STUDENTS' REORGANIZATIONS OF VARIATIONAL, COVARIATIONAL, AND MULTIVARIATIONAL REASONING

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In this paper we examine sixth grade students' constructions and reorganizations of variational, covariational, and multivariational reasoning as they engaged in dynamic digital tasks exploring the science phenomenon of weather. We present case studies of two students from a larger whole-class design experiment to illustrate students' forms of reasoning and the type of design that supported those constructions and reorganizations. We argue that students constructed multivariational relationships by bridging, transforming, and reforming their reasoning and that the nature of the multivariational relationship being constructed affected this process.

Keywords: Algebra and Algebraic Thinking, Design Experiments, Integrated STEM/STEAM

Background

According to the National Research Council (NRC) and Mathematics Learning Study Committee (2001), students "must learn to think mathematically, and they must think mathematically to learn" (p. 1). As the NRC argues, mathematics has facilitated the advancement of science, technology, engineering, business, and government. Mathematics interacts with these disciplines in the form of expressing the variation of multiple quantities. For example, in science, weather forecasters study the variation in air temperatures and dew points to predict the chances of a rainy day. These phenomena usually involve complex relationships between multiple quantities that vary. Although people need to understand this complex variation in many facets of life, school often neglects the study of change in multiple quantities and focuses only on changes in one (variation) or two quantities (covariation). Only one source was found to examine multivariational reasoning, with a focus on undergraduate education (Jones, 2018).

In this paper, we discuss how our project that engaged students in a study of earth and environmental phenomena supported them in reasoning multivariationally. In previous iterations, we found that by engaging with our tasks, simulations, and questioning, students were not only coordinating the change in two quantities but they also reasoned about changes in multiple quantities (e.g., Basu et al., 2020; Panorkou & Germia, 2020a; 2020b; in press). These findings informed our subsequent iterations that aimed to engineer more opportunities to prompt students to study the variation in multiple quantities and reason multivariationally. This paper describes three of those opportunities and discusses how students' thinking progressed from variational, to covariational, and then to multivariational reasoning. Specifically, we explored: 1) How does students' reasoning progress from variation to covariation and then multivariation while engaging with our design? And 2) How does our design support this progression of reasoning?

We use a quantitative reasoning (Thompson, 1994) lens to examine and characterize students' thinking. A quantity is a measurable conceptual attribute that exists in the conception of a situation. Reasoning quantitatively involves constructing the quantities involved in a situation, recognizing which quantities change, and constructing relationships between the changes in pairs of quantities. Thompson and Carlson (2017) define variational reasoning in terms of envisioning "that the quantity's value varies within a setting" (p. 425) while covariational reasoning involves envisioning two quantities' values varying simultaneously.

Our goal was to examine the progression of students' reasoning from variation to covariation and then to multivariation. Because knowledge is dynamically constructed through constructive activity, we aimed to understand how students' meanings about varying quantities could be shaped and reorganized as students interact with our task design, simulations, and questioning. By *meaning*, we refer to "the space of implications that the current understanding mobilizes – actions or schemes that the current understanding implies, that the current understanding brings to mind with little effort" (Thompson et al., 2014, p. 12). By *reorganization* (Piaget, 2001) of students' meanings, we refer to humble inferences we make about their reflections and projections of particular meanings about the quantities and their relationships to a higher conceptual level where these initial meanings become part of a more coherent whole.

Methods

We followed a whole-class design experiment (DE) methodology (Cobb et al., 2003). Our Des were conjecture-driven, in that the research team constructed some initial conjectures about supporting students' quantitative reasoning and these conjectures evolved as the experiment unfolded. In this paper, we present the design of one task focusing on weather, which involves asking students to explore a dynamic simulation and the variation of its quantities.

