($see (20)$\big)$, for $t > 0$ we have $(t, 0, 0, \ldots, 0) \in \mathcal{D}$. Hence $\mathcal{D}$ is a non-empty, open, convex subset of $\mathbb{R}^k$. We will presently prove that the Mellin transform $(M\psi)(s)$ in (32) converges if $\mathrm{Re}(s) \in \mathcal{D}$.

In the following calculation of $(M\psi)(s)$ the reader should initially consider only real s_j, so that the integrand is positive. At the end of the calculation it will become clear that the integral converges for s in the open subset of $\mathbb{C}^k$ where $\mathrm{Re}(s) \in \mathcal{D}$.

$$\begin{aligned}
(M\psi)(s) &= \int_{h \in H} \int_{x \in E_{\mathbb{R}}} h^s \exp(-\|x\sigma(h)\|^2)\, d\mu_{E_{\mathbb{R}}}(x)\, d\mu_H(h) \\
&= \int_{(x,h) \in E_{\mathbb{R}} \times H} h^s \exp(-\|x\sigma(h)\|^2)\, d(\mu_{E_{\mathbb{R}}} \times \mu_H)(x,h) \\
&= \ 2^{r_2} \sqrt{\det(Q^{\mathsf{T}} Q)} \int_{(x,h) \in E_{\mathbb{R}} \times H} \big(\delta(\gamma(x,h))\big)^s \exp(-\|\gamma(x,h)\|^2)\, d(\mu_G \circ \gamma)(x,h),
\end{aligned}$$

where in the last step we used Lemma 3 and $\delta\big(\gamma(x,h)\big)=\delta\big(\sigma(h)x\big)=h$, with δ as in (22) and γ as in (25). Next we substitute $g=\gamma(x,h)$ to get

$$\begin{aligned}
(M\psi)(s) &= 2^{r_2}\sqrt{\det(Q^{\intercal}Q)}\int_{g\in G}\delta(g)^s \mathrm{e}^{-\|g\|^2}\,d\mu_G(g)\\
&= 2^{r_2}\sqrt{\det(Q^{\intercal}Q)}\int_{g\in G}\mathrm{e}^{-\|g\|^2}\prod_{j=1}^{k}\delta(g)_j^{s_j}\,d\mu_G(g)\\
&= 2^{r_2}\sqrt{\det(Q^{\intercal}Q)}\int_{g\in G}\mathrm{e}^{-\|g\|^2}\prod_{j=1}^{k}\Big(\prod_{v\in\mathcal{A}_L}g_v^{e_v q_{jv}}\Big)^{s_j}\,d\mu_G(g)\\
&= 2^{r_2}\sqrt{\det(Q^{\intercal}Q)}\int_{g\in G}\exp\Big(-\sum_{v\in\mathcal{A}_L}e_v g_v^2\Big)\prod_{v\in\mathcal{A}_L}g_v^{e_v\sum_{j=1}^{k}q_{jv}s_j}\prod_{v\in\mathcal{A}_L}\frac{dg_v}{g_v}\\
&= 2^{r_2}\sqrt{\det(Q^{\intercal}Q)}\prod_{v\in\mathcal{A}_L}\int_0^\infty g_v^{e_vS_v(s)}\mathrm{e}^{-e_vg_v^2}\frac{dg_v}{g_v}=\frac{\sqrt{\det(Q^{\intercal}Q)}}{2^{r_1}}\prod_{v\in\mathcal{A}_L}\frac{\Gamma(e_vS_v(s)/2)}{e_v^{e_vS_v(s)/2}},
\end{aligned}\tag{34}$$

where r_1 is the number of real places of L.

Lemma 4. *For any $\sigma\in\mathcal{D}$ $\big($see (33)$\big)$, the Mellin inversion formula holds:*

$$\psi(h)=\frac{1}{(2\pi i)^k}\int_{I_\sigma}(M\psi)(s)\,h^{-s}\,ds \qquad\qquad (h\in\mathbb{R}_+^k),\tag{35}$$

where $s=(s_1,...,s_k)$ and $I_\sigma\subset\mathbb{C}^k$ is the product of the k vertical lines $\mathrm{Re}(s_j)=\sigma_j$, *taken from $\sigma_j-i\infty$ to $\sigma_j+i\infty$.*

Proof. The calculation (34) shows that the Mellin transform $(M\psi)(s)$ is defined for $s\in I_\sigma$. Thus Mellin inversion will work provided that $\int_{I_\sigma}\big|(M\psi)(s)h^{-s}\,ds\big|<\infty$. Since $\big|h^{-s}\big|$ and $\big|e_v^{e_vS_v(s)/2}\big|$ are constant on I_σ, we turn to the factors $|\Gamma(e_vS_v(s)/2)|$ in (34). Write $s=\sigma+iT$, $T\in\mathbb{R}^k$. In a strip $0<C_1\le\mathrm{Re}(z)\le C_2$, we have $|\Gamma(z)|<C_3\exp\big(-|\mathrm{Im}(z)|\big)$.[3] Since $\mathrm{Re}\big(e_vS_v(s)\big)=e_vS_v(\sigma)>0$ for $s\in I_\sigma$,

$$\prod_{v\in\mathcal{A}_L}|\Gamma(e_vS_v(s)/2)|<C_4\exp\Big(-\sum_{v\in\mathcal{A}_L}e_v\big|S_v(T)\big|/2\Big)\le C_4\exp\big(-\|S(T)\|_1/2\big),$$

where $\|(m_v)\|_1:=\sum_{v\in\mathcal{A}_L}|m_v|$ is the L^1-norm on $\mathbb{R}^{\mathcal{A}_L}$, and S is the linear function from (31). Since S is injective, there exists $C_5>0$ such that $\|S(T)\|_1\ge C_5\|T\|_1:=C_5\sum_{j=1}^{k}|T_j|$. Thus $(M\psi)(s)h^{-s}$ is integrable over I_σ and Mellin inversion (35) holds. $\square$

Let

$$\Gamma_v(z):=\begin{cases}\Gamma(z) & \text{if } v \text{ is real,}\\ \Gamma(z)\Gamma(z+\frac{1}{2}) & \text{if } v \text{ is complex,}\end{cases}\tag{36}$$

[3] In fact, $|\Gamma(z)|<C_\varepsilon\exp(-(\pi-\varepsilon)|\mathrm{Im}(z)|/2)$ holds for any $\varepsilon>0$ [1, Cor. 1.4.4].

and

$$\alpha(s) := \sum_{v\in\mathcal{A}_L} \log\Gamma_v\big(S_v(s)\big) = \sum_{v\in\mathcal{A}_L} \log\Gamma_v\big(\textstyle\sum_{j=1}^k q_{jv}s_j\big). \tag{37}$$

We take the branch of $\log\Gamma_v(z)$ which is real when z is real and positive.

Lemma 5. *Let $y=(y_1,\dots,y_k)\in\mathbb{R}^k$ and $\chi := (\mathrm{e}^{y_1/2},\dots,\mathrm{e}^{y_k/2})\in H:=\mathbb{R}_+^k$. Then*

$$\psi(\chi) = \frac{\sqrt{\det(Q^\intercal Q)}}{2^{r_1}(2\sqrt{\pi})^{r_2}(\pi i)^k}\int_{s\in I_\sigma}\exp\big(\alpha(s)-\textstyle\sum_{j=1}^k y_js_j\big)\,ds \qquad (\text{for any } \sigma\in\mathcal{D}), \tag{38}$$

with ψ as in (29), α as in (37), Q as in Lemma 3, I_σ as in Lemma 4, and r_1 (resp. r_2) being the number of real (resp. complex) places of L.

Proof. If v is complex, so $e_v=2$, the duplication formula gives

$$\frac{\Gamma\big(e_vS_v(s)\big)}{e_v^{e_vS_v(s)}} = \frac{\Gamma\big(2S_v(s)\big)}{2^{2S_v(s)}} = \frac{\Gamma\big(S_v(s)\big)\Gamma\big(\frac12+S_v(s)\big)}{2\sqrt{\pi}} = \frac{\Gamma_v\big(S_v(s)\big)}{2\sqrt{\pi}}.$$

If v is real, so $e_v=1$, then

$$\frac{\Gamma\big(e_vS_v(s)\big)}{e_v^{e_vS_v(s)}} = \Gamma\big(S_v(s)\big) = \Gamma_v\big(S_v(s)\big).$$

From (34) and Mellin inversion (35) we get

$$\begin{aligned}
\psi(\chi) &= \frac{1}{(2\pi i)^k}\int_{s\in I_{\sigma/2}}\chi^{-s}\cdot(M\psi)(s)\,ds\\
&= \frac{\sqrt{\det(Q^\intercal Q)}}{2^{r_1}(2\pi i)^k}\int_{s\in I_{\sigma/2}}\prod_{j=1}^k\chi_j^{-s_j}\cdot\prod_{v\in\mathcal{A}_L}\frac{\Gamma\big(\frac{e_vS_v(s)}{2}\big)}{e_v^{e_vS_v(s)/2}}\,ds\\
&= \frac{\sqrt{\det(Q^\intercal Q)}}{2^{r_1}(\pi i)^k}\int_{s\in I_\sigma}\prod_{j=1}^k\chi_j^{-2s_j}\cdot\prod_{v\in\mathcal{A}_L}\frac{\Gamma\big(e_vS_v(s)\big)}{e_v^{e_vS_v(s)}}\,ds\\
&= \frac{\sqrt{\det(Q^\intercal Q)}}{2^{r_1}(2\sqrt{\pi})^{r_2}(\pi i)^k}\int_{s\in I_\sigma}\exp\big(-\textstyle\sum_{j=1}^k y_js_j\big)\displaystyle\prod_{v\in\mathcal{A}_L}\Gamma_v\big(S_v(s)\big)\,ds\\
&= \frac{\sqrt{\det(Q^\intercal Q)}}{2^{r_1}(2\sqrt{\pi})^{r_2}(\pi i)^k}\int_{s\in I_\sigma}\exp\big(\alpha(s)-\textstyle\sum_{j=1}^k y_js_j\big)\,ds. \qquad\square
\end{aligned}$$

Now we apply the lemma to the Basic Inequality (17).

Corollary 6. *For $t>0$ and $a\in L^*$, define $y=y_{a,t}\in\mathbb{R}^k$ by*

$$y_j=(y_{a,t})_j := \begin{cases}\log(t)+\frac{2}{n}\log|\mathrm{Norm}_{L/\mathbb{Q}}(a)| & \text{if } j=1,\\ \frac{2}{n}\sum_{v\in\mathcal{A}_L}e_vq_{jv}\log|a|_v & \text{if } 2\le j\le k.\end{cases} \tag{39}$$

Then, with $\mathcal{L} := \sqrt{\det(Q^\intercal Q)}/\big(2^{r_1}(2\sqrt{\pi})^{r_2}\pi^k\big)$, for any $\sigma\in\mathcal{D}$ we have

$$\int_{x\in E_\mathbb{R}}\mathrm{e}^{-t\,\|ax\|^2}\,d\mu_{E_\mathbb{R}}(x) = \frac{\mathcal{L}}{i^k}\int_{s\in I_\sigma}\exp\big(\alpha(s)-n\textstyle\sum_{j=1}^k y_js_j\big)\,ds, \tag{40}$$

and

$$\frac{2t}{n}\int_{x\in E_{\mathbb{R}}}\|ax\|^2\mathrm{e}^{-t\,\|ax\|^2}\,d\mu_{E_{\mathbb{R}}}(x)=\frac{2\mathcal{L}}{i^k}\int_{s\in I_\sigma}s_1\exp\bigl(\alpha(s)-n\textstyle\sum_{j=1}^k y_js_j\bigr)\,ds. \tag{41}$$

Proof. Define $r=r_{a,t}\in G:=\mathbb{R}_+^{\mathcal{A}_L}$ by $r_v:=t^{1/2}|a|_v$. In view of (30) and Lemma 5, (40) will follow from $\bigl(\delta(r)\bigr)_j=\mathrm{e}^{ny_j/2}$. Indeed, by (22),

$$\bigl(\delta(r)\bigr)_j:=\prod_{v\in\mathcal{A}_L}\bigl(t^{1/2}|a|_v\bigr)^{e_vq_{jv}}=t^{\frac{1}{2}\sum_v e_vq_{jv}}\prod_{v\in\mathcal{A}_L}|a|_v^{e_vq_{jv}}.$$

If $j=1$, then by (20) we have $q_{jv}=1$ for all $v\in\mathcal{M}$. Using (12) and (13) we find

$$\bigl(\delta(r)\bigr)_1=t^{n/2}|\mathrm{Norm}_{L/\mathbb{Q}}(a)|=\mathrm{e}^{ny_1/2}.$$

If $j>1$, then $\sum_v e_vq_{jv}=0$ $\bigl(\text{see (20)}\bigr)$, so

$$\bigl(\delta(r)\bigr)_j=\prod_{v\in\mathcal{A}_L}|a|_v^{e_vq_{jv}}=\mathrm{e}^{ny_j/2},$$

as claimed. To prove (41), apply $-\frac{2t}{n}\frac{d}{dt}$ to (40), noting that $\frac{dy_j}{dt}=0$ for $j\geq 2$. $\square$

4. Existence and uniqueness of the critical point

We shall show that for every $y\in\mathbb{R}^k$ there is a unique $\sigma=\sigma(y)\in\mathcal{D}$ $\bigl(\text{see (33)}\bigr)$ which is a critical point of $F_y\colon\mathcal{D}\to\mathbb{R}$, defined as

$$F_y(\sigma):=\alpha(\sigma)-\sum_{j=1}^k y_j\sigma_j=\alpha(\sigma)-y\cdot\sigma, \tag{42}$$

with α as in (37). The map taking $y\in\mathbb{R}^k$ to the critical point $\sigma(y)\in\mathcal{D}$ is closely related to the Legendre transform of $\alpha\colon\mathcal{D}\to\mathbb{R}$, but we will develop the theory from scratch as ours is an easy case of the general theory of the Legendre transform [14, §E] [20, §1 and §5].

Lemma 7. *Let $\alpha\colon\mathcal{D}\to\mathbb{R}$ be as in* (37)*. Then α is steep* [20, p. 30]*, i.e.,*

$$\lim_{\|\sigma\|\to\infty}\frac{\alpha(\sigma)}{\|\sigma\|}=+\infty,$$

where the limit is taken over $\sigma\in\mathcal{D}$ as its Euclidean norm $\|\sigma\|$ tends to infinity.

Proof. Recall that the linear map S in (31) is injective. Hence there exists $C>0$ such that, for all $\sigma\in\mathcal{D}$,

$$\max_{v\in\mathcal{A}_L}\bigl\{S_v(\sigma)\bigr\}=\max_{v\in\mathcal{A}_L}\bigl\{\bigl|S_v(\sigma)\bigr|\bigr\}=:\|S(\sigma)\|_\infty\geq C\|\sigma\|.$$

For any $\sigma\in\mathcal{D}$, there is a $v_0=v_0(\sigma)\in\mathcal{A}_L$ such that $S_{v_0}(\sigma)=\max_{v\in\mathcal{A}_L}\bigl\{S_v(\sigma)\bigr\}$. The previous inequality says that

$$S_{v_0}(\sigma)\geq C\|\sigma\|. \tag{43}$$

The known behavior of $\Gamma(z)$ for $z>0$ shows that there is a $\kappa<0$ such that

$$\log\Gamma_v(z)>\kappa \qquad\qquad \bigl(\Gamma_v\text{ as in (36)}\bigr), \tag{44}$$

for all $z > 0$ and all $v \in \mathcal{A}_L$ ($\kappa = -1/5$ will do). Also, Stirling's formula shows that

$$\log \Gamma_v(z) > \frac{z \log z}{2} \tag{45}$$

for $z \gg 0$. It follows from (44), (43), and (45) that when $\|\sigma\|$ is large,

$$\alpha(\sigma) := \sum_{v \in \mathcal{A}_L} \log \Gamma_v(S_v(\sigma)) > n\kappa + \log \Gamma_{v_0}\bigl(S_{v_0}(\sigma)\bigr) > n\kappa + \tfrac{1}{2} C \|\sigma\| \log(C\|\sigma\|),$$

and the lemma follows. ∎

The next lemma amounts to the fact that the gradient ∇f of a steep and differentiable strictly convex function f is a bijection. However, in our case the domain $\mathcal{D} \neq \mathbb{R}^k$, which means that we would need to check the boundary behavior of α before citing results from convex analysis. We prefer not to quote and instead adapt the usual proof [20, §1] [14, §E] to our nicely behaved function α.

Lemma 8. *For any $y \in \mathbb{R}^k$ there is a unique $\sigma = \sigma(y) \in \mathcal{D}$ such that $y = \nabla\alpha(\sigma)$.*

Proof. For any $y \in \mathbb{R}^k$, let $F_y \colon \mathcal{D} \to \mathbb{R}$ be defined by $F_y(\tau) := \alpha(\tau) - y \cdot \tau$, and let

$$\alpha^\dagger(y) := \inf_{\tau \in \mathcal{D}} \bigl\{F_y(\tau)\bigr\}, \tag{46}$$

which we will now prove to be finite, i.e., $\alpha^\dagger(y) \neq -\infty$. Let $\tau^{(i)}$ be a sequence in $\mathcal{D}$ such that $F_y(\tau^{(i)})$ converges to $\alpha^\dagger(y)$. By (44), $\alpha(\tau^{(i)})$ is bounded below, so it suffices to check that the sequence $\tau^{(i)}$ is bounded. By Lemma 7, $\alpha(\tau) > (\|y\| + 1)\|\tau\|$ for $\tau \in \mathcal{D}$ with $\|\tau\|$ sufficiently large. For such τ,

$$F_y(\tau) > (\|y\| + 1)\|\tau\| - \|y\|\,\|\tau\| = \|\tau\|,$$

which shows that $\tau^{(i)}$ is bounded.

We now prove that the infimum defining $\alpha^\dagger(y)$ is assumed at a point in the open set $\mathcal{D} \subset \mathbb{R}^k$. Passing to a subsequence of the bounded sequence $\tau^{(i)}$, we may assume that the $\tau^{(i)} \in \mathcal{D}$ converge to a point σ in the closure of $\mathcal{D}$ in $\mathbb{R}^k$. Recall from (33) that $\mathcal{D}$ is the (non-empty) open set consisting of $\tau \in \mathbb{R}^k$ such that $S_v(\tau) > 0$ for all $v \in \mathcal{A}_L$. If $\sigma \notin \mathcal{D}$, then $S_v(\sigma) = 0$ for some $v \in \mathcal{A}_L$. Since $\log \Gamma_v\bigl(S_v(\tau^{(i)})\bigr) \to +\infty$ as $S_v(\tau^{(i)}) \to 0^+$, and the remaining summands in the definition of α remain bounded from below (as does $y \cdot \tau^{(i)}$), we conclude that $\sigma \in \mathcal{D}$. Since σ is an interior minimum of the smooth function F_y, we have $\nabla F_y(\sigma) = 0$. By (42), $y = \nabla\alpha(\sigma)$, as claimed.

To prove the uniqueness of σ, it suffices to prove that F_y is a strictly convex function on $\mathcal{D}$.[4] The strict convexity of F_y follows from the strict convexity of

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[4] That is, $F_y(t\tau + (1-t)\tilde{\tau}) < tF_y(\tau) + (1-t)F_y(\tilde{\tau})$ for all $t \in (0,1)$ and all $\tau \neq \tilde{\tau} \in \mathcal{D}$. Such a function cannot have more than one critical point. To prove this, let $g(t) := F_y\bigl(t\tau + (1-t)\tilde{\tau}\bigr)$. Assuming that F_y is strictly convex, g is a strictly convex function of a single real variable $t \in [0,1]$. Thus, $g'' \geq 0$, so g has an increasing derivative $g'(t) = \nabla F_y(t\tau + (1-t)\tilde{\tau}) \cdot (\tau - \tilde{\tau})$. But $\nabla F_y(\tau) = 0 = \nabla F_y(\tilde{\tau})$ would imply $g'(0) = 0 = g'(1)$, whence g is constant and therefore not strictly convex.

$\log\Gamma(z)$ for $z>0$. Indeed,

$$\begin{aligned}F_y(t\tau+(1-t)\tilde\tau) &= -(t\tau+(1-t)\tilde\tau)\cdot y+\alpha(t\tau+(1-t)\tilde\tau)\\ &= -(t\tau+(1-t)\tilde\tau)\cdot y+\sum_{v\in\mathcal{A}_L}\log\Gamma_v\big(S_v(t\tau+(1-t)\tilde\tau)\big)\\ &\le -(t\tau+(1-t)\tilde\tau)\cdot y+\sum_{v\in\mathcal{A}_L} t\log\Gamma_v\big(S_v(\tau)\big)+(1-t)\log\Gamma_v\big(S_v(\tilde\tau)\big)\\ &= tF_y(\tau)+(1-t)F_y(\tilde\tau),\end{aligned}$$

with strict inequality holding for $t\in(0,1)$ unless $S_v(\tau)=S_v(\tilde\tau)$ for all $v\in\mathcal{A}_L$. But this is impossible because S in (31) is injective. □

The function $\alpha^\dagger$ in (46) is a concave function of $y\in\mathbb{R}^k$, being the infimum over $\tau\in\mathcal{D}$ of the set of concave (in fact, affine) functions $y\mapsto -y\cdot\tau+\alpha(\tau)$. The convex function $-\alpha^\dagger$ is known as the Legendre transform of α.

5. Inequalities at the critical point

To take advantage of the inequality (17), we will later need to drop all terms in (17) corresponding to algebraic integers $a\neq 1$. For this we will need some control of the first coordinate $\sigma_1(y)$ of the function σ in Lemma 8. In this section we take advantage of the concavity of $\Psi:=\Gamma'/\Gamma$ to find a lower bound for $\sigma_1(y)$. Then we use the convexity of $\log\Gamma$ to find a lower bound for $\alpha\big(\sigma(y)\big)$. Let

$$\Psi_{\mathbb{C}}(z):=\Psi(z)+\Psi(z+\tfrac12), \tag{47}$$

$$\Psi_v(z):=\begin{cases}\Psi(z) & \text{if } v \text{ is real,}\\ \Psi_{\mathbb{C}}(z) & \text{if } v \text{ is complex,}\end{cases}\qquad (v\in\mathcal{A}_L). \tag{48}$$

These definitions ensure that $\Psi_v(z)=\frac{d}{dz}\log\Gamma_v(z)=\Gamma_v'(z)/\Gamma_v(z)$ $\big($see (36)$\big)$. Note that $\Psi_v(z)$ is a concave function of z for $z>0$. We also note that $\Psi_v\colon(0,\infty)\to\mathbb{R}$ has an inverse function $\Psi_v^{-1}\colon\mathbb{R}\to(0,\infty)$ since $\Psi(z)$ is strictly increasing when $z>0$, tends to $-\infty$ as $z\to0^+$, and tends to $+\infty$ as $z\to+\infty$.

Writing out the ℓ-th coordinate of the equation $y=\nabla\alpha(\sigma)$ in Lemma 8, we get

$$y_\ell=\sum_{v\in\mathcal{A}_L}\Psi_v\big(S_v(\sigma)\big)q_{\ell v}\qquad\Big(S_v(\sigma)=\textstyle\sum_{j=1}^k q_{jv}\sigma_j,\quad \sigma:=\sigma(y)\Big), \tag{49}$$

which for $\ell=1$ simplifies to

$$y_1=\sum_{v\in\mathcal{A}_L}\Psi_v\big(S_v(\sigma)\big). \tag{50}$$

Lemma 9. *Let L be a number field of degree n, with r_2 complex places. For $y=(y_1,y_2,\dots,y_k)\in\mathbb{R}^k$, let $\sigma_1(y)$ be the first coordinate of the function $\sigma(y)$ defined in Lemma 8. Then*

$$\sigma_1(y_1,y_2,\dots,y_k)\ge\Psi^{-1}\left(\frac{y_1}{n}\right)-\frac{r_2}{2n}. \tag{51}$$

Proof. We prove (51) using the concavity of Ψ. Namely, from (50),

$$\begin{aligned} y_1 &= \sum_{v\in\mathcal{A}_L} \Psi_v\big(S_v(\sigma)\big) = \sum_{v\in\mathcal{A}_L} \Psi\big(S_v(\sigma)\big) + \sum_{v\text{ complex}} \Psi\big(\tfrac{1}{2}+S_v(\sigma)\big) \\ &\le n\Psi\left(\tfrac{1}{n}\Big(\textstyle\sum_{v\in\mathcal{A}_L} S_v(\sigma) + \sum_{v\text{ complex}}\big(\tfrac{1}{2}+S_v(\sigma)\big)\Big)\right) \\ &= n\Psi\big(\tfrac{1}{n}\textstyle\sum_{v\in\mathcal{A}_L} e_v S_v(\sigma) + \tfrac{r_2}{2n}\big) = n\Psi\big(\sigma_1+\tfrac{r_2}{2n}\big), \end{aligned}$$

where the last step uses

$$\frac{1}{n}\sum_{v\in\mathcal{A}_L} e_v S_v(\sigma) = \sigma_1 \qquad \big(\sigma=(\sigma_1,\sigma_2,\dots,\sigma_k)\in\mathbb{C}^k\big), \tag{52}$$

which follows from (20) since

$$\sum_{v\in\mathcal{A}_L} e_v S_v(\sigma) = \sum_{v\in\mathcal{A}_L}\sum_{j=1}^{k} e_v q_{jv}\sigma_j = \sum_{j=1}^{k}\sigma_j \sum_{v\in\mathcal{A}_L} e_v q_{jv} = \sigma_1 \sum_{v\in\mathcal{A}_L} e_v = \sigma_1 n.$$

Inequality (51) now follows, since Ψ^{-1} is an increasing function. □

Our next result is a similar inequality for $\alpha(\sigma)$.

Lemma 10. *With notation as in Lemma 9, we have*

$$\alpha(\sigma) \ge n\log\Gamma\big(\sigma_1+\tfrac{r_2}{2n}\big) \qquad (\sigma=(\sigma_1,\dots,\sigma_k)\in\mathcal{D}). \tag{53}$$

Proof. We compute directly from the definition (37) of α, using the convexity of $z\mapsto\log\Gamma(z)$ for $z>0$ and (52):

$$\begin{aligned} \alpha(\sigma) &= \sum_{v\in\mathcal{A}_L}\log\Gamma\big(S_v(\sigma)\big) + \sum_{v\text{ complex}}\log\Gamma\big(S_v(\sigma)+\tfrac{1}{2}\big) \\ &\ge n\log\Gamma\left(\tfrac{1}{n}\Big(\textstyle\sum_{v\in\mathcal{A}_L} S_v(\sigma) + \sum_{v\text{ complex}}\big(\tfrac{1}{2}+S_v(\sigma)\big)\Big)\right) \\ &= n\log\Gamma\big(\tfrac{1}{n}\textstyle\sum_{v\in\mathcal{A}_L} e_v S_v(\sigma)+\tfrac{r_2}{2n}\big) = n\log\Gamma\big(\sigma_1+\tfrac{r_2}{2n}\big). \end{aligned}$$ □

We now prove a lower bound for $S_v(\sigma)$ in terms of σ_1 and y_1.

Lemma 11. *Let $u\in\mathcal{A}_L$, $y\in\mathbb{R}^k$, and let $\sigma := \sigma(ny)\in\mathcal{D}$ be as in Lemma 8. Assume that $y_1\ge t_0$ for some $t_0\in\mathbb{R}$, and $n:=[L:\mathbb{Q}]\ge 2$. Then $S_u(\sigma)\ge 2/5$ or*

$$S_u(\sigma) \ge \frac{1}{(n-1)\Psi\big(\frac{n\sigma_1}{n-1}+\frac{1}{2}\big)-nt_0} \ge \frac{1}{(n-1)\log(2\sigma_1+\frac{1}{2})-nt_0} > 0. \tag{54}$$

Proof. We shall show below that both denominators in (54) are positive if $S_u(\sigma) < 2/5$, as we may assume. Replacing y with ny in (50), we have

$$ny_1 = \sum_{v\in\mathcal{A}_L}\Psi\big(S_v(\sigma)\big) + \sum_{\substack{v\in\mathcal{A}_L\\ v\text{ complex}}}\Psi\big(\tfrac{1}{2}+S_v(\sigma)\big).$$

Since $-\Psi$ is a monotone decreasing convex function on $(0,\infty)$, we find

$$\begin{aligned}
\Psi\big(S_u(\sigma)\big) &= ny_1 - \sum_{\substack{v\in\mathcal{A}_L\\ v\neq u}} \Psi\big(S_v(\sigma)\big) - \sum_{\substack{v\in\mathcal{A}_L\\ v\ \text{complex}}} \Psi\big(\tfrac{1}{2}+S_v(\sigma)\big)\\
&\geq ny_1 - (n-1)\Psi\bigg(\frac{1}{n-1}\Big(\sum_{\substack{v\in\mathcal{A}_L\\ v\neq u}} S_v(\sigma) + \sum_{\substack{v\in\mathcal{A}_L\\ v\ \text{complex}}} \tfrac{1}{2}+S_v(\sigma)\Big)\bigg)\\
&= ny_1 - (n-1)\Psi\bigg(\frac{1}{n-1}\Big(-S_u(\sigma) + \sum_{v\in\mathcal{A}_L} e_v S_v(\sigma) + \sum_{\substack{v\in\mathcal{A}_L\\ v\ \text{complex}}} \tfrac{1}{2}\Big)\bigg)\\
&= ny_1 - (n-1)\Psi\Big(-\frac{S_u(\sigma)}{n-1} + \frac{n\sigma_1}{n-1} + \frac{r_2}{2(n-1)}\Big) \qquad \big(\text{see } (52)\big)\\
&\geq ny_1 - (n-1)\Psi\Big(\frac{n\sigma_1}{n-1} + \frac{r_2}{2(n-1)}\Big) \geq ny_1 - (n-1)\Psi\Big(\frac{n\sigma_1}{n-1} + \frac{1}{2}\Big).
\end{aligned}$$

From $x\Gamma(x) = \Gamma(x+1)$ and the fact that $\Psi(x) < 0$ for $x < 1.461$,

$$\Psi(x) = -\frac{1}{x} + \Psi(1+x) < -\frac{1}{x} \qquad\qquad (x < 0.461).$$

Hence, as we are assuming $S_u(\sigma) < 2/5$,

$$\frac{-1}{S_u(\sigma)} > \Psi\big(S_u(\sigma)\big) \geq ny_1 - (n-1)\Psi\Big(\frac{n\sigma_1}{n-1} + \frac{1}{2}\Big) \geq nt_0 - (n-1)\Psi\Big(\frac{n\sigma_1}{n-1} + \frac{1}{2}\Big).$$

Since $S_u(\sigma) > 0$, the right-hand side above is negative. Hence the left-most inequality in (54) is proved.