We designed the Hot Air Balloon simulation to show the relationship between the size of the flame in a hot air balloon, the temperature of the air inside the balloon, the density of that air, and the balloon's altitude. We chose to model a hot air balloon to encourage students to reason about the properties of air masses, such as temperature and density, which can affect how air masses interact to form weather patterns. The student can change the temperature of the air inside the balloon using the "turn flame up" and "turn flame down" buttons. Increasing the size of the flame also increases the temperature of the air inside the balloon, which decreases the density of that air, which increases the balloon's altitude (Figure 1).

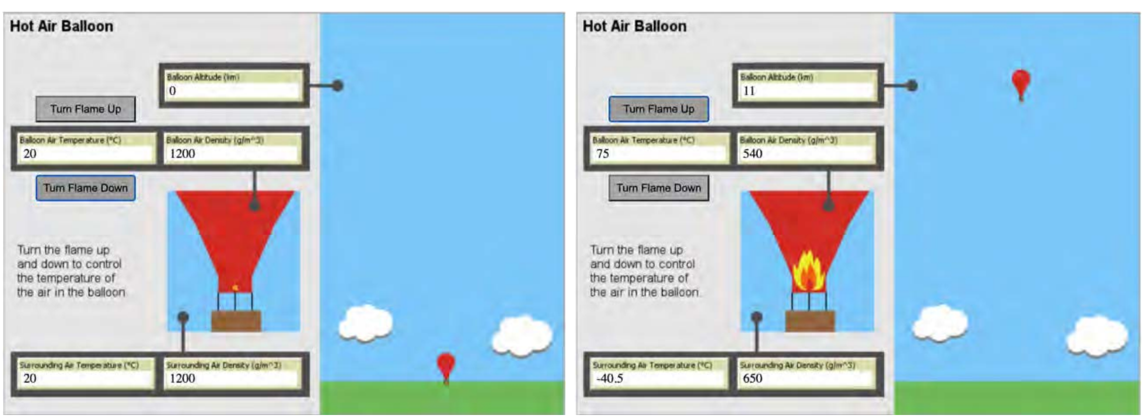

Figure 1: The Hot Air Balloon simulation

We collected data from a sixth-grade classroom from the Northeast of the US. The Des consisted of 15- to 50-minute sessions in which we interviewed the students during their virtual classes in Google Meet. In this paper, we focus on the retrospective analysis (Cobb et al., 2003) of one pair of students, Anne and Violet, to discuss their constructions and reorganizations of variational, covariational, and multivariational reasoning.

Findings

We organize our findings according to how Anne and Violet's constructions and reorganizations took place: by *bridging*, *transforming*, or *reforming*. We also present the type of questioning that might have supported these constructions and reorganizations.

Bridging

Anne and Violet first identified varying quantities as they explore the simulation and its controls. For example, when asked to describe what she noticed in the Hot Air Balloon simulation, Violet clicked to change the flame height and described how altitude, density, and temperature all changed. This showed that Violet constructed variational reasoning about these quantities during her initial explorations of the simulation. Our questioning then turned the students' attention to relationships between these quantities. For example, Anne described relationships between the flame and the balloon's altitude ("when I was turning the flame up, it [the balloon] would like go up") and the flame and the air density ("whenever you turn it [the flame] down, it goes, the density becomes higher"). These excerpts show that she was making connections between pairs of simultaneously changing quantities, thus reorganizing her initial variational reasoning into covariational relationships.

To encourage students to merge the relationships they had reasoned about, we then asked students about the relationships between more than two variables. For example, Anne reasoned, "for the temperature, when you turn it [the flame] down, it gets cooler. And then for the density, it decreases." In this statement, she expressed her reorganization of the covariational relationship she had previously identified into a multivariational envisioning of all three quantities changing at the same time, thereby *bridging* her multiple covariational relationships into a single multivariational relationship.