Next recall [16, §71, eq. (11)],

$$\log(x) - \Psi(x) = \frac{1}{2x} + 2\int_0^\infty \frac{t}{(t^2+x^2)(\mathrm{e}^{2\pi t}-1)}\,dt > 0 \qquad\qquad (x > 0).$$

Whence $\Psi(x) < \log(x)$ for $x > 0$, and so

$$\Psi\Big(\frac{n\sigma_1}{n-1} + \frac{1}{2}\Big) < \log\Big(\frac{n\sigma_1}{n-1} + \frac{1}{2}\Big) \leq \log\big(2\sigma_1 + \tfrac{1}{2}\big).$$

Now the second inequality in (54) follows as before. □

6. Asymptotics

In this section we prove the estimates needed to prove Theorem 1. We will work in a somewhat more abstract setting since the assumption in Theorem 1 that $E \subset \mathcal{O}_L^*$ contain the relative units $E(L/K)$ will be useful only through the following property.

Lemma 12. *Suppose $K \subset L$ is a subfield, $E(L/K) \subset E \subset \mathcal{O}_L^*$, let* Log *be as in* (18)*, and suppose $q = (q_v)_{v\in\mathcal{A}_L} \in \mathrm{Log}(E)^\perp$ lies in the orthogonal complement of* $\mathrm{Log}(E)$ *inside $\mathbb{R}^{\mathcal{A}_L}$ with respect to the inner product* (19)*. Then $q_v = q_{v'}$ whenever v and v' lie above the same Archimedean place of K.*

Proof. It suffices to show that $\mathrm{Log}(E)^\perp$ is contained in the $\mathbb{R}$-span of $\mathrm{Log}(K^*)$ in $\mathbb{R}^{\mathcal{A}_L}$. But, $\operatorname{span}\mathrm{Log}(K^*) = \mathrm{Log}(E(L/K))^\perp$ since

$$\begin{aligned} &\mathrm{Log}(E)^\perp \subset \mathrm{Log}(E(L/K))^\perp, \qquad \mathrm{Log}(K^*) \subset \mathrm{Log}(E(L/K))^\perp, \\ &\dim(\operatorname{span}\mathrm{Log}(K^*)) = \#\mathcal{A}_K = \dim(\mathrm{Log}(E(L/K))^\perp). \end{aligned}$$ □

We formalize the above property as follows.

Definition 13. *For $M, \Omega \geq 1$, a subgroup $E \subset \mathcal{O}_L^*$ is (M,Ω)-dispersed if there is a surjective map of sets $\pi : \mathcal{A}_L \to \mathcal{M}$, where $\mathcal{M}$ has cardinality $\#\mathcal{M} = M$, with the following properties.*

(i) For $w \in \mathcal{M}$, let $r_{1,w}$ and $r_{2,w}$ be respectively the number of real and complex places of L with image w under π, let

$$m_w := r_{1,w} + 2r_{2,w}, \qquad\qquad m := \min_{w\in\mathcal{M}}\{m_w\}. \tag{55}$$

Then $m_w/m \leq \Omega$ for all $w \in \mathcal{M}$.

(ii) If $q = (q_v)_{v\in\mathcal{A}_L} \in \mathrm{Log}(E_\mathbb{R})^\perp$, the orthogonal complement of $\mathrm{Log}(E_\mathbb{R})$ in $\mathbb{R}^{\mathcal{A}_L}$ with respect to $\langle\ ,\ \rangle$ in (19), then $q_v = q_{v'}$ whenever $\pi(v) = \pi(v')$.

This rather *ad hoc* definition is meant mainly to clarify the proof of the estimates below. Being (M,Ω)-dispersed will prove useful if $m \gg M\Omega$, with m as in (55).

Remark 14. *It is easy to see that a subgroup E containing relative units $E(L/K)$, as in Theorem 1, fits in the (M,Ω) framework. Indeed, if K is a subfield of L, let $\mathcal{M} := \mathcal{A}_K$ and let $\pi : \mathcal{A}_L \to \mathcal{M}$ be the restriction map. Then condition (i) of Definition 13 holds with $\Omega = 2$ since $m_w = e_w \cdot [L:K] \leq 2[L:K]$. Lemma 12 shows that condition (ii) holds if the relative units $E(L/K) \subset E \subset \mathcal{O}_L^*$. Thus E in Theorem 1 is $(\#\mathcal{A}_K, 2)$-dispersed. Determining if there are interesting examples of (M,Ω)-dispersion beyond that of Theorem 1 seems to be a difficult question.*

As outlined in §2, the Basic Inequality (17) and Corollary 6 lead us to estimate integrals of the type

$$\frac{1}{i^k}\int_{s\in I_\sigma} \mathrm{e}^{\alpha(s)-ny\cdot s}\,ds = \int_{T\in\mathbb{R}^k} \mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\,dT =: \int_{\mathbb{R}^k}\mathcal{G}(T)\,dT, \tag{56}$$

where $n := [L:\mathbb{Q}],\ y = (y_1,\ldots,y_k) \in \mathbb{R}^k,\ \sigma := \sigma(ny) \in \mathcal{D} \subset \mathbb{R}^k$ as in Lemma 8, and $y\cdot s := \sum_{j=1}^k y_j s_j$. Let $\mathcal{H}(T)$ be the Gaussian approximating $\mathcal{G}(T)$ (see (65) below) in a bounded neighborhood $\Delta \subset \mathbb{R}^k$ of $T = 0$ $\big($see (87)$\big)$. As usual with the saddle point method, we decompose the integral (56) into four pieces

$$\begin{aligned} \int_{\mathbb{R}^k}\mathcal{G}(T)\,dT = \int_{\mathbb{R}^k}\mathcal{H}(T)\,dT + \int_{\mathbb{R}^k-\Delta}\mathcal{G}(T)\,dT - \int_{\mathbb{R}^k-\Delta}\mathcal{H}(T)\,dT \\ + \int_\Delta \big(\mathcal{G}(T)-\mathcal{H}(T)\big)\,dT =: I_1 + I_2 - I_3 + I_4. \end{aligned} \tag{57}$$

The term I_1 in (57) (i.e., $\int_{\mathbb{R}^k}\mathcal{H}$) is readily computed and gives the main term in (57). We shall prove that the terms I_2, I_3 and I_4 are $o(I_1)$ as $m \to \infty$, uniformly in

$y \in \mathbb{R}^k$, provided E is (M,Ω)-dispersed, *as we shall henceforth assume.* We will use $\pi, \mathcal{M}, m, w, m_w, r_{1,w}$ and $r_{2,w}$ as in Definition 13.

Let (cf. [13, p. 134])

$$\alpha_\kappa(z) := \kappa \log\Gamma(z) + (1-\kappa)\log\Gamma(z+\tfrac{1}{2}), \qquad \kappa_w := \frac{r_{1,w}+r_{2,w}}{m_w}. \tag{58}$$

Note that $\frac{1}{2} \le \kappa_w \le 1$. Recall that in (20) we defined a basis $\{q_1,\ldots,q_k\}$ of $\mathrm{Log}(E)^\perp$. For $w \in \mathcal{M}$, define $S_w : \mathbb{C}^k \to \mathbb{C}$ as

$$S_w(s) := S_v(s) := \sum_{j=1}^k q_{jv}s_j =: \sum_{j=1}^k q_{jw}s_j \qquad \big(s \in \mathbb{C}^k,\ v \in \mathcal{A}_L,\ \pi(v)=w\big), \tag{59}$$

where $q_{jw} := q_{jv}$ for any $v \in \pi^{-1}(w)$ (this being well defined by (ii) in Definition 13). We can therefore rewrite α in (37) using (58) as

$$\alpha(s) := \sum_{v\in\mathcal{A}_L} \log\Gamma_v\big(S_v(s)\big) = \sum_{w\in\mathcal{M}}\sum_{v\in\pi^{-1}(w)} \log\Gamma_v\big(S_w(s)\big) = \sum_{w\in\mathcal{M}} m_w\alpha_{\kappa_w}\big(S_w(s)\big). \tag{60}$$

For each $w \in \mathcal{M}$ and $\sigma \in \mathcal{D}$ $\big($see (33)$\big)$, define $\rho_w\colon \mathbb{R}^k \to \mathbb{C}$ by

$$\rho_w(T) := \alpha_{\kappa_w}\big(S_w(\sigma+iT)\big) - \alpha_{\kappa_w}\big(S_w(\sigma)\big) - i\alpha'_{\kappa_w}\big(S_w(\sigma)\big)S_w(T) + \frac{1}{2}\alpha''_{\kappa_w}\big(S_w(\sigma)\big)\big(S_w(T)\big)^2, \tag{61}$$

i.e., ρ_w is the error in the degree-2 Taylor approximation of $T \mapsto \alpha_{\kappa_w}\big(S_w(\sigma+iT)\big)$ at $T=0$. We shall henceforth take any $y \in \mathbb{R}^k$ and let $\sigma := \sigma(ny)$ be the corresponding saddle point in Lemma 8. Thus $\nabla\alpha(\sigma) = ny$. Using this and (60), we find

$$\sum_{j=1}^k ny_jT_j = \sum_{j=1}^k T_j \sum_{w\in\mathcal{M}} m_w\alpha'_{\kappa_w}\big(S_w(\sigma)\big)q_{jw} = \sum_{w\in\mathcal{M}} m_w\alpha'_{\kappa_w}\big(S_w(\sigma)\big)S_w(T). \tag{62}$$

It follows from (60)–(62) that

$$\alpha(\sigma+iT) - ny\cdot(\sigma+iT) = \alpha(\sigma) - ny\cdot\sigma - \frac{1}{2}\sum_{w\in\mathcal{M}} m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)S_w(T)^2 + \rho(T),$$
$$\rho(T) := \sum_{w\in\mathcal{M}} m_w\rho_w(T). \tag{63}$$

The linear terms in T have disappeared precisely because σ is a critical point of $s \mapsto \alpha(s) - ny\cdot s$.

For fixed $y \in \mathbb{R}^k$ and $\sigma := \sigma(ny) \in \mathcal{D}$, define the following functions of $T \in \mathbb{R}^k$:

$$H(T) := \sum_{w\in\mathcal{M}} m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)S_w(T)^2, \tag{64}$$

$$\mathcal{H}(T) := \mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma-\frac{1}{2}H(T)}, \tag{65}$$

$$\mathcal{G}(T) := \mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)} = e^{\rho(T)}\mathcal{H}(T). \tag{66}$$

Although $\mathcal{H}, H, \mathcal{G}$ and ρ depend on $y \in \mathbb{R}^k$, we do not include y in our notation.

6.1. **The main term.** In Lemma 3 we defined the $\#\mathcal{A}_L \times k$ matrix Q of rank k whose coefficients are $Q_{v,j} := q_{jv}$. Recall that we write $q_{jw} := q_{jv}$ for any $v \in \mathcal{A}_L$ with $\pi(v) = w$. We will write $\mathcal{Q}$ for the $M \times k$ matrix with entries $\mathcal{Q}_{wj} := q_{jw}$ and rank k. In the computation of $\psi(\chi)$ in Lemma 5 the term $\det(Q^\intercal Q)$ appears. Using the smaller matrix $\mathcal{Q}$ we have

$$\det(Q^\intercal Q) = \det(\mathcal{Q}^\intercal \mathcal{Q}) \prod_{w\in\mathcal{M}} (r_{1,w} + r_{2,w}) \qquad \big(r_{1,w}, r_{2,w} \text{ as in (58)}\big), \tag{67}$$

as follows from

$$(Q^\intercal Q)_{i,j} = \sum_{v\in\mathcal{A}_L} q_{iv}q_{jv} = \sum_{w\in\mathcal{M}} q_{iw}q_{jw} \sum_{v\in\pi^{-1}(w)} 1 = \sum_{w\in\mathcal{M}} q_{iw}q_{jw}(r_{1,w} + r_{2,w}).$$

For future reference we note that

$$k \le M, \tag{68}$$

as the $M \times k$ matrix $\mathcal{Q}$ has the same rank as Q, namely k.

Let $\mathcal{M}^{[k]}$ be the set of subsets of $\mathcal{M}$ of cardinality k. For $\eta \in \mathcal{M}^{[k]}$, let $\mathcal{Q}_\eta$ be the $k \times k$ submatrix of $\mathcal{Q}$ whose rows are indexed by the elements of η. Next we calculate some integrals such as I_1 in (57), and its derivatives.

Lemma 15. *Let $E \subset \mathcal{O}_L^*$ be (M, Ω)-dispersed (see Definition 13), for $\eta \in \mathcal{M}^{[k]}$ let $\mathcal{Q}_\eta$ be as above, let $(b_w)_{w\in\mathcal{M}} \in \mathbb{R}_+^{\mathcal{M}}$, and define*

$$\mathfrak{D}_\eta := \det\nolimits^2(\mathcal{Q}_\eta) \prod_{w\in\eta} b_w, \qquad \mathfrak{D} := \sum_{\eta\in\mathcal{M}^{[k]}} \mathfrak{D}_\eta. \tag{69}$$

Then, with S_w as in (59),

$$\int_{T\in\mathbb{R}^k} \exp\Big(-\frac{1}{2}\sum_{w\in\mathcal{M}} b_w S_w(T)^2\Big)\, dT = (2\pi)^{k/2}\mathfrak{D}^{-1/2}. \tag{70}$$

Furthermore, for any $w_0 \in \mathcal{M}$ we have

$$\begin{aligned}\int_{\mathbb{R}^k} S_{w_0}(T)^4 \exp\Big(-\frac{1}{2}\sum_{w\in\mathcal{M}} b_w S_w(T)^2\Big)\, dT &= 3(2\pi)^{k/2}\mathfrak{D}^{-5/2} b_{w_0}^{-2}\Big(\sum_{\eta\ni w_0}\mathfrak{D}_\eta\Big)^2 \\ &\le 3(2\pi)^{k/2}\mathfrak{D}^{-1/2} b_{w_0}^{-2}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^k} S_{w_0}(T)^6 \exp\Big(-\frac{1}{2}\sum_{w\in\mathcal{M}} b_w S_w(T)^2\Big)\, dT &= 15(2\pi)^{k/2}\mathfrak{D}^{-7/2} b_{w_0}^{-3}\Big(\sum_{\eta\ni w_0}\mathfrak{D}_\eta\Big)^3 \\ &\le 15(2\pi)^{k/2}\mathfrak{D}^{-1/2} b_{w_0}^{-3}.\end{aligned}$$

Proof. Let $P = (P_{w,j})$ be the $M \times k$ matrix with entries $P_{w,j} := \sqrt{b_w}q_{jw}$ ($w \in \mathcal{M}$, $1 \le j \le k$). Then for $T = (T_1, ..., T_k) \in \mathbb{R}^k$, considered as a $k \times 1$ matrix, $PT \in \mathbb{R}^{\mathcal{M}}$ satisfies $(PT)_w = \sqrt{b_w}S_w(T)$. Hence

$$\sum_{w\in\mathcal{M}} b_w S_w(T)^2 = (PT)^\intercal PT = T^\intercal(P^\intercal P)T = T^\intercal HT \qquad (H := P^\intercal P).$$

The $k\times k$ matrix H is clearly positive semi-definite. The Cauchy-Binet formula gives $\det(H)=\mathfrak{D}$, with $\mathfrak{D}$ as in (69).[5] But $\mathfrak{D}>0$ as $\mathfrak{D}_\eta>0$ for at least one $\eta\in\mathcal{M}^{[k]}$, since $\mathcal{Q}$ has rank k. Hence H is positive definite, and so the integral in (70) is the well-known Gaussian integral attached to a positive definite quadratic form H in k variables, as claimed in (70).

The other equalities in Lemma 15 are obtained by differentiating (70) with respect to b_{w_0} repeatedly. Indeed, noting that the partial derivative $\frac{\partial\mathfrak{D}}{\partial b_{w_0}}=b_{w_0}^{-1}\sum_{\eta\ni w_0}\mathfrak{D}_\eta$ is independent of b_{w_0}, i.e., $\frac{\partial^2\mathfrak{D}}{\partial b_{w_0}^2}=0$, we have

$$-\frac{1}{2}\int_{\mathbb{R}^k}S_{w_0}(T)^2\exp\Big(-\frac{1}{2}\sum_{w\in\mathcal{M}}b_wS_w(T)^2\Big)\,dT=-\frac{1}{2}(2\pi)^{k/2}\mathfrak{D}^{-3/2}\Big(b_{w_0}^{-1}\sum_{\eta\ni w_0}\mathfrak{D}_\eta\Big),$$

$$\frac{1}{4}\int_{\mathbb{R}^k}S_{w_0}(T)^4\exp\Big(-\frac{1}{2}\sum_{w\in\mathcal{M}}b_wS_w(T)^2\Big)\,dT=\frac{3}{4}(2\pi)^{k/2}\mathfrak{D}^{-5/2}\Big(b_{w_0}^{-1}\sum_{\eta\ni w_0}\mathfrak{D}_\eta\Big)^2,$$

$$-\frac{1}{8}\int_{\mathbb{R}^k}S_{w_0}(T)^6\exp\Big(-\frac{1}{2}\sum_{w\in\mathcal{M}}b_wS_w(T)^2\Big)\,dT=-\frac{15}{8}(2\pi)^{k/2}\mathfrak{D}^{-7/2}\Big(b_{w_0}^{-1}\sum_{\eta\ni w_0}\mathfrak{D}_\eta\Big)^3,$$

proving the equalities. The inequalities follow from $\sum_{\eta\ni w_0}\mathfrak{D}_\eta\le\mathfrak{D}$, as $\mathfrak{D}_\eta\ge0$. $\square$

As $\alpha_\kappa''(t)>0$ for $t>0$, we can now evaluate I_1.

Corollary 16. *With $\mathcal{H}(T):=\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma-\frac{1}{2}H(T)}$ as in (65), for $y\in\mathbb{R}^k$ we have*

$$I_1=I_1(ny):=\int_{\mathbb{R}^k}\mathcal{H}(T)\,dT=\frac{(2\pi)^{k/2}\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}}{\sqrt{\det\big(H(\sigma)\big)}},$$

where $\sigma:=\sigma(ny)\in\mathcal{D}$ as in Lemma 8 and

$$\det(H(\sigma))=\sum_{\eta\in\mathcal{M}^{[k]}}\det{}^2(\mathcal{Q}_\eta)\prod_{w\in\eta}m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big).\tag{71}$$

6.2. **The small terms.** We begin with some one-variable estimates.

Lemma 17. *Let $p\ge1000$, $\kappa\in[\frac{1}{2},1]$, and $r>0$, then*

$$\int_{-\infty}^{\infty}|\mathrm{e}^{p\alpha_\kappa(r+it)}|\,dt<\frac{\sqrt{2\pi}\mathrm{e}^{p\alpha_\kappa(r)}}{\sqrt{p\alpha_\kappa''(r)}}\Big(1+\frac{2.31}{p}\Big),\tag{72}$$

$$\int_{-\infty}^{\infty}|t\mathrm{e}^{p\alpha_\kappa(r+it)}|\,dt<0.83\frac{\sqrt{2\pi}\mathrm{e}^{p\alpha_\kappa(r)}}{p\alpha_\kappa''(r)}.\tag{73}$$

Proof. Under the assumptions in the lemma, Sundstrom proved[6]

$$\frac{\int_{-\infty}^{\infty}|\mathrm{e}^{p\alpha_\kappa(r+it)}|\,dt}{\frac{\sqrt{2\pi}\mathrm{e}^{p\alpha_\kappa(r)}}{\sqrt{p\alpha_\kappa''(r)}}}<1+\frac{1}{p}\Big(10^{-76}+(2/\pi)^{1/2}p^{5/6}\mathrm{e}^{-p^{1/3}}+\frac{3}{2}\frac{\mathrm{e}^{p^{1/3}}-1}{8/p^{1/3}}\Big).$$

[5] The Cauchy-Binet formula computes $\det(AB)$, where A is a $k\times\ell$ and B is $\ell\times k$, in terms of the $k\times k$ minors of A and B.

[6] Set $D=2$ in the inequality displayed immediately before Lemma 4.5 in [25, p. 142].

As the quantity in parenthesis is decreasing for $p > (5/2)^3$, we can bound it by $2.3093\cdots$, its value for $p = 1000$. Thus, (72) is proved.

We now prove (73). From [25, Lemma 4.11] we have

$$\int_{-\frac{r}{3\sqrt{2}}}^{\frac{r}{3\sqrt{2}}} |t\mathrm{e}^{p\alpha_\kappa(r+it)}|\,dt < \frac{72\mathrm{e}^{p\alpha_\kappa(r)}}{35p\alpha_\kappa''(r)}, \tag{74}$$

while from [13, Lemma 5.3] we have

$$\int_{|t|>\frac{r}{3\sqrt{2}}} |t\mathrm{e}^{p\alpha_\kappa(r+it)}|\,dt < \frac{2r^2\mathrm{e}^{p\alpha_\kappa(r)}}{p(\kappa-\frac{2}{p})(1+\frac{1}{72})^{p\kappa\lfloor r\rfloor/2}(1+\frac{1}{18})^{(p\kappa-2)/2}}, \tag{75}$$

where $\lfloor r\rfloor$ is the floor of r. Since $0 < r^2\alpha_\kappa''(r) < 1 + r$ [13, p. 141], we have

$$\frac{r^2}{(1+\frac{1}{72})^{p\kappa\lfloor r\rfloor/2}} \le \frac{1}{\alpha_\kappa''(r)}\frac{1+r}{(1+\frac{1}{72})^{p\kappa\lfloor r\rfloor/2}} \le \frac{2}{\alpha_\kappa''(r)}.$$

Indeed, for $0 < r < 1$ the last inequality is obvious, while for $r \ge 1$ a much better inequality follows from $p\kappa \ge 500$. Hence

$$\int_{|t|>\frac{r}{3\sqrt{2}}} |t\mathrm{e}^{p\alpha_\kappa(r+it)}|\,dt < \frac{1}{p\alpha_\kappa''(r)}\frac{4\mathrm{e}^{p\alpha_\kappa(r)}}{(\frac{1}{2}-\frac{2}{1000})(1+\frac{1}{18})^{(500-2)/2}} < \frac{0.00002\mathrm{e}^{p\alpha_\kappa(r)}}{p\alpha_\kappa''(r)}.$$

Combining this with (74) we obtain (73). □

We will need the following inequality, proved by elementary calculus.

$$x^{5/2}\mathrm{e}^{-x} \le \left(\frac{5}{2\mathrm{e}}\right)^{5/2} < 0.8112 \qquad \big(x \ge 0\big). \tag{76}$$

Lemma 18. *Suppose $p \ge 1000$, $\frac{1}{2} \le \kappa \le 1$, $0 < D \le p^{1/3}\sqrt{\kappa}$, and let*

$$\delta := \frac{D}{p^{1/3}\sqrt{\alpha_\kappa''(r)}}. \tag{77}$$

Then, for any $r > 0$,

$$\int_{|t|>\delta} |\mathrm{e}^{p\alpha_\kappa(r+it)}|\,dt < \left(\frac{10^{-76}+\frac{41.43}{D^6}}{p}\right)\frac{\sqrt{2\pi}\mathrm{e}^{p\alpha_\kappa(r)}}{\sqrt{p\alpha_\kappa''(r)}}, \tag{78}$$

and

$$\int_{|t|>\delta} \mathrm{e}^{-\frac{1}{2}p\alpha_\kappa''(r)t^2}\,dt < \frac{3.67}{pD^6}\,\frac{\sqrt{2\pi}}{\sqrt{p\alpha_\kappa''(r)}}. \tag{79}$$

Proof. Inequality (79) follows from

$$\begin{aligned}\int_{|t|>\delta} \mathrm{e}^{-\frac{1}{2}p\alpha_\kappa''(r)t^2}\,dt &\le \frac{2\mathrm{e}^{-p^{1/3}D^2/2}}{p^{2/3}D\sqrt{\alpha_\kappa''(r)}} = \frac{\sqrt{2\pi}}{\sqrt{p\alpha_\kappa''(r)}}\,\frac{8(p^{1/3}D^2/2)^{5/2}\mathrm{e}^{-p^{1/3}D^2/2}}{p\sqrt{\pi}D^6}\\ &< \frac{\sqrt{2\pi}}{\sqrt{p\alpha_\kappa''(r)}}\,\frac{3.67}{pD^6},\end{aligned}$$

where the first inequality is from [13, p. 139] and the last one uses (76) with $x := p^{1/3}D^2/2$. To prove (78) we use [25, Lemma 4.5],

$$\frac{\int_{|t|>\delta} |\mathrm{e}^{p\alpha_\kappa(r+it)}|\,dt}{\frac{1}{p}\frac{\sqrt{2\pi}\mathrm{e}^{p\alpha_\kappa(r)}}{\sqrt{p\alpha_\kappa''(r)}}} < 10^{-76} + \frac{2^{3/2}p^{5/6}\exp(-p^{1/3}D^2/4)}{\sqrt{\pi}D} < 10^{-76} + \frac{41.43}{D^6},$$

where the second inequality again follows from (76). □

Next we deal with the second order remainder term in the Taylor expansion about a of $\log\Gamma(a+ib)$, taking $a = S_w(\sigma)$ and $b = S_w(T)$.

Lemma 19. *Let $E \subset \mathcal{O}_L^*$ be (M,Ω)-dispersed. Then for $w \in \mathcal{M}$, $\sigma \in \mathcal{D}$ (see (33)), $T \in \mathbb{R}^k$ and ρ_w as in (61), we have*

$$\big|\mathrm{Im}\big(\rho_w(T)\big)\big| \le -\frac{\alpha_{\kappa_w}^{(3)}\big(S_w(\sigma)\big)}{3!}|S_w(T)|^3 \le \frac{\sqrt{2}}{3}\alpha_{\kappa_w}''\big(S_w(\sigma)\big)^{3/2}|S_w(T)|^3, \tag{80}$$

$$\big|\mathrm{Re}\big(\rho_w(T)\big)\big| \le \frac{\alpha_{\kappa_w}^{(4)}\big(S_w(\sigma)\big)}{4!}S_w(T)^4 \le \frac{1}{2}\alpha_{\kappa_w}''\big(S_w(\sigma)\big)^2 S_w(T)^4, \tag{81}$$

$$\mathrm{Im}\big(\rho_w(-T)\big) = -\mathrm{Im}\big(\rho_w(T)\big), \qquad \mathrm{Re}\big(\rho_w(-T)\big) = \mathrm{Re}\big(\rho_w(T)\big), \tag{82}$$

$$\text{if } |S_w(T)| \le S_w(\sigma), \text{ then } 0 \le \mathrm{Re}\big(\rho_w(T)\big) \le \frac{\alpha_{\kappa_w}''\big(S_w(\sigma)\big)}{4}S_w(T)^2. \tag{83}$$

Proof. The first inequalities in (80) and (81) are proved in [25, Lemma 4.7], as is also (83). The second inequalities in (80) and (81) follow from [13, Lemma 5.2] and $\kappa_w \ge \frac{1}{2}$. The identities in (82) follow from (61) and $\log\Gamma(\overline{z}) = \overline{\log\Gamma(z)}$. □

Lemma 20. ([13, (5.11)]) *If $u, v \in \mathbb{R}$ with $0 \le u \le R$, then*

$$|\mathrm{Re}(e^{u+iv}-1)| \le \frac{v^2}{2} + u\frac{e^R-1}{R}.$$

We first estimate the easier "outer" terms, I_2 and I_3 in (57), i.e., where the region of integration is $\mathbb{R}^k - \Delta$. For $y \in \mathbb{R}^k$, let $\eta_0 = \eta_0(y) \in \mathcal{M}^{[k]}$ correspond to a maximal summand in (71), so

$$\det{}^2(\mathcal{Q}_\eta)\prod_{w\in\eta} m_w\alpha_{\kappa_w}''\big(S_w(\sigma)\big) \le \det{}^2(\mathcal{Q}_{\eta_0})\prod_{w\in\eta_0} m_w\alpha_{\kappa_w}''\big(S_w(\sigma)\big) \quad (\forall \eta \in \mathcal{M}^{[k]}). \tag{84}$$

Thus,

$$\det\big(H(\sigma)\big) \le \binom{M}{k}\det{}^2(\mathcal{Q}_{\eta_0})\prod_{w\in\eta_0} m_w\alpha_{\kappa_w}''\big(S_w(\sigma)\big),$$

and so

$$\frac{1}{|\det(\mathcal{Q}_{\eta_0})|\prod_{w\in\eta_0}\sqrt{m_w\alpha_{\kappa_w}''\big(S_w(\sigma)\big)}} \le \frac{\sqrt{\binom{M}{k}}}{\sqrt{\det\big(H(\sigma)\big)}}. \tag{85}$$

For $y \in \mathbb{R}^k,\ w \in \eta_0(y)$ and $D > 0$, let $\big(cf.\ (77)\big)$

$$\delta_w := \frac{D}{m_w^{1/3}\sqrt{\alpha''_{\kappa_w}(S_w(\sigma))}}. \tag{86}$$

Define the neighborhood $\Delta \subset \mathbb{R}^k$ of $T = 0 \in \mathbb{R}^k$ as

$$\Delta = \Delta(y) := \big\{T \in \mathbb{R}^k \big|\, |S_w(T)| < \delta_w\ (\forall w \in \eta_0)\big\}. \tag{87}$$

The next lemma shows that I_2 and I_3 are small compared to I_1 in Corollary 16.