Transforming

In one case, we observed Violet expanding a single covariational relationship rather than bridging such relationships together in pairs, instead *transforming* one by including new quantities. Violet originally constructed a covariational relationship between her control of the flame and the resulting changes in the balloon's altitude. Then, when we asked her to describe the changes in the density of the air inside the balloon, she clicked to turn the flame up three times and observed, "What happens is that when I go higher [turn up the flame to lift the balloon], the density inside the balloon gets lower." Then, immediately following this, she clicked to turn the flame up three more times and added, "But the temperature goes higher." We interpret her statements to show that she had added two new quantities to her reasoning, thus *transforming* her single covariational relationship between the flame height and altitude by reorganizing it to construct a multivariational relationship in which changes in the flame height resulted in changes in both the density and the temperature, as well.

Reforming

We also observed students *reforming* their multivariational relationships into relationships with different structures after considering more covariational relationships they found during their explorations. The Hot Air Balloon simulation offers a nested multivariational relationship in which changes in one quantity (flame height) affect the next (air temperature), which affects the next (air density), and then the next (balloon altitude) in a nested sequence. Initially, both Violet and Anne constructed multivariational relationships in which changes in the flame height caused simultaneous changes in the simulation's other variables. However, in subsequent DE sessions, both Violet and Anne further considered other covariational relationships in the simulation and then used these to reorganize their multivariational reasoning.

For example, Anne reasoned that "whenever you turn it [the flame] down, it goes, the density becomes higher." Similarly, Violet argued that "the hotter the air inside the balloon is … the more its density decreases." Then, when we asked Violet to explain her reasoning in this statement, she added, "when you turn up the flame, it gets hotter, the density decreases, and it makes the balloon fly up higher." Violet's wording in this excerpt seems to indicate that she had reorganized her reasoning about the multivariational relationship to construct it as a chain of related dependencies, rather than describing a change in one variable causing simultaneous changes in three other variables as she had before. She had *reformed* her multivariational relationship to include her reasoning about the new covariational relationships.

Similarly, Anne first reasoned that "for the temperature, when you turn it [the flame] down, it gets cooler. And then for the density, it decreases," constructing a multivariational relationship in which a change in one variable caused changes in two others. Later, after she had constructed the covariational relationship between temperature and altitude, we again prompted Anne to reason about all of the quantities. She responded, "When I turn up the temperature, the density starts getting low and then altitude, it shows how like the balloon is going up." We consider this excerpt to show that Anne had reorganized her construction of the multivariational relationship into one in which changes in each of the quantities caused a change in the next in sequence, engaging in *reforming* similar to Violet.

Conclusions

Our analysis shows that the simulations provided opportunities for students to see, control, and reason about multiple changing quantities. As we questioned them about the relationships among higher numbers of these quantities, we observed that the students progressed along a trajectory of first constructing variational reasoning and then reorganizing this into covariation and then into multivariation. Specifically, questions about noticing and describing change such as "What is changing in this simulation?" encouraged students to identify variables and reason variationally about individual quantities. Questions about noticing and describing relationships such as "What is the relationship between depth and temperature?" or "What is the relationship between temperature, dew point, and cloud altitude?" then encouraged students to reorganize their thinking first into covariational and later multivariational relationships.

In this paper we have discussed how students engaged in bridging, transforming, and reforming of their reasoning in different multivariational situations. Specifically, students engaged in a *bridging* form of reorganization in which they first constructed two covariational relationships and then merged these into a single multivariational relationship. However, we also saw Violet engage in *transforming* her existing construction of a single covariational relationship into multivariation by reorganizing it to include the addition of new variables. Moreover, both Violet and Anne engaged in *reforming* their initial multivariational reasoning after considering more of the covariational pairs that make up the larger nested relationship in the simulation. This may indicate that the nature of nested relationships has some effect on students' progressions of multivariational reasoning. This shows that students go through different mental actions, and thus different constructions and reorganizations, based on the type of relationship they have to construct. We thus believe that more research is needed on characterizing students' constructions and reorganizations in different types of multivariational situations.

Acknowledgements

This research was supported by the National Science Foundation (#1742125). We thank Anthony Cuviello for his work on the Hot Air Balloon simulation.

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