Lemma 21. *Suppose $E \subset \mathcal{O}_L^*$ is (M,Ω)-dispersed, $m := \min_{w \in \mathcal{M}}\{m_w\} \geq 1000,\ 0 < D < m^{1/3}/\sqrt{2}$, and $y \in \mathbb{R}^k$. Then*

$$|I_2| = \Big|\int_{\mathbb{R}^k - \Delta} \mathcal{G}(T)\, dT\Big| \leq \frac{(1 + \frac{2.31}{m})^{k-1}\Big(10^{-76} + \frac{41.43}{D^6}\Big)k\sqrt{\binom{M}{k}}}{m} I_1, \tag{88}$$

$$|I_3| = \Big|\int_{\mathbb{R}^k - \Delta} \mathcal{H}(T)\, dT\Big| \leq \frac{3.67k\sqrt{\binom{M}{k}}}{mD^6}\, I_1, \tag{89}$$

with Δ as in (87)*, $\sigma := \sigma(ny) \in \mathcal{D}$ as in Lemma 8, $\mathcal{H}$ and $\mathcal{G}$ as in* (65) *and* (66)*,*

Proof. We first prove (88). Note that $\Gamma(z) = \int_0^\infty x^z \mathrm{e}^{-x} \frac{dx}{x}$ implies

$$|\Gamma(z)| \leq \Gamma\big(\mathrm{Re}(z)\big) \qquad\qquad (\mathrm{Re}(z) > 0). \tag{90}$$

Using this, (66) and (60) we have,

$$\int_{\mathbb{R}^k - \Delta} |\mathcal{G}(T)|\, dT \leq \mathrm{e}^{-ny\cdot\sigma} \prod_{\substack{w \in \mathcal{M}^{[k]} \\ w \notin \eta_0}} \mathrm{e}^{m_w \alpha_{\kappa_w}(S_w(\sigma))} \int_{\mathbb{R}^k - \Delta} \Big|\prod_{w \in \eta_0} \mathrm{e}^{m_w \alpha_{\kappa_w}(S_w(\sigma + iT))}\Big|\, dT.$$

Let $B \subset \mathbb{R}^{\eta_0}$ denote the k-dimensional box

$$B = B(y) := \big\{\tilde{T} \in \mathbb{R}^{\eta_0} \big|\, |\tilde{T}_w| \leq \delta_w \quad (\forall w \in \eta_0)\big\}, \tag{91}$$

and let $B^c := \mathbb{R}^{\eta_0} - B$ denote its complement. Making the change of variables $\tilde{T}_w := S_w(T)$ for $w \in \eta_0$, we have

$$\int_{\mathbb{R}^k - \Delta} \Big|\prod_{w \in \eta_0} \mathrm{e}^{m_w \alpha_{\kappa_w}(S_w(\sigma + iT))}\Big|\, dT = \frac{1}{|\det(\mathcal{Q}_{\eta_0})|} \int_{\tilde{T} \in B^c} \Big|\prod_{w \in \eta_0} \mathrm{e}^{m_w \alpha_{\kappa_w}(S_w(\sigma) + i\tilde{T}_w)}\Big|\, d\tilde{T}.$$

The latter integral is easy to bound using Lemmas 17 and 18. We integrate over k (overlapping) regions, each of which has $k-1$ of the $\tilde{T}_w$ range over all of $\mathbb{R}$, and the remaining $\tilde{T}_{w_0}$ over $|\tilde{T}_{w_0}| > \delta_{w_0}$. Since $m_w \geq m$, we conclude that

$$\int_{\mathbb{R}^k - \Delta} |\mathcal{G}(T)|\, dT \leq \frac{k(2\pi)^{k/2}(1 + \frac{2.31}{m})^{k-1}\big(10^{-76} + \frac{41.43}{D^6}\big)\mathrm{e}^{\alpha(\sigma) - ny\cdot\sigma}}{m|\det(\mathcal{Q}_{\eta_0})| \prod_{w \in \eta_0} \sqrt{m_w \alpha''_{\kappa_w}\big(S_w(\sigma)\big)}}.$$

Now inequality (85) and Corollary 16 prove (88).

Next we prove (89). Changing variables as before, we have

$$\begin{aligned}|I_3| &= \mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}\int_{\mathbb{R}^k-\Delta}\exp\Big(-\frac{1}{2}\sum_{w\in\mathcal{M}} m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)S_w(T)^2\Big)\,dT\\ &\le \mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}\int_{\mathbb{R}^k-\Delta}\exp\Big(-\frac{1}{2}\sum_{w\in\eta_0} m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)S_w(T)^2\Big)\,dT\\ &= \frac{\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}}{|\det(\mathcal{Q}_{\eta_0})|}\int_{B^c}\exp\Big(-\frac{1}{2}\sum_{w\in\eta_0} m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)\tilde{T}_w^2\Big)\,d\tilde{T}.\end{aligned}$$

Once again, we bound $\int_{B^c}$ using k overlapping regions, one for each $w_0\in\eta_0$. The integral over the region given by all $\tilde{T}\in\mathbb{R}^{\eta_0}$ such that $|\tilde{T}_{w_0}|>\delta_{w_0}$ is bounded by

$$\int_{|\tilde{T}_{w_0}|>\delta_{w_0}}\mathrm{e}^{-\frac{1}{2}m_{w_0}\alpha''_{\kappa_{w_0}}(S_{w_0}(\sigma))\tilde{T}_{w_0}^2}\,d\tilde{T}_{w_0}\prod_{\substack{w\in\eta_0\\ w\neq w_0}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2}m_w\alpha''_{\kappa_w}(S_w(\sigma))\tilde{T}_w^2}\,d\tilde{T}_w.$$

We can use (79) to bound the first integral, and the remaining integrals are explicitly known. Hence, summing over the k regions,

$$|I_3|\le\frac{(2\pi)^{k/2}\,\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}}{|\det(\mathcal{Q}_{\eta_0})|}\frac{3.67\,k}{mD^6}\prod_{w\in\eta_0}\frac{1}{\sqrt{m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)}}.$$

We again conclude using (85). □

For the "inner" integral $I_4=\int_\Delta(\mathcal{G}-\mathcal{H})$ in (57), we can only expect estimates of the kind $O(I_1/m)$, whereas I_2 and I_3 are essentially $O\big(I_1\exp(-m^{1/3})\big)$. This allowed us to use simple estimates for the contribution of $w\notin\eta_0$. However, to estimate I_4 we shall need the following geometric result.

Lemma 22. *Let $R=(r_{ij})$ be an $N\times k$ real matrix of rank k, and let $a_i>0$ $(1\le i\le N)$. Define linear maps $P_i\colon\mathbb{R}^k\to\mathbb{R}$ by $P_i(T):=\sum_{j=1}^k r_{ij}T_j$, where $T=(T_1,...,T_k)$. For any k-element subset $\eta=\{i_1,\dots,i_k\}\subset\{1,2,\dots,N\}$, let R_η denote the $k\times k$ submatrix of R given by $\big(R_\eta\big)_{\ell,j}=r_{i_\ell j}$. Define $E_\eta:=|\det(R_\eta)|\prod_{i\in\eta}a_i$, and let η_0 maximize E_η. Then*

$$a_i|P_i(T)|\le\sum_{j\in\eta_0}a_j|P_j(T)|\qquad\qquad(1\le i\le N,\ T\in\mathbb{R}^k).$$

Proof. Replacing r_{ij} by a_ir_{ij}, we may assume $a_i=1$. Hence η_0 simply maximizes $|\det(R_\eta)|$. Fix $i\in\{1,2,\dots,N\}$, and define $\lambda_j\in\mathbb{R}$ for $j\in\eta_0$ by $P_i=\sum_{j\in\eta_0}\lambda_jP_j$. For $j\in\eta_0$, let R_j denote R_η with the j^{th} row of R replaced by the i^{th} row. Then, by Cramer's rule, $|\lambda_j\det(R_\eta)|=|\det(R_j)|\le|\det(R_\eta)|$, so $|\lambda_j|\le 1$. Hence

$$|P_i(T)|=\Big|\sum_{j\in\eta_0}\lambda_jP_j(T)\Big|\le\sum_{j\in\eta_0}|P_j(T)|.$$ □

Remark 23. *The argument of the proof shows Lemma 2 of §2. Namely, let S be the set of rows of R and set $a_i=1$ for all i. The volume of the sign-symmetrized* *parallelotope $P(\eta)$ spanned by the elements of a k-element subset η of S is just $2^k E_\eta$. If this volume is maximized by η_0 over all choices of η, the proof shows every element of S is in $P(\eta_0)$.*

Lemma 24. *Suppose $E \subset \mathcal{O}_L^*$ is (M,Ω)-dispersed. Then, for $y \in \mathbb{R}^k$ and $D > 0$ we have, with notation as in (55) and (57),*

$$|I_4| = \Big| \int_\Delta \big(\mathcal{G}(T) - \mathcal{H}(T)\big)\, dT \Big| \le \frac{M\big(\frac{5}{3}M + \frac{3}{2}Z\big)}{m}\, I_1, \tag{92}$$

where $m := \min_{w\in\mathcal{M}}\{m_w\}$ and $Z := \big(\mathrm{e}^{Mk^4D^4m^{-1/3}} - 1\big)/\big(Mk^4D^4m^{-1/3}\big)$.

Proof. Lemma 22, applied to the matrix $\mathcal{Q}$ and $a_w := \sqrt{m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)}$, shows

$$\sqrt{m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)}\,|S_w(T)| \le \sum_{w_0\in\eta_0} \sqrt{m_{w_0}\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)}\,|S_{w_0}(T)| \tag{93}$$

for $w \in \mathcal{M},\ T \in \mathbb{R}^k$ and η_0 as in (84). Since $x \mapsto x^4$ is convex, we have,

$$\begin{aligned} m_w^2\alpha''_{\kappa_w}\big(S_w(\sigma)\big)^2 S_w(T)^4 &\le \Big(\sum_{w_0\in\eta_0} \sqrt{m_{w_0}\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)}\,|S_{w_0}(T)|\Big)^4 \\ &\le k^3 \sum_{w_0\in\eta_0} m_{w_0}^2\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)^2\, S_{w_0}(T)^4. \end{aligned}$$

For $T \in \Delta$ and $w_0 \in \eta_0$, by (86) and (87) we have

$$m_{w_0}\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)^2 S_{w_0}(T)^4 \le m_{w_0}\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)^2 \delta_{w_0}^4 = D^4 m_{w_0}^{-1/3}.$$

Hence,

$$m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)^2 S_w(T)^4 \le k^3 \sum_{w_0\in\eta_0} \frac{m_{w_0}}{m_w}\frac{D^4}{m^{1/3}} \le k^3 \sum_{w_0\in\eta_0} \frac{2D^4}{m^{1/3}} = \frac{2k^4D^4}{m^{1/3}}.$$

Combining this with Lemma 19 and $\#\mathcal{M} = M$, we conclude that for $T \in \Delta$,

$$\big|\mathrm{Re}\big(\rho(T)\big)\big| = \Big|\sum_{w\in\mathcal{M}} m_w \mathrm{Re}\big(\rho_w(T)\big)\Big| \le \sum_{w\in\mathcal{M}} k^4D^4m^{-1/3} = Mk^4D^4m^{-1/3}.$$

Lemmas 19 and 20 now show that for $T \in \Delta$,

$$\begin{aligned} &\big|\mathrm{Re}\big(\mathrm{e}^{\rho(T)} - 1\big)\big| \le \frac{\mathrm{Im}\big(\rho(T)\big)^2}{2} + \mathrm{Re}\big(\rho(T)\big)Z \\ &\le \frac{1}{2}\Big(\frac{\sqrt{2}}{3}\sum_{w\in\mathcal{M}} m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)^{3/2}|S_w(T)|^3\Big)^2 + \frac{Z}{2}\sum_{w\in\mathcal{M}} m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)^2 S_w(T)^4 \\ &\le \frac{M}{9}\sum_{w\in\mathcal{M}} m_w^2\alpha''_{\kappa_w}\big(S_w(\sigma)\big)^3 S_w(T)^6 + \frac{Z}{2}\sum_{w\in\mathcal{M}} m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)^2 S_w(T)^4, \end{aligned} \tag{94}$$

where in the last step we used the convexity of $x \mapsto x^2$.

By Lemma 19, $\mathrm{Im}\big(e^{\rho(T)}\big)$ is odd, while $\mathrm{Re}\big(e^{\rho(T)}\big)$ is even in T. Furthermore, $\mathcal{H}(T)$ is a real and even function of T, and Δ is mapped to itself by $T \mapsto -T$. Hence, using (66) and (94),

$$\Big|\int_\Delta \big(\mathcal{G}(T)-\mathcal{H}(T)\big)\,dT\Big| = \Big|\int_\Delta (e^{\rho(T)}-1)\mathcal{H}(T)\,dT\Big| = \Big|\int_\Delta \mathrm{Re}(e^{\rho(T)}-1)\mathcal{H}(T)\,dT\Big|$$
$$\leq \sum_{w\in\mathcal{M}} \int_{\mathbb{R}^k} \Big(\frac{M}{9} m_w^2 \alpha''_{\kappa_w}\big(S_w(\sigma)\big)^3 S_w(T)^6 + \frac{Z}{2} m_w \alpha''_{\kappa_w}\big(S_w(\sigma)\big)^2 S_w(T)^4\Big)\mathcal{H}(T)\,dT.$$

Using Lemma 15 and Corollary 16, we find

$$\Big|\int_\Delta \big(\mathcal{G}(T)-\mathcal{H}(T)\big)\,dT\Big| \leq \Big(\sum_{w\in\mathcal{M}} \frac{\frac{5}{3}M+\frac{3}{2}Z}{m_w}\Big)\frac{(2\pi)^{k/2}\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}}{\sqrt{\det(H(\sigma))}}$$
$$\leq \frac{M\big(\frac{5}{3}M+\frac{3}{2}Z\big)}{m} I_1. \qquad \square$$

Our next estimate will let us deal with the term $\int_{E_{\mathbb{R}}} \|ax\|^2 \mathrm{e}^{-t\,\|ax\|^2}\,d\mu(x)$ in the Basic Inequality (17) and (41).

Lemma 25. *If $y\in\mathbb{R}^k$, E is (M,Ω)-dispersed and $m\geq 1000$, then*

$$\int_{T\in\mathbb{R}^k} \big|T_1 \mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\big|dT \leq \frac{1.18\sqrt{\Omega}(1+\frac{2.31}{m})^{k-1}k\sqrt{\binom{M}{k}}}{\sqrt{m}}\sigma_1 I_1, \tag{95}$$

with I_1 as in (57), α as in (60) and $\sigma=(\sigma_1,\dots,\sigma_k):=\sigma(ny)$ as in Lemma 8.

Proof. By (52), for $T\in\mathbb{R}^k$ we have

$$nT_1 = \sum_{v\in\mathcal{A}_L} e_v S_v(T) = \sum_{w\in\mathcal{M}}\sum_{v\in\pi^{-1}(w)} e_v S_w(T) = \sum_{w\in\mathcal{M}} m_w S_w(T). \tag{96}$$

Hence we will need to bound integrals of the kind $\int_{\mathbb{R}^k} |S_w(T)\mathrm{e}^{\alpha(\sigma+iT)}|\,dT$.

Let η_0 be as in (84) and let $w_0\in\eta_0$. Then, using (90) and changing variables as in the proof of Lemma 21,

$$\int_{\mathbb{R}^k} \big|S_{w_0}(T)\mathrm{e}^{\alpha(\sigma+iT)-\alpha(\sigma)}\big|\,dT \leq \int_{\mathbb{R}^k}\Big|S_{w_0}(T)\prod_{w\in\eta_0} \mathrm{e}^{m_w\alpha_{\kappa_w}(S_w(\sigma+iT))-m_w\alpha_{\kappa_w}(S_w(\sigma))}\Big|\,dT$$
$$= \frac{1}{|\det(\mathcal{Q}_{\eta_0})|}\int_{-\infty}^{\infty} |\tilde{T}_{w_0}\mathrm{e}^{m_{w_0}\alpha_{\kappa_{w_0}}(S_{w_0}(\sigma)+i\tilde{T}_{w_0})-m_{w_0}\alpha_{\kappa_{w_0}}(S_{w_0}(\sigma))}|\,d\tilde{T}_{w_0}$$
$$\cdot \prod_{\substack{w\in\eta_0\\ w\neq w_0}} \int_{-\infty}^{\infty} |\mathrm{e}^{m_w\alpha_{\kappa_w}(S_w(\sigma)+i\tilde{T}_w)-m_w\alpha_{\kappa_w}(S_w(\sigma))}|\,d\tilde{T}_w.$$

Using Lemma 17 we obtain,

$$\int_{\mathbb{R}^k} \left|S_{w_0}(T)\mathrm{e}^{\alpha(\sigma+iT)-\alpha(\sigma)}\right| dT \le \frac{1}{|\det(\mathcal{Q}_{\eta_0})|} \frac{0.83\sqrt{2\pi}}{m_{w_0}\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)} \prod_{\substack{w\in\eta_0\\ w\neq w_0}} \frac{(1+\frac{2.31}{m})\sqrt{2\pi}}{\sqrt{m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)}}$$

$$= \frac{0.83\cdot(1+\frac{2.31}{m})^{k-1}}{\sqrt{m_{w_0}\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)}} \frac{(2\pi)^{k/2}}{|\det(\mathcal{Q}_{\eta_0})| \prod_{w\in\eta_0}\sqrt{m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)}}$$

$$\le \frac{0.83\cdot(1+\frac{2.31}{m})^{k-1}\sqrt{\binom{M}{k}}}{\sqrt{m\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)}} \frac{(2\pi)^{k/2}}{\sqrt{\det\big(H(\sigma)\big)}} \qquad \big(\text{see (85)}\big). \tag{97}$$

By inequality (93),

$$\begin{aligned}\sum_{w\in\mathcal{M}} m_w|S_w(T)| &= \sum_{w\in\mathcal{M}} \sqrt{\frac{m_w}{\alpha''_{\kappa_w}\big(S_w(\sigma)\big)}} \sqrt{m_w\alpha''_{\kappa_w}\big(S_w(\sigma)\big)}\, |S_w(T)| \\ &\le \sum_{w\in\mathcal{M}} \sqrt{\frac{m_w}{\alpha''_{\kappa_w}\big(S_w(\sigma)\big)}} \sum_{w_0\in\eta_0} \sqrt{m_{w_0}\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)}\, |S_{w_0}(T)| \\ &\le \sqrt{2\Omega} \sum_{w\in\mathcal{M}} m_w S_w(\sigma) \sum_{w_0\in\eta_0} \sqrt{\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)}\, |S_{w_0}(T)|,\end{aligned}$$

where the last inequality uses $m_{w_0} \le \Omega m \le \Omega m_w$ (valid for any $w, w_0 \in \mathcal{M}$ by Definition 13) and $x^2\alpha''_{\kappa_w}(x) > \kappa_w \ge 1/2$ for $x > 0$ [13, (5.7)]. Hence, by (96),

$$\sum_{w\in\mathcal{M}} m_w|S_w(T)| \le \sqrt{2\Omega} n\sigma_1 \sum_{w_0\in\eta_0} \sqrt{\alpha''_{\kappa_{w_0}}\big(S_{w_0}(\sigma)\big)}\, |S_{w_0}(T)|.$$

It follows that

$$\begin{aligned}\int_{T\in\mathbb{R}^k} \left|T_1\mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\right| dT &= \frac{\mathrm{e}^{-ny\cdot\sigma}}{n} \int_{\mathbb{R}^k} \left|\Big(\sum_{w\in\mathcal{M}} m_w S_w(T)\Big) e^{\alpha(\sigma+iT)}\right| dT \\ &\le \frac{\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}}{n} \cdot \sqrt{2\Omega} n\sigma_1 \sum_{w_0\in\eta_0} \sqrt{\alpha''_{\kappa_{w_0}}(S_{w_0}(\sigma))} \int_{\mathbb{R}^k} \left|S_{w_0}(T)\mathrm{e}^{\alpha(\sigma+iT)-\alpha(\sigma)}\right| dT \\ &\le \sqrt{2\Omega}\sigma_1 \sum_{w_0\in\eta_0} \frac{0.83\cdot(1+\frac{2.31}{m})^{k-1}\binom{M}{k}^{\frac12}}{\sqrt{m}} \frac{(2\pi)^{k/2}\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}}{\sqrt{\det\big(H(\sigma)\big)}} \\ &= \frac{(1.1737\cdots)\sqrt{\Omega}(1+\frac{2.31}{m})^{k-1}k\binom{M}{k}^{\frac12}}{\sqrt{m}}\sigma_1 I_1,\end{aligned}$$

where the last equality uses Corollary 16. ☐

The next lemma will allow us to ensure that each integral in the Basic Inequality (17) is positive.

Lemma 26. *Let $E \subset \mathcal{O}_L$ be (M,Ω)-dispersed,*

$$m \ge N_0 = N_0(M,\Omega,k) := \max\!\Big(10^2k^6M^{3/2}, 10^4k^2\Omega\tbinom{M}{k}, 10^3M^2\Big), \tag{98}$$

and let $a \in \mathcal{O}_L,\ a \neq 0$. Let $t := \exp\big(\Psi(0.51 + \frac{r_2}{2n})\big)$ and $\Psi(x) := \Gamma'(x)/\Gamma(x)$. For $y_{a,t}$ given by Corollary 6 let

$$\mathcal{L} = \frac{\sqrt{\det(\mathcal{Q}^{\intercal}\mathcal{Q})\prod_{w\in\mathcal{M}}(r_{1,w}+r_{2,w})}}{2^{r_1}(2\sqrt{\pi})^{r_2}\pi^k},\quad I_1(ny) = \frac{(2\pi)^{k/2}\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}}{\sqrt{\det\big(H(\sigma)\big)}},\quad \sigma := \sigma(ny_{a,t}).$$

Then the following inequalities hold.

$$\sigma_1(ny_{a,t}) \geq 0.51, \qquad \int_{x\in E_{\mathbb{R}}}\Big(\frac{2t\|ax\|^2}{n}-1\Big)\mathrm{e}^{-t\,\|ax\|^2}\,d\mu_{E_{\mathbb{R}}}(x) > 0.007\mathcal{L}I_1(ny_{a,t}), \tag{99}$$

$$|I_2|+|I_3|+|I_4| \leq 0.01 I_1. \tag{100}$$

Proof. By Corollary 6, for $a \in \mathcal{O}_L,\ a \neq 0$,

$$y_1 := (y_{a,t})_1 = \log(t) + \tfrac{2}{n}\log|\mathrm{Norm}_{L/\mathbb{Q}}(a)| \geq \log(t) = \Psi(0.51 + \tfrac{r_2}{2n}). \tag{101}$$

Applying Lemma 9 to ny, since Ψ^{-1} is increasing we have,

$$\sigma_1 = \sigma_1(ny_{a,t}) \geq \Psi^{-1}(y_1) - \frac{r_2}{2n} \geq \Psi^{-1}\big(\Psi(0.51+\tfrac{r_2}{2n})\big) - \frac{r_2}{2n} = 0.51, \tag{102}$$

as claimed.

We now prove inequality (99). Note that $\mathcal{L}$ is as in Corollary 6, except that we used (67) to express $\mathcal{L}$ in terms of $\mathcal{Q}$ rather than Q. Letting $y := y_{a,t}$, from Corollary 6 we have

$$\begin{aligned}\frac{\int_{E_{\mathbb{R}}}\big(\frac{2t\|ax\|^2}{n}-1\big)\mathrm{e}^{-t\,\|ax\|^2}\,d\mu_{E_{\mathbb{R}}}}{\int_{E_{\mathbb{R}}}\mathrm{e}^{-t\,\|ax\|^2}\,d\mu_{E_{\mathbb{R}}}} &= \frac{\int_{T\in\mathbb{R}^k}(2(\sigma_1+iT_1)-1)\,\mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\,dT}{\int_{T\in\mathbb{R}^k}\mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\,dT} \\ &= 2\sigma_1 - 1 + \frac{2i\int_{\mathbb{R}^k}T_1\,\mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\,dT}{\int_{\mathbb{R}^k}\mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\,dT}.\end{aligned} \tag{103}$$

The numerator in this last quotient is bounded by Lemma 25. As $k \leq k^2 \leq Mk^2 \leq 10^{-4}m$ by (98), we have

$$\Big(1+\frac{2.31}{m}\Big)^{k-1} < \Big(\Big(1+\frac{2.31}{m}\Big)^m\Big)^{k/m} < \mathrm{e}^{2.31/10^4} < 1.0003. \tag{104}$$

Now Lemma 25 and (98) yield

$$2\int_{T\in\mathbb{R}^k}\big|T_1\mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\big|dT \leq \frac{2.361\,\sigma_1 I_1(ny)}{100}. \tag{105}$$

Next we estimate the denominator in (103). By (57) and (56) we have

$$\frac{1}{\mathcal{L}}\int_{E_{\mathbb{R}}}\mathrm{e}^{-t\,\|ax\|^2}\,d\mu_{E_{\mathbb{R}}} = \int_{\mathbb{R}^k}\mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\,dT = I_1 + I_2 - I_3 + I_4, \tag{106}$$

where $I_j := I_j(ny)$ and Δ in (86) and (87) is defined using

$$D := \frac{m^{1/12}}{kM^{1/4}}. \tag{107}$$

The above value of D is chosen so that $D^4k^4Mm^{-1/3}=1$ in Lemma 24. Thus,

$$|I_4| \le \frac{M\left(\frac{5}{3}M+\frac{3}{2}(\mathrm{e}-1)\right)}{m} I_1 \le \frac{M^2\left(\frac{5}{3}+\frac{3}{2}(\mathrm{e}-1)\right)}{m} I_1 < \frac{4.3}{1000} I_1, \tag{108}$$

where the last step used (98).

One easily checks that $m>4,\ M\ge 1$ and $k\ge 1$ imply $D<m^{1/3}/\sqrt{2}$, as required in Lemma 21. Thus,

$$|I_3| \le \frac{3.67k\sqrt{\binom{M}{k}}}{\sqrt{m}} \frac{1}{\sqrt{m}D^6} I_1 = \frac{3.67k\sqrt{\binom{M}{k}}}{\sqrt{m}} \frac{k^6M^{3/2}}{m} I_1 \le \frac{3.67}{10^4} I_1, \tag{109}$$

where the last step used (98) and $\Omega \ge 1$. Similarly, using (88) and (104),

$$|I_2| \le \left(\frac{1.0003}{10^{78}\sqrt{m}} + \frac{41.43}{10^4}\right) I_1 < \frac{4.2}{1000} I_1. \tag{110}$$

Combining the last three bounds we obtain the inequality (100). From (106),

$$\frac{1}{\mathcal{L}} \int_{E_{\mathbb{R}}} \mathrm{e}^{-t\,\|ax\|^2}\, d\mu_{E_{\mathbb{R}}} = \int_{\mathbb{R}^k} \mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\, dT \ge 0.99 I_1. \tag{111}$$

Since $\sigma_1 \ge 0.51$ by (102),

$$2\sigma_1 - 1 + \frac{2i\int_{\mathbb{R}^k} T_1\, \mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\, dT}{\int_{\mathbb{R}^k} \mathrm{e}^{\alpha(\sigma+iT)-ny\cdot(\sigma+iT)}\, dT} \ge 2\sigma_1 - 1 - \frac{0.02361\sigma_1}{0.99} > 1.976\sigma_1 - 1 > 0.0078.$$

Now (103) and (111) conclude the proof of (99). $\square$

We are now ready to prove our main result.

Theorem 27. *Suppose $E \subset \mathcal{O}_L^*$ is a subgroup of the units of the number field L which is (M,Ω)-dispersed in the sense of Definition 13. Let $k := 1+\operatorname{rank}_{\mathbb{Z}}(\mathcal{O}_L^*/E)$, $\binom{M}{k} := \frac{M!}{k!(M-k)!}$, let $\varepsilon_1, \dots, \varepsilon_j$ be independent elements of E, where $j := \operatorname{rank}_{\mathbb{Z}}(E)$, and suppose m in Definition 13 satisfies*

$$m \ge N_0 = N_0(M,\Omega,k) := \max\left(10^2k^6M^{3/2},\, 10^4k^2\Omega\binom{M}{k},\, 10^3M^2\right).$$

Then

$$\|\varepsilon_1 \wedge \cdots \wedge \varepsilon_j\|_1 \ge \|\varepsilon_1 \wedge \cdots \wedge \varepsilon_j\|_2 \ge 1.1^{[L:\mathbb{Q}]}, \tag{112}$$

where the L^1-norm was defined in (4).

Theorem 1 in §1 follows immediately from $j<[L:\mathbb{Q}]$ and Remark 14.

Proof. Note that N_0 is as in Lemma 26 and that

$$\|\varepsilon_1 \wedge \cdots \wedge \varepsilon_j\|_2 = \mu_{E_{\mathbb{R}}}(E_{\mathbb{R}}/E) \ge \frac{\mu_{E_{\mathbb{R}}}(E_{\mathbb{R}}/E)}{|E_{\mathrm{tor}}|}. \tag{113}$$

Take $t := \exp\left(\Psi(0.51+\frac{r_2}{2n})\right)$ as in Lemma 26. In the Basic Inequality (17) take $\mathfrak{a} := \mathcal{O}_L$, so that the sum there includes only nonzero $a \in \mathcal{O}_L$ (modulo E). By Lemma

26, each integral in the sum is positive. Retaining only the term corresponding to $a = 1 \in \mathcal{O}_L$ we have, again by Lemma 26,

$$\frac{\mu_{E_{\mathbb{R}}}(E_{\mathbb{R}}/E)}{|E_{\mathrm{tor}}|} > 0.007 \frac{2^{k/2}\sqrt{\det(\mathcal{Q}^{\intercal}\mathcal{Q})\prod_{w\in\mathcal{M}}(r_{1,w}+r_{2,w})}}{\sqrt{\det\big(H(\sigma)\big)}\pi^{k/2}} \frac{(2/\sqrt{\pi})^{r_2}\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}}{2^n} \tag{114}$$

where $y := y_{1,t}$ and $\sigma := \sigma(ny)$. Corollary 6 applied to $a = 1$ gives

$$y = (\log(t), 0, 0, \ldots, 0) = (\Psi(0.51 + \tfrac{r_2}{2n}), 0, \ldots, 0). \tag{115}$$

We need an upper bound for $\det\big(H(\sigma)\big)$ in (114). In view of (71), we look for an upper bound for $\alpha''_{\kappa_w}\big(S_w(\sigma)\big)$. Note that for $0 \le \kappa \le 1$ and $x > 0$,

$$\alpha''_{\kappa}(x) := \kappa\Psi'(x) + (1-\kappa)\Psi'(x + \tfrac{1}{2}) < \kappa\Psi'(x) + (1-\kappa)\Psi'(x) = \Psi'(x),$$

since $\Psi'(x)$ is decreasing for $x > 0$. Also, $\sigma_1 \ge 0.51$ by (102), so

$$-2 < \Psi(0.51) \le y_1 = \Psi(0.51 + \tfrac{r_2}{2n}) \le \Psi(0.76) < -1. \tag{116}$$

From Lemma 11 we have

$$S_w(\sigma) \ge \frac{1}{(n-1)\log(2\sigma_1 + \frac{1}{2}) - ny_1} \ge \frac{1}{n(\log(3\sigma_1) + 2)} > \frac{1}{n\log(23\sigma_1)}.$$

Estimating the series by an integral, $\Psi'(x) = \sum_{k=0}^{\infty}\frac{1}{(k+x)^2} < \frac{1}{x} + \frac{1}{x^2}$, yields

$$\alpha''_{\kappa_w}\big(S_w(\sigma)\big) < \Psi'\big(S_w(\sigma)\big) < \frac{1}{S_w(\sigma)} + \frac{1}{\big(S_w(\sigma)\big)^2} < 2n^2\log^2(23\sigma_1).$$

From $\det(\mathcal{Q}^{\intercal}\mathcal{Q}) = \sum_{\eta\in\mathcal{M}^{[k]}}\det^2(\mathcal{Q}_\eta)$, $r_{1,w} + r_{2,w} \ge m_w/2$, (68) and from (71),

$$\begin{aligned}\frac{2^{k/2}\sqrt{\det(\mathcal{Q}^{\intercal}\mathcal{Q})\prod_{w\in\mathcal{M}}(r_{1,w}+r_{2,w})}}{\sqrt{\det\big(H(\sigma)\big)}\pi^{k/2}} &\ge \Big(\frac{1}{\sqrt{2\pi}\,n\log(23\sigma_1)}\Big)^k \\ &\ge \Big(\frac{1}{\sqrt{2\pi}\,n\log(23\sigma_1)}\Big)^{(n/100)^{2/15}},\end{aligned} \tag{117}$$

where the last inequality used (98) in the form $n \ge m \ge 100k^6M^{3/2} \ge 100k^6k^{3/2}$.

We now bound the term $\mathrm{e}^{\alpha(\sigma)-ny\cdot\sigma}$ in (114) from below. Using the lower bound for $\alpha(\sigma)$ in Lemma 10, we have

$$\alpha(\sigma) - ny\cdot\sigma \ge n\log\Gamma\big(\sigma_1 + \tfrac{r_2}{2n}\big) - n\sigma_1 y_1. \tag{118}$$

The critical points of $g(r) := \log\Gamma\big(r + \frac{r_2}{2n}\big) - ry_1$ occur where $\Psi\big(r + \frac{r_2}{2n}\big) = y_1 := \Psi(0.51 + \frac{r_2}{2n})$. But $\Psi\colon (0,\infty) \to \mathbb{R}$ is injective, so $r = 0.51$ is the only critical point of g on $(-\frac{r_2}{2n}, \infty)$, and it is a local minimum. On checking the behavior has $r \to -\frac{r_2}{2n}^+$ and as $r \to \infty$, one finds that the minimum of g for $r \in (-\frac{r_2}{2n}, \infty)$ occurs at $r = 0.51$. Using (118) we obtain

$$\alpha(\sigma) - ny\cdot\sigma \ge n\big(\log\Gamma\big(\sigma_1 + \tfrac{r_2}{2n}\big) - \sigma_1 y_1\big) \ge n\big(\log\Gamma\big(0.51 + \tfrac{r_2}{2n}\big) - 0.51\Psi(0.51 + \tfrac{r_2}{2n})\big).$$

Note that $0 \le \frac{r_2}{2n} \le \frac{1}{4}$, $\Psi(r) < -1$ for $0 < r < 0.76$, and $\Psi'(r) > 0$ for $r > 0$. Hence

$$x \mapsto \log\Gamma(0.51 + x) - 0.51\Psi(0.51 + x) + x\log(4/\pi)$$

is decreasing for $0 \le x \le \frac{1}{4}$. We conclude that

$$\begin{aligned} &\alpha(\sigma) - ny \cdot \sigma + r_2 \log(2/\sqrt{\pi}) - n\log(2) \\ &\quad \ge n\big(\log\Gamma(0.76) - 0.51\Psi(0.76) + 0.25\log(4/\pi) - \log(2)\big) > n/10. \end{aligned} \tag{119}$$

Recall that by Lemma 26, we have $\sigma_1 \ge 0.51$. We now distinguish two cases according to how large σ_1 is. If $0.51 \le \sigma_1 < 5$, then $\log(23\sigma_1\sqrt{2\pi}) < 6$. Combining this with (114), (117), and (119), and using $0.1 - \log(1.1) > 0.004$ we obtain

$$\mu_{E_\mathbb{R}}(E_\mathbb{R}/E) \ge 1.1^n \exp\big(0.004n + \log(0.007) - (n/100)^{2/15}(\log(n) + 6)\big) > 1.1^n,$$

since $n \ge \#\mathcal{A}_L \ge m \ge 10^4$.

We now turn to the remaining case, i.e., $\sigma_1 \ge 5$. Here we can be much coarser and use $\log(23\sigma_1) \le \sigma_1$ in (117) and $\Gamma(\sigma_1 + \frac{r_2}{2n}) \ge 24$ in (118). Since $-n\sigma_1 y_1 > n\sigma_1$ by (116), we obtain from (114)

$$\mu_{E_\mathbb{R}}(E_\mathbb{R}/E) \ge 12^n \exp(n\sigma_1 + \log(0.007) - (n/100)^{2/15}(\log(n\sigma_1) + \log\sqrt{2\pi})) > 12^n.$$

□

We note that the proof shows that the 1.1^n appearing in Theorem 27 can be replaced by $\exp\big(nf(r_2/(2n))\big)$, where r_2 is the number of complex places of L and

$$f(x) := \log\Gamma(0.51 + x) - 0.51\Psi(0.51 + x) + x\log(4/\pi) - \log(2).$$

In particular, if L is totally real, we can replace 1.1^n by 2.3^n. After adjusting N_0, we can also replace 0.51 above by $\epsilon + 1/2$ for any $\epsilon > 0$.

Finally, we prove that every element of $\bigwedge^{r_L - 1} \mathrm{LOG}(\mathcal{O}_L^*)$ is represented by a pure wedge, as claimed in the Introduction.

Lemma 28. *Suppose M is a $\mathbb{Z}$-lattice in $\mathbb{R}^n$ of rank $n \ge 1$. Then every element of $w \in \bigwedge^{n-1} M$ has the form $\omega = d\epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_{n-1}$ for some integer d and some basis $\{\epsilon_1, \ldots, \epsilon_n\}$ of M as a $\mathbb{Z}$-module.*

Proof. We may clearly assume $\omega \ne 0$. Define the homomorphism $\wedge_\omega : M \to \bigwedge^n M$ by $\wedge_\omega(m) := \omega \wedge m$. As $\bigwedge^n M \cong \mathbb{Z}$, $M/\ker(\wedge_\omega)$ is torsion-free and so $\ker(\wedge_\omega)$ is a direct summand of M of rank $n-1$. Let $\epsilon_1, \ldots, \epsilon_n$ be a $\mathbb{Z}$-basis of M such that $\epsilon_1, \ldots, \epsilon_{n-1}$ is a $\mathbb{Z}$-basis of $\ker(\wedge_\omega)$, let $\eta := \epsilon_1 \wedge \cdots \wedge \epsilon_{n-1} \in \bigwedge^{n-1} M$, and define $d \in \mathbb{Z}$ by $\omega \wedge \epsilon_n = d\eta \wedge \epsilon_n$. Notice that $\eta \wedge \epsilon_i = 0 = \omega \wedge \epsilon_i$ for $1 \le i \le n-1$.

For $m \in M$, write $m = \sum_{i=1}^n a_i\epsilon_i$ with $a_i \in \mathbb{Z}$. Then

$$\omega \wedge m = \omega \wedge \sum_{i=1}^n a_i\epsilon_i = a_n\omega \wedge \epsilon_n = a_n d\eta \wedge \epsilon_n = d\eta \wedge \sum_{i=1}^n a_i\epsilon_i = d\eta \wedge m.$$

As the $\wedge$-pairing of $\bigwedge^{n-1} M$ with M is non-degenerate, $\omega = d\eta = d\epsilon_1 \wedge \cdots \wedge \epsilon_{n-1}$. □

7. Appendix by Fernando Rodriguez Villegas
Some remarks on Lehmer's conjecture

7.1. The *logarithmic Mahler measure* of a non-zero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is defined as

$$m(P) = \int_0^1 \cdots \int_0^1 \log \left| P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) \right| d\theta_1 \cdots d\theta_n \tag{120}$$

and its *Mahler measure* as $M(P) = e^{m(P)}$, the geometric mean of $|P|$ on the torus

$$T^n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \middle| \, |z_1| = \ldots = |z_n| = 1 \right\}.$$

When $n = 1$ Jensen's formula gives the identity

$$M(P) = |a_0| \prod_{|\alpha_\nu|>1} |\alpha_\nu| \,, \tag{121}$$

where $P(x) = a_0 \prod_{\nu=1}^{d}(x - \alpha_\nu)$, from which we clearly obtain that $M(P) \geq 1$ if $P \in \mathbb{Z}[x]$. By a theorem of Kronecker if $M(P) = 1$ for $P \in \mathbb{Z}[x]$ then P is *cyclotomic*, i.e., P is monic and its roots are either 0 or roots of unity.

In the early 1930's Lehmer famously asked whether there is an absolute lower bound for $M(P)$ when $P \in \mathbb{Z}[x]$ and $M(P) > 1$ [15] [23]. As we recall below, Lehmer's conjecture can be reformulated as a universal lower bound for the L^1-norm of the logarithmic embedding of any (non-torsion) algebraic unit [26, p. 87]. In 1997 Bertrand [5] proposed as a higher-rank version of Lehmer's conjecture that the co-volume under the logarithmic embedding of any rank-ℓ subgroup $E \subset \mathcal{O}_F^*$ of the units of a number field F might be bounded below by some $c_\ell > 0$, independent of E and F ($\ell \geq 2$). This was proved in 1999 by Amoroso and David [2] for $\ell \geq 3$.

Here we refine Bertrand's conjecture by proposing lower bounds that increase exponentially with the rank ℓ. We also consider $m(P)$ for polynomials P in several variables and consider possible generalizations to K-groups.

7.2. We start with some general observations about $m(P)$. First of all, the fact that the integral in (120) is finite for all non-zero P does need a proof. Here is a sketch. Using Jensen's formula we find, as in (121) that

$$m(P) = m(a_0) + \frac{1}{(2\pi i)^n} \sum_{\nu=1}^{d} \int_{T^{n-1}} \log^+ |\alpha_\nu(y)| \frac{dy}{y}, \tag{122}$$

where $y = (y_1, \cdots, y_{n-1})$, $dy/y = dy_1/y_1 \cdots dy_{n-1}/y_{n-1}$, $\log^+(x) = \max\{\log|x|, 0\}$, and $a_0(y), \alpha_v(y), d$ are the leading coefficient, roots and degree, respectively, of P viewed as a polynomial in x_n. The α_ν's are algebraic functions of $y \in \mathbb{C}^{n-1}$, continuous and piecewise smooth, except at those y's where $a_0(y)$ vanishes (where some will go off to infinity).

We can apply the above procedure to any variable x_n on the torus T^n. It is not hard to see that we may change coordinates in such a way that $a_0(y)$ is actually constant, completing the proof by induction on n.

This last remark can be expanded. Let Δ be the Newton polytope of P; i.e., the convex hull of the exponents $m \in \mathbb{Z}^n$ of monomials $x^m = x_1^{m_1} \cdots x_n^{m_n}$ such that if $P = \sum_{m\in\mathbb{Z}^n} c_m x^m$, then $c_m \neq 0$. We define a *face* τ of Δ as the non-empty intersection of Δ with a half-space in $\mathbb{R}^n$. Chose a parameterization $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ of the affine subspace of smallest dimension containing τ; k is the *dimension* of the face τ. Define $P_\tau = \sum_{m\in\mathbb{Z}^k} c_{\phi(m)} x^m$, a polynomial whose own Newton polytope is $\phi^{-1}(\tau)$. We call P_τ the *face polynomial* associated to the face τ. It depends on a choice of ϕ but note that by changing variables in the integral $m(P_\tau)$ is actually independent of that choice.

It is not hard to see that for any facet (co-dimension 1 face) $\tau \subset \Delta$ we can choose ϕ and system of coordinates in T^n so that, in the notation of (122), $a_0(y) = P_\tau$. By (122) and induction on n we conclude [22] that

$$m(P_\tau) \leq m(P), \qquad \text{for all faces} \quad \tau \subset \Delta\,. \tag{123}$$

In particular, $m(P) \geq 0$ for $0 \neq P \in \mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. Also, since clearly $m(PQ) = m(P) + m(Q)$, we have that

$$m(Q) \leq m(P), \qquad \text{if} \quad Q \mid P, \quad 0 \neq P, Q \in \mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]\,. \tag{124}$$

Though Lehmer's conjecture is about polynomials in one variable, polynomials in more variables are also relevant due to the following result [6]. For any $0 \neq P \in \mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ and $0 \neq (a_1, \dots, a_n) \in \mathbb{Z}^n$ we have

$$\lim_{k\to\infty} m(Q_k) = m(P) \qquad \text{where} \quad Q_k(t) = P(t^{a_1 k}, \dots, t^{a_n k}) \tag{125}$$

That is, there are one variable polynomials Q with $m(Q)$ as close to $m(P)$ as desired. (We should note that (125) is not an immediate consequence of general results about integration but requires a somewhat delicate analysis.)

7.3. Let us go back to polynomials in one variable. If we want to find polynomials $P \in \mathbb{Z}[x]$ with positive but small $m(P)$ (namely $m(P) < \log(2)$ [26, p. 87]), we may as well restrict ourselves to minimal polynomials of algebraic units. Let F be a number field of degree n. Let I be the set of embeddings $\sigma : F \longrightarrow \mathbb{C}$ and V the real vector space of formal linear combinations $\sum_{\sigma\in I} \alpha_\sigma[\sigma]$, where $\alpha_\sigma \in \mathbb{R}$. We have the decomposition $V = V^+ \oplus V^-$, where $V^\pm$ is the subspace of V where complex conjugation acts like ± 1. We let $n_\pm = \dim_{\mathbb{R}} V^\pm$ (in terms of the standard notation $n_+ = r_1 + r_2$ and $n_- = r_2$).

By Dirichlet's theorem the image of the unit group $\mathcal{O}_F^*$ by the log map

$$l_1 : \mathcal{O}_F^* \longrightarrow V, \qquad\qquad \epsilon \mapsto \sum_{\sigma\in I} \log|\epsilon^\sigma|\,[\sigma] \tag{126}$$

is a discrete subgroup $L_1 \subset V$ of rank $r = n^+ - 1$.

On V we define the L^1-norm $\left\| \sum_{\sigma\in I} \alpha_\sigma[\sigma] \right\|_1 = \sum_{\sigma\in I} |\alpha_\sigma|$ and we let

$$\mu_{1,1}(F) = \min_{l\in L_1\setminus\{0\}} ||l||_1$$

(the reason for this indexing will become clear shortly). For any unit $\epsilon \in \mathcal{O}_F^*$ we have $|\mathbb{N}_{F/\mathbb{Q}}(\epsilon)| = 1$, hence $\sum_{\sigma\in I} \log|\epsilon^\sigma| = 0$. Thus, if $P \in \mathbb{Z}[x]$ is the (monic) minimal polynomial of ϵ, we have

$$\|l_1(\epsilon)\|_1 = \frac{2[F:\mathbb{Q}]}{[\mathbb{Q}(\epsilon):\mathbb{Q}]}\, m(P)\,.$$

This simple observation allows us to reformulate Lehmer's conjecture as follows.

Conjecture. *(Lehmer) There exists an absolute constant $\delta_1 > 0$ such that*

$$\mu_{1,1}(F) \geq \delta_1, \qquad \textit{for all number fields } F \textit{ with } r \geq 1. \tag{127}$$

The use of the L^1-norm is important in Lehmer's conjecture, as Siegel [19] showed that there is no positive universal lower bound for the L^2-norm of $l_1(\epsilon)$. Indeed, if $p > 2$ is a prime and ϵ is a root of the (irreducible, non cyclotomic) polynomial $x^p - x + 1$, Siegel proved that $\left(\sum_{\sigma\in I} \log^2 |\sigma(\epsilon)|\right)^{1/2} \leq \sqrt{2}\log(p)/\sqrt{p}$ for $F = \mathbb{Q}(\epsilon)$.

7.4. Bertrand suggested a generalization of Lehmer's conjecture by considering a lower bound on the k-dimensional co-volume $V_k(E)$ of the lattice $\mathrm{LOG}(E)$, where $E \subset \mathcal{O}_F^*$ is any subgroup of $\mathbb{Z}$-rank k and LOG is the traditional logarithmic embedding (2). He suggested the existence of a $c_k > 0$ depending only on k such that $V_k(E) \geq c_k$. Since Siegel's examples show this inequality cannot hold for $k = 1$ (Euclidean length is not the right norm), it is somewhat surprising that this measure of size might work for $k \geq 2$. Nonetheless, Amorososo and David [2] proved Bertrand's conjecture for $k \geq 3$ in 1999. A simpler proof was given by Amoroso and Viada in 2012 [4].

7.5. If Bertrand's inequality needs a switch to the L^1-norm when the rank $k = 1$, at the other extreme (i.e., when $k = \mathrm{rank}(\mathcal{O}_F^*)$) it needs to be strengthened as Zimmert's [28] lower bounds for regulators grow exponentially with the rank of $\mathcal{O}_F^*$. Thus, it makes sense to include Lehmer's conjecture by using an L^1-norm and including exponential growth with the rank of E.

Let V be a vector space over $\mathbb{R}$ of dimension n and $L \subset V$ a discrete subgroup of rank $r \geq 1$. A choice of basis $v_1, \ldots, v_n$ for V determines L^1-norms on $\Lambda^k V$ for $k = 1, \ldots, n$ by

$$\Big\| \sum_{1\leq j_1<\cdots<j_k\leq n} a_{j_1,\ldots,j_k} v_{j_1} \wedge \cdots \wedge v_{j_k} \Big\|_1 = \sum_{1\leq j_1<\cdots<j_k\leq n} |a_{j_1,\ldots,j_k}|\,. \tag{128}$$

If $\ell_i \in V$ and $\omega = \ell_1 \wedge \cdots \wedge \ell_k$ is a pure wedge, let A be the $n \times k$ integral matrix whose i-th column consists of the coordinates of ℓ_i in the basis $v_1, \ldots, v_n$. Then it is easily seen that $\left\|\ell_1 \wedge \cdots \wedge \ell_k\right\|_1 = \sum_{A'} |\det A'|$, where A' runs over all $k \times k$ minors of A. For each $1 \leq k \leq r$ we define (with respect to the chosen basis)

$$\mu_k(L) = \min_{\substack{\omega\in\Lambda^k L\\ \omega\neq 0}} \left\{ \|\omega\|_1 \right\}. \tag{129}$$

Returning to the number field situation of the previous section we define the invariants

$$\mu_{1,k}(F) = \mu_k(L_1)\,,$$

where, as before, $L_1 = l_1(\mathcal{O}_F^*)$ is the image of the units of F under the log map in (126). A version of Bertrand's conjecture that includes Lehmer's conjecture and Zimmert's theorem runs as follows.

Conjecture. *There exist absolute constants $c_0 > 0$ and $c_1 > 1$ such that for all number fields F we have $\mu_{1,k}(F) \geq c_0 c_1^k$ for $1 \leq k \leq \mathbb{Z}-\mathrm{rank}(\mathcal{O}_F^*)$.*

Admittedly, the evidence in favor of this conjecture (see §1) is for the case where only pure wedges $\omega = l_1(\epsilon_1) \wedge \cdots \wedge l_1(\epsilon_k)$ of logarithms of units are allowed in (129), but allowing any non-zero $\omega \in \Lambda^k L_1$ seems more natural.

Since the L^2-norm of $v_1 \wedge \cdots \wedge v_k$ coincides with the co-volume of the lattice generated by the v_i, the weaker inequality (without exponential growth in the rank) $\|\omega\|_1 \geq c_k$ for pure k-wedges is a consequence of Bertrand's conjecture and of the general inequality $\|x\|_1 \geq \|x\|_2$ relating L^1- and L^2-norms.

7.6. We may carry these ideas a little further still. Borel proved (see [7, 10]), generalizing Dirichlet's result for units, that for each $j > 1$ there is a regulator map reg_j

$$l_j : K_{2j-1}(F) \longrightarrow V, \qquad \xi \mapsto \sum_{\sigma \in I} \mathrm{reg}_j(\xi^\sigma)\,[\sigma] \tag{130}$$

whose image is a discrete subgroup L_j of $V^\pm$, with $\pm = (-1)^{j-1}$, of rank $n^\pm$ and covolume related to the value of the zeta function ζ_F of F at $s = j$. Here $K_{2j-1}(F)$ are the K groups defined by Quillen.

We now define for $1 \leq k \leq n_\pm$,

$$\mu_{j,k}(F) = \mu_k(L_j)$$

and we may ask: what is the nature of these invariants, how do they depend on the field F? Does the analogue of Lehmer's conjecture hold? Apart from their formal analogy with Lehmer's question, answers to such questions can be quite useful in practice as we now illustrate.

7.7. For general j, not much is known about the groups $K_{2j-1}(F)$ or the map reg_j. For $j = 2$, however, things can be made quite explicit [27] (and of course $j = 1$ corresponds to the case of units). Indeed, up to torsion, $K_3(F)$ is isomorphic to the *Bloch group* $\mathcal{B}(F)$, defined by generators and relations as follows.

For any field F define

$$\mathcal{A}(F) = \Big\{ \sum_i n_i[z_i] \in \mathbb{Z}[F] \mid \sum_i n_i(z_i \wedge (1 - z_i)) = 0 \Big\},$$

where the corresponding term in the sum is omitted if $z_i = 0, 1$ and

$$\mathcal{C}(F) = \left\{ [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] \mid x, y \in F,\ xy \neq 1 \right\}.$$

It is not hard to check that $\mathcal{C}(F) \subset \mathcal{A}(F)$. Finally, let $\mathcal{B}(F) = \mathcal{A}(F)/\mathcal{C}(F)$.

We recall the definition of the *Bloch–Wigner dilogarithm*. Starting with the usual dilogarithm $\mathrm{Li}_2(z) = \sum_{n=1}^\infty \frac{z^n}{n^2}$, one defines for $|z| < 1$,

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1 - z) \log |z|$$

and checks that it extends to a real analytic function on $\mathbb{C} \setminus \{0,1\}$, continuous on $\mathbb{C}$. See [27] for an account of its many wonderful properties. It is obvious that

$$D(\bar{z}) = -D(z)\,. \tag{131}$$

The 5-term relation satisfied by D guarantees that, extended by linearity to $\mathcal{A}(F)$, it induces a well defined function on $\mathcal{B}(\mathbb{C})$ (still denoted by D).

For $j = 2$, (130) can be formulated as follows

$$l_2 : \mathcal{B}(F) \longrightarrow V, \qquad\qquad \xi \mapsto \sum_{\sigma \in I} D(\xi^\sigma)\,[\sigma]$$

(as (131) makes it clear that the image L_2 lies in V^-) whose image L_2 is a discrete subgroup of rank n^-.

An a priori lower bound for $||l_2(\xi)||_1$ even for the simplest case where L_2 is of rank 1 (namely, for a field with only one complex embedding) would be quite useful. For example, in [9] we find that an identity between the Mahler measure of certain two-variable polynomials is equivalent to the following

$$D(7[\alpha] + [\alpha^2] - 3[\alpha^3] + [-\alpha^4]) = 0, \qquad \alpha = (-3 + \sqrt{-7})/4\,. \tag{132}$$

This was proved by Zagier by showing that it is a consequence of series of 5-term relations. Such calculations, however, can be quite hard and at present there is no known algorithm that is guaranteed to exhibit a given element of $\mathcal{A}(F)$ as lying in $\mathcal{C}(F)$. Clearly if we knew a reasonable lower bound for the possible non-zero values of $|D(\xi)|$ for $\xi \in \mathcal{B}(\mathbb{Q}(\sqrt{-7}))$ a simple numerical verification would be enough to prove (132).

Similarly, many identities between the Mahler measure of certain two-variable polynomials and $\zeta_F(2)$ for a corresponding number field F, which by Borel's theorem are known up to an unspecified rational number, could be proved by a numerical check. For example, as outlined in [9],

$$m(x^2 - 2xy - 2x + 1 - y + y^2) = s\frac{1728^{3/2}}{2^6\pi^7}\zeta_F(2)\,,$$

with $s \in \mathbb{Q}^*$, where F is the splitting field $x^4 - 2x^3 - 2x + 1$, of discriminant -1728. However, though numerically s appears to be equal to 1 we cannot prove this at the moment. Again, a reasonable lower bound on $|D(\xi)|$ for non-torsion elements $\xi \in \mathcal{B}(F')$ for number fields F' would allow us to conclude that $s = 1$ by checking it numerically to high enough precision as both sides of the equality are of the form $|D(\xi)|$ for appropriate ξ's (see below for the right hand side).

There is also some evidence that $\mu_{2,1}(F)$ might be universally bounded below, at least for fields with one complex embedding. Indeed, for a such a field [27] one can construct a hyperbolic three dimensional manifold M by taking the quotient of hyperbolic space by a torsion-free subgroup of the group of units of norm 1 in a quaternion algebra over F ramified at all its real places. Its associated Bloch group element $\xi(M)$, obtained from a triangulation of M into ideal tetrahedra, satisfies $D(\xi(M)) = \mathrm{vol}(M)$. On the other hand, the volume of hyperbolic 3-manifolds is known to be universally bounded below. The question becomes then, that of obtaining an upper bound for the index in $\mathcal{B}(F)$ of the subgroup generated by all such $\xi(M)$. This index is likely to be rather small; in fact, if we accept a precise form of Lichtembaum's conjecture, it should be essentially the order of $K_2(\mathcal{O}_F)$, an analogue of a class group. Unfortunately, there is no known upper bound for $|K_2(\mathcal{O}_F)|$ in terms of, say, the degree and discriminant of F.

Finally, to a hyperbolic 3-manifold M with one cusp one may associate [11] a two variable polynomial $A(x,y) \in \mathbb{Z}[x,y]$, called the *$A$-polynomial* of M. Its zero locus parameterizes deformations of the complete hyperbolic structure of M.

It is known that

$$m(A_\tau) = 0$$

for every face polynomial of A and that A is *reciprocal*, i.e. $A(1/x, 1/y) = x^a y^b A(x,y)$ for some $a, b \in \mathbb{Z}$. It is interesting that these two properties, which have a topological and K-theoretic origin, are, for A irreducible, precisely the known necessary conditions for a polynomial in $\mathbb{Z}[x,y]$ to have small Mahler measure (the first, an analogue of being the minimal polynomial of an algebraic unit, because of (123); the second because $m(P)$ is known to be universally bounded below for P non-reciprocal [21]).

Though the whole picture is still not completely clear yet one can prove for many M's identities of the form

$$2\pi m(A) = \|D(\xi(M))\|_1 \ ,$$

where $\xi(M)$ is the Bloch group element associated to M. This suggests a direct link between Lehmer's conjecture and the size of the invariants $\mu_{2,1}$.

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